Zadanie domowe nr 1

Zad. 1

$$f(x)=\frac{\ln(x^2)}{(x^2-7)\cdot\sqrt{|x|}}+|5x^5+x|\cdot\left(\frac{1}{3^x}-3^x\right)\cdot\sin\pi x$$

$D_f=?$
$x^2>0 \Rightarrow x\neq 0$
$\sqrt{|x|}\neq 0 \Rightarrow x\neq 0$
$x^2-7\neq 0 \Rightarrow x\neq\pm\sqrt{7}$

$D_f=\mathbb{R}\setminus\{-\sqrt{7},0,\sqrt{7}\}$ tzn. $x\in D_f \Rightarrow -x\in D_f$

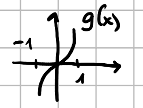
$$f(-x)=\frac{\ln((-x)^2)}{[(-x)^2-7]\cdot\sqrt{|-x|}}+\underbrace{|-5x^5-x|}_{\substack{|(-1)\cdot(5x^5+x)| \\ \| \\ |-1|\cdot|5x^5+x|}}\cdot\Big(\underbrace{\frac{1}{3^{-x}}-3^{-x}}_{\left(3^x-\frac{1}{3^x}\right)}\Big)\cdot\underbrace{\sin(-\pi x)}_{(-\sin\pi x)}=\frac{\ln(x^2)}{(x^2-7)\sqrt{|x|}}+|5x^5+x|\left(\frac{1}{3^x}-3^x\right)\sin(\pi x)=f(x)$$

f jest parzysta

Zad. 2

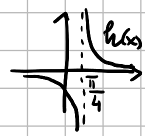
$$\begin{cases} f:\left[-1,\frac{\sqrt{2}}{2}\right)\to\mathbb{R} \\ f(x)=\dfrac{1}{4\arcsin x-\pi}\end{cases}$$

funkcja $g(x)=\arcsin x$ jest rosnąca w $\left[-1,\frac{\sqrt{2}}{2}\right)$



$-\frac{\pi}{2}=\arcsin(-1)$, $\frac{\pi}{4}=\arcsin\frac{\sqrt{2}}{2} \Rightarrow g\left(\left[-1,\frac{\sqrt{2}}{2}\right)\right)=\left[-\frac{\pi}{2},\frac{\pi}{4}\right)$

funkcja $h(x)=\frac{1}{4x-\pi}$ jest malejąca na $\left[-\frac{\pi}{2},\frac{\pi}{4}\right)$



$h\left(-\frac{\pi}{2}\right)=\frac{1}{-2\pi-\pi}=\frac{-1}{3\pi}$

$f=h\circ g$ malejąca (ściśle) w $\left[-1,\frac{\sqrt{2}}{2}\right)$ $\Rightarrow$ różnowartościowa $\Rightarrow$ odwracalna

$$D_{f^{-1}}=f\left(\left[-1,\frac{\sqrt{2}}{2}\right)\right)=\left(-\infty,\frac{-1}{3\pi}\right]$$

$y=\frac{1}{4\arcsin x-\pi}$, $\frac{1}{y}=4\arcsin x-\pi$, $\arcsin x=\frac{1}{4}\left(\frac{1}{y}+\pi\right)$, $x=\sin\frac{1+\pi y}{4y}$ $\Rightarrow f^{-1}(x)=\sin\frac{1+\pi x}{4x}$
$\forall x\leq\frac{-1}{3\pi}$

Zad. 3

a) $x^{\frac{7+\log x}{4}}=10^{1+\log x}\ \Big/\ \log(\dots)$ Zał. $x>0$

$\frac{1}{4}(\log x+7)\cdot\log x=1+\log x$ $t:=\log x$

$\frac{1}{4}(t+7)\cdot t=1+t\ \ /\cdot 4$

$t^2+7t=4+4t$, $t^2+3t-4=0$, $(t-1)(t+4)=0$

$\log x=1 \lor \log x=-4$

Odp. $\underline{x=10} \lor \underline{x=10^{-4}=0{,}0001}$

b) $\log_x(x-1)\leq 1$ Zał. $x>0$, $x\neq 1$, $x-1>0 \Rightarrow \underline{x>1}$

