Zadanie domowe nr 10

Zad. 1 Ekstrema lokalne

a) $f(x,y) = y^2 e^x - x - 3y \qquad D_f = \mathbb{R}^2, \quad f \in C^\infty(\mathbb{R}^2)$

WK $\begin{cases} \frac{\partial f}{\partial x} = y^2 e^x - 1 = 0 \\ \frac{\partial f}{\partial y} = 2ye^x - 3 = 0 \end{cases} \underset{\substack{\text{zał.} \\ y \neq 0}}{\Longrightarrow} \begin{cases} e^x = \frac{1}{y^2} \\ e^x = \frac{3}{2y} \end{cases}$ $\quad \frac{1}{y^2} = \frac{3}{2y}, \quad 2y = 3y^2, \quad y(2-3y) = 0$

$y = 0$ odpada $\qquad y = \frac{2}{3} \Rightarrow e^x = \frac{1}{y^2} = \frac{9}{4} \qquad x = \ln\frac{9}{4}$

Można założyć bez straty dla ogólności można założyć, bowiem gdy $y=0$ to $\frac{\partial f}{\partial x} = -1 \neq 0$

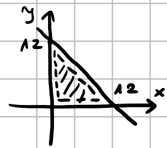
$P_0 = \left(\ln\frac{9}{4}, \frac{2}{3}\right)$

WW $H(x,y) = \begin{bmatrix} y^2 e^x & 2ye^x \\ 2ye^x & 2e^x \end{bmatrix} \qquad H(P_0) = \begin{bmatrix} \frac{4}{9}\cdot\frac{9}{4} & \frac{4}{3}\cdot\frac{9}{4} \\ \frac{4}{3}\cdot\frac{9}{4} & 2\cdot\frac{9}{4} \end{bmatrix} = \frac{9}{4}\begin{bmatrix} \frac{4}{9} & \frac{4}{3} \\ \frac{4}{3} & 2 \end{bmatrix}$

$\det H(P_0) = \left(\frac{9}{4}\right)^2 \cdot \left[\frac{8}{9} - \frac{16}{9}\right] < 0$

brak ekstremów

b) $f(x,y) = 3\ln\frac{x}{6} + 2\ln y + \ln(12-x-y)$ $\qquad$ Zał. $x>0, \ y>0, \ 12-x-y>0$ $\qquad f \in C^2(D_f)$



NK $\begin{cases} \frac{\partial f}{\partial x} = 3\cdot\frac{6}{x}\cdot\frac{1}{6} + \frac{1}{12-x-y}\cdot(-1) = \frac{3}{x} - \frac{1}{12-x-y} = 0 \\ \frac{\partial f}{\partial y} = \frac{2}{y} - \frac{1}{12-x-y} = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{36-4x-3y}{x(12-x-y)} = 0 \\ \frac{24-2x-3y}{y(12-x-y)} = 0 \end{cases} \Rightarrow \begin{cases} 36-4x-3y = 0 \\ 24-2x-3y = 0 \end{cases} \ominus$

$\Rightarrow 12 - 2x = 0, \ x = 6$

$3y = 36 - 24 = 12$

$y = 4$

$P_0 = (6,4) \in D_f$

WW $H(x,y) = \begin{bmatrix} -\frac{3}{x^2} - \frac{1}{(12-x-y)^2} & \frac{-1}{(12-x-y)^2} \\ \frac{-1}{(12-x-y)^2} & -\frac{2}{y^2} - \frac{1}{(12-x-y)^2} \end{bmatrix} \qquad H(P_0) = \begin{bmatrix} -\frac{1}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{3}{8} \end{bmatrix}$

$\Delta_1 = -\frac{1}{3} < 0$
$\Delta_2 = \frac{1}{8} - \frac{1}{16} > 0$ $\Rightarrow$ W P_0 f osiąga maksimum lokalne równe $f(P_0)$

c) $f(x,y) = \left(\frac{\pi}{e+1}\right)^{\cos 3x \cdot \cos y} \qquad D_f = \mathbb{R}^2 \qquad f \in C^\infty(\mathbb{R}^2)$

