Algebra - Zad. domowe nr. 1 - Struktury alg. Liczby zespolone

(1)

a) Czy "o" łączne w zbiorze $\mathbb{Z}$ $\quad \forall x,y \in \mathbb{Z} \quad x \circ y = x + (-1)^x \cdot y$

$x \circ y \in \mathbb{Z} \qquad x \circ y = x \pm y \in \mathbb{Z}$

Dla $x,y,z \in \mathbb{Z}$

$L = (x \circ y) \circ z = (x + (-1)^x \cdot y) \circ z = x + (-1)^x \cdot y + (-1)^{x + (-1)^x \cdot y} \cdot z =$

$= x + (-1)^x \cdot y + (-1)^{x \pm y} \cdot z$

$P = x \circ (y \circ z) = x \circ (y + (-1)^y \cdot z) = x + (-1)^x \cdot (y + (-1)^y \cdot z) =$

$= x + (-1)^x \cdot y + (-1)^{x+y} \cdot z$

$L = P$ bowiem $(-1)^{x+y} = (-1)^{x-y+2y} = (-1)^{x-y} \cdot \underbrace{(-1)^{2y}}_{=1} = (-1)^{x-y}$

TAK ŁĄCZNE

b) Czy $(\mathbb{R} \setminus \{-1\}, \circ)$ to półgrupa / monoid / grupa (przemienna)?

$\forall x,y \in \mathbb{R} \setminus \{\ \} \quad x \circ y := x + y + xy$

"o" wewnętrzne $\quad x \neq -1, y \neq -1 \Rightarrow x + y + xy = x(1+y) + y = x(1+y) + y + 1 - 1$

$= \underbrace{(1+y)}_{\neq 0}\underbrace{(x+1)}_{\neq 0} - 1 \neq -1$

przemienne $\quad x \circ y = x + y + xy = y + x + yx = y \circ x$

łączne $\quad x,y,z \in \mathbb{R} \setminus \{-1\}$

$L = (x \circ y) \circ z = (x + y + xy) \circ z = x + y + xy + z + (x + y + xy) \cdot z =$

$= x + y + z + xy + xz + yz + xyz$

$P = x \circ (y \circ z) = x \circ (y + z + yz) = x + y + z + yz + x \cdot (y + z + yz) =$

$= x + y + z + yz + xy + xz + xyz$

$L = P$

el. neutralny $\quad x \circ e = e \circ x = x$

przemienne! $\quad x + e + xe = x \ , \ e(1+x) = 0 \quad \underline{e = 0}$

$e \in \mathbb{R} \setminus \{-1\}$

(symetryczny)
el. odwrotny $\quad x \circ y = e = 0$

$x + y + xy = 0 \ , \ y(1+x) = -x \ , \ y = \frac{-x}{1+x} \quad x \neq -1$

$y \neq -1$ bo $\frac{-x}{1+x} = -1$, tj. $y \in \mathbb{R} \setminus \{-1\} \Rightarrow \forall x \neq -1 \ \exists y \neq -1 : y \circ x = x \circ y = e$

$\Updownarrow$

$-x = -1 - x$

$0 = -1$ sprzeczność

c) Czy $(\mathbb{R} \setminus \{0\}, \circ)$ to półgrupa / grupa? $\qquad x \circ y := x \cdot |y|$

Działanie jest wewnętrzne $\quad \forall x,y \in \mathbb{R} \quad x \neq 0 \wedge y \neq 0 \Rightarrow x \cdot |y| \neq 0$

Nie jest przemienne $\quad 2 \circ (-1) = 2 \cdot |-1| = 2 \neq (-1) \circ 2 = -1 \cdot |2| = -2$

$\forall x,y,z \in \mathbb{R} \setminus \{0\} \quad L = (x \circ y) \circ z = (x \cdot |y|) \circ z = x \cdot |y| \cdot |z|$

$P = x \circ (y \circ z) = x \circ (y \cdot |z|) = x \cdot |y |z|| = x \cdot |y| |z|$

$L = P \Rightarrow$ działanie jest łączne

Czy istnieje $e \in \mathbb{R} \setminus \{0\}$ t.że $\forall x \in \mathbb{R} \setminus \{0\} \quad x \circ e = e \circ x = x$? $\quad$ NIE! $\rightarrow$ półgrupa nieprzemienna

