Fizyka techniczna - Kartkówka 3 30pkt

Zad. 1

a) $\mathcal{U} = \{(x,y,z) \in \mathbb{R}^3 : xy=0 \vee yz=0\}$ Czy to podprz. liniowa $\mathbb{R}^3$

NIE, bowiem $u = (0,1,1) \in \mathcal{U}$ $v = (1,1,0) \in \mathcal{U}$ $u+v = (1,2,1) \notin \mathcal{U}$

$u_x \cdot u_y = 0 \cdot 1 = 0$ $v_y \cdot v_z = 1 \cdot 0 = 0$ $1 \cdot 2 \neq 0$

b) $p(x) = -2ax+1, \quad q(x) = -x^2-4x+1, \quad r(x) = a^2x^2-ax+1 \in \mathbb{R}_2[x]$

Dla jakich $a \in \mathbb{R}$ układ $\{p,q,r\}$ stanowi bazę $\mathbb{R}_2[x]$?

$\mathcal{B}_k = (1,x,x^2)$ baza kanoniczna $\mathbb{R}_2[x]$ $p = [1,-2a,0]_{\mathcal{B}_k}$, $q = [1,-4,-1]_{\mathcal{B}_k}$, $r = [1,-a,a^2]_{\mathcal{B}_k}$

$\dim \mathbb{R}_2[x] = 3 \Rightarrow$ jeśli $\{p,q,r\}$ liniowo niezależny, to tworzy bazę.

$$\det\begin{bmatrix} 1 & -2a & 0 \\ 1 & -4 & -1 \\ 1 & -a & a^2 \end{bmatrix} \underset{\substack{w_2-w_1 \\ w_3-w_1}}{=} \det\begin{bmatrix} 1 & -2a & 0 \\ 0 & 2a-4 & -1 \\ 0 & a & a^2 \end{bmatrix} = \begin{vmatrix} 2a-4 & -1 \\ a & a^2 \end{vmatrix} = (2a-4)a^2 + a = a(\underbrace{2a^2-4a+1})$$

$\Delta = 16-4\cdot2\cdot1 = 8$

$a = \frac{4 \pm 2\sqrt{2}}{4} = 1 \pm \frac{\sqrt{2}}{2}$

$\det A \neq 0 \Leftrightarrow \mathbb{R} \setminus \{0, 1+\frac{\sqrt{2}}{2}, 1-\frac{\sqrt{2}}{2}\} \Rightarrow$ wówczas (p,q,r) lin. niezależne i tworzą bazę.

c) $\varphi : \mathbb{R} \to \mathbb{R}^3, \quad \varphi(x) = (x+\pi, \pi x, -\pi x)$ Czy liniowe?

NIE $\varphi(0) = (\pi,0,0) \neq \mathbb{0} = (0,0,0)$

Zad. 2

$\mathbb{R}^4 : \mathcal{B} = (b_1 = (1,1,1,1),\ b_2 = (1,1,1,0),\ b_3 = (1,1,0,0),\ b_4 = (1,0,0,0)) \leftarrow$ bazy

$\mathbb{R}^3 : \mathcal{C} = (c_1 = (0,-1,0),\ c_2 = (1,0,0),\ c_3 = (1,1,1))$

$\varphi : \mathbb{R}^4 \to \mathbb{R}^3$ liniowe $\quad A' = M_\varphi(\mathcal{B},\mathcal{C}) = \begin{bmatrix} 1 & -2 & -2 & -2 \\ 4 & 3 & 1 & 1 \\ 2 & 0 & 0 & 0 \end{bmatrix}$

a) $v = (5,0,1,0)$ $\varphi(v) = ?$ za pomocą A'

