Algebra - Zad. domowe nr. 1 - Struktury alg.
Liczby zespolone

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a) Czy "∘" łączne w zbiorze $\mathbb{Z}$ $\quad \forall x,y \in \mathbb{Z} \quad x \circ y = x + (-1)^x \cdot y$

$x \circ y \in \mathbb{Z} \qquad x \circ y = x \pm y \in \mathbb{Z}$

Dla $x, y, z \in \mathbb{Z}$

$L = (x \circ y) \circ z = (x + (-1)^x \cdot y) \circ z = x + (-1)^x \cdot y + (-1)^{x+(-1)^x \cdot y} \cdot z =$
$= x + (-1)^x \cdot y + (-1)^{x \pm y} \cdot z$

$P = x \circ (y \circ z) = x \circ (y + (-1)^y \cdot z) = x + (-1)^x \cdot (y + (-1)^y \cdot z) =$
$= x + (-1)^x \cdot y + (-1)^{x+y} \cdot z$

$L = P$ bowiem $(-1)^{x+y} = (-1)^{x-y+2y} = (-1)^{x-y} \cdot \underbrace{(-1)^{2y}}_{=1} = (-1)^{x-y}$

TAK ŁĄCZNE

b) Czy $(\mathbb{R} \setminus \{-1\}, \circ)$ to półgrupa / monoid / grupa (przemienna)?

$\forall x, y \in \mathbb{R} \setminus \{\ \} \quad x \circ y := x + y + xy$

"∘" wewnętrzne $\quad x \neq -1, y \neq -1 \Rightarrow x + y + xy = x(1+y) + y = x(1+y) + y + 1 - 1$
$= \underset{\neq 0}{(1+y)} \underset{\neq 0}{(x+1)} - 1 \neq -1$

przemienne $\quad x \circ y = x + y + xy = y + x + yx = y \circ x$

łączne $\quad x, y, z \in \mathbb{R} \setminus \{-1\}$

$L = (x \circ y) \circ z = (x + y + xy) \circ z = x + y + xy + z + (x + y + xy) \cdot z =$
$= x + y + z + xy + xz + yz + xyz$

$P = x \circ (y \circ z) = x \circ (y + z + yz) = x + y + z + yz + x \cdot (y + z + yz) =$
$= x + y + z + yz + xy + xz + xyz$

$L = P$

el. neutralny $\quad x \circ e = e \circ x = x$
przemienne! $\quad x + e + xe = x, \quad e(1+x) = 0 \quad \underline{e = 0}$
$e \in \mathbb{R} \setminus \{-1\}$

(symetryczny)
el. odwrotny $\quad x \circ y = e = 0$
$x + y + xy = 0, \quad y(1+x) = -x, \quad y = \frac{-x}{1+x} \quad x \neq -1$

$y \neq -1$ bo $\frac{-x}{1+x} = -1$, tj. $y \in \mathbb{R} \setminus \{-1\} \Rightarrow \forall x \neq -1 \ \exists y \neq -1 : y \circ x = x \circ y = e$
$\Updownarrow$
$-x = -1 - x$
$0 = -1$ sprzeczność

c) Czy $(\mathbb{R} \setminus \{0\}, \circ)$ to półgrupa / grupa? $\qquad x \circ y := x \cdot |y|$

Działanie jest wewnętrzne $\quad \forall x, y \in \mathbb{R} \quad x \neq 0 \wedge y \neq 0 \Rightarrow x \cdot |y| \neq 0$

nie jest przemienne $\quad 2 \circ (-1) = 2 \cdot |-1| = 2 \neq (-1) \circ 2 = -1 \cdot |2| = -2$

$\forall x, y, z \in \mathbb{R} \setminus \{0\} \quad L = (x \circ y) \circ z = (x \cdot |y|) \circ z = x \cdot |y| \cdot |z|$
$P = x \circ (y \circ z) = x \circ (y \cdot |z|) = x \cdot |y|z|| = x \cdot |y| |z|$

