Zadanie domowe nr 6

Zad. 1 Czy to iloczyny skalarne?

a) $g: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ $\quad g(x,y) = 3x_1y_1 - 5x_1y_2 - 5x_2y_1 + 7x_2y_2$

$x=(x_1,x_2),\ y=(y_1,y_2)$

NIE, bowiem $\quad g(x,x) = 3x_1^2 - 10x_1x_2 + 7x_2^2 = x_1^2 + 2(x_1-2x_2)^2 - 2x_1x_2 - x_2^2$

$2(x_1^2 - 4x_1x_2 + 4x_2^2)$

$g(1,1) = 1 + 2 - 2 - 1 = 0$

Nieprawda, że $g(x,x) = 0 \iff x = (0,0)$

b) $g: \mathbb{R}_1[x] \times \mathbb{R}_1[x] \to \mathbb{R}$ $\quad g(p,q) = p(1)q(1) + 2p(2)q(2)$

TAK

liniowość na pierwszą zmienną $\quad \forall p_1,p_2,q \in \mathbb{R}_1[x]\ \forall \alpha,\beta \in \mathbb{R}\ \ g(\alpha p_1 + \beta p_2, q) = (\alpha p_1 + \beta p_2)(1)q(1) + 2(\alpha p_1 + \beta p_2)(2)\cdot q(2) =$

$= [\alpha \cdot p_1(1) + \beta \cdot p_2(1)]\,q(1) + 2\cdot[\alpha \cdot p_1(2) + \beta \cdot p_2(2)]\,q(2) =$

$= \alpha[p_1(1)q(1) + 2p_1(2)q_2] + \beta[p_2(1)q_1 + 2p_2(2)q(2)] = \alpha \cdot g(p_1,q) + \beta \cdot g(p_2,q)$

symetria $\quad g(q,p) = q(1)p(1) + 2q(2)p(2) = p(1)q(1) + 2p(2)q(2) = g(p,q)$

dodatnia określoność $\quad g(p,p)$ $\quad g(p,p) = [p(1)]^2 + 2\cdot[p(2)]^2 \geq 0$

$g(p,p) = 0 \iff p(1) = 0 \wedge p(2) = 0 \qquad p = ax + b$

$\begin{cases} p(1) = a + b = 0 \\ p(2) = 2a + b = 0 \end{cases} \Rightarrow \begin{matrix} a = 0 \\ b = -a = 0 \end{matrix} \Rightarrow p \equiv 0$

Zad. 2 Norma

a) $f \in C([0,2\pi], \mathbb{R}),\ f(x) = 2\sin x - \cos x \qquad \forall f,g \in C([0,2\pi],\mathbb{R})\ \ \langle f,g \rangle = \int_0^{2\pi} f(x)g(x)\,dx$

$\|f\| = \sqrt{\langle f,f \rangle} = \sqrt{\int_0^{2\pi} (2\sin x - \cos x)^2\,dx} = \dots$

$\int (2\sin x - \cos x)^2\,dx = \int (4\sin^2 x - \underbrace{4\sin x\cos x}_{2\sin 2x} + \cos^2 x)\,dx = \int (1 + 3\sin^2 x - 2\sin 2x)\,dx = \int \left(1 + \frac{3}{2}(1-\cos 2x) - 2\sin 2x\right)dx$

$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$= \frac{5}{2}\int dx - \frac{3}{2}\int \cos 2x\,dx - 2\int \sin 2x\,dx =$

$= \frac{5}{2}x - \frac{3}{2}\cdot\frac{1}{2}\sin 2x - 2\left(-\frac{1}{2}\cos 2x\right) + C = \frac{5}{2}x - \frac{3}{4}\sin 2x + \cos 2x + C$

$\|f\|^2 = \left[\frac{5}{2}x - \frac{3}{4}\sin 2x + \cos 2x\right]_0^{2\pi} = 5\pi + 1 - 1 = 5\pi \quad \Rightarrow \quad \|f\| = \sqrt{5\pi}$

b) $C \in M_2(\mathbb{R}),\ C = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \qquad \forall A,B \in M_2(\mathbb{R})\ \ \langle A,B \rangle = tr(A \cdot B^T)$

