IS – Matematyka 1 – KARTK. 4   25pkt

$\underbrace{6+\overset{2,5}{\cancel{2}}+5,5}_{14}+\underbrace{4+7}_{11}$

Gr 2

Zad.1 Oblicz lub uzasadnij, że nie istnieje

a) ~~Oblicz~~ $\lim\limits_{(x,y)\to(0,0)} \frac{x^2+y}{4x^2+3y^3}$ 3p

$(\frac{1}{n},0)\to(0,0) \quad f(\frac{1}{n},0)=\frac{\frac{1}{n^2}}{4\cdot\frac{1}{n^2}}=\frac{1}{4}\xrightarrow[n\to\infty]{}\frac{1}{4}$

$(0,\frac{1}{n})\to(0,0) \quad f(0,\frac{1}{n})=\frac{\frac{1}{n}}{3\cdot\frac{1}{n^3}}=\frac{n^2}{3}\xrightarrow[n\to\infty]{}\infty$

Granica podwójna nie istnieje

b) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+4}-2}=\lim\limits_{(x,y)\to(0,0)}\frac{(x^2+y^2)(\sqrt{\ }+2)}{x^2+y^2+4-4}$ (2,5)

$=2+2=4$

Zad.2 3

$f(x)=\begin{cases} e^{-1/x^2+y^2} & ;(x,y)\neq(0,0) \\ 0 & ;(x,y)=(0,0)\end{cases}$

$\frac{\partial f}{\partial x}(0,0)=\lim\limits_{h\to0}\frac{f(0+h,0)-f(0,0)}{h}=\lim\limits_{h\to0}\frac{1}{h}\cdot\left[e^{-\frac{1}{h^2}}-0\right]$

$=\lim\limits_{h\to0}\frac{1}{h}e^{-\frac{1}{h^2}}=\lim\limits_{h\to0}\frac{\frac{1}{h}}{e^{\frac{1}{h^2}}}\overset{H}{=}\lim\limits_{h\to0}\frac{-\frac{1}{h^2}}{e^{\frac{1}{h^2}}\cdot\frac{-2}{h^3}}$

$=\lim\limits_{h\to0}\frac{h^{\to0}}{2\underset{\downarrow\atop\infty}{e}^{\underset{\downarrow\atop+\infty}{\frac{1}{h^2}}}}=0$

Zad.3 5,5

$f'_{\hat{v}}(P_0)=?\quad P_0=(1,1)\quad \vec{v}=[3,-4]$

$f(x,y)=(x^2+6xy+4)^{\sqrt{xy}}=e^{\sqrt{xy}\ln(x^2+6xy+4)}$ 0,5

$f'_x=e^{\sqrt{xy}\ln(x^2+6xy+4)}\cdot\left[\frac{\sqrt{y}}{2\sqrt{x}}\ln(x^2+6xy+4)+\frac{\sqrt{xy}}{x^2+6xy+4}(2x+6y)\right]$ (1,5)

$f'_x(1,1)=e^{\ln(11)}\cdot\left[\frac{1}{2}\ln(11)+\frac{1}{11}\cdot8\right]=11\left(\frac{8}{11}+\frac{\ln11}{2}\right)$ (0,5)

$f'_y=e^{\sqrt{xy}\ln(x^2+6xy+4)}\left[\frac{\sqrt{x}}{2\sqrt{y}}\ln(x^2+6xy+4)+\frac{\sqrt{xy}}{x^2+6xy+4}(6x)\right]$ (1,5)

$f'_y(1,1)=e^{\ln(11)}\cdot\left[\frac{1}{2}\ln(11)+\frac{1}{11}\cdot6\right]=11\left[\frac{6}{11}+\frac{\ln11}{2}\right]$ (0,5)

$\vec{\nabla}f(P_0)=\left[8+\frac{11\cdot\ln11}{2},\ 6+\frac{11\cdot\ln11}{2}\right]$

$|\vec{v}|=\sqrt{9+16}=5\quad \hat{v}=\left[\frac{3}{5},\frac{-4}{5}\right]$ (0,5)

