

Algebra - Zadanie domowe nr 6

Zad. 1 Diagonalizacja $\varphi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$

Baza.

$$\varphi(x, y, z) = (x + 2y - 2z, x + 3z, x + 3y)$$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & -2 \\ 1 & -\lambda & 3 \\ 1 & 3 & -\lambda \end{vmatrix} = \lambda^2(1-\lambda) - 6 + 6 - 2\lambda - 9(1-\lambda) + 2\lambda = (1-\lambda)(\lambda^2 - 9)$$

$$\text{Spec}(\varphi) = \{-3, 1, 3\} \quad \text{różne proste} \Rightarrow \varphi - \text{diagonalizowalny}$$

• $\lambda_1 = 1$

$$(A - \lambda_1)X = 0$$

$$(A - I)X = 0 \Leftrightarrow \begin{bmatrix} 0 & 2 & -2 \\ 1 & -1 & 3 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & -2 \\ 1 & -1 & 3 \\ 1 & 3 & -1 \end{bmatrix} \xrightarrow{w_2 - w_1} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & -2 \\ 0 & 4 & -4 \end{bmatrix} \xrightarrow{w_2 - w_1, w_3 - w_1} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & -2 \\ 0 & 2 & -2 \end{bmatrix} \xrightarrow{w_3 - w_2} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_1 + w_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} a + 2c = 0 \\ b - c = 0 \end{cases}$$

$$c = b$$

$$b = c = b$$

$$a = -2c = -2b$$

$$E_1 = \{(-2t, t, t), t \in \mathbb{R}\}$$

$$= \lim_{\mathbb{R}} \{(-2, 1, 1)\}$$

• $\lambda_2 = 3$

$$(A - 3I)X = 0$$

$$\begin{bmatrix} -2 & 2 & -2 \\ 1 & -3 & 3 \\ 1 & 3 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 & -2 \\ 1 & -3 & 3 \\ 1 & 3 & -3 \end{bmatrix} \xrightarrow{w_1: (-2)} \begin{bmatrix} 1 & -1 & 1 \\ 1 & -3 & 3 \\ 1 & 3 & -3 \end{bmatrix} \xrightarrow{w_2 - w_1, w_3 - w_1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 2 \\ 0 & 4 & -4 \end{bmatrix} \xrightarrow{w_2: (2)} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_1 + w_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} a = 0 \\ b - c = 0 \end{cases}$$

$$b = c$$

$$E_3 = \{(0, b, b), b \in \mathbb{R}\} = \lim_{\mathbb{R}} \{(0, 1, 1)\}$$

• $\lambda_3 = -3$

$$(A + 3I)X = 0$$

$$\begin{bmatrix} 4 & 2 & -2 \\ 1 & 3 & 3 \\ 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 & -2 \\ 1 & 3 & 3 \\ 1 & 3 & 3 \end{bmatrix} \xrightarrow{w_3 - w_2, w_1/2} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_2 - w_1} \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_2 - w_1 \cdot 2} \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} a + 3b + 3c = 0 \\ -5b - 7c = 0 \end{cases}$$

$$a = -3b - 3c = \frac{21}{5}c - 3c = \frac{6}{5}c$$

$$b = -\frac{7}{5}c$$

$$c = t$$

$$E_{-3} = \{(\frac{6}{5}t, -\frac{7}{5}t, t), t \in \mathbb{R}\}$$

$$\lim_{\mathbb{R}} \{\frac{1}{5}(6, -7, 5)\}$$

baza

$$B = \{(-2, 1, 1), (0, 1, 1), \frac{1}{5}(6, -7, 5)\}$$

$$M_{BB}(\varphi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Zadanie domowe - Diagonalizacja