$L = P \Rightarrow$ działanie jest łączne

Czy istnieje $e \in \mathbb{R} \setminus \{0\}$ t.że $\forall x \in \mathbb{R} \setminus \{0\} \quad x \circ e = e \circ x = x$? NIE! $\longrightarrow$ półgrupa nie przemienna

$\begin{cases} e \circ x = x \\ x \circ e = x \end{cases} \Leftrightarrow \begin{cases} e \cdot |x| = x \\ x \cdot |e| = x \end{cases} \Rightarrow |e| = 1, \ e = \pm 1 \longrightarrow$ wynikałoby $\quad$ x>0 daje 1, x<0 daje -1

(2)   f) analogicznie jak na wykładzie !

e) $9i \cdot z^3 = \bar{z}^5$ $z = re^{i\varphi}$, $r>0$ $\vee$ $z=0$ ← jest rozwiązaniem

$9i\, r^3 e^{3i\varphi} = r^5 e^{-5i\varphi}$, $r^3 \left[9e^{\frac{\pi}{2}i} \cdot e^{3i\varphi} - r^2 e^{-5i\varphi} \right] = 0 \ /: r\neq 0$

$9e^{\frac{\pi}{2}i} e^{3i\varphi} = r^2 e^{-5i\varphi}$

$9e^{8i\varphi} = r^2 e^{-\frac{\pi}{2}i}$

$r^2 = 9 \ \wedge \ 8\varphi = -\frac{\pi}{2} + 2k\pi$

$r = 3$ $\varphi_k = -\frac{\pi}{16} + k \cdot \frac{\pi}{4}$ $k = 0, 1, \dots, 7$

9 rozwiązań: $z = 0$, $z_k = 3e^{\varphi_k i}$ φ_k j.w.

g) $z^3 - 3z^2 + 6z - 4 = 0$

	1	-3	6	-4
1	1	-2	4	0

$z_0 = 1$ ←

$(z-1)(z^2 - 2z + 4) = 0$, $(z-1) \cdot [(z-1)^2 + 3] = 0$

$(z-1)[(z-1)^2 - (-3)] = (z-1)[(z-1)^2 - (\sqrt{3}i)^2] = 0$

$(z-1)(z-1-\sqrt{3}i)(z-1+\sqrt{3}i)$

$z_0 = 1$ $z_1 = 1 + \sqrt{3}i$ $z_2 = 1 - \sqrt{3}i$

a) $z^4 = \dfrac{i-1}{\sin\frac{\pi}{6} + i\cos\frac{\pi}{6}}$ $z^4 = \dfrac{\sqrt{2}\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)}{\cos\frac{\pi}{3} + i \cdot \sin\frac{\pi}{3}} = \dfrac{\sqrt{2}\left(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi\right)}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$

$r^4(\cos 4\varphi + i\sin 4\varphi) = \sqrt{2}\left(\cos\left(\frac{3}{4}\pi - \frac{\pi}{3}\right) + i\sin\left(\frac{3}{4}\pi - \frac{\pi}{3}\right)\right)$

$r^4 = \sqrt{2}$, $4\varphi = \frac{5}{12}\pi + 2k\pi$ $k = 0, 1, 2, 3$

$r = \sqrt[8]{2}$, $\varphi_k = \frac{5}{48}\pi + k \cdot \frac{\pi}{2}$ $k = 0, 1, 2, 3$

$z_k = r \cdot (\cos\varphi_k + i\sin\varphi_k)$ $k = 0, 1, 2, 3$

c) $z^3 = (2-2i)^9$

3 rozw. → $z_0 = (2-2i)^3 = 2^3 + 3 \cdot 2^2(-2i) + 3 \cdot 2(-2i)^2 + (-2i)^3$

$= 8 - 24i - 24 \underbrace{-8i^3}_{8i} = -16 - 16i = -16(1+i)$

$z_1 = z_0\left(\cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi\right) = -16(1+i)\left(-\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = -16(1+i)\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= -8(1+i)(\sqrt{3}i - 1) = -8(\sqrt{3}i - 1 - \sqrt{3} - i)$

