Zadanie domowe nr 6

Zad. 1 Czy to iloczyny skalarne?

a) $g: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ $\quad g(x,y) = 3x_1y_1 - 5x_1y_2 - 5x_2y_1 + 7x_2y_2$

$x=(x_1,x_2),\ y=(y_1,y_2)$

NIE, bowiem $g(x,x) = 3x_1^2 - 10x_1x_2 + 7x_2^2 = x_1^2 + \underbrace{2(x_1-2x_2)^2}_{2(x_1^2-4x_1x_2+4x_2^2)} - 2x_1x_2 - x_2^2$

$g(1,1) = 1 + 2 - 2 - 1 = 0$

Nieprawda, że $g(x,x) = 0 \iff x=(0,0)$

b) $g: \mathbb{R}_1[x] \times \mathbb{R}_1[x] \to \mathbb{R}$ $\quad g(p,q) = p(1)q(1) + 2p(2)q(2)$

TAK

liniowość na pierwszą zmienną $\quad \forall p_1,p_2,q \in \mathbb{R}_1[x]\ \forall \alpha,\beta \in \mathbb{R}\quad g(\alpha p_1+\beta p_2, q) = (\alpha p_1+\beta p_2)(1)q(1) + 2(\alpha p_1+\beta p_2)(2)\cdot q(2) =$

$= [\alpha\cdot p_1(1)+\beta\cdot p_2(1)]\,q(1) + 2\cdot[\alpha\cdot p_1(2)+\beta\cdot p_2(2)]\,q(2) =$

$= \alpha[p_1(1)q(1) + 2p_1(2)q_2] + \beta[p_2(1)q_1 + 2p_2(2)q(2)] = \alpha\cdot g(p_1,q) + \beta\cdot g(p_2,q)$

symetria $\quad g(q,p) = q(1)p(1) + 2q(2)p(2) = p(1)q(1) + 2p(2)q(2) = g(p,q)$

dodatnia określoność $g(p,p)$ $\quad g(p,p) = [p(1)]^2 + 2\cdot[p(2)]^2 \geq 0$

$g(p,p) = 0 \iff p(1)=0 \wedge p(2)=0 \qquad p = ax+b$

$\begin{cases} p(1) = a+b = 0 \\ p(2) = 2a+b = 0 \end{cases} \Rightarrow \begin{array}{l} a=0 \\ b=-a=0 \end{array} \Rightarrow p \equiv 0$

Zad. 2 Norma

a) $f \in C([0,2\pi], \mathbb{R}),\ f(x) = 2\sin x - \cos x \qquad \forall f,g \in C([0,2\pi],\mathbb{R})\quad \langle f,g\rangle = \int_0^{2\pi} f(x)g(x)\,dx$

$\|f\| = \sqrt{\langle f,f\rangle} = \sqrt{\int_0^{2\pi}(2\sin x - \cos x)^2\,dx} = \ldots$

$\int(2\sin x - \cos x)^2\,dx = \int(4\sin^2 x - \underbrace{4\sin x\cos x}_{2\sin 2x} + \cos^2 x)\,dx = \int(1 + 3\sin^2 x - 2\sin 2x)\,dx = \int\left(1 + \frac{3}{2}(1-\cos 2x) - 2\sin 2x\right)dx$

$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$= \frac{5}{2}\int dx - \frac{3}{2}\int\cos 2x\,dx - 2\int\sin 2x\,dx =$

$= \frac{5}{2}x - \frac{3}{2}\cdot\frac{1}{2}\sin 2x - 2\left(-\frac{1}{2}\cos 2x\right) + C = \frac{5}{2}x - \frac{3}{4}\sin 2x + \cos 2x + C$

$\|f\|^2 = \left[\frac{5}{2}x - \frac{3}{4}\sin 2x + \cos 2x\right]_0^{2\pi} = 5\pi + 1 - 1 = 5\pi \quad\Rightarrow\quad \|f\| = \sqrt{5\pi}$

b) $C \in M_2(\mathbb{R}),\ C = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \qquad \forall A,B \in M_2(\mathbb{R})\quad \langle A,B\rangle = tr(A\cdot B^T)$