Zad. 1 b) $\varphi \in \text{End}(\mathbb{C}^3)$, $\varphi(x_1, x_2, x_3) = (ix_1 - x_1 + 2x_2 + x_3, 2ix_2 + 3x_3, ix_3 - x_3)$

$$A = M_{\varphi}(B_K, B_K) \quad A = \begin{bmatrix} i-1 & 2 & 1 \\ 0 & 2i & 3 \\ 0 & 0 & i-1 \end{bmatrix}$$

$$B_K = ((1,0,0), (0,1,0), (0,0,1))$$

$$\chi_A(t) = \det(A - tI) = (2i - t)(i - 1 - t)^2$$

$$\text{Spec}(A) = \{ \lambda_1 = -1+i, \lambda_2 = 2i \}$$

$$k_1 = 2, \quad k_2 = 1$$

$$\lambda_1 = -1+i \quad E_{\lambda_1} = ? \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = (A - \lambda_1 I) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1+i & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow \begin{cases} 2x_2 + x_3 = 0 \\ (1+i)x_2 + 3x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_3 = -2x_2 \\ (1+i)x_2 - 6x_2 = 0 \end{cases}$$

$$x_2 = x_3 = 0$$

$$E_{-1+i} = \{ (x_1, 0, 0); x_1 \in \mathbb{C} \} = \text{lin}_{\mathbb{C}} \{ (1, 0, 0) \} \quad \dim E_{-1+i} = 1 \neq k_1 = 2$$

macierz nie jest diagonalizowalna

$$\lambda_2 = 2i \quad E_{\lambda_2} = ? \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = (A - 2iI) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1-i & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & -1-i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow \begin{cases} x_3 = 0 \\ 2x_2 - (1+i)x_1 = 0 \end{cases}$$

$$E_{2i} = \{ (x_1, -\frac{1}{2}(1+i)x_1, 0); x_1 \in \mathbb{C} \} = \text{lin}_{\mathbb{C}} \{ (1, -\frac{1}{2} - \frac{1}{2}i, 0) \} \quad \dim E_{2i} = 1$$

$$= \text{lin}_{\mathbb{C}} \{ (2, -1-i, 0) \}$$

Diagonalizacja endomorfizmu

Zad. 2 $\varphi: \mathbb{R}_2[x] \rightarrow \mathbb{R}_2[x] \quad \forall p \in \mathbb{R}_2[x] \quad \varphi(p)(x) = 3xp''(x) + (x^2-1)p'(1) + x \cdot p(2)$

$B_x = (1, x, x^2)$ baza $\mathbb{R}_2[x]$

$\varphi(1) = 3x \cdot 0 + (x^2-1) \cdot 1 + x \cdot 1 = x^2 + x - 1 = [-1, 1, 1]_{B_x}$

$\varphi(x) = 3x \cdot 0 + (x^2-1) \cdot 1 + x \cdot 2 = x^2 + 2x - 1 = [-1, 2, 1]_{B_x}$

$\varphi(x^2) = 3x \cdot 2 + (x^2-1) \cdot 1 + x \cdot 4 = x^2 + 10x - 1 = [-1, 10, 1]_{B_x}$

$A = M_{\varphi}(B_x, B_x) = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 2 & 10 \\ 1 & 1 & 1 \end{bmatrix}$

$0 = \det(A - tI) = \begin{vmatrix} -1-t & -1 & -1 \\ 1 & 2-t & 10 \\ 1 & 1 & 1-t \end{vmatrix} \xrightarrow{\substack{W_1+W_2 \\ W_3-W_2}} \begin{vmatrix} -t & 1-t & 9 \\ 1 & 2-t & 10 \\ 0 & -1+t & -9-t \end{vmatrix} \xrightarrow{W_1+W_3} \begin{vmatrix} -t & 0 & -t \\ 1 & 2-t & 10 \\ 0 & t-1 & -9-t \end{vmatrix} =$

$= -t(2-t)(-9-t) - t(t-1) - 10(t-1)(-t) = t[(2-t)(9+t) - t+1 + 10t-10] = t[18+2t-9t-t^2+9t-9]$
 $= t[-t^2+2t+9]$

