Zadanie domowe nr 2

Zad. 1 Zbieżność ? $a_n = \frac{2}{3+1} + \frac{2^2}{3^2+2} + \dots + \frac{2^n}{3^n+n}$

$\forall_{n\in\mathbb{N}}\ a_{n+1} - a_n = \frac{2^{n+1}}{3^{n+1}+n+1} > 0 \Rightarrow a_{n+1} > a_n$ tzn. (a_n) rosnący

a_n rosnący $\Rightarrow \forall_{n\in\mathbb{N}}\ a_1 \le a_n$ ogr. z dołu

Wystarczy pokazać, że jest ograniczony z góry

$$a_n \le \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \dots + \left(\frac{2}{3}\right)^n = \frac{2}{3}\cdot\frac{1-\left(\frac{2}{3}\right)^n}{1-\frac{2}{3}} = 2\left[1-\left(\frac{2}{3}\right)^n\right] < 2$$

monotoniczny i ograniczony, a zatem zbieżny

Zad. 2 $a_1 = \frac{\pi}{10}$, $a_{n+1} = \frac{9+a_n}{10}$

① $\forall_{n\in\mathbb{N}}\ a_n > 0$, więc (a_n) ograniczony z dołu

② $a_1 = \frac{\pi}{10}$, $a_2 = \frac{9+\frac{\pi}{10}}{10} < \frac{90+\pi}{100} < 1$, $a_3 = \frac{9+\frac{90+\pi}{100}}{10} = \frac{900+90+\pi}{1000} < 1$

Hip. $\forall_{n\in\mathbb{N}}\ a_n < 1$

dla $a_1 = \frac{\pi}{10} < 1$ prawda

Założenie $a_k < 1$. Teza $a_{k+1} < 1$

Dowód $a_{k+1} = \frac{9+a_k}{10} < \frac{9+1}{10} = 1$

$\Rightarrow \forall_{n\in\mathbb{N}}\ 0 < a_n < 1$ (a_n) ograniczony

③ $a_{n+1} - a_n = \frac{9+a_n}{10} - a_n = \frac{9+a_n-10a_n}{10} = \frac{9-9a_n}{10} = \frac{9}{10}\underbrace{(1-a_n)}_{>0} > 0 \Rightarrow a_{n+1} > a_n$ dla $n\in\mathbb{N}$

(a_n) rosnący

(a_n) monotoniczny i ograniczony, a zatem zbieżny

$\exists\ A = \lim_{n\to\infty} a_n$

$$\underbrace{a_{n+1}}_{\downarrow A} = \underbrace{\frac{9+a_n}{10}}_{\downarrow \frac{9+A}{10}} \quad (n\to\infty)$$

$A = \frac{9+A}{10}$

$10A = 9 + A$

$9A = 9$

$\underline{A = 1}$

Zad. 3

a) $$\lim_{n\to\infty}\left(\sqrt{n^{10}+3} - 3n^5 - 2n^2\right) \overset{[\infty-\infty]}{=} \lim_{n\to\infty} \underbrace{n^5}_{\downarrow\infty}\underbrace{\left(\underbrace{\sqrt{1+\frac{3}{n^{10}}}}_{\downarrow 1} - 3 - \underbrace{\frac{2}{n^3}}_{\downarrow 0}\right)}_{\downarrow -2} = -\infty$$

b) $$\lim_{n\to\infty}\left(\underbrace{\frac{2n^2}{6n+1}}_{\downarrow\infty} + \underbrace{\frac{4-n^3}{3n^2+7}}_{\downarrow -\infty}\right) \overset{[\infty-\infty]}{=} \lim_{n\to\infty}\frac{\cancel{6n^4}+14n^2+24n-\cancel{6n^4}+4n-n^3}{(6n+1)(3n^2+7)} = \lim_{n\to\infty}\frac{-n^3+14n^2+28n}{18n^3+3n^2+6n+7} = \frac{-1}{18}$$

c) $$\lim_{n\to\infty}\left(\sqrt{9^n+3^n} - \sqrt{3^{2n}+7}\right) \overset{[\infty-\infty]}{=} \lim_{n\to\infty}\frac{\cancel{9^n}+3^n-\cancel{3^{2n}}-7}{\sqrt{9^n+3^n}+\sqrt{3^{2n}+7}} = \lim_{n\to\infty}\frac{\cancel{3^n}\left(1-\underbrace{\frac{7}{3^n}}_{\to 0}\right)}{\cancel{3^n}\left[\sqrt{1+\underbrace{\left(\frac{1}{3}\right)^n}_{\downarrow 0}}+\sqrt{1+\underbrace{\frac{7}{9^n}}_{\downarrow 0\ n\to\infty}}\right]} = \frac{1}{2}$$

d) $$\lim_{n\to\infty}\underbrace{\sqrt[n^2]{3^n+4^n+\dots+15^n}}_{a_n} = 1$$

$$\underbrace{\sqrt[n]{3}}_{\downarrow 1} = \sqrt[n^2]{3^n} \le \underbrace{a_n}_{\downarrow 1} \le \sqrt[n^2]{13\cdot 15^n} = \underbrace{\sqrt[n^2]{13}}_{\downarrow 1}\cdot\underbrace{\sqrt[n]{15}}_{\downarrow 1}$$

