Zadanie domowe nr 7

ZAD. 1

a) $$\int \frac{x+\sqrt{2x-5}}{x-2}dx = \left| \begin{matrix} \sqrt{2x-5}=t \\ 2x-5=t^2 \\ 2dx = 2t\,dt \\ dx = t\,dt \\ x=\frac{t^2+5}{2}\end{matrix}\right| = \int \frac{\frac{t^2+5}{2}+t}{\frac{t^2+5}{2}-2}\, t\,dt = \int \frac{t^2+2t+5}{t^2+1}\, t\,dt = \int \frac{(t^2+1)+(2t+4)}{t^2+1}\cdot t\,dt = \int t\,dt + \int\frac{2t^2+4t}{t^2+1}dt =$$

$$= \int t\,dt + \int \frac{2(t^2+1)+4t-2}{t^2+1}dt = \int (t+2)\,dt + 2\int\frac{2t}{t^2+1}dt - 2\int\frac{dt}{t^2+1} = \frac{t^2}{2}+2t+2\ln|t^2+1| - 2\,\mathrm{arctg}\, t + C =$$

$$= \frac12(2x-5)+2\sqrt{2x-5}+2\ln|2x-4| - 2\,\mathrm{arctg}\sqrt{2x-5}+C$$

b) $$\int\frac{x-5}{\sqrt{5+4x-x^2}}dx = \int \frac{-\frac12\cdot(-2x+4)-3}{\sqrt{-x^2+4x+5}}dx = -\int\frac{-2x+4}{2\sqrt{-x^2+4x+5}}dx - 3\int\frac{dx}{\sqrt{9-(x-2)^2}} = -\sqrt{-x^2+4x+5} - 3\arcsin\frac{x-2}{3}+C$$

$$\int\frac{dx}{\sqrt{9-(x-2)^2}} = \frac13\int\frac{dx}{\sqrt{1-(\frac{x-2}{3})^2}} = \left|\begin{matrix}\frac{x-2}{3}=t\\ x-2=3t\\ dx=3\,dt\end{matrix}\right| = \frac13\int\frac{3\,dt}{\sqrt{1-t^2}} = \arcsin t + C = \arcsin\frac{x-2}{3}+C$$

c) $$\int\frac{5x+2}{\sqrt{2x^2+8x-1}}dx = \int\frac{\frac54(4x+8)-8}{\sqrt{2x^2+8x-1}}dx = \frac52\int\frac{4x+8}{2\sqrt{2x^2+8x-1}}dx - \frac{8}{\sqrt2}\int\frac{dx}{\sqrt{x^2+4x-\frac12}} = \frac52\sqrt{2x^2+8x-1} - 4\sqrt2\ln\left|x+2+\sqrt{x^2+4x-\tfrac12}\right|+C$$

$$\int\frac{dx}{\sqrt{x^2+4x-\frac12}} = \int\frac{dx}{\sqrt{(x+2)^2-\frac92}} = \left|\begin{matrix}x+2=t\\dx=dt\end{matrix}\right| = \int\frac{dt}{\sqrt{t^2-\frac92}} = \ln\left|t+\sqrt{t^2-\tfrac92}\right|+C = \ln\left|x+2+\sqrt{x^2+4x-\tfrac12}\right|+C$$

d) $$I=\int\frac{6x^3-22x^2+21x-7}{\sqrt{x^2-4x+3}}dx = (ax^2+bx+c)\sqrt{x^2-4x+3} + k\cdot\int\frac{dx}{\sqrt{x^2-4x+3}} \Big/ {}' \qquad a,b,c,k\in\mathbb{R}$$

$$\frac{6x^3-22x^2+21x-7}{\sqrt{x^2-4x+3}} = (2ax+b)\sqrt{x^2-4x+3} + (ax^2+bx+c)\cdot\frac{2x-4}{2\sqrt{x^2-4x+3}} + \frac{k}{\sqrt{x^2-4x+3}} \Big/ \cdot\sqrt{x^2-4x+3}$$

$$6x^3-22x^2+21x-7 = (2ax+b)(x^2-4x+3)+(ax^2+bx+c)(x-2)+k =$$
$$= 2ax^3-8ax^2+6ax+bx^2-4bx+3b+ax^3-2ax^2+bx^2-2bx+cx-2c+k =$$