$\Delta = 4+36=40$
 $\sqrt{\Delta} = 2\sqrt{10}$

$\text{Spec}(\varphi) = \{0, 1-\sqrt{10}, 1+\sqrt{10}\}$ tridmo prostc

φ -diagonalizowalny

Zad. 3

Niech $\varphi \in \text{End}(\mathbb{R}^3)$ b.z.c

$\varphi(1, 0, 0) = (1, 0, 0)$

$\varphi(1, 1, 0) = (-1, -1, 0)$

$\varphi(1, 1, 1) = (0, 0, 0)$

Oblicz $\varphi^{100}(3, 6, 9)$.

$\text{Spec}(\varphi) = \{1, -1, 0\}$

Baza wektorow wlasnych

$C = \{c_1 = (1, 0, 0), c_2 = (1, 1, 0), c_3 = (1, 1, 1)\}$

B-baza kanoniczna

$P = P_{B \rightarrow C} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ macierz przejsci

$D = M_{CC}(\varphi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

I sposob:

$A = M_{BB}(\varphi)$

$A = P \cdot D \cdot P^{-1}, A^{100} = P \cdot D^{100} \cdot P^{-1}$

$P^{-1} = ?$

$[P|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\substack{W_1-W_3 \\ W_2-W_3}} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{W_1-W_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{P^{-1}} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$

$\varphi^{100}(3, 6, 9) = ?$

$A^{100} \cdot \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = P \cdot D^{100} \cdot P^{-1} \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{100} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -3 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} -6 \\ -3 \\ 0 \end{bmatrix}$

$\varphi^{100}(3, 6, 9) = (-6, -3, 0)$

II sposob

$(3, 6, 9) = [-3, -3, 9]_C$

$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 9 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 9 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 9 \end{array} \right]$

$D^{100} \cdot \begin{bmatrix} -3 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -3 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 0 \end{bmatrix}$

$\varphi^{100}(3, 6, 9) = [-3, -3, 0]_C = -3(1, 0, 0) - 3(1, 1, 0) = (-6, -3, 0)$

Zad. 4

a) $A = \begin{bmatrix} 1 & -5 \\ 1 & 1 \end{bmatrix} \in M_2(\mathbb{R})$

$\det(A - tI) = \begin{vmatrix} 1-t & -5 \\ 1 & 1-t \end{vmatrix} = (1-t)^2 + 5 > 0$

brak pierwiastków rzeczywistych

$\text{Spec}(A) = \emptyset$ A nie jest diagonalizowalna

b) $A = \begin{bmatrix} 7 & 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

A - trójkątna górna

$\text{Spec}(A) = \{ \lambda_1 = 7, \lambda_2 = 1, \lambda_3 = 2 \}$
 $k_1 = 1, k_2 = 3, k_3 = 1$

$\dim E_{\lambda_1} = k_1 = 1$

$\dim E_{\lambda_3} = k_3 = 1$

Zai $\dim E_{\lambda_2} = 5 - r(A - 1 \cdot I) = 5 - 4 = 1 \neq k_2 = 3$

$A - I = \begin{bmatrix} 6 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$r(A - I) = 4$

NIE jest diagonalizowalna

c) $A = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 \end{bmatrix}$

$\lambda_1 = 0 \in \text{Spec}(A)$
 $k_1 = 3$

$\dim E_{\lambda_1} = ?$

$\dim E_{\lambda_1} = 4 - r(A - 0 \cdot I) = 4 - r(A) = 4 - 1 = 3 = k_1$

TAK, diagonalizowalna

$r(A) = 1$ rząd!

d) $A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & p & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

dla jakich p macierz jest diagonalizowalna?

