MALF - Kartkówka 1 - 25pkt — GRUPA 1 / 5'R

Zad. 1 $G = \mathbb{R}\times\mathbb{R}$ $\quad (x,y)\circ(a,b) = (x+a,\ y\cdot e^a + b)$

Działanie jest:

- wewnętrzne $x+a\in\mathbb{R}$, $y\cdot e^a+b\in\mathbb{R}$
- łączne: $\forall (x,y),(a,b),(c,d)\in G$
$L = [(x,y)\circ(a,b)]\circ(c,d) = (x+a, ye^a+b)\circ(c,d) = (x+a+c, (ye^a+b)\cdot e^c + d)$
$P = (x,y)\circ[(a,b)\circ(c,d)] = (x,y)\circ(a+c, b\cdot e^c+d) = (x+a+c, y\cdot e^{a+c} + be^c + d)$ $\quad L=P$
- nieprzemienne $(1,0)\circ(1,1) = (1+1, 0\cdot e^1+1) = (2,1) \neq (1,1)\circ(1,0) = (1+1, 1\cdot e^1+0) = (2,e)$
- el. neutralny $\bar{e} = (e_1,e_2) = ?$

$$\begin{cases} (x,y)\circ(e_1,e_2) = (x,y) \\ (e_1,e_2)\circ(x,y) = (x,y) \end{cases} \Leftrightarrow \begin{matrix} (x+e_1,\ ye^{e_1}+e_2) = (x,y) \\ (e_1+x,\ e_2\cdot e^x + y) = (x,y) \end{matrix} \Rightarrow \begin{matrix} x+e_1 = x,\ e_1 = 0 \\ ye^{e_1}+e_2 = y \\ e_2\cdot e^x + y = y,\ e_2\cdot e^x = 0,\ e_2 = 0 \end{matrix}$$

Spr. $y\cdot e^0 + 0 = y$ ✓

$\bar{e} = (0,0)\in\mathbb{R}^2$

- el. symetryczny do (x,y): $(x',y') = ?$

$$\begin{cases} (x,y)\circ(x',y') = (0,0) \\ (x',y')\circ(x,y) = (0,0) \end{cases} \Leftrightarrow \begin{matrix} (x+x',\ ye^{x'}+y') = (0,0) \\ (x'+x,\ y'e^x + y) = (0,0) \end{matrix} \Rightarrow \begin{matrix} x'+x=0,\ x' = -x\in\mathbb{R} \\ ye^{x'}+y' = 0,\ y' = -ye^{x'} = -ye^{-x} \end{matrix}$$

Spr. c.z.y. $y'e^x + y = 0$, $-ye^{-x}\cdot e^x + y = 0$ ok.

$(x',y') = (-x, -ye^{-x})\in\mathbb{R}^2$

GRUPA NIEABELOWA

Zad. 2 $z^3 = Re\left(e^{i\pi}\right)\cdot\dfrac{(-1+i)^{10}}{(1+i)^6}$

$Re(e^{i\pi}) = Re(\cos\pi + i\sin\pi) = \cos\pi = -1$

$-1+i = \sqrt{2}\left(-\frac{1}{\sqrt2}+\frac{1}{\sqrt2}i\right) = \sqrt2\, e^{\left(\frac{\pi}{2}+\frac{\pi}{4}\right)i} = \sqrt2\, e^{\frac{3}{4}\pi i}$

$1+i = \sqrt2\left(\frac{1}{\sqrt2}+\frac{1}{\sqrt2}i\right) = \sqrt2\, e^{\frac{\pi}{4}i}$

$$z^3 = -1\,\frac{(\sqrt2)^{10}e^{\frac{15}{2}\pi i}}{(\sqrt2)^6 e^{\frac{3}{2}\pi i}}$$

$z^3 = e^{\pi i}\cdot(\sqrt2)^4\cdot e^{\frac{12}{2}\pi i} = 4e^{\pi i} \Rightarrow$

