Zadanie domowe nr 4

ZAD 1.

a) $$f(x)=\pi\cdot 2^{3\ln^4(5x^6+7)}+\operatorname{arcctg}\frac{1}{\sqrt[8]{\ln\frac{x}{9}}}+\cos^{10}\left(\arccos\left(\frac{11}{12+\sin x}\right)\right)$$

$$f'(x)=\pi\cdot 2^{3\ln^4(5x^6+7)}\cdot\ln 2\cdot 3\cdot 4\ln^3(5x^6+7)\cdot\frac{1}{5x^6+7}\cdot 5\cdot 6x^5+\frac{-1}{1+\sqrt[4]{\ln\frac{x}{9}}}\cdot\left(-\frac{1}{8}\right)\left(\ln\frac{x}{9}\right)^{-9/8}\cdot\frac{9}{x}\cdot\frac{1}{9}+$$

$$+10\cos^9\left(\arccos\left(\frac{11}{12+\sin x}\right)\right)\cdot\left(-\sin\left(\arccos\frac{11}{12+\sin x}\right)\right)\cdot\frac{-1}{\sqrt{1-\frac{121}{(12+\sin x)^2}}}\cdot 11\cdot\frac{-1}{(12+\sin x)^2}\cdot\cos x$$

b) $$f(x)=\left(\log_{\frac{1}{3}}x\right)^{3x}+\operatorname{arctg}(x^2+1)\cdot\operatorname{arctg}\frac{1}{x^2+1}=e^{3x\cdot\ln\left(\log_{\frac{1}{3}}x\right)}+\operatorname{arctg}(x^2+1)\cdot\operatorname{arctg}\frac{1}{x^2+1}$$

$$f'(x)=e^{3x\ln\left(\log_{\frac{1}{3}}x\right)}\cdot\left[3\ln\left(\log_{\frac{1}{3}}x\right)+3x\cdot\frac{1}{\log_{\frac{1}{3}}x}\cdot\frac{1}{x\cdot\ln\frac{1}{3}}\right]+$$

$$+\frac{1}{1+(x^2+1)^2}\cdot 2x\cdot\operatorname{arctg}\frac{1}{x^2+1}+\operatorname{arctg}(x^2+1)\cdot\frac{1}{1+\left(\frac{1}{1+x^2}\right)^2}\cdot\frac{-1}{(x^2+1)^2}\cdot 2x$$

ZAD. 2 $f\in C^1(\mathbb{R})$, $f(0)=2$, $f(1)=1$, $f(2)=-5$, $f'(0)=-3$, $f'(1)=7$, $f'(2)=4$

$g(x)=f(x+f(x))\cdot f(x)$ $\qquad g'(0)=?$

$f\in C^1(\mathbb{R}) \Rightarrow D_f=\mathbb{R} \Rightarrow$ złożenie możliwe

$$g'(x)=f'(x+f(x))\cdot(1+f'(x))\cdot f(x)+f(x+f(x))\cdot f'(x)$$

$$g'(0)=f'(0+\underbrace{f(0)}_{2})\cdot(1+\underbrace{f'(0)}_{-3})\cdot\underbrace{f(0)}_{2}+f(0+\underbrace{f(0)}_{2})\cdot\underbrace{f'(0)}_{-3}=-4f'(2)-3f(2)=-4\cdot 4-3\cdot(-5)=-1$$

ZAD. 3 Uzasadnij, że $\forall x>-1 \quad \operatorname{arctg}x=\frac{\pi}{4}-\operatorname{arctg}\frac{1-x}{1+x}$

Niech $f(x)=\operatorname{arctg}x$, $g(x)=\frac{\pi}{4}-\operatorname{arctg}\frac{1-x}{1+x}$

$D_f=\mathbb{R}$ $\qquad D_g=\mathbb{R}\setminus\{-1\}$ $\qquad$ Zawężamy się do $(-1,+\infty)$

$$f'(x)=\frac{1}{1+x^2}\qquad g'(x)=-\frac{1}{1+\left(\frac{1-x}{1+x}\right)^2}\cdot\frac{-1(1+x)-(1-x)\cdot 1}{(1+x)^2}=-\frac{\cancel{(1+x)^2}}{(1+x)^2+(1-x)^2}\cdot\frac{-1\cancel{-x}-1\cancel{+x}}{\cancel{(1+x)^2}}=\frac{2}{1+2x+x^2+1-2x+x^2}=\frac{2}{2+2x^2}=\frac{1}{1+x^2}$$

