Zadanie domowe nr 6

ZAD. 1

$f'(x)=\frac{tg\sqrt{x}}{\sqrt{x}\cdot\cos^2\sqrt{x}}$, $f(\frac{\pi^2}{16})=0$ $\qquad f(\frac{\pi^2}{36})=?$

$\int f'(x)dx = f(x)+C \qquad \int \frac{tg\sqrt x}{\sqrt x \cdot \cos^2\sqrt x}dx = \left| \begin{array}{l} t=tg\sqrt x \\ dt=\frac{1}{\cos^2\sqrt x}\cdot\frac{1}{2\sqrt x}dx \end{array}\right| = 2\int t\,dt = t^2+C = tg^2\sqrt x + C$; $C\in\mathbb{R}$

Spośród rodziny $\{tg^2\sqrt x + C;\ C\in\mathbb{R}\}$ wybieramy f: $\quad tg^2\sqrt{\frac{\pi^2}{16}}+C=0 \iff tg^2\frac{\pi}{4}+C=1+C=0$

Zatem $C=-1$ $\quad f(x)=tg^2\sqrt x - 1$ $\quad$ oraz $\quad f(\frac{\pi^2}{36}) = tg^2\frac{\pi}{6}-1 = (\frac{\sqrt3}{3})^2-1=-\frac23$

ZAD. 2

a) $\int\frac{\sqrt{arcctg2x}}{1+4x^2}dx = \left|\begin{array}{l} t=arcctg2x \\ dt=-\frac{1}{1+4x^2}dx\end{array}\right| = -\int\sqrt t\,dt = -\frac{t^{3/2}}{\frac32}+C = -\frac23t\sqrt t+C = -\frac23(arcctg2x)^{3/2}+C$

$\int\frac1{x^3}ctg\frac2{x^2}dx = \left|\begin{array}{l}\frac2{x^2}=t \\ -\frac4{x^3}dx=dt\end{array}\right| = -\frac14\int ctg\,t\,dt = -\frac14\int\frac{\cos t}{\sin t}dt = -\frac14\ln|\sin t|+C = -\frac14\ln|\sin\frac2{x^2}|+C$

Odp. $-\frac23(arcctg2x)^{3/2}-\frac14\ln|\sin\frac2{x^2}|+C$

b) $\int\frac{x^5dx}{\sqrt{1-9x^{12}}} = \int\frac{x^5\,dx}{\sqrt{1-(3x^6)^2}} = \left|\begin{array}{l}3x^6=t \\ 18x^5dx=dt\end{array}\right| = \frac1{18}\int\frac{dt}{\sqrt{1-t^2}} = \frac1{18}\arcsin t+C = \frac1{18}\arcsin(3x^6)+C$

c) $\int\frac{ctgx}{(1+ctg^4x)\sin^2x}dx = \left|\begin{array}{l}ctgx=t \\ -\frac1{\sin^2x}dx=dt\end{array}\right| = -\int\frac{t}{1+t^4}dt = \left|\begin{array}{l}t^2=u \\ 2t\,dt=du\end{array}\right| = -\frac12\int\frac{du}{1+u^2} = -\frac12arctg(u)+C = -\frac12arctg(ctg^2x)+C$

d) $\int(\underbrace{x^5+2x^2-1}_{f'})\underbrace{arctg\,x}_{g}\,dx = (\underbrace{\frac{x^6}{6}-\frac23x^3-x}_{f})\underbrace{arctg\,x}_{g} - \int(\underbrace{\frac{x^6}6-\frac23x^3-x}_{f})\cdot\underbrace{\frac1{1+x^2}}_{g'}dx = (\frac{x^6}6-\frac23x^3-x)arctg\,x - \underbrace{\quad}_{(*)}$

$\int(\frac{x^6}6-\frac23x^3-x)\frac1{1+x^2}dx = \frac16\int\frac{x^6-4x^3-6x}{x^2+1}dx = \frac16\int(x^4-x^2-4x+1)dx+\frac16\int\frac{-2x-1}{x^2+1}dx$