$\|C\|^2 = \langle C, C\rangle = tr(C\cdot C^T)$

$$\begin{array}{cc|cc} & & -1 & -1 \\ & & 0 & 1 \\ \hline -1 & 0 & 1 & 1 \\ -1 & 1 & 1 & 2 \end{array}$$

C^T, C, $C\cdot C^T$

$\|C\| = \sqrt{tr(C\cdot C^T)} = \sqrt{1+2} = \sqrt{3}$

Zad. 3 $W = lin\{w_1 = (3,2,0,1,-4),\ w_2 = (1,2,-2,0,1),\ w_3 = (3,-2,6,-2,5)\} \subset \mathbb{R}^5$

Baza $W^\perp$?

$r\begin{bmatrix} 3 & 2 & 0 & 1 & -4 \\ 1 & 2 & -2 & 0 & 1 \\ 3 & -2 & 6 & -2 & 5 \end{bmatrix} \underset{\substack{w_1 - w_2\cdot 3 \\ w_3 - w_2\cdot 3}}{=} r\begin{bmatrix} 0 & -4 & 6 & 1 & -7 \\ 1 & 2 & -2 & 0 & 1 \\ 0 & -8 & 12 & -2 & 2 \end{bmatrix} = r\begin{bmatrix} 1 & 2 & -2 & 0 & 1 \\ 0 & -4 & 6 & 1 & -7 \\ 0 & -8 & 12 & -2 & 2 \end{bmatrix} \underset{w_3 - 2w_2}{=} r\begin{bmatrix} 1 & 2 & -2 & 0 & 1 \\ 0 & -4 & 6 & 1 & -7 \\ 0 & 0 & 0 & -4 & 16 \end{bmatrix} = 3$

$\{w_1, w_2, w_3\}$ baza W

$v \in W^\perp \iff v \perp W \iff \begin{cases} v \perp w_1 \\ v \perp w_2 \\ v \perp w_3 \end{cases} \iff \begin{bmatrix} 3 & 2 & 0 & 1 & -4 \\ 1 & 2 & -2 & 0 & 1 \\ 3 & -2 & 6 & -2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ t \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{cases} x + 2y - 2z + u = 0 \\ -4y + 6z + t - 7u = 0 \\ -4t + 16u = 0 \end{cases}$

$v = (x,y,z,t,u)$

Zad. 3 - ciąg dalszy

$t = 4u \Rightarrow \begin{cases} x + 2y - 2z + u = 0 \\ -4y + 6z - 3u = 0 \end{cases}$ $\quad x = -2y + 2z - u$, $\quad u = -\frac{4}{3}y + 2z$ $\quad x = -2y + 2z + \frac{4}{3}y - 2z = \frac{2}{3}y$, $\quad t = 4u = -\frac{16}{3}y + 8z$

$v \in W^{\perp} \Rightarrow v = \left(\frac{2}{3}y, y, z, -\frac{16}{3}y + 8z, -\frac{4}{3}y + 2z\right)$

$W^{\perp} = \mathrm{lin}\left\{\left(\frac{2}{3}, 1, 0, -\frac{16}{3}, -\frac{4}{3}\right)(0,0,1,8,2)\right\} = \mathrm{lin}\{(2,3,0,-16,-4), (0,0,1,8,2)\}$

BAZA $W^{\perp}$

Zad. 4

$\mathbb{R}_3[x] \quad \forall p, q \in \mathbb{R}_3[x] \quad \langle p, q\rangle = aa_1 + (b-c)(b_1-c_1) + (2c-b)(2c_1-b_1) + dd_1$

$p = ax^3 + bx^2 + cx + d$

$q = a_1x^3 + b_1x^2 + c_1 + d_1$

- $\mathcal{B} = \left(b_1 = 2,\ b_2 = x + x^2,\ b_3 = x + 2x^2,\ b_4 = 3x^3\right)$ baza ortogonalna ?