NK $\begin{cases} \frac{\partial f}{\partial x} = \underbrace{\left(\frac{\pi}{e+1}\right)^{\cos 3x \cos y}}_{\neq 0} \cdot \ln\frac{\pi}{e+1} \cdot (-\sin 3x)\cdot 3\cos y = 0 \\ \frac{\partial f}{\partial y} = \underbrace{\left(\frac{\pi}{e+1}\right)^{\cos 3x \cos y}}_{\neq 0} \cdot \underbrace{\ln\frac{\pi}{e+1}}_{<0} \cdot \cos 3x \cdot (-\sin y) = 0 \end{cases} \Rightarrow \begin{cases} \sin 3x \cos y = 0 \\ \cos 3x \sin y = 0 \end{cases}$

bowiem $\frac{\pi}{e+1} < 1$

Gdy $\sin 3x = 0$ to $\cos 3x = \pm 1 \neq 0$
gdy $\cos y = 0$, to $\sin y \neq 0$

$\Rightarrow \begin{cases} \sin 3x = 0 \\ \sin y = 0 \end{cases} \vee \begin{cases} \cos y = 0 \\ \cos 3x = 0 \end{cases} \Leftrightarrow \begin{cases} 3x = k\pi \\ y = l\pi \end{cases} \vee \begin{cases} y = \frac{\pi}{2} + m\pi \\ 3x = \frac{\pi}{2} + n\pi \end{cases} \Leftrightarrow \begin{cases} x = k\cdot\frac{\pi}{3} \\ y = l\pi \end{cases} \vee \begin{cases} y = \frac{\pi}{2} + m\pi \\ x = \frac{\pi}{6} + n\frac{\pi}{3} \end{cases}$

$k, l \in \mathbb{Z} \qquad m, n \in \mathbb{Z} \qquad k, l, m, n \in \mathbb{Z}$

$P_{kl} = \left(\frac{k\pi}{3}, l\pi\right) \qquad Q_{m,n} = \left(\frac{\pi}{2} + m\pi, \frac{\pi}{6} + \frac{n\pi}{3}\right)$

WW $\frac{\partial^2 f}{\partial x^2} = \left(\frac{\pi}{e+1}\right)^{\cos 3x \cos y} \cdot \left(\ln\frac{\pi}{e+1}\right)^2 \cdot 9\sin^2 3x \cos^2 y + \left(\frac{\pi}{e+1}\right)^{\cos 3x \cos y} \ln\frac{\pi}{e+1} \cdot (-9)\cos 3x \cos y$

$\frac{\partial^2 f}{\partial y \partial x} = \left(\frac{\pi}{e+1}\right)^{\cos 3x \cos y} \left(\ln\frac{\pi}{e+1}\right)^2 3\sin 3x \cos y \cos 3x \sin y + \left(\frac{\pi}{e+1}\right)^{\cos 3x \cos y} \ln\frac{\pi}{e+1} \ 3\sin 3x \sin y$

$\frac{\partial^2 f}{\partial y^2} = \left(\frac{\pi}{e+1}\right)^{\cos 3x \cos y} \left(\ln\frac{\pi}{e+1}\right)^2 \cos^2 3x \sin^2 y + \left(\frac{\pi}{e+1}\right)^{\cos 3x \cos y} \ln\frac{\pi}{e+1} \ (-1)\cos 3x \cos y$

$H(P_{kl}) = \begin{bmatrix} \left(\frac{\pi}{e+1}\right)^{\cos k\pi \cdot \cos l\pi} \ln\frac{\pi}{e+1}(-9)\cos k\pi \cdot \cos l\pi & 0 \\ 0 & \left(\frac{\pi}{e+1}\right)^{\cos k\pi \cdot \cos l\pi} \cdot \ln\frac{\pi}{e+1} \cdot (-1)\cos k\pi \cdot \cos l\pi \end{bmatrix}$

k, l - oba parzyste lub oba nieparzyste

$H(P_{kl}) = \begin{bmatrix} -9 \cdot \frac{\pi}{e+1} \underbrace{\ln\frac{\pi}{e+1}}_{<0} & 0 \\ 0 & \underbrace{-\frac{\pi}{e+1}}_{<0} \underbrace{\ln\frac{\pi}{e+1}}_{<0} \end{bmatrix}$

