

Zadania domowe nr 6

Zad. 1 Czy to iloczynny skalarny?

a) $g: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R} \quad g(x, y) = 3x_1y_1 - 5x_1y_2 - 5x_2y_1 + 7x_2y_2$
 $x = (x_1, x_2), y = (y_1, y_2)$

NIE, bo nie jest $g(x, x) = 3x_1^2 - 10x_1x_2 + 7x_2^2 = x_1^2 + 2(x_1 - 2x_2)^2 - 2x_1x_2 - x_2^2$
 $2(x_1^2 - 4x_1x_2 + 4x_2^2)$

$g(1, 1) = 1 + 2 - 2 - 1 = 0$

Nieprawda, że $g(x, x) = 0 \Leftrightarrow x = (0, 0)$

b) $g: (\mathbb{R}_1, \mathbb{R}_2) \times (\mathbb{R}_1, \mathbb{R}_2) \rightarrow \mathbb{R} \quad g(p, q) = p(1)q(1) + 2p(2)q(2)$

TAK linijność na pierwszej zmiennej
 $\forall p_1, p_2, q \in (\mathbb{R}_1, \mathbb{R}_2) \quad \forall \alpha, \beta \in \mathbb{R} \quad g(\alpha p_1 + \beta p_2, q) = (\alpha p_1 + \beta p_2)(1)q(1) + 2(\alpha p_1 + \beta p_2)(2)q(2) =$
 $= [\alpha p_1(1) + \beta p_2(1)]q(1) + 2[\alpha p_1(2) + \beta p_2(2)]q(2) =$
 $= \alpha [p_1(1)q(1) + 2p_1(2)q(2)] + \beta [p_2(1)q(1) + 2p_2(2)q(2)] = \alpha g(p_1, q) + \beta g(p_2, q)$

symetria $g(q, p) = q(1)p(1) + 2q(2)p(2) = p(1)q(1) + 2p(2)q(2) = g(p, q)$

dodatnia określoność $g(p, p) = [p(1)]^2 + 2[p(2)]^2 \geq 0$

$g(p, p) = 0 \Leftrightarrow p(1) = 0 \wedge p(2) = 0 \quad p = ax + b$

$\begin{cases} p(1) = a + b = 0 \\ p(2) = 2a + b = 0 \end{cases} \Rightarrow \begin{matrix} a = 0 \\ b = -a = 0 \end{matrix} \Rightarrow p = 0$

Zad. 2 Norma

a) $f \in C([0, 2\pi], \mathbb{R}), f(x) = 2\sin x - \cos x \quad \forall f, g \in C([0, 2\pi], \mathbb{R}) \quad \langle f, g \rangle = \int_0^{2\pi} f(x)g(x)dx$

$\|f\| = \sqrt{\langle f, f \rangle} = \sqrt{\int_0^{2\pi} (2\sin x - \cos x)^2 dx} = \dots$

$\int (2\sin x - \cos x)^2 dx = \int (4\sin^2 x - 4\sin x \cos x + \cos^2 x) dx = \int (1 + 3\sin^2 x - 2\sin 2x) dx = \int (1 + \frac{3}{2}(1 - \cos 2x) - 2\sin 2x) dx$
 $= \frac{5}{2} \int dx - \frac{3}{2} \int \cos 2x dx - 2 \int \sin 2x dx =$
 $= \frac{5}{2}x - \frac{3}{2} \cdot \frac{1}{2} \sin 2x - 2(-\frac{1}{2} \cos 2x) + C = \frac{5}{2}x - \frac{3}{4} \sin 2x + \cos 2x + C$

$\|f\|^2 = \left[\frac{5}{2}x - \frac{3}{4} \sin 2x + \cos 2x \right]_0^{2\pi} = 5\pi + 1 - 1 = 5\pi \Rightarrow \|f\| = \sqrt{5\pi}$

b) $C \in M_2(\mathbb{R}), C = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \quad \forall A, B \in M_2(\mathbb{R}) \quad \langle A, B \rangle = \text{tr}(A \cdot B^T)$