$\text{Spec}(A) = \{ \lambda_1 = 5, \lambda_2 = 3, \lambda_3 = 1 \}$
 $k_1 = 2, k_2 = 1, k_3 = 1$

Ważny tw. spektralnego, A diagonalizowalna $\Leftrightarrow \dim E_{\lambda_1} = 2$

$\dim E_{\lambda_1} = 4 - r(A - 5I) = 2 \Leftrightarrow r(A - 5I) = 4 - 2 = 2$

$r(A - 5I) = r \begin{bmatrix} 0 & -2 & 6 & -1 \\ 0 & -2 & p & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -4 \end{bmatrix} = r \begin{bmatrix} 0 & -2 & 6 & -1 \\ 0 & -2 & p & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = r \begin{bmatrix} -2 & 6 & -1 \\ -2 & p & 0 \\ 0 & 0 & 1 \end{bmatrix} \stackrel{w_2 - w_1}{=} r \begin{bmatrix} -2 & 6 & -1 \\ 0 & p-6 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$r(A - 5I) = \begin{cases} 3 & ; p \neq 6 \\ 2 & ; p = 6 \end{cases}$

A diagonalizowalna $\Leftrightarrow p = 6$

Zad. 5 PRAWDA / FAŁSZ

$A = \begin{bmatrix} 7 & -3 & -1 & 0 \\ 0 & 4 & p & 0 \\ 0 & 0 & 7 & \pi \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$A = M_{\mathcal{U}}(B_2^4, B_2^4)$

$A - 7I = \begin{bmatrix} 0 & -3 & -1 & 0 \\ 0 & -3 & p & 0 \\ 0 & 0 & 0 & \pi \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$k_1 = 2, k_2 = 1, k_3 = 1$

$\bullet \text{Spec}(A) = \{ \lambda_1 = 7, \lambda_2 = 4, \lambda_3 = 1 \}$

$\Rightarrow A \text{ diagonalizowalna} \Leftrightarrow \dim E_{\lambda_1} = 4 - r(A - 7I) = 2$

$\bullet \det A = 7 \cdot 4 \cdot 7 = 196 \neq 0 \Rightarrow r(A) = r(4) = 4 \Rightarrow \varphi \text{ surjekcja}$

$\bullet \det(\frac{1}{7} A^T) = \frac{1}{7^4} \cdot 196$

$\Rightarrow \dim \ker \varphi = 4 - 4 = 0 \Rightarrow \varphi \text{ iniekcja}$

$r(A - 7I) = r \begin{bmatrix} 0 & -3 & -1 & 0 \\ 0 & 0 & p+1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{cases} 2 & ; p = -1 \\ 3 & ; p \neq -1 \end{cases}$

$\downarrow$
a) ~~FAŁSZ~~ PRAWDA
c) PRAWDA

b) FAŁSZ

Zadanie domowe - Diagonalizacja

Zad. 7

$$a_1 = a_2 = 2, \quad a_3 = 4 \quad a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3} \quad n \geq 3$$

kasor ogdny?

$$\begin{bmatrix} a_n \\ a_{n-1} \\ a_{n-2} \end{bmatrix} = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n-1} \\ a_{n-2} \\ a_{n-3} \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_A \begin{bmatrix} a_3 \\ a_2 \\ a_1 \end{bmatrix}$$

$$\chi_A(t) = \det(A - tI) = \begin{vmatrix} 2-t & 1 & -2 \\ 1 & -t & 0 \\ 0 & 1 & -t \end{vmatrix} = t^2(2-t) - 2 + t = -t^2(t-2) + (t-2) = (t-2)(1-t^2) = (t-2)(1-t)(1+t)$$

$$\rho_{\text{pec}}(A) = \{-1, 1, 2\} \quad \text{widmo proste} \Rightarrow A \text{ diagonalizowalna}$$

$$\lambda_1 = -1 \quad (A+I) \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 & 1 & -2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} b+c=0 \\ a+b=0 \end{cases} \quad E_{-1} = \{(-b, b, b) \mid b \in \mathbb{R}\} \quad v_1 = (1, -1, 1)$$