$=\frac16\int(x^4-x^2-4x+1)dx - \frac16\int\frac{2x\,dx}{x^2+1} - \frac16\int\frac{dx}{x^2+1} =$

$=\frac16(\frac{x^5}5-\frac{x^3}3-2x^2+x) - \frac16\ln(x^2+1)-\frac16arctg\,x+C$ $\quad(*)$

x^4-x^2-4x+1
$(x^6-4x^3-6x):(x^2+1)$
$-\ \ x^6+x^4$
$-x^4-4x^3-6x$
$-\ \ -x^4-x^2$
$-4x^3+x^2-6x$
$-\ \ -4x^3-4x$
x^2-2x
$-\ \ x^2+1$
$-2x-1$

e) $\int\frac{e^{2x}+4e^x}{\sqrt{9-e^{2x}}}dx = \int\frac{(e^x+4)e^x}{\sqrt{9-e^{2x}}}dx = \left|\begin{array}{l}e^x=t \\ e^xdx=dt\end{array}\right| = \int\frac{t+4}{\sqrt{9-t^2}}dt = \int\frac{(-2t)\cdot(-\frac12)+4}{\sqrt{9-t^2}}dt = -\int\frac{-2t}{2\sqrt{9-t^2}}dt + \frac43\int\frac{dt}{\sqrt{1-(\frac t3)^2}}$

$= -\sqrt{9-t^2}+4\arcsin\frac t3 + C = -\sqrt{9-e^{2x}}+4\arcsin\frac{e^x}{3}+C$

$\int\frac{dt}{\sqrt{1-(\frac t3)^2}} = \left|\begin{array}{ll}\frac t3=u & t=3u \\ & dt=3du\end{array}\right| = \int\frac{3\,du}{\sqrt{1-u^2}} = 3\arcsin(u)+C = 3\arcsin\frac t3+C$

f) $\int 2^{\sin^3x}\cdot\sin^5x\cos x\,dx = \int 2^{\sin^3x}\sin^3x\sin^2x\cos x\,dx = \left|\begin{array}{l}t=\sin^3x \\ dt=3\sin^2x\cos x\,dx\end{array}\right| = \frac13\int \underbrace{2^t}_{f'}\cdot\underbrace{t}_{g}\,dt =$

$=\frac13\left[\underbrace{\frac{2^t}{\ln2}}_{f}\cdot\underbrace{t}_{g} - \int\underbrace{\frac{2^t}{\ln 2}}_{f}\cdot\underbrace{1}_{g'}\,dt\right] = \frac{1}{3\ln2}\left(t\cdot2^t-\int2^tdt\right) = \frac{1}{3\ln2}(t\cdot2^t-\frac{1}{\ln2}2^t)+C = \frac{2^t}{3\ln2}(t-\frac1{\ln2})+C=$

$=\frac{2^{\sin^3x}}{3\ln2}(\sin^3x-\frac1{\ln2})+C$

g) $I=\int\cos(\ln x)dx = \int \underbrace{1}_{f'}\cdot\underbrace{\cos(\ln x)}_{g}dx = = \underbrace{x}_{f}\cdot\underbrace{\cos(\ln x)}_{g'} - \int \underbrace{x}_{f}\cdot\underbrace{(-\sin(\ln x))\cdot\frac1x}_{g'}dx = x\cos(\ln x)+\int\sin(\ln x)dx=$

$= x\cos(\ln x)+x\sin(\ln x) - \int x\cdot\cos(\ln x)\cdot\frac1x\, dx = x\cos(\ln x)+x\sin(\ln x)-I$

$\Rightarrow 2I = x\cos(\ln x)+x\sin(\ln x)+C \quad /:2 \qquad\qquad I=\frac x2[\cos(\ln x)+\sin(\ln x)]+C$