$f'(x)=g'(x) \Rightarrow f(x)=g(x)+C$; $C\in\mathbb{R}$ stała

$f(0)=\operatorname{arctg}0=0$ $\qquad g(0)=\frac{\pi}{4}-\operatorname{arctg}1=\frac{\pi}{4}-\frac{\pi}{4}=0$ $\qquad f(0)=g(0)+C \Leftrightarrow 0=0+C$, $C=0 \Rightarrow f(x)=g(x)$ $\forall x>-1$

ZAD. 4

a) $$\lim_{x\to 1}\frac{e^{x-1}-e^{-x+1}-2x+2}{x-\sin(x-1)-1}\overset{\left[\frac{0}{0}\right]}{\underset{H}{=}}\lim_{x\to 1}\frac{e^{x-1}+e^{-x+1}-2}{1-\cos(x-1)}\overset{\left[\frac{0}{0}\right]}{\underset{H}{=}}\lim_{x\to 1}\frac{e^{x-1}-e^{-x+1}}{\sin(x-1)}\overset{\left[\frac{0}{0}\right]}{\underset{H}{=}}\lim_{x\to 1}\frac{e^{x-1}+e^{-x+1}}{\cos(x-1)}=\frac{2}{1}=2$$

b) $$\lim_{x\to\infty}\left(\sqrt[7]{x^7+6x+1}-x\right)\overset{[\infty-\infty]}{=}\lim_{x\to\infty}x\left(\sqrt[7]{1+\frac{6}{x^6}+\frac{1}{x^7}}-1\right)\overset{[\infty\cdot 0]}{=}\lim_{x\to\infty}\frac{\sqrt[7]{1+\frac{6}{x^6}+\frac{1}{x^7}}-1}{\frac{1}{x}}\overset{\left[\frac{0}{0}\right]}{\underset{H}{=}}$$

$$=\lim_{x\to\infty}\frac{\frac{1}{7}\left(1+\frac{6}{x^6}+\frac{1}{x^7}\right)^{-6/7}\cdot\left(\frac{-36}{x^7}-\frac{7}{x^8}\right)}{-\frac{1}{x^2}}=\lim_{x\to\infty}\frac{\frac{36}{x^5}+\frac{7}{x^6}}{7\cdot\sqrt[7]{\left(1+\frac{6}{x^6}+\frac{1}{x^7}\right)^6}}=\frac{0}{7}=0$$

c) $$\lim_{x\to 1^-}\underset{0^+}{\sqrt{1-x}}\cdot\ln\Big(\underset{-\infty}{\ln}\underset{0}{\,}\frac{1}{\underset{1}{x}}\Big)\overset{[0\cdot\infty]}{=}\lim_{x\to 1^-}\frac{\ln\left(\ln\frac{1}{x}\right)}{\frac{1}{\sqrt{1-x}}}\overset{\left[\frac{\infty}{\infty}\right]}{\underset{H}{=}}\lim_{x\to 1^-}\frac{\frac{1}{\ln\frac{1}{x}}\cdot x\cdot\left(-\frac{1}{x^2}\right)}{-\frac{1}{2}(1-x)^{-3/2}\cdot(-1)}=\lim_{x\to 1^-}\frac{\frac{-1}{x\ln\frac{1}{x}}}{\frac{1}{2}\cdot\frac{1}{\sqrt{(1-x)^3}}}=$$

$$=\lim_{x\to 1^-}\frac{2\cdot\sqrt{(1-x)^3}}{-x\cdot\ln\frac{1}{x}}\overset{\left[\frac{0}{0}\right]}{\underset{H}{=}}\lim_{x\to 1^-}\frac{2\cdot\frac{3}{2}(1-x)^{1/2}\cdot(-1)}{-\ln\frac{1}{x}-x\cdot x\cdot\left(-\frac{1}{x^2}\right)}=\lim_{x\to 1^-}\frac{-3\sqrt{1-x}}{-\ln\frac{1}{x}+1}=\frac{0}{1}=0$$

d) $$\lim_{x\to 0^+}\left(\frac{1}{\sin x}\right)^{\frac{\arccos x}{\ln x}}\overset{[\infty^0]}{=}\lim_{x\to 0^+}e^{\frac{\arccos x}{\ln x}\cdot\ln\left(\frac{1}{\sin x}\right)}=e^{-\pi/2}$$

