Algebra - Zadanie domowe nr 6

Zad. 1 Diagonalizacja Baza. $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$

$\varphi(x,y,z) = (x+2y-2z,\ x+3z,\ x+3y)$

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 2 & -2 \\ 1 & -\lambda & 3 \\ 1 & 3 & -\lambda \end{vmatrix} = \lambda^2(1-\lambda) - 6 + 6 - 2\lambda - 9(1-\lambda) + 2\lambda = (1-\lambda)(\lambda^2-9)$$

$\operatorname{Spec}(\varphi) = \{-3, 1, 3\}$ widmo proste $\Rightarrow \varphi$ - diagonalizowalny

• $\lambda_1 = 1$

$(A-\lambda_1)X = 0$

$$(A-I)X = 0 \iff \begin{bmatrix} 0 & 2 & -2 \\ 1 & -1 & 3 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & -2 \\ 1 & -1 & 3 \\ 1 & 3 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 3 \\ 1 & 3 & -1 \\ 0 & 2 & -2 \end{bmatrix} \xrightarrow[w_2-w_1]{} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 4 & -4 \\ 0 & 2 & -2 \end{bmatrix} \xrightarrow[\substack{w_3-w_2 \\ w_2/:4}]{} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[w_1+w_2]{} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} a + 2c = 0 \\ b - c = 0 \end{cases} \quad \begin{aligned} &c = t \\ &b = c = t \\ &a = -2c = -2t \end{aligned} \quad E_1 = \{(-2t, t, t) : t \in \mathbb{R}\} = \operatorname{lin}_{\mathbb{R}}\{(-2,1,1)\}$$

• $\lambda_2 = 3$ $(A-3I)X = 0$

$$\begin{bmatrix} -2 & 2 & -2 \\ 1 & -3 & 3 \\ 1 & 3 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 & -2 \\ 1 & -3 & 3 \\ 1 & 3 & -3 \end{bmatrix} \xrightarrow[w_1:(-2)]{} \begin{bmatrix} 1 & -1 & 1 \\ 1 & -3 & 3 \\ 1 & 3 & -3 \end{bmatrix} \xrightarrow[\substack{w_2-w_1 \\ w_3-w_1}]{} \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 2 \\ 0 & 4 & -4 \end{bmatrix} \xrightarrow[w_2:(-2)]{} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[w_1+w_2]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} a = 0 \\ b - c = 0 \end{cases} \quad b = c \quad E_3 = \{(0, t, t), t \in \mathbb{R}\} = \operatorname{lin}_{\mathbb{R}}\{(0,1,1)\}$$

• $\lambda_3 = -3$ $(A+3I)X = 0$ $\begin{bmatrix} 4 & 2 & -2 \\ 1 & 3 & 3 \\ 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 4 & 2 & -2 \\ 1 & 3 & 3 \\ 1 & 3 & 3 \end{bmatrix} \xrightarrow[\substack{w_3-w_2 \\ w_1/2}]{} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 3 \\ 2 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[w_2-w_1\cdot 2]{} \begin{bmatrix} 1 & 3 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} a + 3b + 3c = 0 \\ -5b - 7c = 0 \end{cases} \quad \begin{aligned} &a = -3b - 3c = \tfrac{21}{5}c - 3c = \tfrac{6}{5}c \\ &b = -\tfrac{7}{5}c \\ &c = t \end{aligned}$$

$$E_{-3} = \{(\tfrac{6}{5}t, -\tfrac{7}{5}t, t), t \in \mathbb{R}\}$$

$$\operatorname{lin}_{\mathbb{R}}\{\tfrac{1}{5}(6, -7, 5)\}$$

baza

$$\mathcal{B} = \{(-2,1,1), (0,1,1), \tfrac{1}{5}(6,-7,5)\}$$

$$M_{\mathcal{B}\mathcal{B}}(\varphi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Zadanie domowe – Diagonalizacja