$= +8(1 + \sqrt{3} + i - \sqrt{3}i)$

$z_2 = z_0\left(\cos\frac{4}{3}\pi + i\sin\frac{4}{3}\pi\right) =$

$= -16(1+i)\left(-\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right) = -16(1+i)\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 8(1+i)(i\sqrt{3} + 1)$

$= 8(i\sqrt{3} + 1 - \sqrt{3} + i) = 8(1 - \sqrt{3} + i + \sqrt{3}i)$

Zadanie 2

b) $z^3 = \frac{(i-1)^7}{(\sqrt{3}-i)^5}$

$i-1 = \sqrt{2}\left(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right) = \sqrt{2}\, e^{\frac{3}{4}\pi i}$

$\sqrt{3}-i = 2\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right) = 2\, e^{-\frac{\pi}{6}i}$

dla $z \neq 0$

$z^3 = r^3 e^{3i\varphi} = \frac{\sqrt{2}^7 \cdot e^{\frac{21}{4}\pi i}}{2^5 \cdot e^{-\frac{5}{6}\pi i}} = \frac{\sqrt{2}}{4} e^{\left(\frac{21}{4}+\frac{5}{6}\right)\pi i} = \frac{\sqrt{2}}{4} e^{\frac{63+10}{12}\pi i} =$

$= \frac{\sqrt{2}}{4} e^{\frac{73}{12}\pi i} \Rightarrow r^3 = \frac{\sqrt{2}}{4} = \frac{1}{2\sqrt{2}} = \left(\frac{1}{\sqrt{2}}\right)^3, \quad r = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$3\varphi = \frac{73}{12}\pi + 2k\pi, \quad \varphi_k = \frac{73}{36}\pi + \frac{2}{3}k\pi \quad k \in \{0,1,2\}$

$z_k = \frac{\sqrt{2}}{2} e^{i\cdot\varphi_k}, \ k = \{0,1,2\}$

d) $(iz+2)^4 = (z-4i)^4$

Może się zdawać, że to równanie wielomianowe stopnia 4
Ale widać, że "z^4" się zredukuje
To równanie stopnia 3 → będą 3 rozwiązania!

zał. $z-4i \neq 0$
$z \neq 4i$
gdy $z=4i$ $P=0$
$L=(4i\cdot i+2)^4 \neq 0$
to nie rozwiązanie

$\left(\frac{iz+2}{z-4i}\right)^4 = 1 \quad w = \frac{iz+2}{z-4i} \quad w^4 = 1 \Rightarrow w \in \{1, i, -1, -i\}$

(1) $iz+2 = z-4i$
$(i-1)z = -2-4i$
$z = \frac{2+4i}{1-i} = \frac{(2+4i)(1+i)}{2}$
$z = -1+3i$

(i) $iz+2 = i(z-4i)$
$2 = 4$ ↯

(-i) $iz+2 = -i(z-4i)$
$2iz = -6$
$z = \frac{-3}{i} = 3i$

(-1) $iz+2 = -z+4i$
$(i+1)z = -2+4i$
$z = \frac{-2+4i}{1+i}$
$z = \frac{(-2+4i)(1-i)}{2}$
$z = 1+3i$

(3) Zaznacz na płaszczyźnie zbiory

a) $A = \{z \in \mathbb{C}: \ Im(\sqrt{2}\cdot e^{\frac{\pi}{4}i}) \leq |\frac{1}{2}i\cdot\bar{z} - 2+6i| < |\sqrt{3}+i|^2$

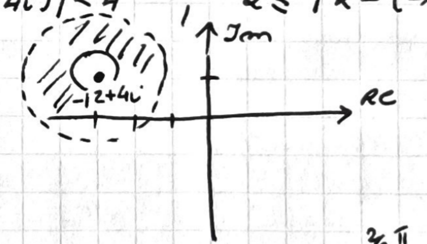
$\sqrt{2}e^{\frac{\pi}{4}i} = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) = \sqrt{2}\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right) = 1+i \qquad Im(\sqrt{2}\cdot e^{\frac{\pi}{4}i}) = 1$