$f'_{\hat{v}}(P_0)=\frac{3}{5}(\cancel{8}+\frac{11}{2}\ln11)-\frac{4}{5}(\cancel{6}+\frac{11}{2}\ln11)=\frac{-11}{10}\ln11$ (0,5)

Zad.4 Równanie stycznej do C w P0

4p $P_0=(1,2,\frac{\pi}{6})$

$C\begin{cases}x=t^2\\ y=2\\ z=\arcsin\frac{2}{t^3+3}\end{cases}$

$\frac{\pi}{6}=\arcsin\frac{1}{2}\to\frac{2}{t^3+3}=\frac{1}{2}\to t^3+3=4$
$t^3=1$

$t^3=1\quad\to \underline{t=1}$ ①

$s(t)=(x'(t),y'(t),z'(t))=\left[2t,0,\frac{1}{\sqrt{1-\frac{4}{(t^3+3)^2}}}\cdot\frac{-2}{(t^3+3)^2}3t^2\right]$ ①

$s(1)=\left[2,0,\frac{1}{\sqrt{\frac{3}{4}}}\cdot\frac{-6}{16}\right]=\left[2,0,-\frac{\sqrt{3}}{4}\right]$ ①

$l:\begin{cases}x=1+2t\\ y=2\\ z=\pi/6-\sqrt{3}/4\,t\end{cases}; t\in\mathbb{R}$ ①

Gr. 3

Zad.1

a) $\lim\limits_{(x,y)\to(0,0)}\frac{x^2+3y^2}{x^2+y}$ 3p

$(\frac{1}{n},0)\to(0,0)\quad f(\frac{1}{n},0)=\frac{\frac{1}{n^2}}{\frac{1}{n^2}}=1\longrightarrow1$

$(0,\frac{1}{n})\to(0,0)\quad f(0,\frac{1}{n})=\frac{3\frac{1}{n^2}}{\frac{1}{n}}=\frac{3}{n}\xrightarrow[n\to\infty]{}0$

Nie istnieje

b) $\lim\limits_{(x,y)\to(0,0)}\frac{\sin(x^4-y^4)}{x^2-y^2}=\lim\limits_{(x,y)\to(0,0)}\underbrace{\frac{\sin(x^4-y^4)}{x^4-y^4}}_{\downarrow\atop1}\underbrace{(x^2+y^2)}_{\downarrow\atop0}=0$ (2,5)

Zad.2 3

$f(x)=\begin{cases}(x^4+y^2)\sin\frac{1}{x^4+y^2} & ;(x,y)\neq(0,0)\\ 0 & ;(x,y)=(0,0)\end{cases}$

$\frac{\partial f}{\partial x}(0,0)=\lim\limits_{h\to0}\frac{1}{h}\left[f(h,0)-f(0,0)\right]=$

$=\lim\limits_{h\to0}\frac{h^4\cdot\sin\frac{1}{h^4}}{h}=\lim\limits_{h\to0}\underset{\downarrow\atop0}{h^3}\cdot\underbrace{\sin\frac{1}{h^4}}_{og}=0$

Zad.3 5,5

$f'_{\vec{v}}(P_0)=?\quad P_0=(1,1,-1)\quad \vec{v}=[\sqrt{3},2,3]$

$f(x,y,z)=\frac{1}{x^2}\cdot\operatorname{arctg}\frac{z+1}{y^2+1}$

$\frac{\partial f}{\partial x}=\frac{-2}{x^3}\operatorname{arctg}\frac{z+1}{y^2+1}$ ① $\quad\frac{\partial f}{\partial x}(P_0)=-2\cdot\operatorname{arctg}0=0$ 0,5

$\frac{\partial f}{\partial y}=\frac{1}{x^2}\cdot\frac{1}{1+\left(\frac{z+1}{y^2+1}\right)^2}$ ① $(z+1)\cdot\frac{-1}{(y^2+1)^2}2y\quad\frac{\partial f}{\partial y}(P_0)=1\cdot\frac{1}{1+0}\cdot0=0$ 0,5

$\frac{\partial f}{\partial z}=\frac{1}{x^2}\cdot\frac{1}{1+\left(\frac{z+1}{y^2+1}\right)^2}\cdot\frac{1}{y^2+1}$ ① $\quad\frac{\partial f}{\partial z}(P_0)=1\cdot\frac{1}{1+0}\cdot\frac{1}{2}=\frac{1}{2}$ 0,5