$\|C\|^2 = \langle C, C \rangle = \text{tr}(C \cdot C^T)$

$\begin{matrix} & \begin{matrix} -1 & -1 \\ 0 & 1 \end{matrix} & (C^T) \\ \begin{matrix} -1 & 0 \\ -1 & 1 \end{matrix} & \begin{matrix} | & \\ \hline & \\ \hline & \end{matrix} & \begin{matrix} 1 & 1 \\ 1 & 2 \end{matrix} \\ (C) & & (C \cdot C^T) \end{matrix} \quad \|C\| = \sqrt{\text{tr}(C \cdot C^T)} = \sqrt{1+2} = \sqrt{3}$

Zad. 3 $W = \text{lin}\{w_1 = (3, 2, 0, 1, -4), w_2 = (1, 2, -2, 0, 1), w_3 = (3, -2, 6, -2, 5)\} \subset \mathbb{R}^5$

Baza W^\perp ?

$r \begin{bmatrix} 3 & 2 & 0 & 1 & -4 \\ 1 & 2 & -2 & 0 & 1 \\ 3 & -2 & 6 & -2 & 5 \end{bmatrix} \xrightarrow{\substack{R_1 - 3R_2 \\ R_3 - 3R_2}} r \begin{bmatrix} 0 & -4 & 6 & 1 & -7 \\ 1 & 2 & -2 & 0 & 1 \\ 0 & -8 & 12 & -2 & 2 \end{bmatrix} \xrightarrow{R_3 - 2R_2} r \begin{bmatrix} 1 & 2 & -2 & 0 & 1 \\ 0 & -4 & 6 & 1 & -7 \\ 0 & 0 & 0 & -4 & 16 \end{bmatrix} = 3$

$\{w_1, w_2, w_3\}$ baza W

$v \in W^\perp \Leftrightarrow v \perp W \Leftrightarrow \begin{cases} v \perp w_1 \\ v \perp w_2 \\ v \perp w_3 \end{cases} \Leftrightarrow \begin{bmatrix} 3 & 2 & 0 & 1 & -4 \\ 1 & 2 & -2 & 0 & 1 \\ 3 & -2 & 6 & -2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{cases} x + 2y - 2z + u = 0 \\ -4y + 6z + t - 7u = 0 \\ -4t + 16u = 0 \end{cases}$

Zad. 3 - avg dalez

$$t = 4u \Rightarrow \begin{cases} x + 2y - 2z + u = 0 \\ -4y + 6z - 3u = 0 \end{cases} \quad \begin{matrix} x = -2y + 2z - u \\ u = -\frac{4}{3}y + 2z \end{matrix} \quad \begin{matrix} x = -2y + 2z + \frac{4}{3}y - 2z = \frac{2}{3}y \\ t = 4u = -\frac{16}{3}y + 8z \end{matrix}$$

$$v \in W^\perp \Rightarrow v = \left(\frac{2}{3}y, y, z, -\frac{16}{3}y + 8z, -\frac{4}{3}y + 2z \right)$$

$$W^\perp = \text{lin} \left\{ \left(\frac{2}{3}, 1, 0, -\frac{16}{3}, -\frac{4}{3} \right), (0, 0, 1, 8, 2) \right\} = \text{lin} \left\{ \begin{matrix} (2, 3, 0, -16, -4) \\ (0, 0, 1, 8, 2) \end{matrix} \right\}$$

$\begin{matrix} \uparrow & \uparrow \\ \text{BAZA} & W^\perp \end{matrix}$

Zad. 4

$$\mathbb{R}_3[x] \quad \forall p, q \in \mathbb{R}_3[x] \quad \langle p, q \rangle = aa_1 + (b-c)(b_1-c_1) + (2c-b)(2c_1-b_1) + dd_1$$

$$p = ax^3 + bx^2 + cx + d$$

$$q = a_1x^3 + b_1x^2 + c_1x + d_1$$

• $\mathcal{B} = (b_1=2, b_2=x+x^2, b_3=x+2x^2, b_4=3x^3)$ baza ortogonalna?

