2024/25 Informatyka stosowana - Analiza - Kartkówka 2 I: 25pkt

Gr 2

Zad.1 $f(x)=\operatorname{arctg}^5(\sqrt[5]{x})$

$f'(0)=?$

$f'(x)=5\operatorname{arctg}^4(\sqrt[5]{x})\cdot\frac{1}{1+\sqrt[5]{x^2}}\cdot\frac{1}{5}\frac{1}{\sqrt[5]{x^4}}\quad x\neq0$

$f'(0)=\lim_{x\to0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0}\frac{\operatorname{arctg}^5(\sqrt[5]{x})-0}{x-0}$

$=\lim_{x\to0}\left(\frac{\operatorname{arctg}\sqrt[5]{x}}{\sqrt[5]{x}}\right)^5=1$

Zad.2 Wklęsłość / wypukłość / p. przegięcia

$f(x)=e^{\sqrt[5]{x}}$

$D_f=\mathbb{R}\quad f'(x)=e^{\sqrt[5]{x}}\cdot\frac{1}{5}x^{-4/5}\quad x\neq0$

$f''(x)=e^{\sqrt[5]{x}}\cdot\frac{1}{25}x^{-8/5}+e^{\sqrt[5]{x}}\cdot\left(\frac{-4}{25}\right)\cdot x^{-9/5}\quad x\neq0$

$=e^{\sqrt[5]{x}}\cdot\frac{1}{25}\left(x^{-8/5}-4x^{-9/5}\right)=\frac{e^{\sqrt[5]{x}}}{25}\frac{1}{\sqrt[5]{x^8}}\left(1-\frac{4}{\sqrt[5]{x}}\right)$

$f''(x)>0\iff1-\frac{4}{\sqrt[5]{x}}>0\iff\frac{\sqrt[5]{x}-4}{\sqrt[5]{x}}>0\iff$

$\iff\sqrt[5]{x}(\sqrt[5]{x}-4)>0\iff x(\sqrt[5]{x}-4)>0\iff$

$\iff\begin{cases}x>0\\ \sqrt[5]{x}>4\end{cases}\vee\begin{cases}x<0\\ \sqrt[5]{x}<4\end{cases}\Rightarrow x>4^5\vee x<0$

f wypukła: $x\in(-\infty,0)$, $x\in(4^5,+\infty)$

f wklęsła: $x\in(0,4^5)$

f ciągła na $\mathbb{R}$ $\Rightarrow$ $(0,f(0))=(0,1)$, $(4^5,f(4^5))=(4^5,e^4)$ p. przegięcia

Zad.3 Oblicz granicę

a) $\lim_{x\to1}\frac{x^x-x}{\ln x-x+1}\overset{[\frac00]}{=}\lim_{x\to1}\frac{e^{x\ln x}-x}{\ln x-x+1}\overset{H}{=}$

$=\lim_{x\to1}\frac{e^{x\ln x}\cdot(\ln x+x\cdot\frac1x)-1}{\frac1x-1}=\lim_{x\to1}\frac{e^{x\ln x}(\ln x+1)-1}{\frac1x-1}$

$\overset{[\frac00]}{\underset{H}{=}}\lim_{x\to1}\frac{e^{x\ln x}(\ln x+1)^2+e^{x\ln x}\cdot\frac1x}{-\frac{1}{x^2}}=\frac{1^2+1}{-1}=-2$

b) $\lim_{x\to0^-}\left(\frac1x-\operatorname{ctg}x\right)\overset{[\infty-\infty]}{=}\lim_{x\to0^-}\left(\frac1x-\frac{1}{\operatorname{tg}x}\right)=$

$=\lim_{x\to0^-}\frac{\operatorname{tg}x-x}{x\cdot\operatorname{tg}x}\overset{[\frac00]}{\underset{H}{=}}\lim_{x\to0^-}\frac{\frac{1}{\cos^2x}-1}{\operatorname{tg}x+\frac{x}{\cos^2x}}=$

$=\lim_{x\to0^+}\frac{1-\cos^2x}{\sin x\cos x+x}=\lim_{x\to0}\frac{\sin^2x}{x+\frac12\sin2x}\overset{[\frac00]}{\underset{H}{=}}$

$=\lim_{x\to0^+}\frac{2\sin x\cos x}{1+\frac12\cdot\cos2x\cdot2}=\frac{0}{1+1}=0$

Zad.4 Wartość najmn. / najwk.