$|\sqrt{3}+i| = \sqrt{3+1} = 2$

tj: $1 \leq |\frac{1}{2}i\bar{z} - 2+6i| < 4$

$|\frac{1}{2}i\bar{z}-2+6i| = |\frac{1}{2}i(\bar{z} - \frac{4}{i} + 12)| = |\frac{1}{2}|\cdot|i|\cdot|\bar{z}+4i+12| =$
$= \frac{1}{2}\cdot|\bar{z}+12+4i| = \frac{1}{2}|z+12-4i| = \frac{1}{2}|z-(-12+4i)|$

$1 \leq \frac{1}{2}|z-(-12+4i)| < 4, \quad 2 \leq |z-(-12+4i)| < 8$



b) $B = \{z \in \mathbb{C}: \ Im(z^6) < 0\}$

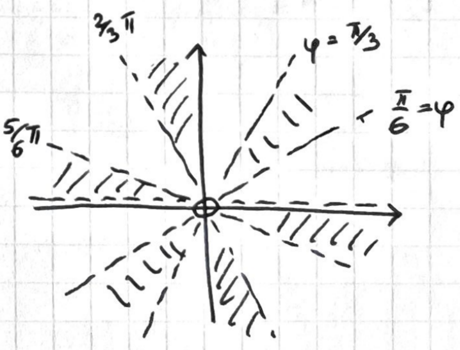
$z^6 = r^6(\cos 6\varphi + i\sin 6\varphi)$

$Im(z^6) = \underbrace{r^6}_{>0} \sin 6\varphi < 0 \iff \sin 6\varphi < 0$

$\pi + 2k\pi < 6\varphi < 2\pi + 2k\pi$

$\frac{\pi}{6} + k\cdot\frac{\pi}{3} < \varphi < \frac{\pi}{3} + k\cdot\frac{\pi}{3}$

$k \in \mathbb{Z}$



Zad. 3 Rozwiąż $\quad i^3\bar{z}\cdot z^6 = \mathrm{Im}(e^{\frac{\pi}{2}i})\cdot\frac{1+i}{1-i}\cdot\frac{(\sqrt{12}-2i)^{15}}{(-\sin\frac{\pi}{5}+i\cos\frac{\pi}{5})^4}$

$i^3=-i$

$\mathrm{Im}(e^{\frac{\pi}{2}i}) = \mathrm{Im}(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}) = \sin\frac{\pi}{2}=1$

$\frac{1+i}{1-i} = \frac{(1+i)^2}{1^2-i^2} = \frac{1+2i-1}{2} = i$

$w=\sqrt{12}-2i \quad |w|=\sqrt{12+4}=4 \quad w=4\left(\frac{2\sqrt3}{4}-\frac{2i}{4}\right)=4\left(\frac{\sqrt3}{2}-\frac{1}{2}i\right)=4e^{-\frac{\pi}{6}i}$

$u=-\sin\frac{\pi}{5}+i\cos\frac{\pi}{5} = 1\cdot\left(\cos(\frac{\pi}{2}+\frac{\pi}{5})+i\sin(\frac{\pi}{2}+\frac{\pi}{5})\right) = e^{\frac{7}{10}\pi i}$

Wstawiając do równania $\quad \underbrace{-\cancel{i}}_{e^{\pi i}}|z|e^{-\varphi i}|z|^6e^{6\varphi i} = 1\cdot\cancel{i}\cdot\frac{4^{15}e^{-\frac{15}{6}\pi i}}{e^{\frac{28}{10}\pi i}}$

Stąd $\quad |z|^7e^{5\varphi i} = 4^{15}e^{-\pi i(1+\frac{5}{2}+\frac{14}{5})}$

$1+\frac{5}{2}+\frac{14}{5}=\frac{10+25+28}{10}=\frac{63}{10}$

$\Rightarrow |z|^7=4^{15} \quad\wedge\quad 5\varphi=-\frac{63}{10}\pi+2k\pi$

$|z|=\sqrt[7]{4^{15}} \qquad \varphi_k=-\frac{63}{50}\pi+\frac{2}{5}k\pi \quad k\in\{0,1,2,3,4\}$