$\mathcal{C} = (1, x, x^2, x^3)$ baza kanoniczna $\mathbb{R}_3[x]$

$$\begin{matrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{matrix} \text{ w } \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \xrightarrow{W_3 - W_2} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = 4 \Rightarrow \mathcal{B} \text{ - to baza } \mathbb{R}_3[x]$$

d c b a

$$\langle b_1, b_2 \rangle = 0 + 0 \cdot 0 + 0 \cdot 1 + 2 \cdot 0 = 0$$

$$\langle b_2, b_3 \rangle = 0 \cdot 0 + 0 \cdot 1 + 1 \cdot 0 + 0 \cdot 0 = 0$$

$$\langle b_1, b_3 \rangle = 0 + 0 \cdot 1 + 0 \cdot 0 + 2 \cdot 0 = 0$$

$$\langle b_2, b_4 \rangle = 0 \cdot 3 + 0 \cdot 0 + 1 \cdot 0 + 0 \cdot 0 = 0$$

$$\langle b_1, b_4 \rangle = 0 \cdot 3 + 0 \cdot 0 + 0 \cdot 0 + 2 \cdot 0 = 0$$

$$\langle b_3, b_4 \rangle = 0 \cdot 3 + 1 \cdot 0 + 0 \cdot 0 + 0 \cdot 0 = 0$$

baza
jest ortogonalna

• $v(x) = x^2 - x + 1$

$$v = [1, -1, 1, 0]_{\mathcal{C}} \quad v = [\alpha, \beta, \gamma, \delta]_{\mathcal{B}}$$

$$\|b_1\|^2 = 0 + 0 + 0 + 4 \quad \|b_2\|^2 = 0 + 0 + 1 + 0 = 1 \quad \|b_3\|^2 = 0 + 1 + 0 + 0 = 1 \quad \|b_4\|^2 = 9$$

$$\alpha = \frac{\langle v, b_1 \rangle}{\|b_1\|^2} = \frac{1}{4} (0 + 0 + 0 + 2) = \frac{1}{2}$$

$$\beta = \frac{\langle v, b_2 \rangle}{\|b_2\|^2} = \langle v, b_2 \rangle = 0 + 0 + 1 \cdot (-3) + 0 = -3$$

$$\gamma = \frac{\langle v, b_3 \rangle}{\|b_3\|^2} = \langle v, b_3 \rangle = 0 + 1 \cdot 2 + 0 + 0 = 2$$

$$\delta = \frac{\langle v, b_4 \rangle}{\|b_4\|^2} = \frac{1}{9} (0 + 0 + 0 + 0) = 0$$

$$v = \left[\frac{1}{2}, -3, 2, 0 \right]_{\mathcal{B}}$$

Zad. 5

$$\mathbb{R}^4 \supset U = \text{lin} \{ b_1 = (1, 1, -1, 0), b_2 = (0, 2, -1, 1), b_3 = (1, 5, -3, 0) \}$$

a) Metoda G-S : $\mathcal{C} = (c_1, c_2, c_3)$ - szukana baza ortogonalna

• $c_1 = b_1 = (1, 1, -1, 0)$

• $c_2 \in \{c_1\}^\perp \wedge \text{lin} \{b_1, b_2\} = \text{lin} \{c_1, c_2\} \Rightarrow \begin{cases} c_2 = b_2 + \alpha c_1 \\ c_2 \circ c_1 = 0 \end{cases}$

$$c_2 \circ c_1 = (b_2 + \alpha c_1) \circ c_1 = b_2 \circ c_1 + \alpha \|c_1\|^2 = (0, 2, -1, 1) \circ (1, 1, -1, 0) + \alpha \cdot (1+1+1) = (2+1) + 3\alpha - 3 + 3\alpha = 0, \underline{\underline{\alpha = -1}}$$

$$c_2 = b_2 - c_1 = (-1, 1, 0, 1)$$

• $c_3 \in \{c_1, c_2\}^\perp \wedge \text{lin} \{b_1, b_2, b_3\} = \text{lin} \{c_1, c_2, c_3\} \Rightarrow c_3 = b_3 + \alpha c_1 + \beta c_2 \wedge c_3 \circ c_1 = 0 \wedge c_3 \circ c_2 = 0$

$$0 = c_3 \circ c_1 = (b_3 + \alpha c_1 + \beta c_2) \circ c_1 = b_3 \circ c_1 + \alpha \|c_1\|^2 + \beta \cdot 0 = (1, 5, -3, 0) \circ (1, 1, -1, 0) + \alpha \cdot 3 = 1+5+3+3\alpha = 9+3\alpha$$