$\|C\|^2 = \langle C, C \rangle = tr(C \cdot C^T)$

$$\begin{array}{cc|cc} & & -1 & -1 \\ & & 0 & 1 \\ \hline -1 & 0 & 1 & 1 \\ -1 & 1 & 1 & 2 \end{array}$$

C^T, C, $C \cdot C^T$

$\|C\| = \sqrt{tr(C \cdot C^T)} = \sqrt{1+2} = \sqrt{3}$

Zad. 3 $W = lin\{w_1 = (3,2,0,1,-4),\ w_2 = (1,2,-2,0,1),\ w_3 = (3,-2,6,-2,5)\} \subset \mathbb{R}^5$

Baza $W^\perp$?

$r\begin{bmatrix} 3 & 2 & 0 & 1 & -4 \\ 1 & 2 & -2 & 0 & 1 \\ 3 & -2 & 6 & -2 & 5 \end{bmatrix} \underset{\substack{w_1 - w_2\cdot 3 \\ w_3 - w_2 \cdot 3}}{=} r\begin{bmatrix} 0 & -4 & 6 & 1 & -7 \\ 1 & 2 & -2 & 0 & 1 \\ 0 & -8 & 12 & -2 & 2 \end{bmatrix} = r\begin{bmatrix} 1 & 2 & -2 & 0 & 1 \\ 0 & -4 & 6 & 1 & -7 \\ 0 & -8 & 12 & -2 & 2 \end{bmatrix} \underset{w_3 - 2w_2}{=} r\begin{bmatrix} 1 & 2 & -2 & 0 & 1 \\ 0 & -4 & 6 & 1 & -7 \\ 0 & 0 & 0 & -4 & 16 \end{bmatrix} = 3$

$\{w_1, w_2, w_3\}$ baza W

$v \in W^\perp \iff v \perp W \iff \begin{cases} v \perp w_1 \\ v \perp w_2 \\ v \perp w_3 \end{cases} \iff \begin{bmatrix} 3 & 2 & 0 & 1 & -4 \\ 1 & 2 & -2 & 0 & 1 \\ 3 & -2 & 6 & -2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ t \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{cases} x + 2y - 2z + u = 0 \\ -4y + 6z + t - 7u = 0 \\ -4t + 16u = 0 \end{cases}$

$v = (x,y,z,t,u)$

Zad. 3 - ciąg dalszy

$t = 4u \Rightarrow \begin{cases} x + 2y - 2z + u = 0 \\ -4y + 6z - 3u = 0 \end{cases}$ $\quad x = -2y + 2z - u$ $\quad u = -\frac{4}{3}y + 2z$ $\quad x = -2y + \cancel{2z} + \frac{4}{3}y - \cancel{2z} = \frac{2}{3}y$ $\quad t = 4u = -\frac{16}{3}y + 8z$

$v \in W^\perp \Rightarrow v = \left(\frac{2}{3}y, y, z, -\frac{16}{3}y + 8z, -\frac{4}{3}y + 2z\right)$

$W^\perp = \text{lin}\{(\frac{2}{3}, 1, 0, -\frac{16}{3}, -\frac{4}{3})\ (0, 0, 1, 8, 2)\} = \text{lin}\{(2, 3, 0, -16, -4), (0, 0, 1, 8, 2)\}$

BAZA $W^\perp$

Zad. 4

$\mathbb{R}_3[x] \qquad \forall p, q \in \mathbb{R}_3[x] \quad \langle p, q\rangle = aa_1 + (b-c)(b_1-c_1) + (2c-b)(2c_1-b_1) + dd_1$

$p = ax^3 + bx^2 + cx + d$
$q = a_1x^3 + b_1x^2 + c_1 + d_1$

- $\mathcal{B} = \left(b_1 = 2,\ b_2 = x + x^2,\ b_3 = x + 2x^2,\ b_4 = 3x^3\right)$ baza ortogonalna ?