x^3: $6=3a \Rightarrow a=2$

x^2: $-22=-10a+2b \Rightarrow b=5a-11=-1$

x^1: $21=6a-6b+c \Rightarrow c=21-6a+6b=21-12-6=3$

x^0: $-7=3b-2c+k \Rightarrow k=2c-3b-7=6+3-7=2$

$$I = (2x^2-x+3)\sqrt{x^2-4x+3} + 2\int\frac{dx}{\sqrt{x^2-4x+3}} = (2x^2-x+3)\cdot\sqrt{x^2-4x+3} + 2\ln\left|x-2+\sqrt{x^2-4x+3}\right|+C$$

$$\int\frac{dx}{\sqrt{x^2-4x+3}} = \int\frac{dx}{\sqrt{(x-2)^2-1}} = \left|\begin{matrix}x-2=t\\dx=dt\end{matrix}\right| = \int\frac{dt}{\sqrt{t^2-1}} = \ln\left|t+\sqrt{t^2-1}\right|+C = \ln\left|x-2+\sqrt{x^2-4x+3}\right|+C$$

e) $$\int\left(x^2+x+\sqrt{1-x^2}\right)^2dx = \int\left[(x^2+x)^2+2(x^2+x)\sqrt{1-x^2}+1-x^2\right]dx = \int(x^4+2x^3+1)\,dx + 2\int x^2\sqrt{1-x^2}\,dx + \int 2x\sqrt{1-x^2}\,dx = \ldots$$

$$\int 2x\sqrt{1-x^2}\,dx = \left|\begin{matrix}1-x^2=t\\-2x\,dx=dt\end{matrix}\right| = -\int\sqrt{t}\,dt = -\frac23 t^{3/2}+C = -\frac23\sqrt{(1-x^2)^3}+C$$

$$\int x^2\sqrt{1-x^2}\,dx = \left|\begin{matrix}x=\sin t\\dx=\cos t\,dt\end{matrix}\right| = \int \sin^2 t\sqrt{1-\sin^2 t}\,\cos t\,dt = \int\sin^2 t\cos^2 t\,dt = \frac14\int(2\sin t\cos t)^2dt = \frac14\int\sin^2 2t\,dt$$
$$= \frac14\int\frac{1-\cos 4t}{2}dt = \frac18\left(t-\frac14\sin 4t\right)+C = \frac18\left(\arcsin x - \frac14\sin(4\arcsin x)\right)+C$$

$1-x^2\geq 0,\ |x|\leq 1 \rightarrow x=\sin t,\ t=\arcsin x$

$$\ldots = \frac{x^5}{5}+\frac{x^4}{2}+x+\frac14\arcsin x - \frac1{16}\sin(4\arcsin x) - \frac23\sqrt{(1-x^2)^3}+C$$

ZAD. 2

$$\int\frac{dx}{3\sin x+4\cos x} = \left|\begin{matrix}t=\mathrm{tg}\frac{x}{2} & \sin x=\frac{2t}{1+t^2}\\ x=2\,\mathrm{arctg}\,t & \\ dx=\frac{2}{1+t^2}dt & \cos x=\frac{1-t^2}{1+t^2}\end{matrix}\right| = \int\frac{\frac{2}{1+t^2}dt}{\frac{6t}{1+t^2}+4\frac{1-t^2}{1+t^2}} = \int\frac{2\,dt}{6t+4-4t^2} = -\int\frac{dt}{2t^2-3t-2} = -\int\frac{dt}{2(t-2)(t+\frac12)}$$

$$= -\frac12\int\left(\frac{A}{t-2}+\frac{B}{t+\frac12}\right)dt = -\frac15\int\frac{dt}{t-2}+\frac15\int\frac{dt}{t+\frac12} = -\frac15\ln|t-2|+\frac15\ln\left|t+\frac12\right|+C = \frac15\ln\left|\frac{\mathrm{tg}\frac{x}{2}+\frac12}{\mathrm{tg}\frac{x}{2}-2}\right|+C$$

$1=A(t+\frac12)+B(t-2)$

$t=-\frac12 \Rightarrow 1=-\frac52 B,\ B=-\frac25 \qquad t=2 \Rightarrow 1=\frac52 A,\ A=\frac25$