na mocy tw. o 3 ciągach

$b_n = \sqrt[n]{13}$

$\lim b_n = 1$

$c_n = \sqrt[n^2]{13}$

(c_n) podciąg (b_n)

zatem $\lim_{n\to\infty} c_n = 1$

e) $$\lim_{n\to\infty}\underbrace{\left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}}\right)}_{a_n} \overset{[\infty\cdot 0]}{=} 1$$ na mocy tw. o 3 ciągach

$$\underbrace{\frac{1}{\sqrt{1+\frac{1}{n}}}}_{\downarrow 1} = \frac{n}{\sqrt{n^2+n}} \le a_n \le \frac{n}{\sqrt{n^2+1}} = \underbrace{\frac{1}{\sqrt{1+\frac{1}{n^2}}}}_{\downarrow 1}$$

f) $$\lim_{n\to\infty}\left(\frac{n^8+2n^3-3}{n^8+n^5}\right)^{\frac{n^3}{\sqrt{3}}} \overset{[1^\infty]}{=} \lim_{n\to\infty}\left(\frac{n^8+n^5-n^5+2n^3-3}{n^8+n^5}\right)^{\frac{n^3}{\sqrt{3}}} =$$

$$= \lim_{n\to\infty}\Bigg[\underbrace{\left(1+\frac{-n^5+2n^3-3}{n^8+n^5}\right)^{\frac{n^8+n^5}{-n^5+2n^3-3}}}_{\downarrow e}\Bigg]^{\underbrace{\frac{(-n^5+2n^3-3)\cdot n^3}{(n^8+n^5)\cdot\sqrt{3}}}_{\downarrow -\frac{1}{\sqrt{3}}}} = e^{-\frac{1}{\sqrt{3}}}$$

Zadanie domowe nr 2

Zad. 3

g) $\lim\limits_{n\to\infty}\left(1+\ln(n^2+3)-2\ln n\right)^{n^2+1} = \lim\limits_{n\to\infty}\left(1+\ln\underbrace{\frac{n^2+3}{n^2}}_{\downarrow 0}\right)^{n^2+1} \underset{[1^\infty]}{=} \lim\limits_{n\to\infty}\Bigg[\underbrace{\left(1+\ln\frac{n^2+3}{n^2}\right)^{\frac{1}{\ln\frac{n^2+3}{n^2}}}}_{\downarrow e}\Bigg]^{(n^2+1)\ln\frac{n^2+3}{n^2}} = e^3$

$\lim\limits_{n\to\infty}\underset{\downarrow\infty}{(n^2+1)}\cdot\underset{\downarrow 0}{\ln\frac{n^2+3}{n^2}} \overset{[\infty\cdot 0]}{=} \lim\limits_{n\to\infty}\ln\left[\left(1+\frac{3}{n^2}\right)^{n^2+1}\right] = \lim\limits_{n\to\infty}\ln\left(\Bigg[\underbrace{\left(1+\frac{3}{n^2}\right)^{\frac{n^2}{3}}}_{\downarrow e}\Bigg]^{\underbrace{\frac{3(n^2+1)}{n^2}}_{\downarrow 3}}\right) = \ln e^3 = 3$

h) $\lim\limits_{n\to\infty}\dfrac{\cos^n 111 + (7n)^{\frac{1}{4n}}}{1-\sqrt[n]{n}} \underset{[\frac{1}{0^-}]}{=} -\infty$

$\sqrt[n]{n}\to 1^+ \Rightarrow 1-\sqrt[n]{n}\xrightarrow[n\to\infty]{} 0^-$

$111 \neq \frac{\pi}{2}+k\pi,\ k\in\mathbb{Z} \Rightarrow |\cos 111|<1 \Rightarrow \cos^n 111 \xrightarrow[n\to\infty]{} 0$

$(7n)^{\frac{1}{4n}} = \sqrt[n]{\sqrt[4]{7}}\cdot\sqrt[4]{\sqrt[n]{n}} \xrightarrow{n\to\infty} 1$

$\sqrt[n]{n}\to 1$

$a>0,\ \sqrt[n]{a}\to 1$

i) $\lim\limits_{n\to\infty}\left(\underbrace{\frac{n}{n+e}}_{\text{ogram.}}\cdot\sin\underset{\downarrow 0}{\frac{1}{n+e}} + n^2\cdot\operatorname{tg}\frac{e}{n}\cdot\left(\cos\frac{\pi}{\sqrt{n}}-1\right)\right) = 0 + \lim\limits_{n\to\infty}\underbrace{\frac{\operatorname{tg}\frac{e}{n}}{\frac{e}{n}}}_{\to 1}\cdot\underbrace{\frac{-2\sin^2\frac{\pi}{2\sqrt{n}}}{\left(\frac{\pi}{2\sqrt{n}}\right)^2}}_{\to 1}\cdot n^2\cdot\frac{e}{n}\cdot\frac{\pi^2}{4n} =$

$= \frac{-2e\pi^2}{4} = \frac{-e\pi^2}{2}$

$0<\frac{n}{n+e}<1$

$\cos 2x = \cos^2 x - \sin^2 x = 1-2\sin^2 x$

$\cos 2x - 1 = -2\sin^2 x$

Zad. 4 $a_n = \dfrac{(1+\cos(n\pi))\cdot\ln 3n+\ln n}{\ln 2n}$ Punkty skupienia?