$\frac{\ln(x-1)}{\ln x}\leq 1\ \ /\cdot(\ln x)^2$

$\ln x\cdot\ln(x-1)\leq(\ln x)^2$, $\ln x\cdot[\ln(x-1)-\ln x]\leq 0$, $\ln x\cdot\ln\frac{x-1}{x}\leq 0$

$$\begin{cases}\ln x\leq 0\\ \ln\frac{x-1}{x}\geq 0\end{cases} \lor \begin{cases}\ln x\geq 0\\ \ln\frac{x-1}{x}\leq 0\end{cases}$$ tj. $$\begin{cases}x\leq 1\\ \frac{x-1}{x}\geq 1\end{cases} \lor \begin{cases}x\geq 1\\ \frac{x-1}{x}\leq 1\end{cases} \Rightarrow x\geq 1 \land \frac{x-1}{x}\leq 1\ \ /\cdot x$$

Odpada!

$x-1\leq x$
$\Downarrow$
$x\geq 1$

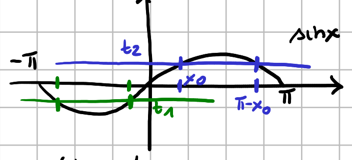
Odp. $x\in(1,\infty)$

c) $5\cos 2x=4\sin x$ $x\in\mathbb{R}$ $\cos 2x=\cos^2x-\sin^2x=1-2\sin^2x$

$5(1-2\sin^2x)=4\sin x$

$10\sin^2x+4\sin x-5=0$ $t:=\sin x$ $10t^2+4t-5=0$ $\Delta=216$ $\sqrt{\Delta}=\sqrt{36\cdot 6}=6\sqrt{6}$

$t_1=\frac{-4-6\sqrt{6}}{20}=\frac{-2-3\sqrt{6}}{10} \lor t_2=\frac{-2+3\sqrt{6}}{10}$



$\sin x=t_1$
$x=x_0+2k\pi$
$\lor\ x=\pi-x_0+2k\pi=-x_0+(2k+1)\pi$

$\sin x=t$
$\arcsin(\sin x)=\arcsin t$
okresowa $T=2\pi$
$x=\arcsin t+2k\pi$

$\Downarrow$

$x=\arcsin\left(\frac{-2-3\sqrt{6}}{10}\right)+2k\pi \lor x=\arcsin\left(\frac{-2+3\sqrt{6}}{10}\right)+2k\pi$

$\lor\ x=-\arcsin\left(\frac{-2-3\sqrt{6}}{10}\right)+(2k+1)\pi \lor x=-\arcsin\left(\frac{-2+3\sqrt{6}}{10}\right)+(2k+1)\pi$

$=\arcsin\frac{2+3\sqrt{6}}{10}+(2k+1)\pi$ $\quad=\arcsin\frac{2-3\sqrt{6}}{10}+(2k+1)\pi$

Zadanie domowe nr 1

Zad. 4

$$f(x)=\ln\left(x+\sqrt{x^2+1}\right)+\sqrt{\cos(\sin x)}+\sqrt[4]{\arcsin(\log_6 x)}+\log_5\log_3\log_2 x+\arccos\frac{3}{4+2\sin x}$$

$D_f = ?$

① $x+\sqrt{x^2+1}>0$, $\sqrt{x^2+1}>-x$ $\Downarrow$ $x\in\mathbb{R}$

$x\geq 0$ spełnione zawsze

$x<0$ można obustronnie podnieść do kwadratu $x^2+1>(-x)^2=x^2$ spełnione zawsze

② $\cos(\sin x)\geq 0$ $\Downarrow$ $x\in\mathbb{R}$ $\quad \forall x\in\mathbb{R}$ $\sin x\in[-1,1]\subset[-\frac{\pi}{2},\frac{\pi}{2}] \Rightarrow \cos(\sin x)\geq 0$