$\Delta_1 = \frac{-9\pi}{e+1} \ln\frac{\pi}{e+1} > 0$
$\Delta_2 = \left(\frac{\pi}{e+1}\right)^2 \ln^2\frac{\pi}{e+1} > 0$

minima lokalne

k, l - różnej parzystości

$H(P_{kl}) = \begin{bmatrix} 9 \cdot \frac{e+1}{\pi} \ln\frac{\pi}{e+1} & 0 \\ 0 & \frac{e+1}{\pi} \ln\frac{\pi}{e+1} \end{bmatrix}$

$\Delta_1 < 0$
$\Delta_2 = 9\left(\frac{e+1}{\pi}\right)^2 \ln^2\frac{\pi}{e+1} > 0$

maksima lokalne

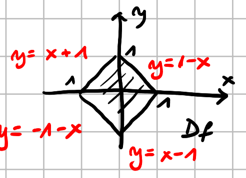
$H(Q_{mn}) = \begin{bmatrix} 0 & * \\ \underset{\uparrow *}{3\ln\frac{\pi}{e+1} \sin\left(\frac{\pi}{2} + m\pi\right)\sin\left(\frac{\pi}{2} + m\pi\right)} & 0 \end{bmatrix}$

$\Delta_2 < 0 \Rightarrow$ brak ekstremów

ZAD. 2 Wartość najmniejsza i największa

a) $f(x,y) = \arcsin(x+y) + \arccos(x-y)$ w jej dziedzinie naturalnej

$D_f = ?$ $\begin{cases} |x+y| \le 1 \\ |x-y| \le 1 \end{cases}$



D_f - zbiór zwarty (domknięty, ograniczony)

$f \in C(D_f)$

Na mocy tw. Weierstrassa f osiąga w zbiorze D_f wartość najmniejszą i największą

- wnętrze:

$\text{int}(D_f) = \{(x,y) \in \mathbb{R}^2 : |x+y| < 1 \wedge |x-y| < 1\}$

$$\begin{cases} f'_x = \dfrac{1}{\sqrt{1-(x-y)^2}} - \dfrac{1}{\sqrt{1-(x-y)^2}} = 0 \\ f'_y = \underbrace{\dfrac{1}{\sqrt{1-(x-y)^2}}}_{>0} - \underbrace{\dfrac{1}{\sqrt{1-(x-y)^2}} \cdot (-1)}_{>0} = 0 \end{cases} \Rightarrow \forall (x,y) \in \text{int}(D_f) \quad f'_y(x,y) > 0$$

brak rozwiązań układu

- Brzeg: $\partial(D_f)$

$f(1,0) = \arcsin 1 + \arccos 1 = \frac{\pi}{2} + 0 = \frac{\pi}{2}$

$f(0,1) = \arcsin 1 + \arccos(-1) = \frac{\pi}{2} + \pi = \frac{3}{2}\pi$

$f(0,-1) = \arcsin(-1) + \arccos 1 = -\frac{\pi}{2} + 0 = -\frac{\pi}{2}$

$f(-1,0) = \arcsin(-1) + \arccos(-1) = -\frac{\pi}{2} + \pi = \frac{\pi}{2}$

$y = x+1$; $x \in (-1,0)$ $\quad g(x) = f(x, x+1) = \arcsin(2x+1) + \pi \quad g'(x) = \dfrac{1}{\sqrt{1-(2x+1)^2}} \cdot 2 \ne 0$ brak

