MALF – Kartkówka 3 — 30pkt

Zad. 1 Prawda / fałsz $\qquad \mathcal{B}=(1,x,x^2,x^3,x^4)$

$p=x^4-x^2+x$, $\quad q=x^4+2x^3+x^2+1$, $\quad r=x^3+x+10$, $\quad s=1$ $\quad \in \mathbb{R}_4[x]$

$$r\begin{bmatrix}0&1&-1&0&1\\1&0&1&2&1\\10&1&0&1&0\\1&0&0&0&0\end{bmatrix} = r\begin{bmatrix}1&0&0&0&0\\0&1&-1&0&1\\1&0&1&2&1\\10&1&0&1&0\end{bmatrix} \underset{w_4-10w_1}{\overset{w_3-w_1}{=}} r\begin{bmatrix}1&0&0&0&0\\0&1&-1&0&1\\0&0&1&2&1\\0&1&0&1&0\end{bmatrix} \overset{w_4-w_2}{=} r\begin{bmatrix}1&0&0&0&0\\0&1&-1&0&1\\0&0&1&2&1\\0&0&1&1&-1\end{bmatrix} \overset{w_5-w_4}{=} r\begin{bmatrix}1&0&0&0&0\\0&1&-1&0&1\\0&0&1&2&1\\0&0&0&-1&-2\end{bmatrix} = 4$$

a) $\{p,q,r,s\}$ lin. zależne — FAŁSZ, bo rank $=4$

b) $\{p,q,r,s\}$ baza $\mathbb{R}_4[x]$ — FAŁSZ $\quad \dim \mathbb{R}_4[x]=5$

c) $w=-x-1 \in \operatorname{lin}\{p,q,r,s\}$

FAŁSZ

$$\begin{vmatrix}-1&-1&0&0&0\\1&0&0&0&0\\0&1&-1&0&1\\0&0&1&2&1\\0&0&0&-1&-2\end{vmatrix} = 1\cdot(-1)^3\begin{vmatrix}-1&0&0&0\\1&-1&0&1\\0&1&2&1\\0&0&-1&-2\end{vmatrix} = \begin{vmatrix}-1&0&1\\1&2&1\\0&-1&-2\end{vmatrix}$$

$$= 4-1+0-0-1-0 \neq 0$$

Zad. 2

a) $\mathcal{U}=\{f\in\mathbb{R}_2[x] : f(1)=0 \vee f'(2)=0\}$

To nie podprz. liniowa $\qquad$ Np. $f(x)=(x-2)^2 \qquad f\in\mathcal{U}$ bo $f'(x)=2(x-2)$, $f'(2)=0$

$g(x)=x-1 \qquad g\in\mathcal{U}$, bo $g(1)=0$

$(f+g)(x)=(x-2)^2+x-1 \qquad (f+g)'(x)=2(x-2)+1$

$(f+g)(1)=1+1-1=1\neq 0 \qquad (f+g)'(2)=0+1=1\neq 0$

b) $\varphi:\mathbb{R}\to\mathbb{R}^4 \qquad \varphi(x)=(0,\ x+1,\ \pi x,\ -x)$ czy liniowe?

NIE $\quad \varphi(0)=(0,1,0,0)\neq(0,0,0,0)$

Zad. 3 $\quad \varphi:\mathbb{R}^2\to\mathbb{R}^3 \qquad \mathcal{B}=(b_1=(1,1),\ b_2=(2,1)) \qquad \mathcal{C}=(c_1=(1,2,0),\ c_2=(2,3,0),\ c_3=(0,0,1))$

$$A'=M_\varphi(\mathcal{B},\mathcal{C})=\begin{bmatrix}-6&-7\\4&5\\1&1\end{bmatrix}$$

a) $\varphi(5,0)=?$ $\qquad (5,0)=5(b_2-b_1)=5(2,1)-5(1,1)=(10,5)-(5,5)$

$(5,0)=[-5,\ 5]_{\mathcal{B}}$

$$A'\begin{bmatrix}5\\-5\end{bmatrix}=\begin{bmatrix}-6&-7\\4&5\\1&1\end{bmatrix}\begin{bmatrix}-5\\5\end{bmatrix}=\begin{bmatrix}30-35\\-20+25\\5-5\end{bmatrix}=\begin{bmatrix}-5\\5\\0\end{bmatrix}$$