$v = [\alpha,\beta,\gamma,\delta]_{\mathcal{B}}$ $\quad P = P_{\mathcal{B}_k^4 \to \mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ $\quad P \cdot \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $\quad \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = P^{-1} \begin{bmatrix} 5 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $\quad P^{-1} = ?$

$$[P|I] = \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{w_2-w_1 \\ w_3-w_1 \\ w_4-w_1}]{} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & -1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[/\cdot(-1)]{w_2 \leftrightarrow w_4} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[\substack{w_1-w_4 \\ w_2-w_4 \\ w_3-w_4}]{} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 \end{array}\right] \xrightarrow[\substack{w_1-w_3 \\ w_2-w_3}]{} \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 \end{array}\right] \xrightarrow[w_1-w_2]{} \left[\begin{array}{c|cccc} & 0 & 0 & 0 & 1 \\ I_4 & 0 & 0 & 1 & -1 \\ & 0 & 1 & -1 & 0 \\ & 1 & -1 & 0 & 0 \end{array}\right]$$

$\underbrace{\qquad}_{P^{-1}}$

$$P^{-1} \cdot \begin{bmatrix} 5 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 5 \end{bmatrix}_{\mathcal{B}} \qquad A' \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} -10 \\ 7 \\ 0 \end{bmatrix}_{\mathcal{C}}$$

$\varphi(v) = [-10,7,0]_{\mathcal{C}} = -10 \cdot c_1 + 7 \cdot c_2 = 10(0,-1,0) + 7(1,0,0)$

$\varphi(v) = (7, 10, 0)$

b) wzór φ Niech $A = M_\varphi(\mathcal{B}_k^4, \mathcal{B}_k^3)$

$A' = Q^{-1}AP$ $\quad Q = P_{\mathcal{B}_k^3 \to \mathcal{C}}$ $\quad A = QA'P^{-1}$ $\quad Q = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -2 & -2 \\ 4 & 3 & 1 & 1 \\ 2 & 0 & 0 & 0 \end{bmatrix} \cdot P^{-1} = \begin{bmatrix} 6 & 3 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 2 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$A \cdot \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} x+2z+3t \\ 2x-t \\ 2t \end{bmatrix} \qquad \varphi(x,y,z,t) = (x+2z+3t,\ 2x-t,\ 2t)$$

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Zad. 3 $\varphi : \mathbb{R}^3 \to \mathbb{R}^4$ $\varphi(x,y,z) = (5x-y+z,\ x+z,\ y-z,\ -x-y-z)$

a) $\operatorname{Ker}\varphi = ?$ $\varphi(x,y,z)=0 \Leftrightarrow \begin{cases} 5x-y+z=0 \\ x+z=0 \\ y-z=0 \\ -x-y-z=0 \end{cases} \Rightarrow \begin{matrix} x=-z \\ y=z \end{matrix}$

$z-z-z=0 \Rightarrow z=0 \Rightarrow x=y=0$

$\Uparrow$

$\operatorname{Ker}\varphi = \{(0,0,0)\}$

$\dim \operatorname{Ker}\varphi = 0$

$\operatorname{Im}\varphi = ?$ $\dim \operatorname{Im}\varphi = \dim \mathbb{R}^3 - \dim \operatorname{Ker}\varphi = 3-0=3$

b) $\left.\begin{matrix} \operatorname{Ker}\varphi = \{0\} \Rightarrow \varphi\text{- monomorfizm} \\ \operatorname{Im}\varphi \neq \mathbb{R}^4 \Rightarrow \varphi \text{ nie jest epimorfizmem} \end{matrix}\right\} \Rightarrow$ nie jest izomorfizmem

Zad. 4 $A, B \in M_2(\mathbb{R})$ $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 5 & -4 \\ 7 & 1 \end{bmatrix}$ Układ $\{A,B\}$ uzupełnij do bazy $M_2(\mathbb{R})$

$$\begin{bmatrix} 1 & 3 & 2 & 1 \\ 5 & -4 & 7 & 1 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \xrightarrow{w_2-w_1} \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & -19 & -3 & -4 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

(A, B, E_{21}, E_{22}) $E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$