$z=0$ nie spełnia, mocen $z\neq0$, $z = |z|e^{i\varphi}$

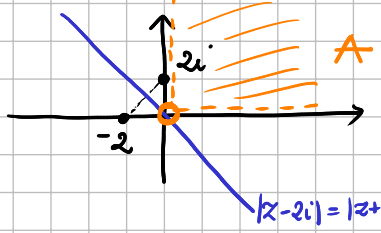
$|z|^3 e^{3i\varphi} = 4e^{\pi i} \Rightarrow |z|^3 = 4,\ |z| = \sqrt[3]{4}$

$3\varphi = \pi + 2k\pi$, $\varphi_k = \frac{\pi}{3} + \frac{2}{3}k\pi$, $k\in\{0,1,2\}$

$z_k = \sqrt[3]{4}\, e^{\varphi_k i}$, $k\in\{0,1,2\}$

Zad. 3 $A = \{z\in\mathbb{C}:\ \mathrm{Im}(z^2) > 0 \ \wedge\ |z-2i| \le |z+2|\}$

$z = x+iy$ $\quad z^2 = x^2 + 2ixy - y^2$ $\quad \mathrm{Im}(z^2) = 2xy > 0 \Leftrightarrow xy > 0$ — I i III ćw. bez brzegu



Zad. 4 $0 = z^5 - z^4 - 2z^3 + 2z^2 + z - 1 = z^4(z-1) - 2z^2(z-1) + (z-1) = (z-1)(z^4 - 2z^2 + 1)$

$= (z-1)(z^2-1)^2 = (z-1)^3(z+1)^2$

$z_1 = 1$ krotność 3
$z_2 = -1$ krotność 2
} 5 liczą z krotnościami.

Zad. 4 Czy $\left(\cos\frac{2}{81}\pi + i\sin\frac{2}{81}\pi\right)^{27} = -1$

FAŁSZ

$L = \cos\frac{2\cdot27}{3\cdot27}\pi + i\sin\frac{2\cdot27}{3\cdot27}\pi = \cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi \neq -1$

MALF - Kartkówka 1 - 25 pkt GRUPA 2 / Wtorek

Zad. 1 $G = (\mathbb{R}\setminus\{0\}) \times \mathbb{R}$ $\quad (a,b)*(c,d) = (ac,\ b\cdot c^2+d)$

Działanie jest

- wewnętrzne $(a,b), (c,d) \in G$ tzn. $a\neq 0, c\neq 0 \Rightarrow ac\neq 0 \Rightarrow (a,b)*(c,d)\in G$
- łączne $\forall (x,y), (a,b), (c,d) \in G$

$L = (x,y)*[(a,b)*(c,d)] = (x,y)*(ac, bc^2+d) = (xac, y\cdot a^2c^2 + bc^2 + d)$

$P = [(x,y)*(a,b)]*(c,d) = (xa, y\cdot a^2+b)*(c,d) = (xac, (ya^2+b)\cdot c^2 + d)$ $\quad L=P$

- nieprzemienne

$(2,1)*(1,1) = (2\cdot 1,\ 1\cdot 1^2+1) = (2,2) \neq (1,1)*(2,1) = (1\cdot 2,\ 1\cdot 2^1+1) = (2,3)$

- el. neutralny $\bar{e} = (e_1,e_2) = ?$

$$\begin{cases} (a,b)*(e_1,e_2) = (a,b) \\ (e_1,e_2)*(a,b) = (a,b) \end{cases} \Leftrightarrow \begin{matrix} (ae_1,\ be_1^2+e_2) = (a,b) \\ (e_1a,\ e_2a^2+b) = (a,b) \end{matrix} \Rightarrow \begin{matrix} ae_1 = a \overset{a\neq 0}{\Rightarrow} e_1 = 1 \\ be_1^2+e_2 = b \\ (e_1=1)\ b+e_2 = b,\ e_2 = 0 \end{matrix}$$