$y = x-1$; $x \in (0,1)$ $\quad h(x) = f(x, x-1) = \arcsin(2x-1) + 0 \quad h'(x) = \dfrac{1}{\sqrt{1-(2x-1)^2}} \cdot 2 \ne 0$ brak

$y = 1-x$; $x \in (0,1)$ $\quad p(x) = f(x, 1-x) = \arcsin 1 + \arccos(2x-1) \quad p'(x) = \dfrac{-1}{\sqrt{1-(2x-1)^2}} \cdot 2 \ne 0$ brak

$y = -x-1$; $x \in (-1,0)$ $\quad q(x) = f(x, -1-x) = \arcsin(-1) + \arccos(2x+1)$

$q'(x) = \dfrac{-1}{\sqrt{1-(2x+1)^2}} \cdot 2 \ne 0$ brak

wartość najmniejsza $f(0,-1) = -\frac{\pi}{2}$

wartość największa $f(0,1) = \frac{3}{2}\pi$

b) Metoda mnożników Lagrange'a: $f(x,y) = x^2 - 8x - 3y^2$ w zbiorze $D = \{(x,y) : x^2+y^2 \le 4\}$

$f \in C^\infty(\mathbb{R}^2) \wedge$ zbiór zwarty (domknięty i ograniczony)

Na mocy tw. Weierstrassa f osiąga w D swoje kresy

- wnętrze $\text{int}\, D = \{(x,y) : x^2+y^2 < 4\}$

$$\begin{cases} \frac{\partial f}{\partial x} = 2x - 8 = 0 \\ \frac{\partial f}{\partial y} = -6y = 0 \end{cases} \qquad P_0 = (4,0) \notin \text{int}\, D$$

! $\vec{\nabla} g = (2x, 2y) \ne \vec{0}$ dla $(x,y) \ne (0,0)$; $(0,0) \notin \partial D$ można pominąć

- brzeg $\partial D = \{(x,y) : x^2+y^2 = 4\}$

$\mathcal{L}(x,y,\lambda) := x^2 - 8x - 3y^2 + \lambda(x^2+y^2-4)$

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x} = 2x - 8 + 2\lambda x = 0 \\ \frac{\partial \mathcal{L}}{\partial y} = -6y + 2\lambda y = 0 \iff 2y(\lambda - 3) = 0 \Rightarrow y = 0 \vee \lambda = 3 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = x^2 + y^2 - 4 = 0 \end{cases}$$

$y = 0 \Downarrow$ $x^2 = 4$, $x = \pm 2$; $P_1 = (2,0)$ $P_2 = (-2,0)$

$\lambda = 3$: $2x - 8 + 6x = 0$, $x = 1$ $\Downarrow$ $y^2 = 3$, $y = \pm\sqrt{3}$; $P_3 = (1,\sqrt{3})$, $P_4 = (1,-\sqrt{3})$

$f(2,0) = 4 - 16 - 0 = -12$

$f(-2,0) = 4 + 16 = 20$ — wartość największa

$f(1,\sqrt{3}) = 1 - 8 - 9 = -16$

$f(1,-\sqrt{3}) = 1 - 8 - 9 = -16$

wartość najmniejsza w punktach P_3, P_4

ZAD. 3 Ekstrema lokalne $f(x,y) = x^8 + y^8 - 2x^4$

$D_f = \mathbb{R}^2$ $\quad f \in C^\infty(\mathbb{R}^2)$

WK $\begin{cases} f'_x = 8x^7 - 8x^3 = 0 \iff 8x^3(x^4-1) = 0 \Rightarrow x \in \{-1, 0, 1\} \\ f'_y = 8y^7 = 0 \Rightarrow y = 0 \end{cases}$

$P_0 = (0,0)$, $P_1 = (1,0)$, $P_2 = (-1,0)$

$$H(x,y) = \begin{bmatrix} 56x^6 - 24x^2 & 0 \\ 0 & 56y^7 \end{bmatrix}$$

$H(P_1) = H(P_2) = \begin{bmatrix} 32 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad \Delta_2 = 0$ Nie rozstrzyga

