FT - MALF - Kartkówka 4 30pkt

<u>Zad.1</u> Zdiagonalizować $\varphi \in End(\mathbb{R}^3)$, $\varphi(x,y,z) = (2x+2y+z,\ 2x+5y+2z,\ x+2y+2z)$

$D = ?$, $P = ?$

$$A = M_\varphi(\mathcal{B}_{st}^3, \mathcal{B}_{st}^3) = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

$$0 = \det(A - tI) = \begin{vmatrix} 2-t & 2 & 1 \\ 2 & 5-t & 2 \\ 1 & 2 & 2-t \end{vmatrix} \overset{k_1-k_3}{=} \begin{vmatrix} 1-t & 2 & 1 \\ 0 & 5-t & 2 \\ t-1 & 2 & 2-t \end{vmatrix} \overset{w_3-w_1}{=} \begin{vmatrix} 1-t & 2 & 1 \\ 0 & 5-t & 2 \\ 2t-2 & 0 & 1-t \end{vmatrix}$$

$$\overset{k_3-\frac{1}{2}k_2}{=} \begin{vmatrix} 1-t & 2 & 0 \\ 0 & 5-t & -\frac{1}{2}+\frac{t}{2} \\ 2t-2 & 0 & 1-t \end{vmatrix} = (1-t)^2(5-t) + (2t-2)(t-1) = (1-t)^2[5-t+2] = (7-t)(1-t)^2$$

$Spec(A) = \{\lambda_1 = 1,\ \lambda_2 = 7\}$ $\quad$ $\dim E_{\lambda_2} = 1$

$k_1 = 2,\ k_2 = 1$ $\quad$ $1 \le \dim E_{\lambda_1} \le 2$

- $E_{\lambda_1} = E_1 = \{v = (x,y,z) : \varphi(v) = v\}$
  $(A - I)X = 0$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ \cancel{2} & \cancel{4} & \cancel{2} & 0 \\ \cancel{1} & \cancel{2} & \cancel{1} & 0 \end{array}\right] \Rightarrow x + 2y + z = 0, \quad z = -x - 2y$$

$E_{\lambda_1} = \{(x, y, -x-2y);\ x,y \in \mathbb{R}\} = lin\{(1,0,-1), (0,1,-2)\}$

$\dim E_{\lambda_1} = 2$

- $E_{\lambda_2} = E_7 = \{v : \varphi(v) = 7v\}$
  $(A - 7I)X = 0$

$$\left[\begin{array}{ccc|c} -5 & 2 & 1 & 0 \\ 2 & -2 & 2 & 0 \\ 1 & 2 & -5 & 0 \end{array}\right] \xrightarrow{w_1+2w_2} \left[\begin{array}{ccc|c} \cancel{-1} & \cancel{-2} & \cancel{5} & 0 \\ 2 & -2 & 2 & 0 \\ 1 & 2 & -5 & 0 \end{array}\right] \quad \begin{cases} x - y + z = 0 \\ x + 2y - 5z = 0 \end{cases} \Rightarrow 3x - 3z = 0$$

$x = z$

$y = x + z = 2x$

$E_7 = \{(x, 2x, x);\ x \in \mathbb{R}\} = lin\{(1,2,1)\}$ $\quad$ $\dim E_{\lambda_2} = 1$

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ -1 & -2 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

<u>Zad.2</u> $f \in End(\mathbb{R}^3)$ $\quad$ $\varphi(2,1,2) = (0,0,0)$, $\varphi(-1,1,1) = (-12,12,12)$, $\varphi(1,2,-2) = (-12,-24,24)$ $\quad$ $f^{100}(1,2,1) = ?$

$$\begin{vmatrix} 2 & -1 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & -2 \end{vmatrix} = -2-4+1 - 2-4-1 = -12 \neq 0$$

$Spec\ A = \{\lambda_1 = 0,\ \lambda_2 = 12,\ \lambda_3 = -12\}$ $\quad$ mamy pr. k $\Rightarrow$ diagonalizowalny

$$M_\varphi(C,C) = D = \begin{bmatrix} 0 & & \\ & 12 & \\ & & -12 \end{bmatrix} \qquad P = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & -2 \end{bmatrix} \qquad X = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$v = (1,2,1) = [\alpha, \beta, \gamma]_C$ $\quad$ $X' = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$

$PX' = X$

$C = (c_1 = (2,1,2),\ c_2 = (-1,1,1),\ c_3 = (1,2,-2))$
baza wektorów wł.