$$\lim_{x\to 0^+}\underset{\frac{\pi}{2}}{\arccos x}\cdot\frac{\ln\left(\frac{1}{\sin x}\right)}{\ln x}=?\qquad\left[\frac{\infty}{\infty}\right]$$

$$\lim_{x\to 0^+}\frac{\ln\left(\frac{1}{\sin x}\right)}{\ln x}\overset{\left[\frac{\infty}{\infty}\right]}{\underset{H}{=}}\lim_{x\to 0^+}\frac{\sin x\cdot\frac{-1}{\sin^2 x}\cdot\cos x}{\frac{1}{x}}=\lim_{x\to 0^+}\underset{\to -1}{\frac{-x}{\sin x}}\cdot\underset{\to 1}{\cos x}=-1$$

e) $$\lim_{x\to\infty}(1+2^x)^{\frac{1}{x}}\underset{[\infty^0]}{=}\lim_{x\to\infty}e^{\frac{1}{x}\ln(1+2^x)}=e^{\ln 2}=2$$

$$\lim_{x\to\infty}\frac{\ln(1+2^x)}{x}\overset{\left[\frac{\infty}{\infty}\right]}{\underset{H}{=}}\lim_{x\to\infty}\frac{\frac{1}{1+2^x}\cdot 2^x\ln 2}{1}=\lim_{x\to\infty}\frac{\ln 2}{\underset{0}{\frac{1}{2^x}}+1}=\ln 2$$

Zadanie domowe nr 4

ZAD. 5

$$\lim_{x\to\infty}\frac{3^{\ln(x^3+1)}}{x^3+5}\ \overset{[\frac{\infty}{\infty}]}{\underset{H}{=}}\ \lim_{x\to\infty}\frac{3^{\ln(x^3+1)}\ln 3\cdot\frac{1}{x^3+1}\cdot\cancel{3x^2}}{\cancel{3x^2}}=\lim_{x\to\infty}\ln 3\cdot\frac{3^{\ln(x^3+1)}}{x^3+1}\ \overset{[\frac{\infty}{\infty}]}{=}$$

$$=\ln 3\cdot\lim_{x\to\infty}\frac{3^{\ln(x^3+1)}}{x^3+1}\ \underset{H}{=}\ \ln 3\cdot\lim_{x\to\infty}\frac{3^{\ln(x^3+1)}\cdot\ln 3\cdot\frac{1}{x^3+1}\cdot\cancel{3x^2}}{\cancel{3x^2}}=(\ln 3)^2\cdot\lim_{x\to\infty}\frac{3^{\ln(x^3+1)}}{x^3+1}$$

itd...

Obliczamy

$$\lim_{x\to\infty}\frac{3^{\ln(x^3+1)}}{x^3+5}=\lim_{x\to\infty}\frac{e^{\ln(x^3+1)\cdot\ln 3}}{x^3+5}=\lim_{x\to\infty}\frac{(x^3+1)^{\ln 3}}{x^3+5}=\infty$$

$3>e \Rightarrow \ln 3>\ln e=1$

ZAD. 6

$$\lim_{x\to\infty}\left(\sqrt[5]{(x+1)(x+2)(x+3)(x+4)(x+5)}-x\right)\overset{[\infty-\infty]}{=}\left|\begin{matrix}t=x+3\\x=t-3\\t\to\infty\end{matrix}\right|=\lim_{t\to\infty}\left(\sqrt[5]{(t-2)(t-1)t(t+1)(t+2)}-t+3\right)=$$

$$=3+\lim_{t\to\infty}\left(\sqrt[5]{(t^2-4)(t^2-1)t}-t\right)=3+\lim_{t\to\infty}\left(\sqrt[5]{t^5-5t^3+4t}-t\right)=3+\lim_{t\to\infty}t\cdot\left(\sqrt[5]{1-\frac{5}{t^2}+\frac{4}{t^4}}-1\right)\overset{[\infty\cdot 0]}{=}$$

$$=3+\lim_{t\to\infty}\frac{\sqrt[5]{1-\frac{5}{t^2}+\frac{4}{t^4}}-1}{\frac{1}{t}}\ \overset{[\frac{0}{0}]}{\underset{H}{=}}\ 3+\lim_{t\to\infty}\frac{\frac{1}{5}\left(1-\frac{5}{t^2}+\frac{4}{t^4}\right)^{-4/5}\cdot\left(\frac{10}{t^3}-\frac{16}{t^5}\right)}{-\frac{1}{t^2}}=3+\lim_{t\to\infty}\frac{-\frac{10}{t}+\frac{16}{t^3}}{5\cdot\sqrt[5]{\left(1-\frac{5}{t^2}+\frac{4}{t^4}\right)^3}}=3+0=3$$