Zad. 1 b) $\varphi \in End(\mathbb{C}^3)$, $\varphi(z_1, z_2, z_3) = (iz_1 - z_1 + 2z_2 + z_3,\ 2iz_2 + 3z_3,\ iz_3 - z_3)$

$A = M_\varphi(\mathcal{B}_k, \mathcal{B}_k)$
$\mathcal{B}_k = ((1,0,0),(0,1,0),(0,0,1))$

$$A = \begin{bmatrix} i-1 & 2 & 1 \\ 0 & 2i & 3 \\ 0 & 0 & i-1 \end{bmatrix}$$

$\chi_A(t) = \det(A - tI) = (2i - t)(i - 1 - t)^2$

$Spec(A) = \{\lambda_1 = -1+i,\ \lambda_2 = 2i\}$
$k_1 = 2,\ k_2 = 1$

$\lambda_1 = -1+i$ $\quad E_{\lambda_1} = ?$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = (A - \lambda_1 I)\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1+i & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \Rightarrow \begin{array}{l} 2z_2 + z_3 = 0 \\ (1+i)z_2 + 3z_3 = 0 \end{array} \Rightarrow \begin{array}{l} z_3 = -2z_2 \\ (1+i)z_2 - 6z_2 = 0 \\ z_2 = z_3 = 0 \end{array}$$

$E_{-1+i} = \{(z_1, 0, 0);\ z_1 \in \mathbb{C}\} = \mathrm{lin}_{\mathbb{C}}\{(1,0,0)\}$ $\quad \dim E_{-1+i} = 1 \neq k_1 = 2$

Macierz **nie** jest diagonalizowalna

$\lambda_2 = 2i$ $\quad E_{2i} = ?$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = (A - 2iI)\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} -1-i & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & -1-i \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \Rightarrow \begin{array}{l} z_3 = 0 \\ 2z_2 - (1+i)z_1 = 0 \end{array}$$

$E_{2i} = \{(z_1, -\frac{1}{2}(1+i)z_1, 0);\ z_1 \in \mathbb{C}\} = \mathrm{lin}_{\mathbb{C}}\{(1, -\frac{1}{2} - \frac{1}{2}i, 0)\}$ $\quad \dim E_{2i} = 1$
$= \mathrm{lin}_{\mathbb{C}}\{(2, -1-i; 0)\}$

Algebra – Zadanie domowe nr 5

Diagonalizacja endomorfizmu

Zad. 2 $\varphi: \mathbb{R}_2[x] \to \mathbb{R}_2[x]$ $\quad \forall p \in \mathbb{R}_2[x] \quad \varphi(p)(x) = 3xp''(x) + (x^2-1)p(1) + x \cdot p(2)$

$\mathcal{B}_k = (1, x, x)$ baza $\mathbb{R}_2[x]$

$$\varphi(1) = 3x \cdot 0 + (x^2-1) \cdot 1 + x \cdot 1 = x^2 + x - 1 = [-1, 1, 1]_{\mathcal{B}_k}$$
$$\varphi(x) = 3x \cdot 0 + (x^2-1) \cdot 1 + x \cdot 2 = x^2 + 2x - 1 = [-1, 2, 1]_{\mathcal{B}_k}$$
$$\varphi(x^2) = 3x \cdot 2 + (x^2-1) \cdot 1 + x \cdot 4 = x^2 + 10x - 1 = [-1, 10, 1]_{\mathcal{B}_k}$$

$$A = M_\varphi(\mathcal{B}_k, \mathcal{B}_k) = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 2 & 10 \\ 1 & 1 & 1 \end{bmatrix}$$