Zadanie domowe nr 7

ZAD. 3

a) $\displaystyle\int \frac{\sin 6x}{\sqrt{7-4\cos 3x}}\,dx = \int \frac{2\sin 3x\cos 3x}{\sqrt{7-4\cos 3x}}\,dx = \left|\begin{array}{l} \cos 3x = t \\ -3\sin 3x\,dx = dt \end{array}\right| = -\frac{1}{3}\int \frac{2t\,dt}{\sqrt{7-4t}} = \left|\begin{array}{l} u=\sqrt{7-4t} \\ u^2 = 7-4t \\ 2u\,du = -4dt \\ dt = -\frac{1}{2}u\,du \\ t=\frac{1}{4}(7-u^2)\end{array}\right| = -\frac{2}{3}\int \frac{\frac{1}{4}(7-u^2)}{\cancel{u}}\cdot\frac{-\cancel{u}}{2}\,du$

$\displaystyle = \frac{1}{12}\int (7-u^2)\,du = \frac{7}{12}u - \frac{1}{36}u^3 + C = \frac{7}{12}\cdot\sqrt{7-4\cos 3x} - \frac{1}{36}(7-4\cos 3x)^{3/2} + C$

b) $\displaystyle\int \sin^3 x\cos^3 x\,dx = \int\left(\frac{1}{2}\cdot 2\cdot\sin x\cos x\right)^3 dx = \frac{1}{8}\int \sin^3 2x\,dx = \frac{1}{8}\int (1-\cos^2 2x)\sin 2x\,dx = \left|\begin{array}{l} \cos 2x = t \\ -2\sin 2x\,dx = dt \\ \sin 2x\,dx = -\frac{1}{2}dt\end{array}\right| =$

$\displaystyle = -\frac{1}{16}\int (1-t^2)\,dt = -\frac{1}{16}\left(t-\frac{t^3}{3}\right)+C = \frac{1}{16}\left(\frac{1}{3}\cos^3 2x - \cos 2x\right)+C$

c) $\displaystyle\int \frac{dx}{26\cos^2 x + \sin^2 x - 10\sin x\cos x} = \left|\begin{array}{l} t = \operatorname{tg}x \\ x = \operatorname{arctg}t \\ dx = \frac{dt}{1+t^2}\end{array}\right.$  $\left.\begin{array}{l} \sin x = \frac{t}{\sqrt{1+t^2}} \\ \cos x = \frac{1}{\sqrt{1+t^2}}\end{array}\right| = \int \frac{\frac{dt}{1+t^2}}{\frac{26}{1+t^2}+\frac{t^2}{1+t^2}-10\frac{t}{1+t^2}} = \int \frac{dt}{t^2-10t+26}$

$\displaystyle = \int \frac{dt}{(t-5)^2+1} = \operatorname{arctg}(t-5)+C = \operatorname{arctg}(\operatorname{tg}x-5)+C$

d) $\displaystyle\int \frac{dx}{\sin x + \operatorname{tg}x} = \int \frac{\cos x\,dx}{\sin x\cos x + \sin x} = \left|\begin{array}{l} t = \cos x \\ x = \arccos t \\ dx = \frac{-1}{\sqrt{1-t^2}}dt\end{array}\ \ \sin x = \sqrt{1-t^2}\right| = \int \frac{t\cdot\frac{-1}{\sqrt{1-t^2}}\,dt}{\sqrt{1-t^2}\,(t+1)} = -\int \frac{t\,dt}{(t+1)(1-t^2)} = \int \frac{t\,dt}{(t+1)^2(t-1)} = \ldots$

$R(-\sin x, \cos x) = -R(\sin x, \cos x)$

Albo

$\displaystyle\int \frac{\cos x\,dx}{\sin x(\cos x+1)} = \int \frac{\cos x\cdot\sin x}{\sin^2 x\,(\cos x+1)} = \int \frac{\cos x\,\sin x}{(1-\cos^2 x)(1+\cos x)} = \left|\begin{array}{l} t=\cos x \\ dt = -\sin x\,dx\end{array}\right| = -\int \frac{t\,dt}{(1-t^2)(1+t)}$ jak wyżej