ZAD 7

$a,b\in\mathbb{R}$ $\quad a=?\quad b=?$

$$\lim_{x\to 0}\Big(\overbrace{\frac{\operatorname{tg}2x}{x^3}+\frac{a}{x^2}+\frac{\sin bx}{x}}^{f(x)}\Big)=0$$

$$\lim_{x\to 0}\frac{\operatorname{tg}2x}{x^3}=\lim_{x\to 0}\underbrace{\frac{\operatorname{tg}2x}{2x}}_{\to 1}\cdot\underbrace{\frac{2}{x^2}}_{\to\infty}=\infty$$

$$\lim_{x\to 0}\frac{a}{x^2}=\begin{cases}\infty & a>0\\-\infty & a<0\end{cases}\ \Rightarrow$$

Jeśli $a>0$ granica wyjściowa wynosi ∞ nie zaś 0

$$\lim_{x\to 0}\frac{\sin bx}{x}=\lim_{x\to 0}\frac{\sin bx}{bx}\cdot b=b$$

Zatem $a<0$

Zał. $a<0$ $\Rightarrow$

$$\lim_{x\to 0}\left(\frac{\operatorname{tg}2x}{x^3}+\frac{a}{x^2}\right)\overset{[\infty-\infty]}{=}\lim_{x\to 0}\frac{\operatorname{tg}2x+ax}{x^3}\ \overset{[\frac{0}{0}]}{\underset{H}{=}}\ \lim_{x\to 0}\frac{\frac{1}{\cos^2 2x}\cdot 2+a}{3x^2}\ \overset{[\frac{2+a}{0^+}]}{=}\begin{cases}+\infty & a+2>0\\-\infty & a+2<0\\ ? & a=-2\end{cases}$$

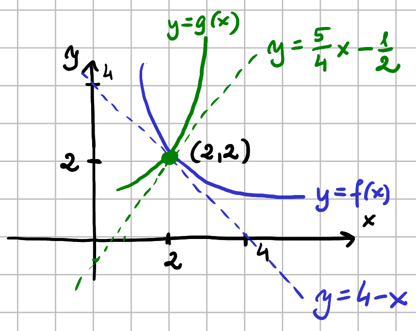
Niech $a=-2$

$$\lim_{x\to 0}\frac{\frac{1}{\cos^2 2x}\cdot 2-2}{3x^2}=\frac{2}{3}\cdot\lim_{x\to 0}\frac{\frac{1}{\cos^2 2x}-1}{x^2}\ \overset{[\frac{0}{0}]}{\underset{H}{=}}\ \frac{2}{3}\cdot\lim_{x\to 0}\frac{\frac{-2}{\cos^3 2x}(-\sin 2x)\cdot\cancel{2}}{\cancel{2}x}=$$

$$=\frac{4}{3}\cdot\lim_{x\to 0}\frac{\sin 2x}{x}\cdot\frac{1}{\cos^3 2x}=\frac{4}{3}\lim_{x\to 0}\underbrace{\frac{\sin 2x}{2x}}_{\to 1}\cdot 2\cdot\underbrace{\frac{1}{\cos^3 2x}}_{\to 1}=\frac{8}{3}$$

$$\lim_{x\to 0}f(x)=b+\frac{8}{3}=0\ \Rightarrow\ b=-\frac{8}{3},\ a=-2$$

ZAD 8



Zał. $f,g\in C^1(\mathbb{R})$

$$\lim_{x\to 2}\frac{f'(x)}{g'(x)}=\ ?\ \ldots$$

$f'(2)=-1$ $\rightarrow$ stosując tw. o arytmetyce granic

$g'(2)=\frac{5}{4}$

$$\ldots=\lim_{x\to 2}\frac{f'(x)}{g'(x)}=\frac{\lim_{x\to 2}f'(x)}{\lim_{x\to 2}g(x)}=\frac{f'(2)}{g'(2)}=-\frac{4}{5}$$

oraz korzystając z ciągłości f', g'