$C = (1, x, x^2, x^3)$ baza kanoniczna $\mathbb{R}_3[x]$

$$\begin{array}{c} b_1 \\ b_2 \\ b_3 \\ b_4 \\ \\ \end{array} r\begin{array}{c}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \\ \begin{matrix} d & c & b & a \end{matrix}\end{array} \xlongequal{w_3 - w_2} r\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = 4 \Rightarrow \mathcal{B} \text{ - to baza } \mathbb{R}_3[x]$$

$\langle b_1, b_2\rangle = 0 + 0\cdot 0 + 0\cdot 1 + 2\cdot 0 = 0$ $\qquad \langle b_2, b_3\rangle = 0\cdot 0 + 0\cdot 1 + 1\cdot 0 + 0\cdot 0 = 0$

$\langle b_1, b_3\rangle = 0 + 0\cdot 1 + 0\cdot 0 + 2\cdot 0 = 0$ $\qquad \langle b_2, b_4\rangle = 0\cdot 3 + 0\cdot 0 + 1\cdot 0 + 0\cdot 0 = 0$

$\langle b_1, b_4\rangle = 0\cdot 3 + 0\cdot 0 + 0\cdot 0 + 2\cdot 0 = 0$ $\qquad \langle b_3, b_4\rangle = 0\cdot 3 + 1\cdot 0 + 0\cdot 0 + 0\cdot 0 = 0$

baza jest ortogonalna

- $v(x) = x^2 - x + 1$

$v = [1, -1, 1, 0]_C \qquad v = [\alpha, \beta, \gamma, \delta]_{\mathcal{B}}$

$\|b_1\|^2 = 0 + 0 + 0 + 4 \qquad \|b_2\|^2 = 0 + 0 + 1 + 0 = 1 \qquad \|b_3\|^2 = 0 + 1 + 0 + 0 = 1 \qquad \|b_4\|^2 = 9$

$\alpha = \frac{\langle v, b_1\rangle}{\|b_1\|^2} = \frac{1}{4}(0 + 0 + 0 + 2) = \frac{1}{2} \qquad \beta = \frac{\langle v, b_2\rangle}{\|b_2\|^2} = \langle v, b_2\rangle = 0 + 0 + 1\cdot(-3) + 0 = -3$

$\gamma = \frac{\langle v, b_3\rangle}{\|b_3\|^2} = \langle v, b_3\rangle = 0 + 1\cdot 2 + 0 + 0 = 2 \qquad \delta = \frac{\langle v, b_4\rangle}{\|b_4\|^2} = \frac{1}{9}(0 + 0 + 0 + 0) = 0$

$v = \left[\frac{1}{2}, -3, 2, 0\right]_{\mathcal{B}}$

Zad. 5

$\mathbb{R}^4 \supset U = \mathrm{lin}\{b_1 = (1,1,-1,0),\ b_2 = (0,2,-1,1),\ b_3 = (1,5,-3,0)\}$

a) Metoda G-S : $C = (c_1, c_2, c_3)$ - szukana baza ortogonalna

- $c_1 := b_1 = (1,1,-1,0)$
- $c_2 \in \{c_1\}^{\perp} \wedge \mathrm{lin}\{b_1, b_2\} = \mathrm{lin}\langle c_1, c_2\rangle \Rightarrow \begin{cases} c_2 = b_2 + \alpha c_1 \\ c_2 \circ c_1 = 0 \end{cases}$

$c_2 \circ c_1 = (b_2 + \alpha c_1) \circ c_1 = b_2 \circ c_1 + \alpha \cdot \|c_1\|^2 = (0,2,-1,1)\circ(1,1,-1,0) + \alpha\cdot(1+1+1) = (2+1) + 3\alpha = 3 + 3\alpha = 0$, $\underline{\underline{\alpha = -1}}$