$z=0$ nie spełnia równania

$z\neq0$

$z=|z|e^{\varphi i}$

Rozw. $\quad z_k=\sqrt[7]{4^{15}}e^{\varphi_k i}$

$k\in\{0,1,2,3,4\}$

Zad. 3 Rozwiąż $\quad \frac{|4-3i|}{i}\cdot\bar{z}^2 = \frac{4+12i}{3-i}\cdot\frac{(\sin\frac{\pi}{6}-i\cos\frac{\pi}{3})^7}{(\sin\frac{\pi}{3}-i\cos\frac{\pi}{3})^5}\cdot z^5$

$|4-3i|=\sqrt{16+9}=5$

$\frac{1}{i}=\frac{i}{i^2}=-i$

$\frac{4+12i}{3-i}=4\cdot\frac{1+3i}{3-i}=4\,\frac{(1+3i)(3+i)}{9+1}=\frac{4}{10}(\cancel{3}+i+9i-\cancel{3})=\frac{4}{10}\cdot10i=4i$

$\sin\frac{\pi}{6}-i\cos\frac{\pi}{3}=\frac12-\frac12 i=\frac12(1-i)=\frac{\sqrt2}{2}\left(\frac{\sqrt2}{2}-\frac{\sqrt2}{2}i\right)=\frac{1}{\sqrt2}e^{-\frac{\pi}{4}i}$

$\sin\frac{\pi}{3}-i\cos\frac{\pi}{3}=\frac{\sqrt3}{2}-\frac12 i=\cos(-\frac{\pi}{6})+i\sin(-\frac{\pi}{6})=e^{-\frac{\pi}{6}i}$

$z=0$ jest rozwiązaniem

Zał. $z\neq0$

$z=|z|\cdot e^{\varphi i}$

Wstawiając $\quad \underbrace{-5\cancel{i}}_{e^{\pi i}}\cdot|z|^2e^{-2\varphi i}=4\cancel{i}\,\frac{(\frac{1}{\sqrt2})^7\cdot e^{-\frac{7}{4}\pi i}}{e^{-\frac{5}{6}\pi i}}\cdot|z|^5e^{5i\varphi} \quad /:|z|^2$

$5e^{\pi i}e^{-2\varphi i}=\frac{4}{8\sqrt2}\cdot e^{-\pi i(\frac{7}{4}+\frac{5}{6})}\cdot|z|^3e^{5i\varphi} \quad /\cdot2\sqrt2 \;/\cdot e^{2\varphi i}$

$\frac{7}{4}-\frac{5}{6}=\frac{42-20}{24}=\frac{22}{24}=\frac{11}{12}$

$10\sqrt2\;e^{\pi i}e^{\frac{11}{12}\pi i}=|z|^3e^{7\varphi i} \Rightarrow |z|^3=10\sqrt2 \quad\wedge\quad 7\varphi=\frac{23}{12}\pi+2k\pi$

$|z|=\sqrt[3]{10\sqrt2} \qquad \varphi_k=\frac{23}{84}\pi+\frac{2}{7}k\pi$

$k\in\{0,1,\dots,6\}$

Rozw: $\quad z=0 \;\vee\; z_k=\sqrt[3]{10\sqrt2}\cdot e^{\varphi_k i}$

Zad. 4 Zaznacz na płaszczyźnie zespolonej $\quad A=\{z\in\mathbb{C}: |3\bar z+2i|\geq6 \wedge |z|<|z+2| \wedge \frac{\pi}{4}<\arg(\bar z)<\frac{3}{4}\pi\}$

- $|3\bar z+2i|\geq6$
  $|3(\bar z+\frac23 i)|\geq6$
  $|\bar z+\frac23 i|\geq2$
  $|\bar z+\frac23 i|=|\overline{\bar z+\frac23 i}|=|z-\frac23 i|\geq2$

- $|z|<|z+2|$
  $|z-0|<|z-(-2)|$

- $\frac{\pi}{4}<\arg(\bar z)<\frac34\pi$
  $\frac{\pi}{4}<2\pi-\arg(z)<\frac34\pi$
  $\arg(z)<\frac74\pi \;\wedge\; \arg z>\frac54\pi$