Kartkówka 2 - Grupa 2 - MALG, 2025

1 $U=[A|B]=\begin{bmatrix} p-2 & 2 & p-1 & 1 & \big| & 2 \\ 0 & p-1 & 0 & 5 & \big| & 6-p \\ 3p-6 & 6 & 4p-4 & 5 & \big| & 8 \\ p-2 & 2 & 4p-3 & 5 & \big| & 7 \end{bmatrix} \xrightarrow[w_4-w_1]{w_3-3w_1} \begin{bmatrix} p-2 & 2 & p-1 & 1 & \big| & 2 \\ 0 & p-1 & 0 & 5 & \big| & 6-p \\ 0 & 0 & p-1 & 2 & \big| & 2 \\ 0 & 0 & 3p-2 & 4 & \big| & 5 \end{bmatrix} \xrightarrow{k_3 \leftrightarrow k_4}$

$$\begin{array}{ccccc} x_1 & x_2 & x_4 & x_3 & \\ \end{array}$$

$$\rightarrow \begin{bmatrix} p-2 & 2 & 1 & p-1 & \big| & 2 \\ 0 & p-1 & 5 & 0 & \big| & 6-p \\ 0 & 0 & 2 & p-1 & \big| & 2 \\ 0 & 0 & 4 & 3p-2 & \big| & 5 \end{bmatrix} \xrightarrow{w_4-2w_3} \begin{bmatrix} p-2 & 2 & 1 & p-1 & \big| & 2 \\ 0 & p-1 & 5 & 0 & \big| & 6-p \\ 0 & 0 & 2 & p-1 & \big| & 2 \\ 0 & 0 & 0 & p & \big| & 1 \end{bmatrix}$$

WNIOSKI: $p \in \mathbb{R}\setminus\{2,1,0\}$ $r(U)=r(A)=4$, $n=4$ ukł. oznaczony (1 rozw.)

- $p=0$ ukł. sprzeczny (ostatnie równanie sprzeczne)
- $p=1$ $\begin{bmatrix} -1 & 2 & 1 & 0 & \big| & 2 \\ \cancel{0} & \cancel{0} & \cancel{5} & \cancel{0} & \big| & \cancel{5} \\ 0 & 0 & 2 & 0 & \big| & 2 \\ 0 & 0 & 0 & 1 & \big| & 1 \end{bmatrix}$ $r(A)=r(U)=3$, $n=4$ nieoznaczony (1 parametr)
- $p=2$ $\begin{bmatrix} 0 & 2 & 1 & 1 & \big| & 2 \\ 0 & 1 & 5 & 0 & \big| & 4 \\ 0 & 0 & 2 & 1 & \big| & 2 \\ 0 & 0 & 0 & 2 & \big| & 1 \end{bmatrix} \xrightarrow{w_1 \leftrightarrow w_2} \begin{bmatrix} 0 & 1 & 5 & 0 & \big| & 4 \\ 0 & 2 & 1 & 1 & \big| & 2 \\ 0 & 0 & 2 & 1 & \big| & 2 \\ 0 & 0 & 0 & 2 & \big| & 1 \end{bmatrix} \xrightarrow{w_2-2w_1}$

$$\rightarrow \begin{bmatrix} 0 & 1 & 5 & 0 & \big| & 4 \\ 0 & 0 & -9 & 1 & \big| & -6 \\ 0 & 0 & 2 & 1 & \big| & 2 \\ 0 & 0 & 0 & 2 & \big| & 1 \end{bmatrix} \xrightarrow{w_2+5w_3} \begin{bmatrix} 0 & 1 & 5 & 0 & \big| & 4 \\ 0 & 0 & 1 & 6 & \big| & 4 \\ 0 & 0 & 2 & 1 & \big| & 2 \\ 0 & 0 & 0 & 2 & \big| & 1 \end{bmatrix} \xrightarrow{w_3-2w_1} \begin{bmatrix} 0 & 1 & 5 & 0 & \big| & 4 \\ 0 & 0 & 1 & 6 & \big| & 4 \\ 0 & 0 & 0 & -11 & \big| & -6 \\ 0 & 0 & 0 & 2 & \big| & 1 \end{bmatrix}$$

sprzeczny

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2 $l: \begin{cases} 2x-y+5=0 \\ x+2y+z=0 \end{cases}$ $\quad k: \frac{x-1}{2}=\frac{y+2}{-3}=z$

- $\vec{a} \parallel l$ $\quad \vec{b} \parallel k$

$\vec{a}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 0 \\ 1 & 2 & 1 \end{vmatrix}=[-1,-2,5]$ $\quad \vec{b}=[2,-3,1]$

$l \parallel k \iff \vec{a} \parallel \vec{b} \iff \vec{a}\times\vec{b}=\vec{0}$

$\vec{a}\times\vec{b}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -2 & 5 \\ 2 & -3 & 1 \end{vmatrix}=[13,11,7] \neq \vec{0}$