$c_2 = b_2 - c_1 = (-1, 1, 0, 1)$

- $c_3 \in \{c_1, c_2\}^{\perp} \wedge \mathrm{lin}\{b_1, b_2, b_3\} = \mathrm{lin}\{c_1, c_2, c_3\} \Rightarrow c_3 = b_3 + \alpha c_1 + \beta c_2 \wedge c_3 \circ c_1 = 0 \wedge c_3 \circ c_2 = 0$

$0 = c_3 \circ c_1 = (b_3 + \alpha c_1 + \beta c_2)\circ c_1 = b_3 \circ c_1 + \alpha\cdot\|c_1\|^2 + \beta\cdot 0 = (1,5,-3,0)\circ(1,1,-1,0) + \alpha\cdot 3 = 1 + 5 + 3 + 3\alpha = 9 + 3\alpha$ $\quad \underline{\alpha = -3}$

$0 = c_3 \circ c_2 = b_3 \circ c_2 + \beta\|c_2\|^2 = (1,5,-3,0)\circ(-1,1,0,1) = -1 + 5 + 3\beta = 4 + 3\beta \Rightarrow \beta = -4/3$

$c_3 = (1,5,-3,0) - 3\cdot(1,1,-1,0) - \frac{4}{3}(-1,1,0,1) = \left(-\frac{2}{3}, \frac{2}{3}, 0, -\frac{4}{3}\right) = \frac{2}{3}(-1,1,0,-1)$

Zad. 5 b) Rzut ortogonalny $v = (1,0,1,0) \in \mathbb{R}^4$ na $\mathcal{U}$

$u = \Pi_{\mathcal{U}}(v) = ?$ $\quad w = v - u \perp \mathcal{U} = \text{lin}\{c_1, c_2, c_3\}$ <=>

$u = [\alpha, \beta, \gamma]_C$ $\quad \alpha = \frac{\langle v, c_1 \rangle}{\|c_1\|^2} = \frac{1}{3} \cdot (1,0,1,0) \circ (1,1,-1,0) = \frac{1}{3} \cdot 0 = 0$

$\beta = \frac{\langle v, c_2 \rangle}{\|c_2\|^2} = \frac{1}{3} \cdot (1,0,1,0) \circ (-1,1,0,1) = -\frac{1}{3}$

$\gamma = \frac{\langle v, c_3 \rangle}{\|c_3\|^2} = \frac{(1,0,1,0) \circ (-1,1,0,-2) \cdot \frac{2}{3}}{(\frac{2}{3})^2 \cdot 6} = (-1) \cdot \frac{3}{2} \cdot \frac{1}{6} = -\frac{1}{4}$

$v = [0, -\frac{1}{3}, -\frac{1}{4}]_C = -\frac{1}{3} c_2 - \frac{1}{4} c_3 = -\frac{1}{3}(-1,1,0,1) - \frac{1}{4}(-1,1,0,-2) \cdot \frac{2}{3} = (\frac{1}{3}, -\frac{1}{3}, 0, -\frac{1}{3}) + (\frac{1}{6}, -\frac{1}{6}, 0, \frac{1}{3})$
$= (\frac{1}{2}, -\frac{1}{2}, 0, 0)$

Zad. 6. (u_1, u_2) uzupełnić do bazy $\mathbb{R}^4$ $\quad u_1 = (1,1,1,0)$, $u_2 = (0,1,-1,1)$

$u_1 \circ u_2 = 0 + 1 - 1 + 0 = 0 \quad u_1 \perp u_2$

$r \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 4 \quad u_3 = e_3,\ u_4 = e_4 \Rightarrow (u_1, u_2, u_3, u_4)$ baza $\mathbb{R}^4$

$u_3 \perp u_4 \quad u_1 \circ u_3 = 1 \neq 0$ nie ortogonalna

$C = (c_1, c_2, c_3, c_4)$ szukana baza ortogonalna

$c_1 = u_1$, $c_2 = u_2$ bo $u_1 \circ u_2 = 0$

$c_3 = u_3 + \alpha c_1 + \beta c_2 \ \wedge\ 0 = c_1 \circ c_3,\ 0 = c_2 \circ c_3$