$$0 = \det(A - tI) = \begin{vmatrix} -1-t & -1 & -1 \\ 1 & 2-t & 10 \\ 1 & 1 & 1-t \end{vmatrix} \underset{\substack{w_1+w_2 \\ w_3-w_2}}{=} \begin{vmatrix} -t & 1-t & 9 \\ 1 & 2-t & 10 \\ 0 & -1+t & -9-t \end{vmatrix} \underset{w_1+w_3}{=} \begin{vmatrix} -t & 0 & -t \\ 1 & 2-t & 10 \\ 0 & t-1 & -9-t \end{vmatrix} =$$

$$= -t(2-t)(-9-t) - t(t-1) - 10(t-1)(-t) = t\left[(2-t)(9+t) - t + 1 + 10t - 10\right] = t\left[18 + 2t - \cancel{9t} - t^2 + \cancel{9t} - 9\right]$$
$$= t\left[-t^2 + 2t + 9\right]$$

$\Delta = 4 + 36 = 40$
$\sqrt{\Delta} = 2\sqrt{10}$

$\mathrm{Spec}(\varphi) = \{0, 1-\sqrt{10}, 1+\sqrt{10}\}$ widmo proste
$\Downarrow$
φ - diagonalizowalny

Zad. 3 Niech $\varphi \in \mathrm{End}(\mathbb{R}^3)$ t.że
$$\varphi(1,0,0) = (1,0,0)$$
$$\varphi(1,1,0) = (-1,-1,0)$$
$$\varphi(1,1,1) = (0,0,0)$$

Oblicz $\varphi^{100}(3,6,9)$.

$\mathrm{Spec}(\varphi) = \{1, -1, 0\}$

Baza wektorów własnych $C = \{c_1 = (1,0,0), c_2 = (1,1,0), c_3 = (1,1,1)\}$

$\mathcal{B}$ - baza kanoniczna

$$P = P_{\mathcal{B} \to C} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{macierz przejścia}$$

$$D = M_{CC}(\varphi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

I SPOSÓB:

$A = M_{BB}(\varphi) \quad A = P \cdot D \cdot P^{-1}, \quad A^{100} = P \cdot D^{100} \cdot P^{-1}$

$P^{-1} = ?$

$$[P|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[w_2-w_3]{w_1-w_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[w_1-w_2]{} \Bigg[I \,\Bigg|\, \underbrace{\begin{matrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{matrix}}_{P^{-1}}\Bigg]$$

$\varphi^{100}(3,6,9) = ?$

$$A^{100} \cdot \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = P \cdot D^{100} \cdot P^{-1} \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{100} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -3 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} -6 \\ -3 \\ 0 \end{bmatrix}$$

$$\varphi^{100}(3,6,9) = (-6,-3,0)$$

II SPOSÓB

$(3,6,9) = [-3,-3,9]_C$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 9 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & -6 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 9 \end{array}\right] \to \left[\begin{array}{c|c} I & \begin{matrix} -3 \\ -3 \\ 9 \end{matrix} \end{array}\right]$$

$$D^{100} \cdot \begin{bmatrix} -3 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -3 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ 0 \end{bmatrix}$$

$$\varphi^{100}(3,6,9) = [-3,-3,0]_C = -3(1,0,0) - 3(1,1,0) = (-6,-3,0)$$

ALGEBRA – Zadanie domowe nr 6 – Diagonalizacja

Zad. 4

a) $A = \begin{bmatrix} 1 & -5 \\ 1 & 1 \end{bmatrix} \in M_2(\mathbb{R})$

$\det(A - tI) = \begin{vmatrix} 1-t & -5 \\ 1 & 1-t \end{vmatrix} = (1-t)^2 + 5 > 0$

brak pierwiastków rzeczywistych

$Spec(A) = \emptyset$ A nie jest diagonalizowalna

b) $A = \begin{bmatrix} 7 & 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$ A - trójkątna górna

$Spec(A) = \{ \lambda_1 = 7,\ \lambda_2 = 1,\ \lambda_3 = 2 \}$, $k_1 = 1$, $k_2 = 3$, $k_3 = 1$

$\dim E_{\lambda_1} = k_1 = 1$

$\dim E_{\lambda_3} = k_3 = 1$

Zaś $\dim E_{\lambda_2} = 5 - r(A - 1 \cdot I) = 5 - 4 = 1 \neq k_2 = 3$

$A - I = \begin{bmatrix} 6 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 3 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$ $r(A - I) = 4$

NIE jest diagonalizowalna

c) $A = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 \end{bmatrix}$ $\lambda_1 = 0 \in Spec(A)$, $k_1 = 3$

$r(A) = 1$ widać!