$C = (1, x, x^2, x^3)$ baza kanoniczna $\mathbb{R}_3[x]$

$$\begin{array}{c c} \begin{array}{c} b_1 \\ b_2 \\ b_3 \\ b_4 \end{array} & r\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \\ & \begin{array}{cccc} d & c & b & a \end{array} \end{array} \xlongequal{w_3 - w_2} r\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = 4 \Rightarrow \mathcal{B} \text{ - to baza } \mathbb{R}_3[x]$$

$\langle b_1, b_2\rangle = 0 + 0\cdot 0 + 0\cdot 1 + 2\cdot 0 = 0$
$\langle b_1, b_3\rangle = 0 + 0\cdot 1 + 0\cdot 0 + 2\cdot 0 = 0$
$\langle b_1, b_4\rangle = 0\cdot 3 + 0\cdot 0 + 0\cdot 0 + 2\cdot 0 = 0$

$\langle b_2, b_3\rangle = 0\cdot 0 + 0\cdot 1 + 1\cdot 0 + 0\cdot 0 = 0$
$\langle b_2, b_4\rangle = 0\cdot 3 + 0\cdot 0 + 1\cdot 0 + 0\cdot 0 = 0$
$\langle b_3, b_4\rangle = 0\cdot 3 + 1\cdot 0 + 0\cdot 0 + 0\cdot 0 = 0$

baza jest ortogonalna

- $v(x) = x^2 - x + 1$

$v = [1, -1, 1, 0]_C \qquad v = [\alpha, \beta, \gamma, \delta]_\mathcal{B}$

$\|b_1\|^2 = 0 + 0 + 0 + 4 \qquad \|b_2\|^2 = 0 + 0 + 1 + 0 = 1 \qquad \|b_3\|^2 = 0 + 1 + 0 + 0 = 1 \qquad \|b_4\|^2 = 9$

$\alpha = \frac{\langle v, b_1\rangle}{\|b_1\|^2} = \frac{1}{4}(0 + 0 + 0 + 2) = \frac{1}{2} \qquad \beta = \frac{\langle v, b_2\rangle}{\|b_2\|^2} = \langle v, b_2\rangle = 0 + 0 + 1\cdot(-3) + 0 = -3$

$\gamma = \frac{\langle v, b_3\rangle}{\|b_3\|^2} = \langle v, b_3\rangle = 0 + 1\cdot 2 + 0 + 0 = 2 \qquad \delta = \frac{\langle v, b_4\rangle}{\|b_4\|^2} = \frac{1}{9}(0 + 0 + 0 + 0) = 0$

$v = [\frac{1}{2}, -3, 2, 0]_\mathcal{B}$

Zad. 5

$\mathbb{R}^4 \supset U = \text{lin}\{b_1 = (1, 1, -1, 0),\ b_2 = (0, 2, -1, 1),\ b_3 = (1, 5, -3, 0)\}$

a) Metoda G-S : $C = (c_1, c_2, c_3)$ - szukana baza ortogonalna

- $c_1 := b_1 = (1, 1, -1, 0)$
- $c_2 \in \{c_1\}^\perp \wedge \text{lin}\{b_1, b_2\} = \text{lin}\langle c_1, c_2\rangle \Rightarrow \begin{cases} c_2 = b_2 + \alpha c_1 \\ c_2 \circ c_1 = 0 \end{cases}$

$c_2 \circ c_1 = (b_2 + \alpha c_1)\circ c_1 = b_2 \circ c_1 + \alpha\cdot\|c_1\|^2 = (0, 2, -1, 1)\circ(1, 1, -1, 0) + \alpha\cdot(1 + 1 + 1) = (2 + 1) + 3\alpha - 3 + 3\alpha = 0,\ \underline{\underline{\alpha = -1}}$

