Kartkówka 4 (30 pkt)

Zad. 1 $\varphi: \mathbb{C}^2 \to \mathbb{C}^2$ $\quad \varphi(z,w) = (iz,\ 2z + w - iw)$

Zagadnienie własne:

$A = M_\varphi(\mathcal{B}_k, \mathcal{B}_k)$ $\quad \mathcal{B}_k = \{(1,0),(0,1)\}$ $\quad A = \begin{bmatrix} i & 0 \\ 2 & 1-i \end{bmatrix} \in M_2(\mathbb{C})$

$\chi_A(t) = \det(A - tI) = \begin{vmatrix} i-t & 0 \\ 2 & 1-i-t \end{vmatrix} = (i-t)(1-i-t)$

$Spec(A) = \{\lambda_1 = i,\ \lambda_2 = 1-i\}$ $\quad k_1 = k_2 = 1$

Widmo proste $\Rightarrow$ φ-diagonalizowalny

$E_{\lambda_1} = ?$ $\quad \lambda_1 = i$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = (A - \lambda_1 I)\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 1-2i \end{bmatrix}\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 2z + (1-2i)w \end{bmatrix} \Rightarrow z = \frac{1-2i}{-2} w$

$E_i = \{((-\frac{1}{2} + i)w, w);\ w \in \mathbb{C}\}$

$= lin_{\mathbb{C}}\{(-\frac{1}{2}+i, 1)\}$ $\quad \dim E_i = 1$

$E_{\lambda_2} = ?$ $\quad \lambda_2 = 1-i$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = (A - \lambda_2 I)\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} 2i-1 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} z \\ w \end{bmatrix} \Rightarrow z = 0$, $\quad E_{1-i} = \{(0,w), w \in \mathbb{C}\} = lin_{\mathbb{C}}\{(0,1)\}$, $\dim E_{1-i} = 1$

$P = \begin{bmatrix} -\frac{1}{2}+i & 0 \\ 1 & 1 \end{bmatrix}$ macierz diagonalizująca

$D = \begin{bmatrix} i & 0 \\ 0 & 1-i \end{bmatrix}$

Zad. 2 $f \in End(\mathbb{R}^3)$, $f(1,0,0) = (-3,0,0)$, $f(2,2,0) = (0,0,0)$, $f(0,1,1) = (0,3,3)$

$f^{314}(2,7,2) = ?$ $\quad Spec(f) = \{\lambda_1 = -3,\ \lambda_2 = 0,\ \lambda_3 = 3\}$, $k_1 = 1$, $k_2 = 1$, $k_3 = 1$ $\to$ widmo proste $\to$ f-diagonalizowalny

Baza wektorów własnych $\mathcal{B} = \{v_1 = (1,0,0),\ v_2 = (2,2,0),\ v_3 = (0,1,1)\}$

$P_{\mathcal{B}_k^3 \to \mathcal{B}} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad D = M_f(\mathcal{B},\mathcal{B}) = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $\quad D = P^{-1}AP$

$P^{-1} = ?$

$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{w_2 - w_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[w_2 \cdot \frac{1}{2}]{w_1 - w_2} \left[I \,\middle|\, \overbrace{\begin{array}{ccc} 1 & -1 & 1 \\ 0 & 1/2 & -1/2 \\ 0 & 0 & 1 \end{array}}^{P^{-1}} \right]$

$w = (2,7,2)_{\mathcal{B}_k^3} = [\alpha, \beta, \gamma]_{\mathcal{B}} = ?$

$PX' = X$ $\quad X' = P^{-1} \cdot X = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1/2 & -1/2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 5/2 \\ 2 \end{bmatrix}$

$D^{314} \cdot X' = \begin{bmatrix} (-3)^{314} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3^{314} \end{bmatrix}\begin{bmatrix} -3 \\ 5/2 \\ 2 \end{bmatrix} = \begin{bmatrix} (-3)^{315} \\ 0 \\ 2 \cdot 3^{314} \end{bmatrix}$

$f^{314}(w) = -3^{315} \cdot v_1 + 2 \cdot 3^{314} v_3$

$= -3^{315} \cdot (1,0,0) + 2 \cdot 3^{314}(0,1,1) = (-3^{315}, 2 \cdot 3^{314}, 2 \cdot 3^{314})$ w bazie kanon.

Zad. 3 $A = M_\varphi(\mathcal{B}_k^4, \mathcal{B}_k^4)$ $\quad \varphi \in End(\mathbb{R}^4)$ $\quad A = \begin{bmatrix} 7 & -3 & -1 & 0 \\ 0 & 4 & p^3 & 0 \\ 0 & 0 & 7 & \pi \\ 0 & 0 & 0 & 1 \end{bmatrix}$

a) $p = ?$ by $\dim E_7 = 2$

$Spec(A) = \{1, 4, 7\}$ $\quad \dim E_7 = 4 - \underbrace{r(A - 7I)}_{2} = 2$

$r(A - 7I) = r\begin{bmatrix} 0 & -3 & -1 & 0 \\ 0 & -3 & p^3 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{\pi} \\ 0 & 0 & 0 & -6 \end{bmatrix}$ $\quad$ dla $p^3 = -1$ tj. $p = -1$

b) $B^5 = ?$ $\quad$ tw. C-H

$B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ -1 & -1 & 0 \end{bmatrix}$ $\quad \chi_B(t) = \begin{vmatrix} -t & 0 & 1 \\ 0 & -t & 2 \\ -1 & -1 & -t \end{vmatrix} = -t^3 - t - 2t = -t^3\ 3t \Rightarrow -B^3 - 3B = 0$, $\quad B^3 = -3B$

$B^5 = B^3 \cdot B^2 = (-3B) \cdot B^2 = -3 \cdot B^3 = -3 \cdot (-3B) = 9B = \begin{bmatrix} 0 & 0 & 9 \\ 0 & 0 & 18 \\ -9 & -9 & 0 \end{bmatrix}$

Zad. 4

$\mathbb{R}_2[x] \quad \langle p,q\rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$

$\mathcal{U} = \operatorname{lin}\{\, b_1 = x \quad b_2 = -1-x+x^2 \,\}$

a) $C = (c_1, c_2) \qquad c_1 = b_1 = x$

$c_2 = b_2 + \alpha c_1 \quad \wedge \quad \langle c_2, c_1\rangle = 0$

$0 = \langle b_2 + \alpha c_1, c_1\rangle = \langle b_2, c_1\rangle + \alpha \cdot \|c_1\|^2 = 1\cdot(-1) + (-1)\cdot 0 + (-1)\cdot 1 + \alpha\cdot[1+0+1]$

$2\alpha - 2 = 0 \quad \alpha = 1$

$\underline{c_1 = x} \qquad c_2 = b_2 + c_1 = \underline{-1+x^2}$

b) $r = 5 + x^2 \qquad r = (5,0,1)_{B_k} = [\alpha, \beta]_C$

$\alpha = \frac{\langle r, c_1\rangle}{\|c_1\|^2} = \frac{6\cdot 1 + 5\cdot 0 + 6\cdot 1}{1+0+1} = 6$

$\beta = \frac{\langle r, c_2\rangle}{\|c_2\|^2} = \frac{6\cdot 0 + 5\cdot(-1) + 6\cdot 0}{0\cdot 0 + (-1)\cdot(-1) + 0\cdot 0} = \frac{-5}{1} = -5$

$r = [6, -5]_C$

$r = 6c_1 - 5c_2 = 6\cdot x - 5(-1+x^2) = -5x^2 + 6x + 5$

$= (5, 6, 5)_{B_k}$