$\varphi(5,0)=[-5,\ 5,0]_{\mathcal{C}}=-5c_1+5c_2=-5(c_1-c_2)$

$=-5[(1,2,0)-(2,3,0)]=-5(-1,-1,0)$

$=(5,5,0)$

b) wzór $\varphi=?$

$A'=Q^{-1}AP \qquad A=QA'P^{-1} \qquad Q=P_{st^3\to\mathcal{C}}=\begin{bmatrix}1&2&0\\2&3&0\\0&0&1\end{bmatrix} \qquad P=P_{st^2\to\mathcal{B}}=\begin{bmatrix}1&2\\1&1\end{bmatrix}$

$P^{-1}=?\qquad \det P=1-2=-1 \qquad P^{-1}=\frac{1}{-1}\begin{bmatrix}1&-1\\-2&1\end{bmatrix}^T=-\begin{bmatrix}1&-2\\-1&1\end{bmatrix}=\begin{bmatrix}-1&2\\1&-1\end{bmatrix}$

$$A=\begin{bmatrix}1&2&0\\2&3&0\\0&0&1\end{bmatrix}\begin{bmatrix}-6&-7\\4&5\\1&1\end{bmatrix}\begin{bmatrix}-1&2\\1&-1\end{bmatrix}=\begin{bmatrix}1&2&0\\2&3&0\\0&0&1\end{bmatrix}\begin{bmatrix}-1&-5\\1&3\\0&1\end{bmatrix}=\begin{bmatrix}1&1\\1&-1\\0&1\end{bmatrix}$$

$$A\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x+y\\x-y\\y\end{bmatrix}$$

$$\varphi(x,y)=(x+y,\ x-y,\ y)$$

Zad. 4 $\varphi: \mathbb{R}_2[x] \to \mathbb{R}_2[x]$ $\varphi(p)(x) = 2x\,p'(x) + x^2 p(0) + p(2)$

a) $A = M_\varphi(\mathcal{B}, \mathcal{B}) = ?$ $\mathcal{B} = (1, x, x^2)$

$$\varphi(1) = 2x \cdot 0 + x^2 \cdot 1 + 1 = x^2 + 1$$
$$\varphi(x) = 2x \cdot 1 + x^2 \cdot 0 + 2 = 2x + 2$$
$$\varphi(x^2) = 2x \cdot 2x + x^2 \cdot 0 + 4 = 4x^2 + 4$$

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 0 \\ 1 & 0 & 4 \end{bmatrix}$$

b) Ker φ, Im φ - bazy, wymiary, własności φ

$p(x) = ax^2 + bx + c$
$p'(x) = 2ax + b$

$$\varphi(p)(x) = 2x(2ax+b) + x^2 \cdot c + 4a + 2b + c$$
$$= 4ax^2 + 2bx + cx^2 + 4a + 2b + c = (4a+c)x^2 + 2bx + 4a + 2b + c$$

• Ker φ = ? $\varphi(p)(x) = 0 = 0x^2 + 0 \cdot x + 0 \iff \begin{cases} 4a + c = 0 & \to c = -4a \\ 2b = 0 & \to b = 0 \\ 4a + 2b + c = 0 & \to 4a + 2 \cdot 0 - 4a = 0 \end{cases}$

Ker $\varphi = \{ p = ax^2 - 4a \mid a \in \mathbb{R} \}$
Ker φ = lin$\{x^2 - 4\}$ dim Ker $\varphi = 1$
baza

φ - nie jest monomorfizmem

dim Im φ = dim $\mathbb{R}_2[x]$ - dim Ker φ = $3 - 1 = 2 \neq$ dim $\mathbb{R}_2[x]$
φ - nie jest epimorfizmem

Zad. 5 $U = \{ p \in \mathbb{R}_3[x] : p''(1) = 0 \}$ baza U, dim U ?

$p \in \mathbb{R}_3[x]$
$$p = ax^3 + bx^2 + cx + d$$
$$p' = 3ax^2 + 2bx + c$$
$$p'' = 6ax + 2b$$

$p''(1) = 0$
$6a + 2b = 0$ $b = -3a$

$$U \ni p(x) = ax^3 - 3ax^2 + cx + d = a(x^3 - 3x^2) + cx + d$$

U = lin$\{x^3 - 3x^2, x, 1\}$ dim $U = 3$
baza lub $(0,0,-3,1)$, $(0,1,0,0)$, $(1,0,0,0)$

$$r\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix} = 3$$