Zadanie domowe nr 3

Zad. 1

a) $$\lim_{x\to 9}\frac{x^3-729}{x-\sqrt{x}-6}\overset{[\frac{0}{0}]}{=}\lim_{x\to 9}\frac{(x-9)(x^2+9x+8)}{(\sqrt{x}-3)(\sqrt{x}+2)}=\lim_{x\to 9}\frac{\sqrt{x}+3}{\sqrt{x}+2}\cdot(x^2+9x+8)=\frac{6}{5}\cdot 3\cdot 81=\frac{1458}{5}$$

b) $$\lim_{x\to-\infty}\left(\sqrt{x^2+2x}+6x\right)\underset{[\infty-\infty]}{=}\lim_{x\to-\infty}\frac{x^2+2x-36x^2}{\sqrt{x^2+2x}-6x}=\lim_{x\to-\infty}\frac{x(2-35x)}{|x|\cdot\sqrt{1+\frac{2}{x}}-6x}=\lim_{x\to-\infty}\frac{x(2-35x)}{-x\sqrt{1+\frac{2}{x}}-6x}=$$
$$=\lim_{x\to-\infty}\frac{2-35x}{-\sqrt{1+\underbrace{\tfrac{2}{x}}_{\to 0}}-6}\underset{[\frac{+\infty}{-7}]}{=}-\infty$$

c) $$\lim_{x\to 0}\underbrace{\sin\sqrt[5]{x}}_{\to 0}\cdot\underbrace{\sin\frac{5}{x}}_{\text{ogram.}}=0$$

d) $$\lim_{x\to\infty}\frac{\ln x}{\sin^4x+\cos^4x}=\infty$$ na mocy tw. o dwóch funkcjach

$$0\le\sin^4x+\cos^4x\le 2$$
$$\frac{\ln x}{\sin^4x+\cos^4x}\ge\frac{\ln x}{2}\xrightarrow[x\to\infty]{}\infty$$

e) $$\lim_{x\to\infty}\frac{\ln(2+e^{3x})}{\ln(3+e^{2x})}\overset{[\frac{\infty}{\infty}]}{=}\frac{3}{2}$$ na mocy tw. o trzech funkcjach

$$\frac{3}{2}\xleftarrow[x\to\infty]{}\frac{3x}{\ln 3+2x}=\frac{\ln(e^{3x})}{\ln(3\cdot e^{2x})}\le\frac{\ln(2+e^{3x})}{\ln(3+e^{2x})}\le\frac{\ln(2\cdot e^{3x})}{\ln(e^{2x})}=\frac{\ln 2+3x}{2x}\xrightarrow[x\to\infty]{}\frac{3}{2}$$

f) $$\lim_{x\to 4\pi^2}\frac{\operatorname{tg}\sqrt{x}}{(\sqrt{x}-2\pi)(\sqrt{x}+2\pi)}\overset{[\frac{0}{0}]}{=}\lim_{x\to 4\pi^2}\underbrace{\frac{\operatorname{tg}(\sqrt{x}-2\pi)}{(\sqrt{x}-2\pi)}}_{1}\cdot\frac{1}{(\sqrt{x}+2\pi)}=\frac{1}{4\pi}$$

g) $$\lim_{x\to 0}\Big(1+\underbrace{\underset{\downarrow\,0}{x}\,\underset{\downarrow\,0}{e^{-1/x^2}}\,\underset{\text{ogr.}}{\sin\frac{1}{x^4}}}_{\to 0}\Big)^{\overset{\to+\infty}{e^{1/x^2}}}\underset{[1^\infty]}{=}\lim_{x\to 0}\Big[\underbrace{\Big(1+xe^{-\frac{1}{x^2}}\sin\frac{1}{x^4}\Big)^{\frac{1}{xe^{-\frac{1}{x^2}}\sin\frac{1}{x^4}}}}_{\to e}\Big]^{\overset{\to 0}{x}\,\cancel{e^{-1/x^2}}\,\underset{\text{ogram.}}{\sin\frac{1}{x^4}}\,\cancel{e^{1/x^2}}}=e^0=1$$

h) $$\lim_{x\to 0}\frac{(e^x-3^{-x})^3\cdot\left(1-\cos\frac{x}{5}\right)}{\operatorname{arctg}^5(\pi x)}\overset{[\frac{0}{0}]}{=}\lim_{x\to 0}\frac{\left[(e^x-1)-\left(\left(\frac{1}{3}\right)^x-1\right)\right]^3\cdot 2\sin^2\frac{x}{10}}{\operatorname{arctg}^5(\pi x)}=$$