$\begin{cases} e \circ x = x \\ x \circ e = x \end{cases} \Leftrightarrow \begin{cases} e \cdot |x| = x \\ x \cdot |e| = x \end{cases} \Rightarrow |e| = 1 \ , \ e = \pm 1 \quad \rightarrow$ wynika z (1) $\quad x > 0$ daje 1, $x < 0$ daje -1

② f) analogicznie jak na wykładzie!

e) $9i \cdot z^3 = \bar{z}^5$ $\quad z = re^{i\varphi}$, $r>0$ $\quad \vee \; z=0$ ← jest rozwiązaniem

$$9i\, r^3 e^{3i\varphi} = r^5 e^{-5i\varphi}, \quad r^3\left[9e^{\frac{\pi}{2}i}\cdot e^{3i\varphi} - r^2 e^{-5i\varphi}\right]=0 \;/ : r\neq 0$$

$$9e^{\frac{\pi}{2}i}e^{3i\varphi} = r^2 e^{-5i\varphi}$$

$$9e^{8i\varphi} = r^2 e^{-\frac{\pi}{2}i}$$

$r^2=9 \wedge 8\varphi = -\frac{\pi}{2}+2k\pi$

$r=3$ $\quad \varphi_k = -\frac{\pi}{16} + k\cdot\frac{\pi}{4}$ $\quad k=0,1,\dots,7$

9 rozwiązań: $z=0$, $z_k = 3e^{\varphi_k i}$ $\quad \varphi_k$ j.w.

g) $z^3-3z^2+6z-4=0$

	1	-3	6	-4
1	1	-2	4	0

$z_0=1$

$(z-1)(z^2-2z+4)=0$, $\quad (z-1)\cdot[(z-1)^2+3]=0$

$(z-1)[(z-1)^2-(-3)] = (z-1)[(z-1)^2-(\sqrt{3}i)^2]=0$

$(z-1)(z-1-\sqrt{3}i)(z-1+\sqrt{3}i)$

$z_0=1$ $\quad z_1=1+\sqrt{3}i$ $\quad z_2 = 1-\sqrt{3}i$

a) $z^4 = \frac{i-1}{\sin\frac{\pi}{6}+i\cos\frac{\pi}{6}}$ $\quad z^4=\frac{\sqrt{2}\left(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)}{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}} = \frac{\sqrt{2}\left(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi\right)}{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}$

$$r^4(\cos4\varphi+i\sin4\varphi)=\sqrt{2}\left(\cos\left(\tfrac{3}{4}\pi-\tfrac{\pi}{3}\right)+i\sin\left(\tfrac{3}{4}\pi-\tfrac{\pi}{3}\right)\right)$$

$r^4=\sqrt{2}$, $\quad 4\varphi=\frac{5}{12}\pi+2k\pi$ $\quad k=0,1,2,3$

$r=\sqrt[8]{2}$, $\quad \varphi_k = \frac{5}{48}\pi + k\cdot\frac{\pi}{2}$ $\quad k=0,1,2,3$

$z_k = r\cdot(\cos\varphi_k + i\sin\varphi_k)$ $\quad k=0,1,2,3$

c) $z^3 = (2-2i)^9$

3 rozw. → $z_0 = (2-2i)^3 = 2^3+3\cdot2^2(-2i)+3\cdot2(-2i)^2+(-2i)^3$

$= 8-24i-24\underbrace{-8i^3}_{8i} = -16-16i = -16(1+i)$

$z_1 = z_0\left(\cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi\right) = -16(1+i)\left(-\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right) = -16(1+i)\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$

$= -8(1+i)(\sqrt{3}i-1) = -8(\sqrt{3}i-1-\sqrt{3}-i)$

$= +8(1+\sqrt{3}+i-\sqrt{3}i)$

$z_2 = z_0\left(\cos\frac{4}{3}\pi+i\sin\frac{4}{3}\pi\right) =$

$= -16(1+i)\left(-\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right) = -16(1+i)\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right) = 8(1+i)(i\sqrt{3}+1)$

$= 8(i\sqrt{3}+1-\sqrt{3}+i) = 8(1-\sqrt{3}+i+\sqrt{3}i)$

Zadanie 2

b) $z^3 = \dfrac{(i-1)^7}{(\sqrt{3}-i)^5}$

$i-1 = \sqrt{2}\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = \sqrt{2}\, e^{\frac{3}{4}\pi i}$