$\vec{\nabla}f(P_0)=\left[0,0,\frac{1}{2}\right]\quad|\vec{v}|=\sqrt{3+4+9}=4$ 0,5

$\hat{v}=\left[\frac{\sqrt{3}}{4},\frac{1}{2},\frac{3}{4}\right)\quad f'_{\hat{v}}(P_0)=\left[0,0,\frac{1}{2}\right]\circ\left[\frac{\sqrt{3}}{4},\frac{1}{2},\frac{3}{4}\right]=\frac{3}{8}$ 0,5

Zad.4 4p Równanie stycznej do C w P0

$C\begin{cases}x=\ln(\cos^2t)\\ y=1\\ z=4t\end{cases}\quad P_0=\left(\ln\frac{1}{2},1,\pi\right)$

$P_0=\left(x(\frac{\pi}{4}),y(\frac{\pi}{4}),z(\frac{\pi}{4})\right)$ ①

$\vec{s}=(x'(t),y'(t),z'(t))=\left[\frac{2\cos t(-\sin t)}{\cos^2t},0,4\right]=$ ①
$=\left[-2\operatorname{tg}t,0,4\right]$

$s(\frac{\pi}{4})=\left[-2\cdot1,0,4\right]$ ①

$l:\begin{cases}x=\ln\frac{1}{2}-2t\\ y=1\\ z=\pi+4t\end{cases}t\in\mathbb{R}$ ①

Gr. 2

Zad. 5 Ekstrema lokalne

7p

$f(x,y) = 3xe^{y} - x^{3} - e^{3y}$

$D_f = \mathbb{R}^2$

NK $\begin{cases} f'_x = 3e^{y} - 3x^{2} = 0 \\ f'_y = 3xe^{y} - 3e^{3y} = 0 \end{cases}$ ① $\Rightarrow \begin{cases} x^{2} = e^{y} \\ xe^{y} = e^{3y}, \ x = e^{2y} \end{cases}$ ①

$x^2 = e^y, \ (e^{2y})^2 = e^y, \ e^{4y} = e^y, \ e^y(e^{3y}-1) = 0$

$e^{3y} = 1, \ 3y = 0, \ y = 0 \rightarrow x = e^{2\cdot 0} = 1$

$P_0 = (1,0)$ ①

WW $H(x,y) = \begin{bmatrix} -6x & 3e^{y} \\ 3e^{y} & 3xe^{y} - 9e^{3y} \end{bmatrix}$ ~~④~~ ②

$H(1,0) = \begin{bmatrix} -6 & 3 \\ 3 & -6 \end{bmatrix}$ ①

$\Delta_1 = -6 < 0$

$\Delta_2 = 36 - 9 > 0$

w P_0 max lokalne

równe $f(1,0) = 1$ ①

Gr. 3

Zad. 5

7p

$f(x,y) = e^{1+x} + e^{y-x} + e^{2-y}$

$D_f = \mathbb{R}^2$

NK $\begin{cases} f'_x = e^{1+x} - e^{y-x} = 0 \\ f'_y = e^{y-x} - e^{2-y} = 0 \end{cases}$ ① $\Leftrightarrow \begin{cases} e^{1+x} = e^{y-x} \\ e^{y-x} = e^{2-y} \end{cases}$ ①

$\Leftrightarrow \begin{cases} 1+x = y-x \\ y-x = 2-y \end{cases} \Rightarrow \begin{matrix} 1+x = 2-y \\ y = 1-x \end{matrix}$

$1+x = 1-x-x$

$x = 0$

$P_0 = (0,1)$ ①

WW $H(x,y) = \begin{bmatrix} e^{1+x} + e^{y-x} & -e^{y-x} \\ -e^{y-x} & e^{y-x} + e^{2-y} \end{bmatrix}$ ②

$H(0,1) = \begin{bmatrix} 2e & -e \\ -e & 2e \end{bmatrix}$ ①

$\Delta_1 = 2e > 0$

$\Delta_2 = 4e^2 - e^2 = 3e^2 > 0$

w P_0 min lok $f(0,1) = 3e$ ①