$\dim E_{\lambda_1} = ?$

$\dim E_{\lambda_1} = 4 - r(A - 0 \cdot I) = 4 - r(A) = 4 - 1 = 3 = k_1$

TAK, diagonalizowalna

d) $A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & p & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ dla jakich p macierz jest diagonalizowalna?

$Spec(A) = \{ \lambda_1 = 5,\ \lambda_2 = 3,\ \lambda_3 = 1 \}$, $k_1 = 2$, $k_2 = 1$, $k_3 = 1$

Na mocy tw. spektralnego, A diagonalizowalna $\iff \dim E_{\lambda_1} = 2$

$\dim E_{\lambda_1} = 4 - r(A - 5I) = 2 \iff r(A - 5I) = 4 - 2 = \underline{\underline{2}}$

$r(A - 5I) = r \begin{bmatrix} 0 & -2 & 6 & -1 \\ 0 & -2 & p & 0 \\ 0 & 0 & 0 & 4 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{-4} \end{bmatrix} \begin{matrix} \\ \\ /:4 \\ \\ \end{matrix} = r \begin{bmatrix} 0 & -2 & 6 & -1 \\ 0 & -2 & p & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = r \begin{bmatrix} -2 & 6 & -1 \\ -2 & p & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{w_2 - w_1}{=} r \begin{bmatrix} -2 & 6 & -1 \\ 0 & p-6 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$r(A - 5I) = \begin{cases} 3 & ; p \neq 6 \\ 2 & ; p = 6 \end{cases}$

A - diagonalizowalna $\iff p = 6$

Zad. 5 PRAWDA / FAŁSZ

$A = \begin{bmatrix} 7 & -3 & -1 & 0 \\ 0 & 4 & p^3 & 0 \\ 0 & 0 & 7 & \pi \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$A = M_\varphi(\mathcal{B}_e^4, \mathcal{B}_e^4)$

- $Spec(A) = \{ \lambda_1 = 7,\ \lambda_2 = 4,\ \lambda_3 = 1 \}$ ($k_1 = 2$, $k_2 = 1$, $k_3 = 1$) $\Rightarrow$ A diagonalizowalna $\iff \dim E_{\lambda_1} = 4 - r(A - 7I) = 2 \iff r(A - 7I) = 2$
- $\det A = 7 \cdot 4 \cdot 7 = 196 \neq 0 \Rightarrow r(A) = r(\varphi) = 4 \Rightarrow \varphi$ surjekcja $\Rightarrow \dim \ker\varphi = 4 - 4 = 0 \Rightarrow \varphi$ injekcja
- $\det(\frac{1}{7} A^T) = \frac{1}{7^2} \cdot 196$

$A - 7I = \begin{bmatrix} 0 & -3 & -1 & 0 \\ 0 & -3 & p^3 & 0 \\ 0 & 0 & 0 & \pi \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$r(A - 7I) = r \begin{bmatrix} 0 & -3 & -1 & 0 \\ 0 & 0 & p^3 + 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{cases} 2 & ; p = -1 \\ 3 & p \neq -1 \end{cases}$ $\longrightarrow$

a) ~~FAŁSZ~~ PRAWDA

b) FAŁSZ

c) PRAWDA

Zadanie domowe – Diagonalizacja

Zad. 7

$a_1 = a_2 = 2$, $a_3 = 4$ $\quad a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ $\quad k=3$