$\sqrt{3}-i = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = 2\, e^{-\frac{\pi}{6}i}$

dla $z \neq 0$

$z^3 = r^3 e^{3i\varphi} = \dfrac{\sqrt{2}^7 \cdot e^{\frac{21}{4}\pi i}}{2^5 \cdot e^{-\frac{5}{6}\pi i}} = \dfrac{\sqrt{2}}{4}\, e^{\left(\frac{21}{4}+\frac{5}{6}\right)\pi i} = \dfrac{\sqrt{2}}{4}\, e^{\frac{63+10}{12}\pi i} =$

$= \dfrac{\sqrt{2}}{4}\, e^{\frac{73}{12}\pi i} \Rightarrow r^3 = \frac{\sqrt{2}}{4} = \frac{1}{2\sqrt{2}} = \left(\frac{1}{\sqrt{2}}\right)^3, \quad r = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$3\varphi = \frac{73}{12}\pi + 2k\pi, \quad \varphi_k = \frac{73}{36}\pi + \frac{2}{3}k\pi$

$k \in \{0,1,2\}$

$z_k = \frac{\sqrt{2}}{2} e^{i \cdot \varphi_k}, \; k = \{0,1,2\}$

d) $(iz+2)^4 = (z-4i)^4$

Może się zdawać, że to równanie wielomianowe stopnia 4
Ale widać, że "z^4" się zredukuje
To równanie stopnia 3 → będą 3 rozwiązania!

zał. $z-4i \neq 0$
$z \neq 4i$
gdy $z=4i$ $P=0$
$L=(4i\cdot i+2)^4 \neq 0$
to nie rozwiązanie

$\left(\frac{iz+2}{z-4i}\right)^4 = 1 \qquad w = \frac{iz+2}{z-4i} \qquad w^4=1 \Rightarrow w \in \{1, i, -1, -i\}$

(1) $iz+2 = z-4i$
$(i-1)z = -2-4i$
$z = \frac{2+4i}{1-i} = \frac{(2+4i)(1+i)}{2}$
$z = -1+3i$

(i) $iz+2 = i(z-4i)$
$2 = 4$ ↯

(−i) $iz+2 = -i(z-4i)$
$2iz = -6$
$z = \frac{-3}{i} = 3i$

(−1) $iz+2 = -z+4i$
$(i+1)z = -2+4i$
$z = \frac{-2+4i}{1+i}$
$z = \frac{(-2+4i)(1-i)}{2}$
$z = 1+3i$

(3) Zaznacz na płaszczyźnie zbiory

a) $A = \{z \in \mathbb{C}: \; Im\left(\sqrt{2}\cdot e^{\frac{\pi}{4}i}\right) \leq \left|\frac{1}{2}i\cdot\bar{z} - 2+6i\right| < |\sqrt{3}+i|^2$

$\sqrt{2}\, e^{\frac{\pi}{4}i} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = 1+i \qquad Im\left(\sqrt{2}\cdot e^{\frac{\pi}{4}i}\right) = 1$

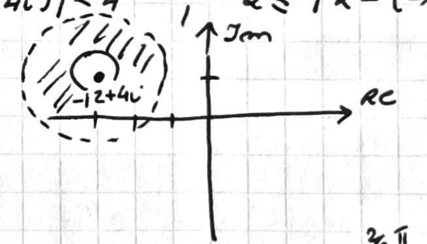
$|\sqrt{3}+i| = \sqrt{3+1} = 2$

tj: $1 \leq \left|\frac{1}{2}i\bar{z} - 2+6i\right| < 4$

$\left|\frac{1}{2}i\bar{z} - 2+6i\right| = \left|\frac{1}{2}i\left(\bar{z} - \frac{4}{i} + 12\right)\right| = \left|\frac{1}{2}\right| \cdot |i| \cdot |\bar{z} + 4i + 12| =$

$= \frac{1}{2}\cdot|\bar{z} + 12 + 4i| = \frac{1}{2}|z + 12 - 4i| = \frac{1}{2}|z - (-12+4i)|$

$1 \leq \frac{1}{2}|z - (-12+4i)| < 4, \qquad 2 \leq |z - (-12+4i)| < 8$



b) $B = \{z \in \mathbb{C}: \; Im(z^6) < 0\}$

$z^6 = r^6(\cos 6\varphi + i\sin 6\varphi)$

$Im(z^6) = \underbrace{r^6}_{>0} \sin 6\varphi < 0 \iff \sin 6\varphi < 0$

$\pi + 2k\pi < 6\varphi < 2\pi + 2k\pi$

$\frac{\pi}{6} + k\cdot\frac{\pi}{3} < \varphi < \frac{\pi}{3} + k\cdot\frac{\pi}{3}$

$k \in \mathbb{Z}$