$c_2 = b_2 - c_1 = (-1, 1, 0, 1)$

- $c_3 \in \{c_1, c_2\}^\perp \wedge \text{lin}\{b_1, b_2, b_3\} = \text{lin}\{c_1, c_2, c_3\} \Rightarrow c_3 = b_3 + \alpha c_1 + \beta c_2 \wedge c_3 \circ c_1 = 0 \wedge c_3 \circ c_2 = 0$

$0 = c_3 \circ c_1 = (b_3 + \alpha c_1 + \beta c_2)\circ c_1 = b_3 \circ c_1 + \alpha\cdot\|c_1\|^2 + \beta\cdot 0 = (1, 5, -3, 0)\circ(1, 1, -1, 0) + \alpha\cdot 3 = 1 + 5 + 3 + 3\alpha = 9 + 3\alpha$

$\underline{\alpha = -3}$

$0 = c_3 \circ c_2 = b_3 \circ c_2 + \beta\|c_2\|^2 = (1, 5, -3, 0)\circ(-1, 1, 0, 1) = -1 + 5 \overset{+3\beta}{=} 4 + 3\beta \Rightarrow \beta = -4/3$

$c_3 = (1, 5, -3, 0) - 3\cdot(1, 1, -1, 0) - \frac{4}{3}(-1, 1, 0, 1) = (-\frac{2}{3}, \frac{2}{3}, 0, -\frac{4}{3}) = \frac{2}{3}(-1, 1, 0, -1)$

Zad. 5 b) Rzut ortogonalny $v=(1,0,1,0)\in\mathbb{R}^4$ na U

$u=\pi_U(v)=?$ $\qquad w=v-u\perp U=\text{lin}\{c_1,c_2,c_3\}\iff$

$u=[\alpha,\beta,\gamma]_C$ $\qquad \alpha=\dfrac{\langle v,c_1\rangle}{\|c_1\|^2}=\dfrac{1}{3}\cdot(1,0,1,0)\circ(1,1,-1,0)=\dfrac{1}{3}\cdot 0=0$

$\beta=\dfrac{\langle v,c_2\rangle}{\|c_2\|^2}=\dfrac{1}{3}\cdot(1,0,1,0)\circ(-1,1,0,1)=-\dfrac{1}{3}$

$\gamma=\dfrac{\langle v,c_3\rangle}{\|c_3\|^2}=\dfrac{(1,0,1,0)\circ(-1,1,0,-2)\cdot\frac{2}{3}}{\left(\frac{2}{3}\right)^2\cdot 6}=(-1)\cdot\dfrac{3}{2}\cdot\dfrac{1}{6}=-\dfrac{1}{4}$

$v=[0,-\frac{1}{3},-\frac{1}{4}]_C=-\frac{1}{3}c_2-\frac{1}{4}c_3=-\frac{1}{3}(-1,1,0,1)-\frac{1}{\cancel{4}_2}(-1,1,0,-2)\cdot\frac{\cancel{2}}{3}=\left(\frac{1}{3},-\frac{1}{3},0,-\frac{1}{3}\right)+\left(\frac{1}{6},-\frac{1}{6},0,\frac{1}{3}\right)$

$=\underline{\left(\frac{1}{2},-\frac{1}{2},0,0\right)}$

Zad. 6. (u_1,u_2) uzupełnić do bazy $\mathbb{R}^4$ $\qquad u_1=(1,1,1,0),\ u_2=(0,1,-1,1)$

$u_1\circ u_2=0+1-1+0=0 \qquad u_1\perp u_2$

$r\begin{bmatrix}1&1&1&0\\0&1&-1&1\\0&0&1&0\\0&0&0&1\end{bmatrix}=4 \qquad u_3=e_3,\ u_4=e_4 \Rightarrow (u_1,u_2,u_3,u_4)$ baza $\mathbb{R}^4$