Wzór ogólny?
$$\begin{bmatrix} a_n \\ a_{n-1} \\ a_{n-2}\end{bmatrix} = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix} a_{n-1} \\ a_{n-2} \\ a_{n-3}\end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0\end{bmatrix}}_{A}^{\overset{n-k}{n-3}} \cdot \begin{bmatrix} a_3 \\ a_2 \\ a_1\end{bmatrix}$$

$$\chi_A(t) = \det(A - tI) = \begin{vmatrix} 2-t & 1 & -2 \\ 1 & -t & 0 \\ 0 & 1 & -t\end{vmatrix} = t^2(2-t) - 2 + t = -t^2(t-2) + (t-2) = (t-2)(1-t^2) = (t-2)(1-t)(1+t)$$

$Spec(A) = \{-1, 1, 2\}$ widmo proste $\Rightarrow$ A diagonalizowalna

$\lambda_1 = -1 \quad (A+I)\begin{bmatrix} a \\ b \\ c\end{bmatrix} = \begin{bmatrix} 3 & 1 & -2 \\ 1 & 1 & 0 \\ 0 & 1 & 1\end{bmatrix}\begin{bmatrix} a \\ b \\ c\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0\end{bmatrix} \Rightarrow \begin{matrix} b+c=0 \\ a+b=0\end{matrix} \quad E_{-1} = \{(-b, b, -b)\ b \in \mathbb{R}\} \quad v_1 = (1,-1,1)$

$\lambda_2 = 1 \quad \begin{bmatrix} 0 \\ 0 \\ 0\end{bmatrix} = (A-I)\begin{bmatrix} a \\ b \\ c\end{bmatrix} = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 0 \\ 0 & 1 & -1\end{bmatrix}\begin{bmatrix} a \\ b \\ c\end{bmatrix} \Rightarrow a = b = c \quad E_1 = \{(a,a,a);\ a \in \mathbb{R}\} \quad v_2 = (1,1,1)$

$\lambda_3 = 2 \quad \begin{bmatrix} 0 \\ 0 \\ 0\end{bmatrix} = (A-2I)\begin{bmatrix} a \\ b \\ c\end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 0 \\ 0 & 1 & -2\end{bmatrix}\begin{bmatrix} a \\ b \\ c\end{bmatrix} \Rightarrow \begin{matrix} a - 2b = 0 \\ b - 2c = 0\end{matrix} \quad E_2 = \{(4c, 2c, c);\ c \in \mathbb{R}\} \quad v_3 = (4,2,1)$

$$\begin{bmatrix} a_n \\ a_{n-1} \\ a_{n-2}\end{bmatrix} = P \cdot D^{n-3} \cdot P^{-1}\begin{bmatrix} 4 \\ 2 \\ 2\end{bmatrix} \qquad P = \begin{bmatrix} 1 & 1 & 4 \\ -1 & 1 & 2 \\ 1 & 1 & 1\end{bmatrix} \qquad P^{-1} = ?$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 4 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1\end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 4 & 1 & 0 & 0 \\ 0 & 2 & 6 & 1 & 1 & 0 \\ 0 & 0 & -3 & -1 & 0 & 1\end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1/3 & 0 & 4/3 \\ 0 & 2 & 0 & -1 & 1 & 2 \\ 0 & 0 & -3 & -1 & 0 & 1\end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1/3 & 0 & 4/3 \\ 0 & 1 & 0 & -1/2 & 1/2 & 1 \\ 0 & 0 & 1 & 1/3 & 0 & -1/3\end{array}\right] \to \left[\begin{array}{c|ccc} I_3 & 1/6 & -1/2 & 4/3 \\ & -1/2 & 1/2 & 1 \\ & 1/3 & 0 & -1/3\end{array}\right] \quad \underbrace{}_{P^{-1}}$$

$$P^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & 6 \\ 2 & 0 & -2\end{bmatrix}$$