$f(x)=\frac{\sqrt3}{3}\cos x+\sin x\quad x\in[2\pi,\frac52\pi]$ f ciągła w przedziale domkn. $\to$ osiąga kresy

BRZEG: $f(2\pi)=\frac{\sqrt3}{3}\quad f(\frac52\pi)=1$

WNĘTRZE: $f'(x)=\frac{\sqrt3}{3}(-\sin x)+\cos x=0$, $\frac{\sqrt3}{3}\sin x=\cos x$, $\operatorname{tg}x=\sqrt3$

$x=\frac\pi3+k\pi,\ k\in\mathbb{Z}\ \wedge\ x\in[2\pi,\frac52\pi]\Rightarrow x=\frac73\pi$

$f(\frac73\pi)=\frac{\sqrt3}{3}\cdot\frac12+\frac{\sqrt3}{2}=\frac{\sqrt3}{2}\cdot\frac43=\frac23\sqrt3=\frac{2}{\sqrt3}$

W. NAJMN. $f(2\pi)$

W. NAJWIĘK. $f(\frac73\pi)$

Gr 3

Zad.1 $f(x)=\sin^7(\sqrt[7]{x-1})\qquad f'(1)=?$

$f'(x)=7\sin^6\sqrt[7]{x-1}\cdot\cos\sqrt[7]{x-1}\cdot\frac17\frac{1}{\sqrt[7]{(x-1)^6}}\quad x\neq1$

$f'(1)=\lim_{x\to1}\frac{f(x)-f(1)}{x-1}=\lim_{x\to1}\frac{\sin^7(\sqrt[7]{x-1})-0}{x-1}=$

$=\lim_{x\to1}\left(\frac{\sin\sqrt[7]{x-1}}{\sqrt[7]{x-1}}\right)^7=1$

Zad.2 Wypukłość / wklęsłość / p.p.

$f(x)=\frac{1}{\sqrt x}\log_{\frac12}x\qquad D_f:(0,\infty)$

$f'(x)=-\frac12x^{-3/2}\cdot\log_{\frac12}x+\frac{1}{\sqrt x}\frac{1}{x\cdot\ln\frac12}\quad x>0$

$f''(x)=\frac34x^{-5/2}\cdot\log_{\frac12}x-\frac12x^{-3/2}\frac{1}{x\cdot\ln\frac12}+\quad x>0$

$+\left(-\frac12\right)x^{-3/2}\cdot\frac{1}{x\cdot\ln\frac12}+\frac{1}{\sqrt x}\cdot\frac{-1}{x^2\cdot\ln\frac12}$

$f''(x)=\frac{-1}{\ln\frac12}\cdot\frac{1}{x^2\sqrt x}-\frac{1}{\ln\frac12}\cdot\frac{1}{x^2\sqrt x}+\frac34\cdot\frac{1}{x^2\sqrt x}\log_{\frac12}x$

$=\underbrace{\frac{1}{x^2\sqrt x}}_{>0}\left(\frac{-2}{\ln\frac12}+\frac34\log_{\frac12}x\right)>0\iff\frac34\log_{\frac12}x>\frac{2}{\ln\frac12}$

$\log_{\frac12}x>\frac{8}{3\ln\frac12}=\log_{\frac12}\left(\frac12\right)^{\frac{8}{3\ln\frac12}},\quad x<\frac12^{\frac{8}{3\ln\frac12}}$

f. wypukła: $x\in\left(0,\left(\frac12\right)^{\frac{8}{3\ln\frac12}}\right)$

f. wklęsła: $x\in\left(\left(\frac12\right)^{\frac{8}{3\ln\frac12}},\infty\right)$

$P=\left(\left(\frac12\right)^{\frac{8}{3\ln\frac12}},f\left(\left(\frac12\right)^{\frac{8}{3\ln\frac12}}\right)\right)$ p. przegięcia

Zad.3 a) $\lim_{x\to0^+}[\ln(1+x)]^x\overset{[0^0]}{=}\lim_{x\to0^+}e^{x\cdot\ln(\ln(1+x))}$

$\lim_{x\to0^+}\underset{\to0}{x}\cdot\underset{-\infty}{\ln}[\ln(1+x)]\overset{[0\cdot\infty]}{=}\lim_{x\to0^+}\frac{\ln[\ln(1+x)]}{\frac1x}\overset{[\frac\infty\infty]}{\underset{H}{=}}$

$=\lim_{x\to0^+}\frac{\frac{1}{\ln(1+x)}\cdot\frac{1}{1+x}}{-\frac{1}{x^2}}=\lim_{x\to0^+}\frac{-x^2}{(1+x)\cdot\ln(1+x)}\overset{[\frac00]}{\underset{H}{=}}$