$u_3\perp u_4 \qquad u_1\circ u_3=1\neq 0$ nie ortogonalna

$C=(c_1,c_2,c_3,c_4)$ szukana baza ortogonalna

$c_1=u_1,\ c_2=u_2$ bo $u_1\circ u_2=0$

$c_3=u_3+\alpha c_1+\beta c_2 \ \wedge\ 0=c_1\circ c_3,\ 0=c_2\circ c_3$

$0=c_1\circ c_3=c_1\circ(u_3+\alpha c_1+\beta c_2)=c_1\circ u_3+\alpha\cdot\|c_1\|^2=(1,1,1,0)\circ(0,0,1,0)+\alpha\cdot 3=1+3\alpha \qquad \alpha=-\frac{1}{3}$

$0=c_2\circ c_3=c_2\circ u_3+\beta\cdot\|c_2\|^2=(0,1,-1,1)\circ(0,0,1,0)+3\beta=-1+3\beta \Rightarrow \beta=1/3$

$c_3=u_3-\frac{1}{3}c_1+\frac{1}{3}c_2=(0,0,1,0)+(-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},0)+(0,\frac{1}{3},-\frac{1}{3},\frac{1}{3})=(-\frac{1}{3},0,\frac{1}{3},\frac{1}{3})$

$c_4=u_4+\alpha c_1+\beta c_2+\gamma c_3$

$0=c_4\circ c_1=u_4\circ c_1+\alpha\cdot\|c_1\|^2=(0,0,0,1)\circ(1,1,1,0)+3\alpha=3\alpha,\ \alpha=0$

$0=c_4\circ c_2=u_4\circ c_2+\beta\cdot\|c_2\|^2=(0,0,0,1)\circ(0,1,-1,1)+3\beta=3\beta+1,\ \beta=-\frac{1}{3}$

$0=c_4\circ c_3=u_4\circ c_3+\gamma\cdot\|c_3\|^2=(0,0,0,1)\circ(-\frac{1}{3},0,\frac{1}{3},\frac{1}{3})+\frac{3}{9}\gamma=\frac{1}{3}+\frac{1}{3}\gamma,\ \gamma=-1$

$\Rightarrow c_4=u_4-c_3-\frac{1}{3}c_2$

$c_4=(0,0,0,1)-(-\frac{1}{3},0,\frac{1}{3},\frac{1}{3})+(0,-\frac{1}{3},\frac{1}{3},-\frac{1}{3})=(\frac{1}{3},-\frac{1}{3},0,\frac{1}{3})$

Zad. 7 $\quad\varphi=\varphi_3\circ\varphi_1\circ\varphi_2$ w $\mathbb{R}^2$

φ_1 – obrót o $\frac{\pi}{6}$ ↺

φ_2 – odbicie względem prostej tworzącej z Ox kąt $\frac{2}{3}\pi$

φ_3 – obrót o $\frac{7}{6}\pi$ ↺

$\varphi_1\sim A_1=\begin{bmatrix}\cos\frac{\pi}{6}&-\sin\frac{\pi}{6}\\ \sin\frac{\pi}{6}&\cos\frac{\pi}{6}\end{bmatrix}=\begin{bmatrix}\sqrt{3}/2&-1/2\\1/2&\sqrt{3}/2\end{bmatrix}$

$\varphi_2\sim A_2=\begin{bmatrix}\cos\frac{4}{3}\pi&\sin\frac{4}{3}\pi\\ \sin\frac{4}{3}\pi&\cos\frac{4}{3}\pi\end{bmatrix}=\begin{bmatrix}-1/2&-\sqrt{3}/2\\-\sqrt{3}/2&-1/2\end{bmatrix}$

$\varphi_3\sim A_3=\begin{bmatrix}\cos\frac{7}{6}\pi&-\sin\frac{7}{6}\pi\\ \sin\frac{7}{6}\pi&\cos\frac{7}{6}\pi\end{bmatrix}=\begin{bmatrix}-\sqrt{3}/2&1/2\\-1/2&-\sqrt{3}/2\end{bmatrix}$

$\varphi\sim A=A_3A_2A_1=\begin{bmatrix}1/2&\sqrt{3}/2\\ \sqrt{3}/2&-1/2\end{bmatrix}$

Zad. 8

$A=\begin{bmatrix}-4/5&-3/5\\-3/5&4/5\end{bmatrix}\in M_2(\mathbb{R})$ $\qquad$ Jaką izometrię liniową reprezentuje?

