Informatyka stosowana – ANALIZA – Kartkówka 3

$\Sigma: 25$ pkt

Gr. 2

Zad. 1 Długość łuku krzywej

$$\Gamma: \begin{cases} x(t) = 5\cos^3\frac{t}{2} \\ y(t) = 5\sin^3\frac{t}{2} \end{cases} ; \ t \in [0, 2\pi]$$

$$x'(t) = 5\cdot 3\cos^2\tfrac{t}{2}\left(-\sin\tfrac{t}{2}\right)\cdot\tfrac{1}{2} = -\tfrac{15}{2}\cos^2\tfrac{t}{2}\sin\tfrac{t}{2}$$

$$y'(t) = 5\cdot 3\sin^2\tfrac{t}{2}\cdot\cos\tfrac{t}{2}\cdot\tfrac{1}{2} = \tfrac{15}{2}\sin^2\tfrac{t}{2}\cos\tfrac{t}{2}$$

$$(x')^2+(y')^2 = \frac{225}{4}\left(\cos^4\tfrac{t}{2}\sin^2\tfrac{t}{2} + \sin^4\tfrac{t}{2}\cos^2\tfrac{t}{2}\right) =$$

$$= \frac{225}{4}\cos^2\tfrac{t}{2}\sin^2\tfrac{t}{2}\underbrace{\left(\cos^2\tfrac{t}{2}+\sin^2\tfrac{t}{2}\right)}_{1} = \frac{225}{16}\left(2\sin\tfrac{t}{2}\cos\tfrac{t}{2}\right)^2$$

$$= \frac{225}{16}\sin^2 t$$

$$L = \int_0^{2\pi}\sqrt{(x')^2+(y')^2}\,dt = \frac{15}{4}\int_0^{2\pi}\sqrt{\sin^2 t}\,dt = \frac{15}{4}\int_0^{2\pi}|\sin t|\,dt$$

$$= \frac{15}{4}\int_0^{\pi}\sin t\,dt - \frac{15}{4}\int_{\pi}^{2\pi}\sin t\,dt = \frac{15}{4}(-\cos t)\Big|_0^{\pi} - \frac{15}{4}(-\cos t)\Big|_{\pi}^{2\pi} =$$

$$= \frac{15}{4}(1+1) - \frac{15}{4}(-1+(-1)) = 15$$

Zad. 2

a) $$\int\frac{x^3\,dx}{\sqrt{36-x^4}} = \int\frac{x^2 x\,dx}{\sqrt{36-(x^2)^2}} = \left|\begin{matrix} x^2 = t \\ 2x\,dx = dt\end{matrix}\right| =$$

$$= \frac{1}{2}\int\frac{t\,dt}{\sqrt{36-t^2}} = -\frac{1}{2}\int\frac{-2t}{2\sqrt{36-t^2}}\,dt = -\frac{1}{2}\sqrt{36-t^2}+C =$$

$$= -\frac{1}{2}\sqrt{36-x^4}+C$$

$$\int_0^{\sqrt{6}}\frac{x^3\,dx}{\sqrt{36-x^4}} = \lim_{T\to\sqrt{6}^-}\int_0^T\frac{x^3\,dx}{\sqrt{36-x^4}} = \lim_{T\to\sqrt{6}^-}\left[-\frac{1}{2}\sqrt{36-x^4}\right]_0^T =$$

$$= \lim_{T\to\sqrt{6}^-}\Big[-\frac{1}{2}\underbrace{\sqrt{36-T^4}}_{\to 0}+\frac{1}{2}\cdot 6\Big] = 3$$

b) $$\int\frac{1}{x^3}\ln^2(2x)\,dx = \left|\begin{matrix} f' = \frac{1}{x^3} & g = \ln^2(2x) \\ f = \frac{-1}{2x^2} & g' = 2\ln(2x)\cdot\frac{1}{2x}\cdot 2 = \frac{2\ln(2x)}{x}\end{matrix}\right|$$