$\boxed{l \nparallel k}$

- $l \cap k = ?$

$l: x=0 \Rightarrow y=5 \Rightarrow z=-10$

$A \in l$ $\quad A=(0,5,-10) \in l$

$B \in k$ $\quad B=(1,-2,0) \in k$

$\vec{AB}=[1,-7,10]$ $\quad (\vec{a}\times\vec{b})\circ\vec{AB}=[13,11,7]\circ[1,-7,10]=13-77+70=13-7=6 \neq 0$

Proste skośne

$$d(l,k)=\frac{|(\vec{a}\times\vec{b})\circ\vec{AB}|}{|\vec{a}\times\vec{b}|}=\frac{|6|}{\sqrt{13^2+11^2+7^2}}$$

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3 $A,B,C,D \in M_4\,\mathbb{R}$

$A^TA=I \Rightarrow 1=\det I=\det A\cdot\det A^T=(\det A)^2 \Rightarrow \det A=\pm 1$

$B=-A^T \Rightarrow \underline{\det B}=(-1)^4\det A^T=\underline{\det A}$

$\det(ABA)=(\det A)^3<0 \Rightarrow \underline{\det A=-1}$

$C=-3\det B=-3\cdot 1=-3$

$$\det(2AB^{-1}C)^{-1}=\frac{1}{\det(2AB^{-1}C)}=\frac{1\cdot\cancel{\det B}}{2^4\cancel{\det A}\cdot\det C}=\frac{1}{16\cdot\det C}=\frac{-1}{48}$$

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4 $P=(1,2,-2)$ $\quad \vec{w}=[4,4,1]$ $\quad \pi: \begin{cases} x=t+2s \\ y=1+2t \\ z=2+s \end{cases}$ $t,s\in\mathbb{R}$

a) $P \in \pi$ FAŁSZ

$\begin{cases} 1=t+2s \\ 2=1+2t \Rightarrow t=\frac{1}{2} \\ -2=2+s \Rightarrow s=-4 \end{cases}$ $\quad 1 \neq \frac{1}{2}+2(-4)$ sprzeczny $\quad P \notin \pi$

b) Rzut ukośny P na π w kierunku $\vec{w}$ nie istnieje gdy $\vec{w} \parallel \pi$ tzn. $\vec{w} \perp \vec{m}$ $\quad \vec{m} \perp \pi$

PRAWDA

$\vec{m}=\vec{a}\times\vec{b}$

$\vec{w}\circ\vec{m} \overset{?}{=} 0 \iff \vec{w}\circ(\vec{a}\times\vec{b})=\begin{vmatrix} 4 & 4 & 1 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{vmatrix}=$

$\pi \parallel \vec{a}=[1,2,0]$

$\pi \parallel \vec{b}=[2,0,1]$

$=8+0+0-4-0-4=0$

$\vec{w} \parallel \pi$

Rozwiązania - Kart. 2 - grupa 2

① $p \in \mathbb{R}$

$$U = [A|B] = \left[\begin{array}{cccc|c} p & 0 & 2p & p+1 & p+3 \\ 0 & p-1 & 1 & p+3 & p+4 \\ 0 & 2p-2 & 4 & 2p+6 & 2p+10 \\ 3p & 1-p & 6p+1 & 3p+3 & 3p+10 \end{array}\right] \xrightarrow[w_3-2w_2]{w_4-3w_1} \left[\begin{array}{cccc|c} p & 0 & 2p & p+1 & p+3 \\ 0 & p-1 & 1 & p+3 & p+4 \\ 0 & 0 & 2 & 0 & 2 \\ 0 & 1-p & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\frac{1}{2}w_3]{w_4+w_2} \left[\begin{array}{cccc|c} p & 0 & 2p & p+1 & p+3 \\ 0 & p-1 & 1 & p+3 & p+4 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & p+3 & p+5 \end{array}\right] \xrightarrow{w_4-2w_3} \left[\begin{array}{cccc|c} p & 0 & 2p & p+1 & p+3 \\ 0 & p-1 & 1 & p+3 & p+4 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & p+3 & p+3 \end{array}\right]$$

WNIOSKI 1) $p \in \mathbb{R} \setminus \{0, 1, -3\}$ ukł. oznaczony, jedno rozw.
$r(A) = r(U) = 4 = n = 4$

2) $p = 0$
$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 3 \\ 0 & -1 & 1 & 3 & 4 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 3 & 3 \end{array}\right]$$
ukł. sprzeczny
równanie 1-sze i 4-te sprzeczności

3) $p = -3$
$$\left[\begin{array}{cccc|c} -3 & 0 & -6 & -2 & 0 \\ 0 & -4 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right]$$
$r(A) = r(U) = 3$
$n = 4$
ukł. nieoznaczony (1 parametr.)

