Fizyka techniczna – MALF – Kartkówka 1 – 2024r. | 20pkt

Zad.1 Czy to (pół)grupa (przemienna)?

$(A,\circ)$ $\quad A=(\mathbb{R}\setminus\{0\})\times\mathbb{Z}=\{(a,b):\ a\in\mathbb{R},\ a\neq 0,\ b\in\mathbb{Z}\}$

$\forall (a,b),(c,d)\in A \quad (a,b)\circ(c,d)=(ac,\ b+d)$

- działanie wewnętrzne, bowiem $a,c\in\mathbb{R}\setminus\{0\}\Rightarrow ac\in\mathbb{R}\setminus\{0\}$; $b,d\in\mathbb{Z}\Rightarrow b+d\in\mathbb{Z}$
- przemienne $ac=ca \wedge b+d=d+b \Rightarrow (a,b)\circ(c,d)=(c,d)\circ(a,b)\quad \forall (a,b),(c,d)\in A$
- łączne: Niech $(a,b),(c,d),(u,v)\in A$ dowolne

$L=[(a,b)\circ(c,d)]\circ(u,v)=(ac,b+d)\circ(u,v)=(acu,\ b+d+v)$

$P=(a,b)\circ[(c,d)\circ(u,v)]=(a,b)\circ(cu,d+v)=(acu,b+d+v)$

$L=P$

- element neutralny?

$(a,b)\circ(e_1,e_2)=(a,b) \Leftrightarrow (ae_1,b+e_2)=(a,b) \Rightarrow e_1=1,\ e_2=0$

$(e_1,e_2)\in A$ bowiem $e_1=1\in\mathbb{R}\setminus\{0\}$, $e_2=0\in\mathbb{Z}$

$\forall (a,b)\in A \quad (a,b)\circ(1,0)=(a,b)$ ponadto $(1,0)\in A$, bowiem $1\neq 0,\ 0\in\mathbb{Z}$

zatem $(1,0)$ to el. neutralny

- el. symetryczny do $(a,b)\in A$

$(a,b)\circ(a',b')=(1,0) \Leftrightarrow (aa',b+b')=(1,0) \Rightarrow \begin{cases} aa'=1 \\ b+b'=0\end{cases} \Rightarrow \begin{matrix} a'=1/a \\ b'=-b\end{matrix}$ $\quad$ z zał. $a\neq 0$; $b\in\mathbb{Z}\Rightarrow -b\in\mathbb{Z}$

$\forall (a,b)\in A\ \exists (a',b')=(\frac{1}{a},-b)\in A$

bowiem $a\neq 0 \Rightarrow \frac{1}{a}\neq 0$, $b\in\mathbb{Z}\Rightarrow -b\in\mathbb{Z}$ $\quad$ WNIOSEK: grupa przemienna

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Zad. 2 Rozwiąż $\quad i^3\bar{z}\cdot z^6=\mathrm{Im}(e^{\frac{\pi}{2}i})\cdot\frac{1+i}{1-i}\cdot\frac{(\sqrt{12}-2i)^{15}}{(-\sin\frac{\pi}{5}+i\cos\frac{\pi}{5})^4}$

$i^3=-i$

$\mathrm{Im}(e^{\frac{\pi}{2}i})=\mathrm{Im}(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})=\sin\frac{\pi}{2}=1$

$\frac{1+i}{1-i}=\frac{(1+i)^2}{1^2-i^2}=\frac{1+2i-1}{2}=i$

$w=\sqrt{12}-2i \quad |w|=\sqrt{12+4}=4 \quad w=4\left(\frac{2\sqrt{3}}{4}-\frac{2i}{4}\right)=4\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)=4e^{-\frac{\pi}{6}i}$

$u=-\sin\frac{\pi}{5}+i\cos\frac{\pi}{5}=1\cdot\left(\cos(\frac{\pi}{2}+\frac{\pi}{5})+i\sin(\frac{\pi}{2}+\frac{\pi}{5})\right)=e^{\frac{7}{10}\pi i}$

$z=0$ nie spełnia równania

$z\neq 0$

$z=|z|e^{\varphi i}$

Wstawiając do równania $\quad \underbrace{-\cancel{i}}_{e^{\pi i}}|z|e^{-\varphi i}|z|^6e^{6\varphi i}=1\cdot\cancel{i}\cdot\frac{4^{15}e^{-\frac{15}{6}\pi i}}{e^{\frac{28}{10}\pi i}}$

Stąd $\quad |z|^7\cdot e^{5\varphi i}=4^{15}e^{-\pi i(1+\frac{5}{2}+\frac{14}{5})}$ $\qquad 1+\frac{5}{2}+\frac{14}{5}=\frac{10+25+28}{10}=\frac{63}{10}$

$\Rightarrow |z|^7=4^{15} \quad\wedge\quad 5\varphi=-\frac{63}{10}\pi+2k\pi$

$|z|=\sqrt[7]{4^{15}} \qquad \varphi_k=-\frac{63}{50}\pi+\frac{2}{5}k\pi \quad k\in\{0,1,2,3,4\}$

Rozw. $\quad z_k=\sqrt[7]{4^{15}}e^{\varphi_k i}$ $\quad k\in\{0,1,2,3,4\}$

Zad. 3 Zaznacz na płaszczyźnie $\quad A=\{z\in\mathbb{C}:\ \mathrm{Im}(z^4)<0 \wedge |z|<|z+2i|\}$

- dla $z=0 \quad \mathrm{Im}(z)\not<0$

Zał. $z\neq 0 \quad z=|z|(\cos\varphi+i\sin\varphi)$

$z^4=|z|^4(\cos 4\varphi+i\sin 4\varphi)$

$\mathrm{Im}(z^4)=|z|^4\sin 4\varphi<0 \Leftrightarrow \sin 4\varphi<0$

$\pi+2k\pi<4\varphi<2\pi+2k\pi, \quad k\in\mathbb{Z}$

$\frac{\pi}{4}+k\frac{\pi}{2}<\varphi<\frac{\pi}{2}+k\frac{\pi}{2}$

- $|z|<|z+2i|$

$|z-0|<|z-(-2i)|$