$$\cos 2\alpha=\cos^2\alpha-\sin^2\alpha=1-2\sin^2\alpha$$
$$2\sin^2\alpha=1-\cos 2\alpha$$

$$=\lim_{x\to 0}\Big[\underbrace{\frac{(e^x-1)}{x}}_{\ln e=1}-\underbrace{\frac{\left(\left(\frac{1}{3}\right)^x-1\right)}{x}}_{\ln\frac{1}{3}}\Big]^3\cdot 2\cdot\underbrace{\frac{\sin^2\frac{x}{10}}{\left(\frac{x}{10}\right)^2}}_{1}\cdot\underbrace{\frac{(\pi x)^5}{\operatorname{arctg}^5(\pi x)}}_{1}\cdot\frac{\cancel{x^3}\cdot\frac{\cancel{x^2}}{100}}{\pi^5\cancel{x^5}}=\left(1-\ln\frac{1}{3}\right)^3\cdot 2\cdot\frac{1}{100\pi^5}=\frac{(1+\ln 3)^3}{50\pi^5}$$

i) $$\lim_{x\to 3}\frac{3^{x-3}-x^2+8}{x-3}\overset{[\frac{0}{0}]}{=}\left|\begin{array}{l}x-3=t\\x\to 3\\t\to 0\\x=3+t\end{array}\right|=\lim_{t\to 0}\frac{3^t-(3+t)^2+8}{t}=\lim_{t\to 0}\frac{3^t-9-6t-t^2+8}{t}=\lim_{t\to 0}\left(\frac{3^t-1}{t}-\frac{t^2+6t}{t}\right)$$
$$=\lim_{t\to 0}\left(\frac{3^t-1}{t}-t-6\right)=\ln 3-6$$

j) $$\lim_{x\to 1}(1-x)\cdot\operatorname{tg}\frac{\pi x}{2}\underset{[0\cdot\infty]}{=}\left|\begin{array}{l}x-1=t\\x\to 1\\t\to 0\\x=1+t\end{array}\right|=\lim_{t\to 0}(-t)\cdot\operatorname{tg}\left[\frac{\pi}{2}(1+t)\right]=-\lim_{t\to 0}t\cdot\operatorname{tg}\left(\frac{\pi}{2}+\frac{\pi}{2}t\right)=-\lim_{t\to 0}t\cdot\left(-\operatorname{ctg}\frac{\pi}{2}t\right)$$
$$=\lim_{t\to 0}t\cdot\operatorname{ctg}\frac{\pi}{2}t=\lim_{t\to 0}\frac{t}{\operatorname{tg}\frac{\pi t}{2}}=\lim_{t\to 0}\underbrace{\frac{\frac{\pi t}{2}}{\operatorname{tg}\frac{\pi t}{2}}}_{1}\cdot\frac{2}{\pi}=\frac{2}{\pi}$$

k) $$\lim_{x\to\infty}\left(x^{\sqrt{x}}-(\sqrt{x})^x\right)\overset{[\infty-\infty]}{=}\lim_{x\to\infty}\left(e^{\sqrt{x}\ln x}-e^{x\ln\sqrt{x}}\right)=\lim_{x\to\infty}\left(e^{\sqrt{x}\ln x}-e^{\frac{x}{2}\ln x}\right)=$$
$$=\lim_{x\to\infty}\underset{+\infty}{e^{\left(\underset{+\infty}{\frac{x}{2}\ln x}\right)}}\cdot\Big[\underbrace{\underset{\downarrow\,0}{e^{\overset{-\infty}{(\sqrt{x}-\frac{x}{2})\ln x}}}-1}_{\to -1}\Big]=-\infty$$

$$\lim_{x\to\infty}\left(\sqrt{x}-\frac{x}{2}\right)=\lim_{x\to\infty}\underset{\infty}{x}\Big(\underbrace{\underset{0}{\frac{1}{\sqrt{x}}}-\frac{1}{2}}_{\to-\frac{1}{2}}\Big)=-\infty$$

l) $$\lim_{x\to 3^+}\sin\frac{x-3}{x^2-6x+9}=\lim_{x\to 3^+}\sin\frac{x-3}{(x-3)^2}=\lim_{x\to 3^+}\sin\underbrace{\frac{1}{x-3}}_{+\infty}$$