$$0 = c_3 \circ c_2 = b_3 \circ c_2 + \beta \|c_2\|^2 = (1, 5, -3, 0) \circ (-1, 1, 0, 1) = -1+5 = 4+3\beta \Rightarrow \beta = -\frac{4}{3}$$

$$\underline{\underline{\alpha = -3}}$$

$$c_3 = (1, 5, -3, 0) - 3 \cdot (1, 1, -1, 0) - \frac{4}{3} \cdot (-1, 1, 0, 1) = \left(-\frac{2}{3}, \frac{2}{3}, 0, -\frac{4}{3} \right) = \frac{2}{3} (-1, 1, 0, -1)$$

Zad. 5 b) Rzut ortogonalny $v = (1, 0, 1, 0) \in \mathbb{R}^4$ na \mathcal{U}

$u = \Pi_{\mathcal{U}}(v) = ?$ $W = v - u \perp \mathcal{U} = \text{lin}\{c_1, c_2, c_3\} \Leftrightarrow$

$u = [\alpha, \beta, \gamma]_{\mathcal{C}}$ $\alpha = \frac{\langle v, c_1 \rangle}{\|c_1\|^2} = \frac{1}{3} \cdot (1, 0, 1, 0) \cdot (1, 1, -1, 0) = \frac{1}{3} \cdot 0 = 0$

$\beta = \frac{\langle v, c_2 \rangle}{\|c_2\|^2} = \frac{1}{3} \cdot (1, 0, 1, 0) \cdot (-1, 1, 0, 1) = -\frac{1}{3}$

$\gamma = \frac{\langle v, c_3 \rangle}{\|c_3\|^2} = \frac{(1, 0, 1, 0) \cdot (-1, 1, 0, -2) \cdot \frac{2}{3}}{(\frac{2}{3})^2 \cdot 6} = (-1) \cdot \frac{3}{2} \cdot \frac{1}{6} = -\frac{1}{4}$

$v = [0, -\frac{1}{3}, -\frac{1}{4}]_{\mathcal{C}} = -\frac{1}{3}c_2 - \frac{1}{4}c_3 = -\frac{1}{3}(-1, 1, 0, 1) - \frac{1}{4}(-1, 1, 0, -2) \cdot \frac{2}{3} = (\frac{1}{3}, -\frac{1}{3}, 0, -\frac{1}{3}) + (\frac{1}{6}, -\frac{1}{6}, 0, \frac{1}{3})$
 $= (\frac{1}{2}, -\frac{1}{2}, 0, 0)$

Zad. 6. (u_1, u_2) uzupełnić do bazy \mathbb{R}^4 $u_1 = (1, 1, 1, 0), u_2 = (0, 1, -1, 1)$

$u_1 \circ u_2 = 0 + 1 - 1 + 0 = 0$ $u_1 \perp u_2$

$\alpha \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 4$ $u_3 = c_3, u_4 = c_4 \Rightarrow (u_1, u_2, u_3, u_4)$ baza \mathbb{R}^4
 $u_3 \perp u_4$ $u_1 \circ u_3 = 1 \neq 0$ nie ortogonalna

$\mathcal{C} = (c_1, c_2, c_3, c_4)$ szukana baza ortogonalna

$c_1 = u_1, c_2 = u_2$ bo $u_1 \circ u_2 = 0$

$c_3 = u_3 + \alpha c_1 + \beta c_2 \wedge 0 = c_1 \circ c_3, 0 = c_2 \circ c_3$

$0 = c_1 \circ c_3 = c_1 \circ (u_3 + \alpha c_1 + \beta c_2) = c_1 \circ u_3 + \alpha \cdot \|c_1\|^2 = (1, 1, 1, 0) \cdot (0, 0, 1, 0) + \alpha \cdot 3 = 1 + 3\alpha$
 $0 = c_2 \circ c_3 = c_2 \circ u_3 + \beta \cdot \|c_2\|^2 = (0, 1, -1, 1) \cdot (0, 0, 1, 0) + 3\beta = -1 + 3\beta \Rightarrow \beta = \frac{1}{3}$ $\alpha = -\frac{1}{3}$