$A\cdot A^T=\frac{1}{5}\begin{bmatrix}-4&-3\\-3&4\end{bmatrix}\cdot\frac{1}{5}\begin{bmatrix}-4&-3\\-3&4\end{bmatrix}=\frac{1}{25}\begin{bmatrix}25&0\\0&25\end{bmatrix}=I$ $\qquad$ A-ortogonalna $\Rightarrow$ reprezentuje izom. liniową

$\det A=-\frac{16}{25}-\frac{9}{25}=-1 \Rightarrow A$ reprezentuje symetrię ortogonalną

Punkty (x,y) na osi symetrii spełniają $\quad A\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\y\end{bmatrix}\Rightarrow\begin{cases}-4x-3y=5x\\-3x+4y=5y\end{cases}\Rightarrow$ oś symetrii $y=-3x$

II SPOSOB $\quad u = \Pi_{\mathcal{U}}(v)$ rzut v na $\mathcal{U}$, $w = v - u \perp \mathcal{U}$

$w = ?$ $\quad v = (1,0,1,0) \quad u \in \mathcal{U}$, $u = \alpha \cdot b_1 + \beta \cdot b_2 + \gamma \cdot b_3$

$= (\alpha + \gamma, \alpha + 2\beta + 5\gamma, -\alpha - \beta - 3\gamma, \beta)$

$w = (1 - \alpha - \gamma, -\alpha - 2\beta - 5\gamma, 1 + \alpha + \beta + 3\gamma, -\beta)$

$0 = b_1 \circ w = (1,1,-1,0) \circ w = \cancel{1} - \alpha - \gamma - \alpha - 2\beta - 5\gamma - \cancel{1} - \alpha - \beta - 3\gamma = -3\alpha - 3\beta - 9\gamma \Rightarrow \alpha + \beta + 3\gamma = 0$

$0 = b_2 \circ w = (0,2,-1,1) \circ w = -2\alpha - 4\beta - 10\gamma - 1 - \alpha - \beta - 3\gamma - \beta = -1 - 3\alpha - 6\beta - 13\gamma \Rightarrow 3\alpha + 6\beta + 13\gamma = -1$

$0 = b_3 \circ w = (1,5,-3,0) \circ w = 1 - \alpha - \gamma - 5\alpha - 10\beta - 25\gamma - 3 - 3\alpha - 3\beta - 9\gamma = -2 - 9\alpha - 13\beta - 35\gamma \quad 9\alpha + 13\beta + 35\gamma = -2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 3 & 6 & 13 & -1 \\ 9 & 13 & 35 & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 3 & 4 & -1 \\ 0 & 4 & 8 & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 4/3 & -1/3 \\ 0 & 2 & 4 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 4/3 & -1/3 \\ 0 & 0 & 4/3 & -1/3 \end{array}\right]$$

$\frac{4}{3}\gamma = -\frac{1}{3}$, $\gamma = -\frac{1}{4}$

$\beta = -\frac{1}{3} - \frac{4}{3}\gamma = -\frac{1}{3} + \frac{1}{3} = 0$

$\alpha = -\beta - 3\gamma = -3 \cdot (-\frac{1}{4}) = \frac{3}{4}$

$u = [\alpha, \beta, \gamma]_{\mathcal{B}} = [\frac{3}{4}, 0, -\frac{1}{4}]_{\mathcal{B}} =$

$= \frac{3}{4}(1,1,-1,0) - \frac{1}{4}(1,5,-3,0) = \frac{1}{4}(2,-2,0,0) = \underline{(\frac{1}{2}, -\frac{1}{2}, 0, 0)}$