granica nie istnieje bowiem

Korzystamy z definicji Heinego granicy funkcji

$$x_n=3+\frac{1}{2n\pi}\xrightarrow[n\to\infty]{}3^+\qquad f(x)=\sin\frac{1}{x-3}\qquad f(x_n)=\sin 2n\pi=0\longrightarrow 0$$
$$y_n=3+\frac{1}{\frac{\pi}{2}+2n\pi}\xrightarrow[n\to\infty]{}3^+\qquad f(y_n)=\sin\left(\frac{\pi}{2}+2n\pi\right)=\sin\frac{\pi}{2}=1\to 1$$

Zadanie domowe nr 3

Zad. 2

$$f(x) = \begin{cases} x + \dfrac{\sin x}{2x - \frac{12}{x-1}} & ; x \in \mathbb{R} \setminus \{-2, 1, 3\} \\ 1 & ; x \in \{-2, 1, 3\} \end{cases}$$

$x \neq 1$

$2x - \frac{12}{x-1} \neq 0 \quad / \cdot (x-1)^2 \qquad 2x(x-1) - 12 \neq 0, \ x(x-1) - 6 \neq 0, \ x^2 - x - 6 \neq 0, \ (x+2)(x-3) \neq 0 \qquad x \neq 2, \ x \neq 3$

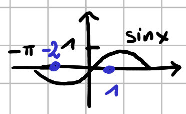
Zatem $D_f = \mathbb{R}$

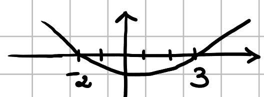
$$f(x) = x + \frac{x-1}{2(x+2)(x-3)} \cdot \sin x$$

① $f(-2) = 1$

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} \Big(\underbrace{x}_{\to -2} + \underbrace{\frac{x-1}{2\underbrace{(x+2)}_{0^-}\underbrace{(x-3)}_{3/5}}}_{-\infty} \cdot \sin x \Big) = +\infty$$

$\sin(-2) = -\sin 2 < 0$





$x \to -2^-$ | $x \to -2^+$

$x - 2 \to 0^-$ | $(x+2)(x-3) \to 0^-$

$$\lim_{x \to -2^+} \Big(\underbrace{x}_{-2} + \underbrace{\frac{\overbrace{x-1}^{\to -3}}{2(x+2)(x-3)}}_{+\infty} \cdot \underbrace{\sin x}_{\sin(-2) < 0} \Big) = -\infty$$

$(x+2)(x-3) \to 0^-$

f nie jest ciągła w $x = -2$

f nie jest jednostronnie ciągła w $x = -2$

$x = -2$ punkt nieciągłości II rodzaju

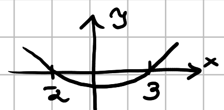
② $f(1) = 1$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \Big(\underbrace{x}_{1} + \frac{\overbrace{\sin x}^{\to \sin 1 > 0}}{\underbrace{2x}_{2} - \underbrace{\frac{12}{\underbrace{x-1}_{0^-}}}_{-\infty}} \Big) = 1 + 0 = 1$$

f jest ciągła w $x = 1$

$$\lim_{x \to 1^+} \Big(x + \frac{\sin x}{\underbrace{2x}_{2} - \underbrace{\frac{12}{\underbrace{x-1}_{0^+}}}_{+\infty}} \Big) = 1 + 0 = 1$$

$0 < 3 < \pi$



③ $f(3) = 1$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \Big(\underbrace{x}_{3^+} + \underbrace{\frac{\overbrace{x-1}^{\to 2^-}}{2\underbrace{(x+2)(x-3)}_{0^-}}}_{-\infty} \underbrace{\sin x}_{\sin 3 > 0} \Big) = -\infty$$