4) $p = 1$
$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 2 & 4 \\ 0 & 0 & 1 & 4 & 5 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 4 & 4 \end{array}\right] \xrightarrow{w_3-w_2} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 2 & 4 \\ 0 & 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & -4 & -4 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{4} & \cancel{4} \end{array}\right]$$
$r(A) = r(U) = 3 < n = 4$
ukł. nieoznaczony (1 parametr)

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② $P' = ?$ P' - punkt symetryczny do P względem $\Pi : x+y+z+2=0$

gdzie $\{P\} = k \cap l$ $k: \frac{x-2}{2} = -y-3 = \frac{z-1}{3}$ $l: \begin{cases} x = 1-t \\ y = 2+2t \\ z = 1-t \end{cases} ; t \in \mathbb{R}$

$P = ?$ $k: \frac{x-2}{2} = \frac{y-(-3)}{-1} = \frac{z-1}{3}$

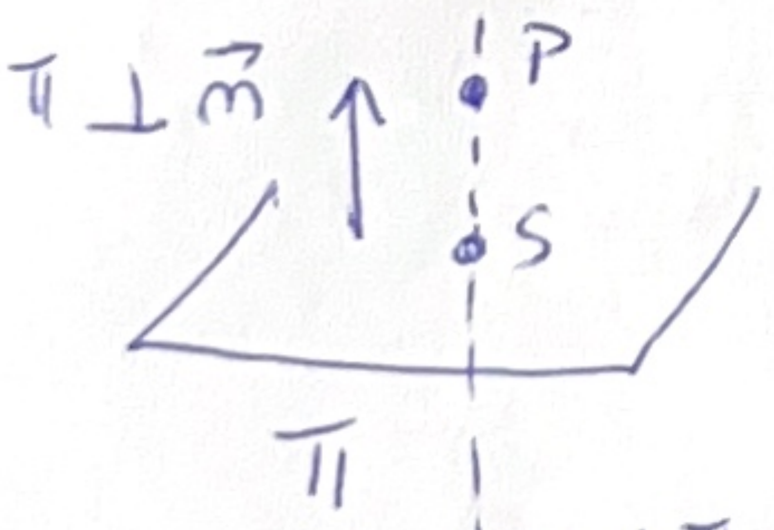
$[2,-1,3] \parallel k$
$(2,-3,1) \in k$

$k: \begin{cases} x = 2+2s \\ y = -3-s \\ z = 1+3s \end{cases}$ $s \in \mathbb{R}$

$k \cap l = ?$
$\begin{cases} 2+2s = 1-t \\ -3-s = 2+2t \\ 1+3s = 1-t \end{cases}$ $\rightarrow t = -1-2s$, $\Rightarrow 2+2s = 1+3s$, $s = 1$, $t = -3$

spr. $-3-1 = 2+2(-3)$
$-4 = 2-6$

$\downarrow s = 1$
$P = (4,-4,4)$



$\vec{n} = [1,1,1] \perp \Pi$, $m \perp \Pi \Rightarrow \vec{n} \parallel \Pi$, $P \in m$ $\Rightarrow m = \begin{cases} x = 4+r \\ y = -4+r \\ z = 4+r \end{cases}$ $r \in \mathbb{R}$

$m \perp \Pi = \{S\}$ $x+y+z+2=0$
$4+r-4+r+4+r+2=0$
$3r+6=0$ $r=-2$ $\rightarrow S = (2,-6,2)$

$P' = T_{\vec{PS}}(S)$
$\vec{PS} = [-2,-2,-2]$ $P' = (2-2, -6-2, -2+2) = (0,-8,0)$

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③ $A, B, C, D, E \in M_4(\mathbb{R})$ nieosobliwe $M = 3BCD^TET$

$\det M = \underbrace{3^4}_{81} \cdot \det B \cdot \underbrace{\det C}_{-3\det B} \cdot \underbrace{\det(ED)}_{= \frac{1}{12}} = -\frac{3 \cdot 81}{12} (\det B)^2$

$\det(A^{-1}BAB) = 4$
$\det(B^2) = (\det B)^2 = 4$
$\det(B^T) = \det B < 0 \Rightarrow \det B = -2$

$\det M = -\frac{3 \cdot 81}{12} \cdot 4 = -81$

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④ a) $\vec{a} = [1,4,0]$ $\vec{b} = [2,-1,50]$ $\vec{a} \circ \vec{b} = 2-4+0 < 0$ $\cos\measuredangle(\vec{a},\vec{b}) < 0$
kąt rozwarty   FAŁSZ

b) $\det A = \begin{vmatrix} 1 & p & 2 & p \\ 0 & 0 & 2 & -p \\ 0 & 0 & p & 2 \\ 0 & 2 & 0 & 0 \end{vmatrix} = 1 \cdot (-1)^2 \begin{vmatrix} 0 & 2 & -p \\ 0 & p & 2 \\ 2 & 0 & 0 \end{vmatrix} = 8 + p^2 \cdot 2 \geq 8$

$\forall p \in \mathbb{R}$ $\det A \neq 0$ PRAWDA