$f(x,y) = (x^4-1)^2 + y^8 - 1 \ge -1$

$f(P_1) = f(P_2) = -1$

$\Rightarrow$ w P_1 i P_2 f osiąga minima lokalne

ZAD.3 Ciąg dalszy $H(P_0)=\begin{bmatrix}0&0\\0&0\end{bmatrix}$ nie rozstrzyga

Zawężamy f do dwóch prostych: $y=x$ $\quad f(x,x)=2x^8-2x^4=2x^4(x^4-1)<0$ dla małych x blisko 0

$x=0$ $\quad f(0,y)=y^8>0$

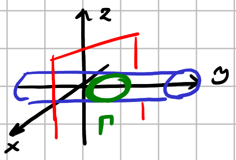
Zatem w P_0 brak ekstremum

ZAD.4 Wartość najmniejsza i największa $f(x,y,z)=3x-y-3z$

w zbiorze $\Gamma=\{(x,y,z):\ \underbrace{x+y-z=0}_{g_1}\ \wedge\ x^2+2z^2=1\}$ $\quad g_2=x^2+2z^2-1=0$

$x+y-z=0$ rów. płaszczyzny

$x^2+2z^2=1$, $\left(\frac{x}{1}\right)^2+\left(\frac{z}{\frac{1}{\sqrt{2}}}\right)^2=1$ walec eliptyczny



Γ - elipsa

$\Gamma\subset\mathbb{R}^3$ domknięty, ograniczony

$D_f=\mathbb{R}^3$; $f\in C^\infty(\mathbb{R}^3)$

Na mocy tw. Weierstrassa f osiąga kresy w Γ.

$\mathcal{L}(x,y,z;\lambda,\mu):=f+\lambda g_1+\mu g_2=3x-y-3z+\lambda(x+y-z)+\mu(x^2+2z^2-1)$

Czy $\vec\nabla g_1,\ \vec\nabla g_2$ są liniowo niezależne?

$G':=\begin{bmatrix}\vec\nabla g_1\\ \vec\nabla g_2\end{bmatrix}=\begin{bmatrix}1&1&-1\\2x&0&4z\end{bmatrix}$ $\quad r(G')=2$ gdy $x\neq0\ \vee\ z\neq0$

Gdy $x=z=0$, to punkt $(0,0,0)$ nie spełnia $g_2=0$

można zatem pominąć

WK

$$\begin{cases}\mathcal{L}'_x=3+\lambda+2\mu x=0\\ \mathcal{L}'_y=-1+\lambda=0 \Rightarrow \boxed{\lambda=1}\\ \mathcal{L}'_z=-3-\lambda+4\mu z=0\\ \mathcal{L}'_\lambda=x+y-z=0\\ \mathcal{L}'_\mu=x^2+2z^2-1=0\end{cases}$$

$$\left.\begin{matrix}4+2\mu x=0\\ -4+4\mu z=0\end{matrix}\right\}\oplus \Rightarrow 2\mu x+4\mu z=2\mu(x+2z)=0$$

$\mu=0 \quad \vee \quad x=-2z$

$\mu=0$ nie spełnia $4+2\mu x=0$

$x=-2z \Rightarrow y=3z \ \wedge\ 4z^2+2z^2=1$, $\quad z=\pm\frac{1}{\sqrt{6}}$

dla $z=\frac{1}{\sqrt6}$: $P_1=\left(\frac{-2}{\sqrt6},\frac{3}{\sqrt6},\frac{1}{\sqrt6}\right)$; dla $z=-\frac{1}{\sqrt6}$: $P_2=\left(\frac{2}{\sqrt6},\frac{-3}{\sqrt6},\frac{-1}{\sqrt6}\right)$

$f(x,y,z)=3x-y-3z$ $\quad f(P_1)=-\frac{12}{\sqrt6}=-2\sqrt6$ w. najmniejsza (i min. warunkowe) $\quad f(P_2)=\frac{12}{\sqrt6}=2\sqrt6$ w. największa (i max. warunkowe)