$x \to 3^-$

$(x+2)(x-3) \to 0^-$

$$\lim_{x \to 3^+} \Big(\underbrace{x}_{3^+} + \underbrace{\frac{\overbrace{x-1}^{\to 2^+}}{2\underbrace{(x+2)(x-3)}_{0^+}}}_{+\infty} \underbrace{\sin x}_{\sin 3 > 0} \Big) = +\infty$$

f nie jest ciągła w $x = 3$

f nie jest jednostronnie ciągła w $x = 3$

$x = 3$ punkt nieciągłości II rodzaju

Zad. 3

a) $$\lim_{x \to \frac{\pi}{4}} \frac{\ln(\operatorname{tg} x)}{1 - \operatorname{ctg} x} \overset{\left[\frac{0}{0}\right]}{=} \lim_{x \to \frac{\pi}{4}} \frac{1}{1 - \operatorname{ctg} x} \cdot \ln(\operatorname{tg} x) \overset{[\infty \cdot 0]}{=} \lim_{x \to \frac{\pi}{4}} \ln\left[(\operatorname{tg} x)^{\frac{1}{1 - \operatorname{ctg} x}} \right] =$$

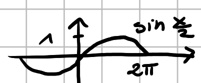
$$= \lim_{x \to \frac{\pi}{4}} \ln\left[\Big(\underbrace{[1 + (\operatorname{tg} x - 1)]^{\frac{1}{\operatorname{tg} x - 1}}}_{e} \Big)^{\frac{\operatorname{tg} x - 1}{1 - \operatorname{ctg} x}} \right] = \ln(e^1) = 1$$

$$\lim_{x \to \frac{\pi}{4}} \frac{\operatorname{tg} x - 1}{1 - \operatorname{ctg} x} = \lim_{x \to \frac{\pi}{4}} \frac{\operatorname{tg} x - 1}{1 - \frac{1}{\operatorname{tg} x}} = \lim_{x \to \frac{\pi}{4}} \operatorname{tg} x = 1$$

b) $$\lim_{x \to 0^-} \frac{e^x - \cos x}{\sqrt{1 - \cos x}} \overset{\left[\frac{0}{0}\right]}{=} \lim_{x \to 0^-} \frac{e^x - 1 - (\cos x - 1)}{\sqrt{1 - \cos x}} = \lim_{x \to 0^-} \left(\frac{e^x - 1}{\sqrt{1 - \cos x}} + \frac{1 - \cos x}{\sqrt{1 - \cos x}} \right) = \lim_{x \to 0^-} \left(\frac{e^x - 1}{\sqrt{2 \sin^2 \frac{x}{2}}} + \sqrt{1 - \cos x} \right) =$$

$$= \lim_{x \to 0^-} \left(\frac{e^x - 1}{\sqrt{2} \cdot |\sin \frac{x}{2}|} + \sqrt{1 - \cos x} \right) = \lim_{x \to 0^-} \left(\frac{e^x - 1}{\sqrt{2}(-\sin \frac{x}{2})} + \sqrt{1 - \cos x} \right) =$$

$$= \lim_{x \to 0^-} \Big(\underbrace{\frac{e^x - 1}{x}}_{1} \cdot \underbrace{\frac{\frac{x}{2}}{\sin \frac{x}{2}}}_{1} \cdot \frac{-1}{\sqrt{2}} \cdot 2 + \underbrace{\sqrt{1 - \cos x}}_{0} \Big) = \frac{-2}{\sqrt{2}} = -\sqrt{2}$$



$x \to 0^- \quad \sin \frac{x}{2} \to 0^-$

$\cos x = 1 - 2\sin^2 \frac{x}{2}$

$2\sin^2 \frac{x}{2} = 1 - \cos x$

Zad. 4

$$\lim_{x \to 2} \frac{a + b \cdot \ln(x-1)}{x - 2} = 7 \qquad \text{dla jakich } a, b \in \mathbb{R}\ ?$$

$$\lim_{x \to 2} \frac{a + b \overbrace{\ln(x-1)}^{\to 0}}{\underbrace{x - 2}_{\to 0}} \ \overset{\left[\frac{a}{0}\right]}{=}$$

$a \neq 0 \Rightarrow$ granica nie istnieje

lewo- i prawostronna granica są różne równe $+\infty$ lub $-\infty$

Niech $a = 0$.

$$\lim_{x \to 2} \frac{b \cdot \ln(x-1)}{x - 2} \overset{\left[\frac{0}{0}\right]}{=} \lim_{x \to 2} \frac{b}{x-2} \ln(x-1) = \lim_{x \to 2} \ln\Big[\underbrace{(x-1)^{\frac{b}{x-2}}}_{[1^\infty]} \Big] =$$

$$= \lim_{x \to 2} \ln\left(\left[1 + (x-2)\right]^{\frac{1}{x-2}} \right)^b = \ln e^b = b$$

Zatem $\underline{b = 7}$, $\underline{a = 0}$