$$\begin{bmatrix} a_n \\ a_{n-1} \\ a_{n-2}\end{bmatrix} = \begin{bmatrix} 1 & 1 & 4 \\ -1 & 1 & 2 \\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix} (-1)^{n-3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^{n-3}\end{bmatrix}\frac{1}{6}\begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & 6 \\ 2 & 0 & -2\end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 2\end{bmatrix} = \frac{1}{6}\begin{bmatrix} (-1)^{n-3} & 1 & 2^{n-1} \\ (-1)^{n-2} & 1 & 2^{n-2} \\ (-1)^{n-3} & 1 & 2^{n-3}\end{bmatrix}\begin{bmatrix} 2 \\ 6 \\ 4\end{bmatrix} = \left[\ ?\right.$$

$$a_n = \frac{1}{6} \cdot \left[2 \cdot (-1)^{n-3} + 6 + 4 \cdot 2^{n-1}\right] = \frac{1}{3} \cdot (-1)^{n-3} + \frac{1}{3} \cdot 2^n + 1$$

Zad. 8

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 2\end{bmatrix} \qquad A^5 - A^4 = ? \qquad B = \begin{bmatrix} i & 1 \\ 0 & i\end{bmatrix} \qquad B^{77} = ?$$

$$\chi_A(t) = \begin{vmatrix} 2-t & 3 & 1 \\ 0 & -t & 0 \\ 0 & -1 & 2-t\end{vmatrix} = -t(2-t)^2 = -t(4 - 4t + t^2) = -t^3 + 4t^2 - 4t \quad \Rightarrow / -A^3 + 4A^2 - 4A = 0$$

$$\begin{array}{l} \quad t^2 + 3t + 8 \\ \hline (t^5 - t^4) : (t^3 - 4t^2 + 4t) \\ \underline{t^5 - 4t^4 + 4t^3} \\ \quad 3t^4 - 4t^3 \\ \quad \underline{3t^4 - 12t^3 + 12t^2} \\ \qquad 8t^3 - 12t^2 \\ \qquad \underline{8t^3 - 32t^2 + 32t} \\ \qquad\quad 20t^2 - 32t \end{array}$$

$$A^5 - A^4 = 24A^2 - 32A$$

$B^2 - 2iB - I = 0 \ / \cdot B^{-1} \quad \det B \neq 0 \Rightarrow \exists B^{-1}$ policzyć...

$B - 2iI - B^{-1} = 0$

$B^{-1} = \underline{B - 2iI}$

$\chi_B(t) = (i - t)^2 \quad \Rightarrow (i \cdot I - B)^2 = 0 \quad -I - 2iB + B^2 = 0$

$$\begin{cases} t^{77} = q(t) \cdot (i-t)^2 + \overbrace{at + b}^{r(t)} \\ 77t^{76} = q'(t) \cdot (i-t)^2 + q(t) \cdot 2(i-t) \cdot (-1) + a \end{cases}$$

$t = i \quad \begin{cases} i^{77} = ai + b \\ 77i^{76} = a \end{cases} \qquad \begin{matrix} a = (i^4)^{19} \cdot 77 = 77 \\ b = i^{77} - ai = i - 77i = -76i \end{matrix}$

$\underline{B^{77} = 77B - 76iI} = \ldots$ policzyć

Zad. 6

$A \in M_3(\mathbb{C})$, $\operatorname{tr} A = 2$, $\lambda_1 = 7 \in Spec(A)$, $v = (1,2,1)$, $w = (1,1,0)$ odpow. λ_1

v, w – lin. niezależne $\Rightarrow \dim E_7 \geq 2$
$\qquad k_1 \geq 2$

$2 = \operatorname{tr} A = 7 + 7 + \lambda_2 \Rightarrow \lambda_2 = -12$, $k_2 = 1$

$\det A = 7 \cdot 7 \cdot (-12) = -588$