$c_3 = u_3 - \frac{1}{3}c_1 + \frac{1}{3}c_2 = (0, 0, 1, 0) + (-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, 0) + (0, \frac{1}{3}, -\frac{1}{3}, \frac{1}{3}) = (\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3})$

$c_4 = u_4 + \alpha c_1 + \beta c_2 + \gamma c_3$

$0 = c_4 \circ c_1 = u_4 \circ c_1 + \alpha \cdot \|c_1\|^2 = (0, 0, 0, 1) \cdot (1, 1, 1, 0) + 3\alpha = 3\alpha, \alpha = 0$

$0 = c_4 \circ c_2 = u_4 \circ c_2 + \beta \cdot \|c_2\|^2 = (0, 0, 0, 1) \cdot (0, 1, -1, 1) + 3\beta = 3\beta + 1, \beta = -\frac{1}{3}$

$0 = c_4 \circ c_3 = u_4 \circ c_3 + \gamma \cdot \|c_3\|^2 = (0, 0, 0, 1) \cdot (\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}) + \frac{2}{3}\gamma = \frac{1}{3} + \frac{2}{3}\gamma, \gamma = -\frac{1}{2}$

$\Rightarrow c_4 = u_4 - \frac{1}{3}c_2$

$c_4 = (0, 0, 0, 1) - (-\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}) = (\frac{1}{3}, -\frac{1}{3}, 0, \frac{1}{3})$

II SPOSÓB

$u = \Pi_{\mathcal{U}}(v)$ rzut v na \mathcal{U} , $W = v - u \perp \mathcal{U}$

$W = ?$ $v = (1, 0, 1, 0) \in \mathcal{U}, u = \alpha \cdot b_1 + \beta \cdot b_2 + \gamma \cdot b_3$

$W = (1 - \alpha - \gamma, -\alpha - 2\beta - 5\gamma, 1 + \alpha + \beta + 3\gamma, -\beta)$

$0 = b_1 \circ W = (1, 1, -1, 0) \circ W = 1 - \alpha - \gamma - \alpha - 2\beta - 5\gamma - 1 + \alpha + \beta + 3\gamma = -3\alpha - 3\beta - 9\gamma \Rightarrow \alpha + \beta + 3\gamma = 0$

$0 = b_2 \circ W = (0, 2, -1, 1) \circ W = -2\alpha - 4\beta - 10\gamma - 1 - \alpha - \beta - 3\gamma - \beta = -1 - 3\alpha - 6\beta - 13\gamma \Rightarrow 3\alpha + 6\beta + 13\gamma = -1$

$0 = b_3 \circ W = (1, 5, -3, 0) \circ W = 1 - \alpha - \gamma - 5\alpha - 10\beta - 25\gamma - 3 - 3\alpha - 3\beta - 9\gamma = -2 - 9\alpha - 13\beta - 35\gamma = -2$ $9\alpha + 13\beta + 35\gamma = -2$

$\begin{bmatrix} 1 & 1 & 3 & | & 0 \\ 3 & 6 & 13 & | & -1 \\ 9 & 13 & 35 & | & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 3 & | & 0 \\ 0 & 3 & 4 & | & -1 \\ 0 & 4 & 8 & | & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 3 & | & 0 \\ 0 & 1 & \frac{4}{3} & | & -\frac{1}{3} \\ 0 & 2 & 4 & | & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 3 & | & 0 \\ 0 & 1 & \frac{4}{3} & | & -\frac{1}{3} \\ 0 & 0 & \frac{4}{3} & | & -\frac{4}{3} \end{bmatrix}$

$4\gamma = -\frac{4}{3} \Rightarrow \gamma = -\frac{1}{4}$

$\beta = -\frac{1}{3} - \frac{4}{3}\gamma = -\frac{1}{3} + \frac{1}{3} = 0$

$\alpha = -\beta - 3\gamma = -3 \cdot (-\frac{1}{4}) = \frac{3}{4}$

$u = [\alpha, \beta, \gamma]_{\mathcal{B}} = [\frac{3}{4}, 0, -\frac{1}{4}]_{\mathcal{B}} =$

$= \frac{3}{4}(1, 1, -1, 0) - \frac{1}{4}(1, 5, -3, 0) = \frac{1}{4}(2, -2, 0, 0) = (\frac{1}{2}, -\frac{1}{2}, 0, 0)$