$=\lim_{x\to0^+}\frac{-2x}{\ln(1+x)+1}=\frac01=0$ Odp. $e^0=1$

b) $\lim_{x\to0}\left(\frac{e^x}{e^x-1}-\frac1x\right)\overset{[\infty-\infty]}{=}\lim_{x\to0}\frac{xe^x-e^x+1}{xe^x-x}\overset{[\frac00]}{\underset{H}{=}}$

$=\lim_{x\to0}\frac{\cancel{e^x}+xe^x-\cancel{e^x}}{e^x+xe^x-1}\overset{[\frac00]}{\underset{H}{=}}\lim_{x\to0}\frac{e^x+xe^x}{e^x+e^x+xe^x}=\frac12$

Zad.4 $f(x)=-\arcsin\frac{1}{\sqrt x}+\arccos(1-x)$ w całej dziedzinie

$x\neq0$

$\left|\frac{1}{\sqrt x}\right|\le1,\ 1\le\sqrt x,\ 1\le x$

$|1-x|\le1,\ -1\le1-x\le1,\ x\le2\wedge x\ge0$

$\Rightarrow[1,2]=D_f$ domkn. f. ciągła $\to$ osiąga kresy

BRZEG

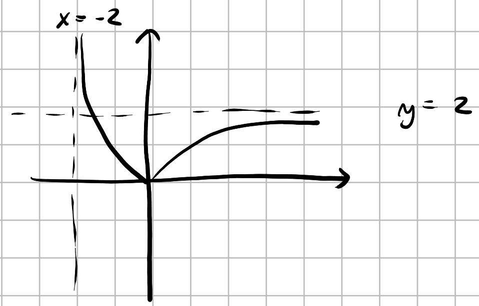
$f(1)=-\arcsin1+\arccos0=-\frac\pi2+\frac\pi2=0$ ← w. najmn.

$f(2)=-\arcsin\frac{1}{\sqrt2}+\arccos(-1)=-\frac\pi4+\pi=\frac34\pi$ ← w. najwk.

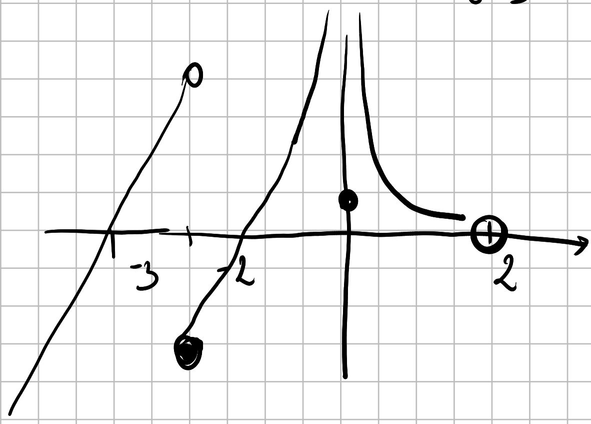
WNĘTRZE

$f'(x)=-\frac{1}{\sqrt{1-\frac1x}}\cdot\frac{-1}{2}\frac{1}{x\sqrt x}+\frac{-1}{\sqrt{1-(1-x)^2}}\cdot(-1)>0$ na $[1,2]$ $\quad\forall x\in[1,2]\ f'(x)\neq0$

Gr. 2



Gr 3



Zad. 4 / Gr. 3

$$f(x) = + \arcsin \frac{1}{\sqrt{x}} + \arccos(1-x)$$

$$D_f = [1,2]$$

dla $x \in (1,2)$: $f'(x) = \dfrac{1}{\sqrt{1-\frac{1}{x}}} \cdot \dfrac{-1}{2} \dfrac{1}{x\sqrt{x}} + \dfrac{-1}{\sqrt{1-(1-x)^2}} \cdot (-1)$

$$f'(x) = \frac{-1}{2x\sqrt{x-1}} + \frac{1}{\sqrt{2x-x^2}} = 0 \iff 2x\sqrt{x-1} = \sqrt{2x-x^2} \quad /^2 \quad \begin{smallmatrix} x>0 \\ \text{z zał} \end{smallmatrix}$$

$$4x^2(x-1) = 2x - x^2 \quad /:x \quad \begin{smallmatrix} x \neq 0 \\ \text{z zał} \end{smallmatrix}$$

$$4x^2 - 3x - 2 = 0$$

$\Delta = ?$ $x_1 = ?$ $x_2 = ?$