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 1 & -2 & 1 \end{array}\right]$$

$$\circlearrowleft\left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 1 & -2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 2 & -1 & 1 & 1 \\ 2 & 1 & -2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & -3 & -3 & -3 \\ 0 & -1 & -6 & -3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 6 & 3 \end{array}\right] \xrightarrow{w_3 - w_2}$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 5 & 2 \end{array}\right]$$

$5\gamma = 2, \ \gamma = \frac{2}{5}$

$\beta + \gamma = 1, \ \beta = 1 - \gamma = \frac{3}{5}$

$\alpha + \beta + 2\gamma = 2 \qquad \alpha = 2 - 2\gamma - \beta = 2 - \frac{4}{5} - \frac{3}{5} = \frac{3}{5}$

$$v = \left[\frac{3}{5}, \frac{3}{5}, \frac{2}{5}\right]_e$$

$$D^{100} X' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 12^{100} & 0 \\ 0 & 0 & 12^{100} \end{bmatrix} \begin{bmatrix} 3/5 \\ 3/5 \\ 2/5 \end{bmatrix} = \begin{bmatrix} 0 \\ 12^{100} \cdot \frac{3}{5} \\ 12^{100} \cdot \frac{2}{5} \end{bmatrix}$$

$$f^{100}(v) = 12^{100} \cdot \frac{3}{5} c_2 + 12^{100} \cdot \frac{2}{5} c_3 = \frac{1}{5} \cdot 12^{100} \left[3c_2 + 2c_3\right] =$$
$$= \frac{1}{5} 12^{100} \left[(-3, 3, 3) + (2, 4, -4)\right] = \frac{1}{5} \cdot 12^{100} (-1, 7, -1)$$

Zad. 3 $\quad A = \begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix}$ tw. C-H $\to$ oblicz A^5 za pomocą A oraz I

$$0 = \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -1 \\ 2 & -1-\lambda \end{vmatrix} = (3-\lambda)(-1-\lambda) + 2 = -3 - 3\lambda + \lambda + \lambda^2 + 2$$
$$= \lambda^2 - 2\lambda - 1$$

$\Rightarrow A^2 - 2A - I = 0$

$A^2 = 2A + I$

$$A^5 = A \cdot A^2 \cdot A^2 = A(2A+I)(2A+I)$$
$$= A(4A^2 + 4A + I) = A(8A + 4I + 4A + I)$$
$$= A(12A + 5I) = 12A^2 + 5A = 12(2A+I) + 5A$$
$$= 29A + 12I$$

Zad. 4

$\mathbb{R}_2[x] \qquad \mathcal{B}_k = (1, x, x^2) \qquad \langle p, q \rangle = \int_{-1}^{1} p(x) q(x)\, dx$

a) Czy $\mathcal{B}_k$ ortogonalna? $\quad \langle 1, x^2 \rangle = \int_{-1}^{1} x^2 dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{1}{3} - (-\frac{1}{3}) = \frac{2}{3} \neq 0$

b) zortogonalizować $\mathcal{B}_k$ metodą G-S

Ozn. $b_1 = 1, \ b_2 = x, \ b_3 = x^2 \qquad C = (c_1, c_2, c_3)$ szukana baza ortog.

$c_1 = b_1 = 1$

$c_2 \perp c_1 \ \wedge \ \mathrm{lin}(b_1, b_2) = \mathrm{lin}(c_1, c_2) \Rightarrow c_2 = b_2 + \alpha c_1$

$$0 = \langle c_2, c_1 \rangle = \langle b_2 + \alpha c_1, c_1 \rangle = \langle b_2, c_1 \rangle + \alpha \langle c_1, c_1 \rangle = \int_{-1}^{1} x\, dx + \alpha \int_{-1}^{1} dx =$$
$$= \frac{x^2}{2} + \alpha x \Big|_{-1}^{1} = \frac{1}{2} + \alpha - \left(\frac{1}{2} - \alpha\right) = 2\alpha \Rightarrow \alpha = 0, \quad c_2 = b_2$$

$c_3 \perp c_1, \ c_3 \perp c_2 \ \wedge \ c_3 = b_3 + \alpha c_1 + \beta c_2$

$$\begin{cases} 0 = \langle c_3, c_1 \rangle = \langle b_3, c_1 \rangle + \alpha \langle c_1, c_1 \rangle + \beta \overset{0}{\langle c_2, c_1 \rangle} = \int_{-1}^{1} x^2 dx + \alpha \int_{-1}^{1} dx = \frac{x^3}{3} + \alpha x \Big|_{-1}^{1} = \frac{1}{3} + \alpha + \frac{1}{3} + \alpha \\ 0 = \langle c_3, c_2 \rangle = \langle b_3, c_2 \rangle + \alpha \underset{0}{\langle c_1, c_2 \rangle} + \beta \langle c_2, c_2 \rangle = \int_{-1}^{1} x^3 dx + \beta \int_{-1}^{1} x^2 dx = \frac{x^4}{4} + \beta \frac{x^3}{3} \Big|_{-1}^{1} = \frac{1}{4} + \frac{\beta}{3} - \frac{1}{4} + \beta/3 \end{cases}$$

$$\to \begin{cases} 2\alpha + \frac{2}{3} = 0 \\ \frac{2}{3}\beta = 0 \end{cases} \qquad \begin{matrix} \alpha = -\frac{1}{3} \\ \beta = 0 \end{matrix} \qquad c_3 = b_3 - \frac{1}{3} c_1 = x^2 - \frac{1}{3}$$

c)

norma $r = x^2 + x + 1 \qquad \|r\|^2 = \langle r, r \rangle = \int (x^2 + 5)^2 dx = \int_{-1}^{1} (x^4 + 10x^2 + 25)\, dx$

$= \frac{x^5}{5} + \frac{10}{3}x^3 + 25x = 2\left[\frac{1}{5} + \frac{10}{3} + 25\right]$