$0 = c_1 \circ c_3 = c_1 \circ (u_3 + \alpha c_1 + \beta c_2) = c_1 \circ u_3 + \alpha \cdot \|c_1\|^2 = (1,1,1,0) \circ (0,0,1,0) + \alpha \cdot 3 = 1 + 3\alpha$ $\quad \alpha = -\frac{1}{3}$

$0 = c_2 \circ c_3 = c_2 \circ u_3 + \beta \cdot \|c_2\|^2 = (0,1,-1,1) \circ (0,0,1,0) + 3\beta = -1 + 3\beta \Rightarrow \beta = 1/3$

$c_3 = u_3 - \frac{1}{3} c_1 + \frac{1}{3} c_2 = (0,0,1,0) + (-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, 0) + (0, \frac{1}{3}, -\frac{1}{3}, \frac{1}{3}) = (-\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3})$

$c_4 = u_4 + \alpha c_1 + \beta c_2 + \gamma c_3$

$0 = c_4 \circ c_1 = u_4 \circ c_1 + \alpha \cdot \|c_1\|^2 = (0,0,0,1) \circ (1,1,1,0) + 3\alpha = 3\alpha$, $\alpha = 0$

$0 = c_4 \circ c_2 = u_4 \circ c_2 + \beta \cdot \|c_2\|^2 = (0,0,0,1) \circ (0,1,-1,1) + 3\beta = 3\beta + 1$, $\beta = -\frac{1}{3}$

$0 = c_4 \circ c_3 = u_4 \circ c_3 + \gamma \cdot \|c_3\|^2 = (0,0,0,1) \circ (-\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}) + \frac{3}{9}\gamma = \frac{1}{3} + \frac{1}{3}\gamma$, $\gamma = -1$

$\Rightarrow c_4 = u_4 - c_3 - \frac{1}{3} c_2$

$c_4 = (0,0,0,1) - (-\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}) + (0, -\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}) = (\frac{1}{3}, -\frac{1}{3}, 0, \frac{1}{3})$

Zad. 7 $\quad \varphi = \varphi_3 \circ \varphi_1 \circ \varphi_2$ w $\mathbb{R}^2$

φ_1 - obrót o $\frac{\pi}{6}$

φ_2 - odbicie względem prostej tworzącej z Ox kąt $\frac{2}{3}\pi$

φ_3 - obrót o $\frac{7}{6}\pi$

$\varphi_1 \sim A_1 = \begin{bmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$

$\varphi_2 \sim A_2 = \begin{bmatrix} \cos\frac{4}{3}\pi & \sin\frac{4}{3}\pi \\ \sin\frac{4}{3}\pi & \cos\frac{4}{3}\pi \end{bmatrix} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$

$\varphi_3 \sim A_3 = \begin{bmatrix} \cos\frac{7}{6}\pi & -\sin\frac{7}{6}\pi \\ \sin\frac{7}{6}\pi & \cos\frac{7}{6}\pi \end{bmatrix} = \begin{bmatrix} -\sqrt{3}/2 & 1/2 \\ -1/2 & -\sqrt{3}/2 \end{bmatrix}$

$\varphi \sim A = A_3 A_2 A_1 = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$

Zad. 8

$A = \begin{bmatrix} -4/5 & -3/5 \\ -3/5 & 4/5 \end{bmatrix} \in M_2(\mathbb{R})$ $\quad$ Jaką izometrię liniową reprezentuje?

$A \cdot A^T = \frac{1}{5}\begin{bmatrix} -4 & -3 \\ -3 & 4 \end{bmatrix} \cdot \frac{1}{5}\begin{bmatrix} -4 & -3 \\ -3 & 4 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix} = I$ $\quad$ A-ortogonalna => reprezentuje izom. liniową

$\det A = -\frac{16}{25} - \frac{9}{25} = -1$ => A reprezentuje symetrię ortogonalną

Punkty (x,y) na osi symetrii spełniają $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{cases} -4x - 3y = 5x \\ -3x + 4y = 5y \end{cases} \Rightarrow$ oś symetrii $y = -3x$