$\displaystyle\frac{t}{(t+1)(t-1)^2} = \frac{A}{t+1} + \frac{B}{(t+1)^2} + \frac{C}{t-1} \quad /\cdot (t+1)^2(t-1) \Rightarrow t = A(t^2-1)+B(t-1)+C(t^2+2t+1) =$

$= (A+C)t^2 + (B+2C)t + (-A-B+C)$

$\begin{array}{l} t^2: \\ t^1: \\ t^0: \end{array}\left\{\begin{array}{l} 0 = A+C \\ 1 = B+2C \\ 0 = C-A-B \end{array}\right. \quad \begin{array}{l} \Rightarrow A = -C \\ B = 1-2C \\ 0 = C+C-1+2C \Rightarrow C = \frac{1}{4},\ A = -\frac{1}{4},\ B = \frac{1}{2}\end{array}$

$\displaystyle\ldots = -\frac{1}{4}\int\frac{dt}{t+1} + \frac{1}{2}\int\frac{dt}{(t+1)^2} + \frac{1}{4}\int\frac{dt}{t-1} = -\frac{1}{4}\ln|t+1| - \frac{1}{2(t+1)} + \frac{1}{4}\ln|t-1| + C = \frac{1}{4}\ln\left|\frac{\cos x-1}{\cos x+1}\right| - \frac{1}{2(\cos x+1)} + C$

e) $\displaystyle\int \frac{\cos^3 x}{1+\sin^2 x}\,dx = \int \frac{(1-\sin^2 x)\cos x}{1+\sin^2 x}\,dx = \left|\begin{array}{l} \sin x = t \\ \cos x\,dx = dt\end{array}\right| = \int \frac{1-t^2}{1+t^2}\,dt = -\int \frac{t^2+1-2}{t^2+1}\,dt = -\int dt + 2\int\frac{dt}{t^2+1} =$

$= -t + 2\operatorname{arctg}t + C = -\sin x + 2\operatorname{arctg}(\sin x)\,dx$

Albo $R(\sin x, -\cos x) = -R(\sin x, \cos x)$ $\displaystyle\int \frac{\cos^3 x}{1+\sin^2 x}\,dx = \left|\begin{array}{l} t = \sin x \\ x = \arcsin t \\ dx = \frac{dt}{\sqrt{1-t^2}}\end{array}\ \ \cos x = \sqrt{1-t^2}\right| = \int \frac{(1-t^2)\cancel{\sqrt{1-t^2}}}{1+t^2}\cdot\frac{dt}{\cancel{\sqrt{1-t^2}}}$ itd...

ZAD. 4

a) $\displaystyle\int \frac{x^3+2}{x^2\cdot\sqrt{9-x^2}}\,dx = \left|\begin{array}{l} x = 3\sin t,\ t\in[-\frac{\pi}{2},\frac{\pi}{2}] \\ dx = 3\cos t\,dt \\ 9-x^2 = 9\cos^2 t \\ t = \arcsin\frac{x}{3}\end{array}\right| = \int \frac{27\sin^3 t + 2}{9\sin^2 t\cdot\cancel{3|\cos t|}\ \cos t}\,\cancel{3\cos t}\,dt = \frac{2}{9}\int\frac{dt}{\sin^2 t} + 3\int \sin t\,dt$

$\displaystyle = -\frac{2}{9}\operatorname{ctg}t - 3\cos t + C = -\frac{2}{9}\operatorname{ctg}\left(\arcsin\frac{x}{3}\right) - 3\cos\left(\arcsin\frac{x}{3}\right) + C$

b) $\displaystyle\int \frac{dx}{(1+x^2)\sqrt{1+x^2}} = \left|\begin{array}{l} x = \operatorname{tg}t \\ t = \operatorname{arctg}x \\ dt = \frac{dx}{1+x^2}\end{array}\right| = \int \frac{dt}{\sqrt{1+\operatorname{tg}^2 t}} = \int \sqrt{\cos^2 t}\,dt = \int \cos t\,dt = \sin t + C = \sin(\operatorname{arctg}x) + C$

$t\in[-\frac{\pi}{2},\frac{\pi}{2}]$

$\displaystyle 1+\operatorname{tg}^2 t = 1 + \frac{\sin^2 t}{\cos^2 t} = \frac{\cos^2 t + \sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t}$