$$\lambda_2 = 1 \quad (A-I) \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow a=b=c \quad E_1 = \{(a, a, a) \mid a \in \mathbb{R}\} \quad v_2 = (1, 1, 1)$$

$$\lambda_3 = 2 \quad (A-2I) \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 0 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow \begin{cases} a-2b=0 \\ b-2c=0 \end{cases} \quad E_2 = \{(4c, 2c, c) \mid c \in \mathbb{R}\} \quad v_3 = (4, 2, 1)$$

$$\begin{bmatrix} a_n \\ a_{n-1} \\ a_{n-2} \end{bmatrix} = P \cdot D^{n-3} \cdot P^{-1} \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 & 4 \\ -1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \quad P^{-1} = ?$$

$$\begin{bmatrix} 1 & 1 & 4 & | & 1 & 0 & 0 \\ -1 & 1 & 2 & | & 0 & 1 & 0 \\ 1 & 1 & 1 & | & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 4 & | & 1 & 0 & 0 \\ 0 & 2 & 6 & | & 1 & 1 & 0 \\ 0 & 0 & -3 & | & -1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & | & -\frac{1}{3} & 0 & \frac{4}{3} \\ 0 & 2 & 0 & | & -\frac{1}{3} & 1 & 1 \\ 0 & 0 & -3 & | & -1 & 0 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{6} \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & 6 \\ 2 & 0 & -2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & | & -\frac{1}{6} & 0 & \frac{4}{3} \\ 0 & 1 & 0 & | & -\frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 0 & 1 & | & \frac{1}{3} & 0 & -\frac{1}{3} \end{bmatrix} \rightarrow \begin{bmatrix} I_3 & | & \frac{1}{6} \begin{bmatrix} -1 & -1 & 4 \\ -3 & 3 & 6 \\ 2 & 0 & -2 \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} a_n \\ a_{n-1} \\ a_{n-2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 4 \\ -1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} (-1)^{n-3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^{n-3} \end{bmatrix} \frac{1}{6} \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & 6 \\ 2 & 0 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} (-1)^{n-3} & 1 & 2^{n-1} \\ (-1)^{n-2} & 1 & 2^{n-2} \\ (-1)^{n-3} & 1 & 2^{n-3} \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$$

$$a_n = \frac{1}{6} \cdot [2 \cdot (-1)^{n-3} + 6 + 4 \cdot 2^{n-1}] = \frac{1}{3} \cdot (-1)^{n-3} + \frac{1}{3} \cdot 2^n + 1$$

Zad. 8

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 2 \end{bmatrix}$$

$$A^5 - A^4 = ?$$

$$B = \begin{bmatrix} i & 1 \\ 0 & i \end{bmatrix} \quad B^{77} = ?$$

$$\chi_A(t) = \begin{vmatrix} 2-t & 3 & 1 \\ 0 & -t & 0 \\ 0 & -1 & 2-t \end{vmatrix} = -t(2-t)^2 = -t(4-4t+t^2) = -t^3+4t^2-4t \Rightarrow -A^3+4A^2-4A=0$$

$$\begin{array}{r} t^2+3t+8 \\ (t^5-t^4):(t^3-4t^2+4t) \\ \hline t^5-4t^4+4t^3 \\ \hline 3t^4-4t^3 \\ 3t^4-12t^3+12t^2 \\ \hline 8t^3-12t^2 \\ 8t^3-32t^2+32t \\ \hline 20t^2-32t \end{array}$$

$$A^5 - A^4 = 24A^2 - 32A$$

$$B^2 - 2iB - I = 0 \quad \text{dla } B \neq 0 \Rightarrow \exists B^{-1}$$

$$B - 2iI - B^{-1} = 0 \quad B^{-1} = \frac{B - 2iI}{B^2 - 2iB - I}$$

$$\chi_B(t) = (i-t)^2 \Rightarrow (i \cdot I - B)^2 = 0 \quad -I - 2iB + B^2 = 0$$