$$= \frac{-1}{2x^2}\cdot\ln^2(2x) + \int\frac{1}{2x^2}\cdot\frac{2\ln(2x)}{x}\,dx = \frac{-1}{2x^2}\ln^2(2x)$$

$$+\int\frac{1}{x^3}\ln(2x)\,dx = \left|\begin{matrix} u' = \frac{1}{x^3} & v = \ln(2x) \\ u = \frac{-1}{2x^2} & v' = \frac{1}{2x}\cdot 2 = \frac{1}{x}\end{matrix}\right| =$$

$$= -\frac{1}{2x^2}\ln^2(2x) - \frac{1}{2x^2}\ln(2x) + \frac{1}{2}\int\frac{dx}{x^3} =$$

$$= -\frac{1}{2x^2}\left[\ln^2(2x)+\ln(2x)+\frac{1}{2}\right]+C$$

c) $$\underbrace{\int\frac{x^2+4x-6}{x^2(x^2-4x+6)}\,dx}_{I} = \int\left[\frac{A}{x}+\frac{B}{x^2}+\frac{C(2x-4)+D}{x^2-4x+6}\right]dx$$

$$x^2+4x-6 = Ax(x^2-4x+6)+B(x^2-4x+6)+$$
$$+C(2x-4)x^2+Dx^2 = A(x^3-4x^2+6x)+Bx^2-4Bx+6B$$
$$+2Cx^3-4Cx^2+Dx^2$$

$x^3: \quad 0 = A+2C \quad\to\quad \underline{C=0}$

$x^2: \quad 1 = -4A+B-4C+D \quad\to\quad 1 = -1-4C+D \qquad \underline{D=2}$

$x^1: \quad 4 = 6A-4B \quad\to\quad 6A = 4+4B = 0, \ \underline{A=0}$

$x^0: \quad -6 = 6B \quad\to\quad \underline{B=-1}$

$$I = -\int\frac{dx}{x^2}+2\int\frac{dx}{x^2-4x+6} = \frac{1}{x}+2\int\frac{dx}{(x-2)^2+2} = \frac{1}{x}+\int\frac{dx}{1+\left(\frac{x-2}{\sqrt{2}}\right)^2}$$

$$\left|\begin{matrix} x-2 = \sqrt{2}t \\ dx = \sqrt{2}\,dt\end{matrix}\right| = \frac{1}{x}+\sqrt{2}\int\frac{dt}{1+t^2} = \frac{1}{x}+\sqrt{2}\,\mathrm{arctg}\,\frac{x-2}{\sqrt{2}}+C$$

Gr. 3

Zad. 1 Długość łuku krzywej

$$\Gamma: \begin{cases} x(t) = \cos^3 4t \\ y(t) = -\sin^3 4t \end{cases} \qquad t \in \left[0, \tfrac{\pi}{4}\right]$$

$$x'(t) = 3\cos^2 4t(-\sin 4t)\cdot 4 = -12\cos^2 4t\sin 4t$$

$$y'(t) = -3\sin^2 4t\cos 4t\cdot 4 = -12\sin^2 4t\cos 4t$$

$$(x')^2+(y')^2 = 144\left[\cos^4 4t\sin^2 4t+\sin^4 4t\cos^2 4t\right] =$$

$$= 144\cos^2 4t\sin^2 4t\underbrace{(\cos^2 4t+\sin^2 4t)}_{1} = 36\cdot\sin 8t$$

$$L = \int_0^{\pi/4}\sqrt{36\sin 8t}\,dt = 6\int_0^{\pi/4}|\sin 8t|\,dt = 6\int_0^{\pi/8}\sin 8t\,dt - 6\int_{\pi/8}^{\pi/4}\sin 8t\,dt$$