Zadanie domowe nr 6

ZAD. 3

$$f(x)=\frac{2x+3}{x^2(x+5)^3(x^2+7)^2\underset{\Delta<0}{(x^2+2x+15)}}$$

Ułamki podstawowe

$$f(x)=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+5}+\frac{D}{(x+5)^2}+\frac{E}{(x+5)^3}+\frac{F\cdot 2x}{x^2+7}+\frac{G}{x^2+7}+\frac{H\cdot 2x}{(x^2+7)^2}+\frac{I}{(x^2+7)^2}+\frac{J\cdot(2x+2)}{x^2+2x+15}+\frac{K}{x^2+2x+15}$$

Ułamki proste

$$f(x)=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+5}+\frac{D}{(x+5)^2}+\frac{E}{(x+5)^3}+\frac{Fx+G}{x^2+7}+\frac{Hx+I}{(x^2+7)^2}+\frac{Jx+K}{x^2+2x+15}$$

ZAD. 4

a) $$\int\frac{2x-1}{x^2-6x+9}dx=\int\frac{2x-1}{(x-3)^2}dx=\int\frac{2(x-3)+5}{(x-3)^2}dx=2\int\frac{dx}{x-3}+5\int\frac{dx}{(x-3)^2}=2\ln|x-3|-\frac{5}{x-3}+C$$

b) $$\int\frac{x^6-8x^3+x^2-4}{x^4-8x}dx=\int\frac{x^2(x^4-8x)+x^2-4}{x^4-8x}dx=\int x^2dx+\int\frac{x^2-4}{x(x^3-8)}dx=\int x^2dx+\int\frac{\cancel{(x-2)}(x+2)}{x\cancel{(x-2)}(x^2+2x+4)}dx=\ldots$$

$$\frac{x+2}{x(x^2+2x+4)}=\frac{A}{x}+\frac{B(2x+2)+C}{x^2+2x+4}\quad\Big/\cdot x(x^2+2x+4)$$

$$x+2=A(x^2+2x+4)+B(2x^2+2x)+Cx$$

$x^2:\ 0=A+2B$

$x^1:\ 1=2A+2B+C$

$x^0:\ 2=4A \implies A=\frac12$

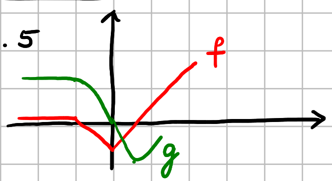
$2B=-\frac12 \quad B=-\frac14$

$C=1-2A-2B=1-1+\frac12=\frac12$

$$\ldots=\int x^2dx+\frac12\int\frac{dx}{x}-\frac14\int\frac{2x+2}{x^2+2x+4}dx+\frac12\int\frac{dx}{(x+1)^2+3}=\frac{x^3}{3}-\frac14\ln|x^2+2x+4|+\frac12\cdot\frac{\sqrt3}{3}\operatorname{arctg}\frac{x+1}{\sqrt3}+C$$

$$\int\frac{dx}{(x+1)^2+3}=\frac13\int\frac{dx}{\left(\frac{x+1}{\sqrt3}\right)^2+1}=\left|\begin{array}{ll}\frac{x+1}{\sqrt3}=t & x+1=\sqrt3\,t\\ & dx=\sqrt3\,dt\end{array}\right|=\frac{\sqrt3}{3}\int\frac{dt}{t^2+1}=\frac{\sqrt3}{3}\operatorname{arctg}t+C=\frac{\sqrt3}{3}\operatorname{arctg}\frac{x+1}{\sqrt3}+C$$

ZAD. 5



g maleje w przedziale w którym $f(x)<0$

rośnie — // — $f(x)>0$

$g=\mathrm{const}$ gdy $f(x)=0$

$\implies$ Zatem g jest pierwotną f

ZAD. 6

$y=w(x)$ f. wielomianowa $\implies w\in C^\infty(\mathbb{R}) \implies \exists\, w''(2)=0$ bo $(2,w(2))$ to punkt przeg.