③ $x>0$

④ $\arcsin(\log_6 x)\geq 0$, $\log_6 x\geq 0$

⑤ $-1\leq \log_6 x\leq 1$

④, ⑤ $\Rightarrow$ $0\leq \log_6 x\leq 1$, $1\leq x\leq 6$

⑥ $\log_2 x>0$, $x>1$

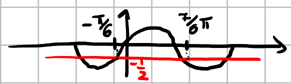
⑦ $\log_3\log_2 x>0$, $\log_2 x>1$, $x>2$

⑧ $4+2\sin x\neq 0$, $\sin x\neq -2$ zawsze

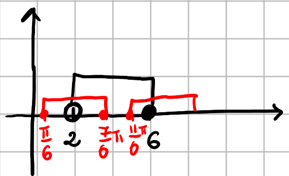
⑨ $-1\leq\frac{3}{4+2\sin x}\leq 1$ $\quad \forall x\in\mathbb{R}$ $4+2\sin x>0 \Rightarrow -1\leq\frac{3}{4+2\sin x}$ zawsze spełnione

$3\leq 4+2\sin x$, $-\frac{1}{2}\leq\sin x$

$-\frac{\pi}{6}+2k\pi\leq x\leq\frac{7}{6}\pi+2k\pi$



$x\in[2,6]$



$\pi\approx 3{,}14$ $\quad 2\pi-\frac{\pi}{6}=\frac{11}{6}\pi<6$

$\frac{\pi}{6}\approx 0{,}52$

Odp. $x\in(2,\frac{7}{6}\pi]\cup[\frac{11}{6}\pi,6]$

Zad. 5

$\cos(\operatorname{arctg}(-1))=\cos(-\frac{\pi}{4})=\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}$ $\qquad \arccos(-\frac{\sqrt{3}}{2})=\pi-\arccos\frac{\sqrt{3}}{2}=\pi-\frac{\pi}{6}=\frac{5}{6}\pi$

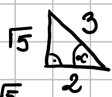
$\arcsin(\sin(-5))=\arcsin(-\sin 5)=-\arcsin(\sin 5)=-\arcsin(\sin(\underbrace{5-2\pi}_{\approx -1{,}28}))=-(5-2\pi)=2\pi-5$

$\operatorname{arctg}(\operatorname{tg}\frac{17}{16}\pi)=\operatorname{arctg}(\operatorname{tg}\frac{\pi}{16})=\frac{\pi}{16}$

$\operatorname{tg}(\operatorname{arctg}\frac{17}{16}\pi)=\frac{17}{16}\pi$

$\operatorname{tg}(\arccos(-\frac{2}{3}))=-\frac{\sqrt{5}}{2}$ $\longrightarrow$ $\arccos(-\frac{2}{3})=\alpha$

$\cos\alpha=-\frac{2}{3}$ II ćw.



II ćw. $\Rightarrow \operatorname{tg}\alpha<0$, $\operatorname{tg}\alpha=-\frac{\sqrt{5}}{2}$

Zad. 6

$f(x)=\sin(\operatorname{arctg}4x)$

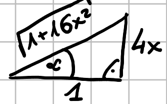
$\operatorname{arctg}4x=\alpha$ $\qquad \sin\alpha=?$

$\operatorname{tg}\alpha=4x \wedge \alpha\in(-\frac{\pi}{2},\frac{\pi}{2})$

gdy $x\geq 0$, to $\operatorname{arctg}4x\geq 0$ tzn. $\alpha\in[0,\frac{\pi}{2})$, $\sin\alpha\geq 0$

gdy $x<0$, to $\operatorname{arctg}4x<0$ tzn. $\alpha\in(-\frac{\pi}{2},0)$, $\sin\alpha<0$

$\operatorname{tg}\alpha=\frac{\sin\alpha}{\cos\alpha}$, $\cos\alpha>0$ gdy $\alpha\in(-\frac{\pi}{2},\frac{\pi}{2})$ $\Rightarrow$ $\operatorname{tg}\alpha=\frac{\sin\alpha}{\sqrt{1-\sin^2\alpha}}$ $\qquad$ Znak $\operatorname{tg}\alpha$ jest taki jak znak $\sin\alpha$