$$T = \frac{2\pi}{8} = \frac{\pi}{4}$$

$$= \left[-\frac{6}{8}\cos 8t\right]_0^{\pi/8} - \left[-\frac{6}{8}\cos 8t\right]_{\pi/8}^{\pi/4} = \left[-\frac{6}{8}(-1)+\frac{6}{8}\cdot 1\right] - \left[-\frac{6}{8}\cdot 1+\frac{6}{8}(-1)\right]$$

$$= \frac{6}{8}\cdot 2\cdot 2 = \frac{6}{2} = 3$$

Zad. 2

a) $$\int\frac{x^5\,dx}{4+x^6} = \int\frac{x^3x^2\,dx}{4+(x^3)^2} = \left|\begin{matrix} x^3 = t \\ 3x^2\,dx = dt\end{matrix}\right|$$

$$= \frac{1}{3}\int\frac{t\,dt}{4+t^2} = \frac{1}{6}\int\frac{2t}{4+t^2}\,dt = \frac{1}{6}\ln|4+t^2|+C$$
$$= \frac{1}{6}\ln|4+x^6|+C$$

$$\int_{-\infty}^{1}\frac{x^5\,dx}{4+x^6} = \lim_{T\to-\infty}\left[\frac{1}{6}\ln|4+x^6|\right]_T^1 =$$

$$= \lim_{T\to-\infty}\Big[\frac{1}{6}\ln 5 - \frac{1}{6}\underbrace{\ln\underbrace{|4+T^6|}_{\infty}}_{\infty}\Big] = -\infty$$

b) $$\int\sqrt{x}\ln^2 x\,dx = \left|\begin{matrix} f' = \sqrt{x} & g = \ln^2 x \\ f = \frac{2}{3}x^{3/2} & g' = 2\ln x\cdot\frac{1}{x}\end{matrix}\right|$$

$$= \frac{2}{3}x^{3/2}\cdot\ln^2 x - \frac{4}{3}\int\sqrt{x}\ln x\,dx = \left|\begin{matrix} u' = \sqrt{x} & v = \ln x \\ u = \frac{2}{3}x^{3/2} & v' = \frac{1}{x}\end{matrix}\right|$$

$$= \frac{2}{3}x\sqrt{x}\ln^2 x - \frac{4}{3}\left[\frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\int\sqrt{x}\,dx\right] =$$

$$= \frac{2}{3}x\sqrt{x}\left(\ln^2 x - \frac{4}{3}\ln x + \frac{8}{3}\right)+C$$

c) $$\underbrace{\int\frac{4x^2-7x-5}{\sqrt{-x^2+4x+5}}\,dx}_{I} = (ax+b)\sqrt{-x^2+4x+5}+\lambda\int\frac{dx}{\sqrt{-x^2+4x+5}}$$

$$\frac{4x^2-7x-5}{\sqrt{-x^2+4x+5}} = a\sqrt{-x^2+4x+5}+(ax+b)\frac{(-2x+4)}{2\sqrt{-x^2+4x+5}}+\frac{\lambda}{\sqrt{-x^2+4x+5}}$$

$$4x^2-7x-5 = a(-x^2+4x+5)+(ax+b)(-x+2)+\lambda$$
$$= -ax^2+4ax+5a-ax^2+2ax-bx+2b+\lambda$$

$x^2: \quad 4 = -2a \quad\to\quad a = -2$

$x^1: \quad -7 = 6a-b \quad\to\quad b = 6a+7 = -12+7 = -5$

$x^0: \quad -5 = 5a+2b+\lambda \quad\to\quad \lambda = -5-5a-2b = -5+10+10 = 15$

$$I = (-2x-5)\sqrt{-x^2+4x+5}+15\int\frac{dx}{\sqrt{-x^2+4x+5}} = \ldots \quad \left(= \arcsin\frac{x-2}{3}+C\right)$$

$$\int\frac{dx}{\sqrt{-x^2+4x+5}} = \int\frac{dx}{\sqrt{9-(x-2)^2}} = \frac{1}{3}\int\frac{dx}{\sqrt{1-\left(\frac{x-2}{3}\right)^2}} = \left|\begin{matrix} x-2 = 3t \\ dx = 3\,dt\end{matrix}\right| = \int\frac{dt}{\sqrt{1-t^2}} =$$