Niech $w''(x)=x-2$ (możliwie niskiego stopnia) $\int(x-2)\,dx=\frac{x^2}{2}-2x+C$; $C\in\mathbb{R}$

$w'(2)=2$ $\quad \frac{2^2}{2}-2\cdot2+C=2 \iff 2-4+C=2$, $C=4$ $\implies w'(x)=\frac{x^2}{2}-2x+4$

$\int\left(\frac{x^2}{2}-2x+4\right)dx=\frac16x^3-x^2+4x+D$, $D\in\mathbb{R}$

$w(2)=1$ $\quad \frac16 2^3-2^2+4\cdot2+D=1 \iff \frac83-4+8+D=1 \iff D=-\frac{13}{3}$ $\implies w(x)=\frac{x^3}{6}-x^2+4x-\frac{13}{3}$

ZAD. 7

$$\int\frac{dx}{x}=\int1\cdot\frac1x\,dx=x\cdot\frac1x-\int x\cdot\left(-\frac1{x^2}\right)dx=1+\int\frac{dx}{x}\implies 0=1$$

błąd wnioskowania

Całka nieoznaczona to RODZINA funkcji pierwotnych.
Funkcje pierwotne różnią się o stałą.

$\int\frac{dx}{x}=\ln|x|+C$ tj. rodzina $\{\ln|x|+C\ ;\ C\in\mathbb{R}\}$ oraz rodzina $\{\ln|x|+1+D\ ;\ D\in\mathbb{R}\}$ są identyczne

ZAD. 8

a) $$\int\frac{x\cos^3x-\sin x}{\cos^2x}e^{\sin x}dx=\underbrace{\int x\cos x\,e^{\sin x}dx}_{I_1}-\underbrace{\int\frac{\sin x}{\cos^2x}e^{\sin x}dx}_{I_2}=xe^{\sin x}-\frac{e^{\sin x}}{\cos x}+C=\left(x-\frac1{\cos x}\right)e^{\sin x}+C$$

$$I_1=\int x\cdot(e^{\sin x}\cos x)\,dx=xe^{\sin x}-\int1\cdot e^{\sin x}dx$$

$$I_2=\int\frac{\sin x}{\cos^2x}e^{\sin x}dx=\frac1{\cos x}\cdot e^{\sin x}-\int\frac1{\cos x}\cdot e^{\sin x}\cos x\,dx=\frac{e^{\sin x}}{\cos x}-\int e^{\sin x}dx$$

b) $$\int\frac{\sqrt{x^2+1}\,(\ln(x^2+1)-2\ln x)}{x^4}dx\overset{x>0}{=}\int\frac{x\sqrt{1+\frac1{x^2}}}{x^4}\left(\ln(x^2+1)-\ln x^2\right)dx=\int\sqrt{1+\frac1{x^2}}\cdot\frac1{x^3}\ln\left(1+\frac1{x^2}\right)dx=\left|\begin{array}{l}1+\frac1{x^2}=t\\ -\frac2{x^3}dx=dt\quad \frac{dx}{x^3}=-\frac12dt\end{array}\right|$$

$$=-\frac12\int\sqrt t\ln t\,dt=-\frac12\left[\frac23t^{3/2}\ln t-\int\frac23t^{3/2}\cdot\frac1t\,dt\right]=-\frac13t\sqrt t\cdot\ln t+\frac13\int\sqrt t\,dt=-\frac13t\sqrt t\ln t+\frac29t\sqrt t+C=\frac13\left(1+\frac1{x^2}\right)^{3/2}\left(\frac23-\ln\left(1+\frac1{x^2}\right)\right)+C$$

ZAD. 9

$$\int\frac{1+2x^2}{x^5(1+x^2)^3}dx=\int\frac{x+2x^3}{x^6(1+x^2)^3}dx=\int\frac{x+2x^3}{(x^2+x^4)^3}dx=\left|\begin{array}{l}t=x^2+x^4\\ dt=(2x+4x^3)dx\\ \frac12dt=(x+2x^3)dx\end{array}\right|=\frac12\int\frac{dt}{t^3}=\frac12\,\frac{t^{-2}}{-2}+C=\frac{-1}{4t^2}+C$$

$$=-\frac{1}{4(x^2+x^4)^2}+C$$