$$B^{77} = 77B - 76iI = \dots \text{polaryz}$$

$$\begin{cases} t^{77} = q(t) \cdot (i-t)^2 + r(t) \\ 77t^{76} = q'(t) \cdot (i-t)^2 + q(t) \cdot 2(i-t) \cdot (-1) + a \end{cases} \quad \begin{cases} i^{77} = ai+b \\ 77i^{76} = a \end{cases} \quad \begin{cases} a = (i^4)^{19} \cdot 77 = 77 \\ b = i^{77} - ai = i - 77i = -76i \end{cases}$$

Zad. 6

$$A \in M_3(\mathbb{C}), \quad \text{tr } A = 2, \quad \lambda_1 = 7 \in \text{Spec}(A), \quad v = (1, 2, 1), \quad w = (1, 1, 0) \text{ odpow. } \lambda_1$$

$$v, w - \text{lin. niezalczne} \Rightarrow \dim E_7 \geq 2$$

$$2 = \text{tr } A = 7 + \lambda_2 \Rightarrow \lambda_2 = -12, \quad \lambda_3 = 1$$

$$\det A = 7 \cdot 7 \cdot (-12) = -588$$

Próba

ZAD.

$$A = \begin{bmatrix} p & 2 & -1 \\ 1 & p & -1 \\ 0 & 0 & 2 \end{bmatrix} \in M_3(\mathbb{R})$$

Dla jakich $p \in \mathbb{R}$ macierz A jest diagonalizowalna?

$$\chi_A(t) = \det(A - tI) = \begin{vmatrix} p-t & 2 & -1 \\ 1 & p-t & -1 \\ 0 & 0 & 2-t \end{vmatrix} = (2-t) [(p-t)^2 - 2]$$

$$= (2-t)(p-t-\sqrt{2})(p-t+\sqrt{2}) = -(t-2)(t-p-\sqrt{2})(t-p+\sqrt{2})$$

$$\forall p \in \mathbb{R} \quad p-\sqrt{2} \neq p+\sqrt{2}$$

① $\begin{matrix} p+\sqrt{2}=2 \Rightarrow p=2-\sqrt{2} \\ p-\sqrt{2}=2 \Rightarrow p=2+\sqrt{2} \end{matrix} \Rightarrow p \neq 2 \pm \sqrt{2}$ Wtedy proste A -diagonalizowalna

② $p=2-\sqrt{2}$ $A = \begin{bmatrix} 2-\sqrt{2} & 2 & -1 \\ 1 & 2-\sqrt{2} & -1 \\ 0 & 0 & 2 \end{bmatrix}$

$$\lambda_1 = 2 \quad k_1 = 2 \quad \dim E_{\lambda_1} = 3 - r(A - (2-\sqrt{2})I) = 3 - r \begin{bmatrix} -\sqrt{2} & 2 & -1 \\ 1 & -\sqrt{2} & -1 \\ 0 & 0 & 0 \end{bmatrix} = 3 - 2 = 1 \neq k_1 = 2$$

NIE DIAGONALIZOWALNA

③ $p=2+2\sqrt{2}$ $A = \begin{bmatrix} 2+\sqrt{2} & 2 & -1 \\ 1 & 2+\sqrt{2} & -1 \\ 0 & 0 & 2 \end{bmatrix}$

$$\lambda_1 = 2 \quad k_1 = 2 \quad \dim E_{\lambda_1} = 3 - r(A - 2I) = 3 - r \begin{bmatrix} \sqrt{2} & 2 & -1 \\ 1 & \sqrt{2} & -1 \\ 0 & 0 & 0 \end{bmatrix} = 3 - 2 = 1 \neq k_1 = 2$$

NIE DIAGONALIZOWALNA