Gr. 2

Zad. 2

d)
$$\int \frac{4\,\mathrm{tg}\,x+9}{\sqrt{2\,\mathrm{tg}^2 x+8\,\mathrm{tg}\,x-1}}\cdot\frac{dx}{\cos^2 x} = \left| \begin{array}{l} t=\mathrm{tg}\,x \\ \frac{dx}{\cos^2 x}=dt \end{array}\right.$$

$$= \int \frac{4t+9}{\sqrt{2t^2+8t-1}}\,dt = \int \frac{4t+8-1}{\sqrt{2t^2+8t-1}}\,dt =$$

$$= 2\int \frac{4t+8}{2\sqrt{2t^2+8t-1}}\,dt - \underbrace{\int \frac{dt}{\sqrt{2t^2+8t-1}}}_{I} = \ldots$$

$$I = \frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{t^2+4t-\frac{1}{2}}} = \frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{(t+2)^2-4-\frac{1}{2}}} =$$

$$= \frac{\sqrt{2}}{2}\ln\left|t+2+\sqrt{t^2+4t-\tfrac{1}{2}}\right| + C$$

$$\ldots = 2\sqrt{2t^2+8t-1} - \frac{\sqrt{2}}{2}\ln\left|t+2+\sqrt{t^2+4t-\tfrac{1}{2}}\right| + C$$

$$= 2\sqrt{2\,\mathrm{tg}^2x+8\,\mathrm{tg}\,x-1} - \frac{\sqrt{2}}{2}\ln\left|\mathrm{tg}\,x+2+\sqrt{\mathrm{tg}^2x+4\,\mathrm{tg}\,x-\tfrac{1}{2}}\right| + C$$

Gr. 3

Zad. 2

d)
$$\int \frac{dx}{2+\sin x-\cos x} = \left| \begin{array}{ll} t=\mathrm{tg}\,\frac{x}{2} & dx=\frac{2}{1+t^2}\,dt \\ x=2\,\mathrm{arctg}\,t & \cos x=\frac{1-t^2}{1+t^2} \\ & \sin x=\frac{2t}{1+t^2} \end{array}\right|$$

$$= \int \frac{\frac{2}{1+t^2}\,dt}{2+\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}} = 2\int\frac{dt}{2+2t^2+2t-1+t^2} = 2\int \frac{dt}{3t^2+2t+1}$$

$$= \frac{2}{3}\int\frac{dt}{t^2+\frac{2}{3}t+\frac{1}{3}} = \frac{2}{3}\int\frac{dt}{(t+\frac{1}{3})^2+\frac{1}{3}-\frac{1}{9}} = \frac{2}{3}\int\frac{dt}{(t+\frac{1}{3})^2+\frac{2}{9}}$$

$$= \frac{2}{3}\cdot\frac{9}{2}\int\frac{dt}{1+\frac{9}{2}(t+\frac{1}{3})^2} = 3\int\frac{dt}{1+\left(\frac{3t+1}{\sqrt{2}}\right)^2} = \left| \begin{array}{l} \frac{3t+1}{\sqrt{2}}=u \\ 3\,dt=\sqrt{2}\,du \\ dt=\frac{\sqrt{2}}{3}\,du \end{array}\right.$$

$$= 3\int\frac{\frac{\sqrt{2}}{3}\,du}{1+u^2} = \sqrt{2}\,\mathrm{arctg}(u)+C = \sqrt{2}\,\mathrm{arctg}\left(\frac{3\,\mathrm{tg}\frac{x}{2}+1}{\sqrt{2}}\right)+C$$