$a_n = \dfrac{(1+(-1)^n)\cdot\ln 3n+\ln n}{\ln 2n}$

$a_{2k} = \dfrac{2\cdot\ln 6k+\ln 2k}{\ln 4k} = \dfrac{2(\ln 6+\ln k)+\ln 2+\ln k}{\ln 4+\ln k} = \dfrac{2\ln 6+\ln 2+3\ln k}{\ln 4+\ln k} = \dfrac{3+\frac{2\ln 6+\ln 2}{\ln k}}{1+\frac{\ln 4}{\ln k}} \to \frac{3}{1} = 3$

$\frac{2\ln 6+\ln 2}{\ln k}\to 0,\quad \frac{\ln 4}{\ln k}\to 0,\quad k\to\infty$

$a_{2k+1} = \dfrac{0+\ln(2k+1)}{\ln 2+\ln(2k+1)} = \dfrac{1}{\frac{\ln 2}{\ln(2k+1)}+1} \longrightarrow 1$

Zbiór p. skupienia $S=\{1,3\}$, $\limsup\limits_{n\to\infty} a_n = 3$, $\liminf\limits_{n\to\infty} a_n = 1$

Inne podciągi zbieżne różnią się od (a_{2k}), (a_{2k+1}) skończoną liczbę wyrazów albo są ich podciągami

---

Zad. 5

a) $\lim\limits_{n\to\infty}\sqrt[3]{\underset{\downarrow\infty}{\frac{n^6}{8}}+n^4+\frac{1}{n+1}}\cdot\ln\Big(1+\pi\cdot\operatorname{arctg}\frac{1}{n^2}\Big) \overset{[\infty\cdot 0]}{=} \lim\limits_{n\to\infty} n^2\cdot\sqrt[3]{\frac{1}{8}+\frac{1}{n^2}+\frac{1}{n^6(n+1)}}\cdot\ln\left(1+\pi\cdot\operatorname{arctg}\frac{1}{n^2}\right) = \ldots$

(*) $b_n\to 0$ $\quad\lim\limits_{n\to\infty}\dfrac{\ln(1+b_n)}{b_n} = \lim\limits_{n\to\infty}\ln\left[(1+b_n)^{\frac{1}{b_n}}\right] = \ln e = 1$

$\ldots = \lim\limits_{n\to\infty}\sqrt[3]{\frac{1}{8}+\underset{\downarrow 0}{\frac{1}{n^2}}+\underset{\downarrow 0}{\frac{1}{n^6(n+1)}}}\cdot\underbrace{\frac{\ln\left(1+\pi\cdot\operatorname{arctg}\frac{1}{n^2}\right)}{\pi\cdot\operatorname{arctg}\frac{1}{n^2}}}_{\to 1}\cdot\pi\cdot\underbrace{\frac{\operatorname{arctg}\frac{1}{n^2}}{\frac{1}{n^2}}}_{\to 1} = \sqrt[3]{\frac{1}{8}}\cdot\pi = \frac{\pi}{2}$

b) $\lim\limits_{n\to\infty}\left(\frac{5}{6}+\frac{13}{36}+\ldots+\frac{2^n+3^n}{36^n}\right) = \lim\limits_{n\to\infty}\left(\frac{2^1+3^1}{6^1}+\frac{2^2+3^2}{6^2}+\ldots+\frac{2^n+3^n}{6^n}\right) =$

$= \lim\limits_{n\to\infty}\left[\left(\frac{1}{3}\right)^1+\left(\frac{1}{2}\right)^1+\left(\frac{1}{3}\right)^2+\left(\frac{1}{2}\right)^2+\ldots+\left(\frac{1}{3}\right)^n+\left(\frac{1}{2}\right)^n\right] = \lim\limits_{n\to\infty}\left[\left(\frac{1}{3}\right)^1+\left(\frac{1}{3}\right)^2+\ldots+\left(\frac{1}{3}\right)^n+\left(\frac{1}{2}\right)^1+\left(\frac{1}{2}\right)^2+\ldots+\left(\frac{1}{2}\right)^n\right] =$

$= \lim\limits_{n\to\infty}\left[\frac{1}{3}\cdot\frac{1-\left(\frac{1}{3}\right)^n}{1-\frac{1}{3}}+\frac{1}{2}\cdot\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}\right] = \lim\limits_{n\to\infty}\left[\frac{1}{2}\left(1-\underset{\downarrow 0}{\left(\tfrac{1}{3}\right)^n}\right)+1-\underset{\downarrow 0}{\left(\tfrac{1}{2}\right)^n}\right] = \frac{1}{2}\cdot 1+1 = \frac{3}{2}$