Spr. czy $e_2a^2+b=b$, $0\cdot a^2+b=b$ ✓

$\bar{e} = (1,0) \in G$ bo $1\neq 0$

- el. symetryczny do (a,b) : $(a',b') = ?$

$$\begin{cases} (a,b)*(a',b') = (1,0) \\ (a',b')*(a,b) = (1,0) \end{cases} \Leftrightarrow \begin{cases} (a\cdot a',\ b\cdot a'^2+b') = (1,0) \\ (a'a,\ b'a^2+b) = (1,0) \end{cases} \Rightarrow \begin{matrix} aa' = 1 \overset{a\neq 0}{\Rightarrow} a' = \frac{1}{a} \neq 0 \\ b'+ba'^2 = 0 \quad b' = -ba'^2 = -\frac{b}{a^2} \end{matrix} \quad a\neq 0$$

Spr. czy $b'a^2+b=0$, $-\frac{b}{a^2}\cdot a^2+b=0$ TAK

GRUPA NIEABELOWA

Zad. 2 $z^4 = \operatorname{Im}\left(e^{i\frac{\pi}{2}}\right)\cdot \frac{(1-i\sqrt{3})^9}{i^{13}}$

$\operatorname{Im}\left(e^{i\frac{\pi}{2}}\right) = \operatorname{Im}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$

$i^{13} = (i^4)^3\cdot i = 1\cdot i = i = e^{\frac{\pi}{2}i}$

$1-i\sqrt{3} = 2\left(\frac{1}{2}+\frac{-\sqrt{3}}{2}i\right) = 2e^{-\frac{\pi}{3}i}$

$z^4 = 1\cdot \frac{2^9 e^{-\frac{9}{3}\pi i}}{e^{\frac{\pi}{2}i}}$

$z^4 = 2^9\cdot \frac{e^{-3\pi i}}{e^{\frac{\pi}{2}i}} = 2^9 e^{-\frac{7}{2}\pi i}$

$z^4 = 2^9 e^{\frac{\pi}{2}i} = |z|^4 e^{4i\varphi} = 2^9 e^{\frac{\pi}{2}i}$

$z\neq 0 \rightarrow z = |z|e^{i\varphi}$

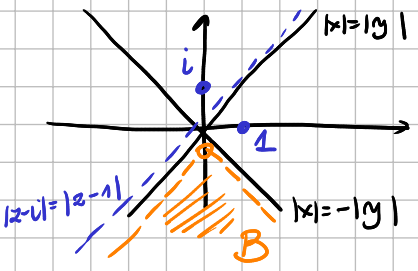
$|z^4| = \sqrt[4]{2^9} = 2^2\cdot 2^{\frac{1}{4}} = 4\sqrt[4]{2}$

$4\varphi = \frac{\pi}{2}+2k\pi$, $\varphi_k = \frac{\pi}{8}+k\frac{\pi}{2}$

$z_k = |z|\cdot e^{i\varphi_k} = 4\sqrt[4]{2}\, e^{\frac{\pi}{8}+k\frac{\pi}{2}}$, $k\in\{0,1,2,3\}$

Zad. 3 $B = \{z\in\mathbb{C} : \operatorname{Re}(z^2) < 0 \wedge |z-i| > |z-1|\}$

$z = x+iy$ $\quad z^2 = x^2+2ixy-y^2$ $\quad \operatorname{Re}(z^2) = x^2-y^2 < 0$ $\quad x^2 < y^2$, $|x| < |y|$



Zad. 4 $0 = z^5+z^4-6z^3-6z^2+6z+9 = z^4(z+1)-6z^2(z+1)+9(z+1) =$

$= (z+1)(z^4-6z^2+9) = (z+1)(z^2-3)^2 = (z+1)(z-\sqrt{3})^2(z+\sqrt{3})^2$

$z_1 = -1$, krotność 1 $\quad z_2 = \sqrt{3}$, krotność 2 $\quad z_3 = -\sqrt{2}$

/5 liczące z krotnościami.

Zad. 5 Czy $(1-i)^{20} = 0$?

$1-i = \sqrt{2}\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right) = \sqrt{2}\,e^{-\frac{\pi}{4}i}$

$(1-i)^{20} = (\sqrt{2})^{20} e^{-\frac{20}{4}\pi i} = 2^{10}e^{-5\pi i} = 2^{10}\cdot e^{\pi i} = 2^{10}(\underbrace{\cos\pi}_{-1}+i\underbrace{\sin\pi}_{0}) = -2^{10} \neq 0$

FAŁSZ