$\operatorname{tg}\alpha = 4x = \frac{4x}{1}$

$\sin\alpha=\frac{4x}{\sqrt{1+16x^2}}$ $\Rightarrow$ $f(x)=\frac{4x}{\sqrt{1+16x^2}}$

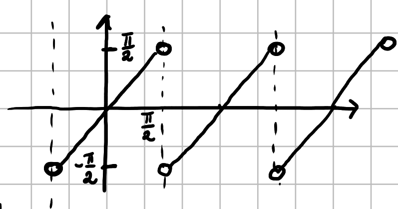
Zad. 7

a) $f(x)=\operatorname{arctg}(\operatorname{tg}x)$

$D_f=\mathbb{R}\setminus\{\frac{\pi}{2}+k\pi\}$, $k\in\mathbb{Z}$

f - okresowa , $f(x)=f(x+\pi)$

$\forall x\in(-\frac{\pi}{2},\frac{\pi}{2})$ $f(x)=x$



b) $g(x)=\frac{x^2-16}{(x+4)(\sqrt{x}+2)(\sqrt{x}-2)}\arcsin\frac{2-|x|}{3}$

$g(x)=\frac{x^2-16}{x^2-16}\cdot\arcsin\frac{2-|x|}{3}$

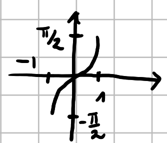
$D_g=?$ $\quad x\neq 4$, $x\neq -4$

$-1\leq\frac{2-|x|}{3}\leq 1$, $-3\leq 2-|x|\leq 3$, $-5\leq -|x|\leq 1$

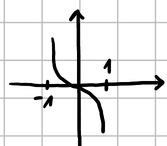
tj. $-1\leq|x|\leq 5$, $|x|\leq 5$

$D_g=[-5,5]\setminus\{-4,4\}$

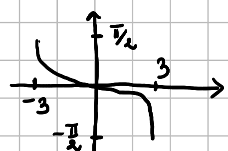
g - parzysta



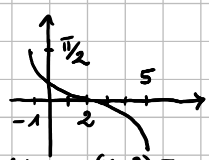
$g_1(x)=\arcsin x$



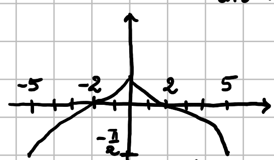
$g_2(x)=g_1(-x)=\arcsin(-x)$



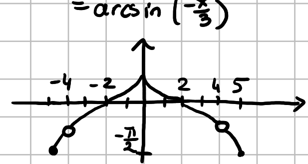
$g_3(x)=g_2(\frac{x}{3})=\arcsin(-\frac{x}{3})$



$g_4(x)=g_3(x-2)=\arcsin(-\frac{(x-2)}{3})=\arcsin\frac{2-x}{3}$



$g_5(x)=g_4(|x|)=\arcsin\frac{2-|x|}{3}$



$g(x)$

Zadanie domowe nr 1

Zad. 8 $\quad ctg\left[\frac{1}{2}\cdot \arccos\left(-\frac{4}{7}\right)\right] = ?$

$\alpha := \arccos\left(-\frac{4}{7}\right) \qquad ctg\frac{\alpha}{2} = ?$

$\alpha \in [0,\pi]$

$\cos\alpha = -\frac{4}{7} < 0 \Rightarrow \alpha \in [\frac{\pi}{2},\pi], \ \frac{\alpha}{2} \in [\frac{\pi}{4},\frac{\pi}{2}] \Rightarrow ctg\frac{\alpha}{2} \geq 0$

$ctg^2\frac{\alpha}{2} = \frac{\cos^2\frac{\alpha}{2}}{\sin^2\frac{\alpha}{2}} \underset{(*)}{=} \frac{1+\cos\alpha}{1-\cos\alpha}, \quad ctg\frac{\alpha}{2} \geq 0 \Rightarrow ctg\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}} = \sqrt{\frac{1+(-4/7)}{1-(-4/7)}} = \sqrt{\frac{3/7}{11/7}} = \sqrt{\frac{3}{11}}$

$(*) \quad \cos 2\beta = \cos^2\beta - \sin^2\beta$
$= 1 - 2\sin^2\beta$
$= 2\cos^2\beta - 1$