ALGEBRA – Zadanie domowe nr 5 – Odwzorowania liniowe

Zad. 1 Czy f jest liniowe?

a) $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (7x+y+7,\ 7x-y)$

NIE

$$L = f(\alpha(x,y)) = f(\alpha x, \alpha y) = (7\alpha x + \alpha y + \underline{7},\ 7\alpha x - \alpha y)$$
$$P = \alpha \cdot f(x,y) = \alpha(7x+y+7,\ 7x-y) = (7\alpha x + \alpha y + \underline{\alpha 7},\ 7\alpha x - \alpha y)$$

$L \neq P$

b) $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x+y+z$ TAK

$\forall (x_1,y_1,z_1), (x_2,y_2,z_2) \in \mathbb{R}^3$ $\forall \alpha \in \mathbb{R}$

$$\alpha \cdot f(x,y,z) = \alpha \cdot (x+y+z) = \alpha x + \alpha y + \alpha z = f(\alpha x, \alpha y, \alpha z) = f(\alpha \cdot (x,y,z))$$
$$f(x_1,y_1,z_1) + f(x_2,y_2,z_2) = (x_1+y_1+z_1) + (x_2+y_2+z_2) = (x_1+x_2) + (y_1+y_2) + (z_1+z_2) =$$
$$= f(x_1+x_2,\ y_1+y_2,\ z_1+z_2) = f[(x_1,y_1,z_1) + (x_2,y_2,z_2)]$$

c) $f: M_3(\mathbb{C}) \to \mathbb{C}$ $f(A) = \det A$ NIE

$$L = 2 \cdot f(I_3) = 2 \cdot \det(I_3) = 2 \cdot 1$$
$$P = f(2I_3) = \det \begin{bmatrix} 2&0&0 \\ 0&2&0 \\ 0&0&2 \end{bmatrix} = 2^3$$

$L \neq P$

d) $f: \mathbb{R}_2[x] \to M_2(\mathbb{R})$, $f(ax^2+bx+c) = \begin{bmatrix} a & b \\ -b & c \end{bmatrix}$ TAK

$\forall$ $a_1x^2 + b_1x + c_1 \in \mathbb{R}_2[x]$

$\forall$ $a_2x^2 + b_2x + c_2 \in \mathbb{R}_2[x]$

$\forall$ $\alpha \in \mathbb{R}$

$$\alpha \cdot f(a_1x^2 + b_1x + c_1) = \alpha \begin{bmatrix} a_1 & b_1 \\ -b_1 & c_1 \end{bmatrix} = \begin{bmatrix} \alpha a_1 & \alpha b_1 \\ -\alpha b_1 & \alpha c_1 \end{bmatrix} = f(\alpha a_1 x^2 + \alpha b_1 x + \alpha c_1) = f[\alpha \cdot (a_1x^2 + b_1x + c_1)]$$

$$f[(a_1x^2+b_1x+c_1) + (a_2x^2+b_2x+c_2)] = f[(a_1+a_2)x^2 + (b_1+b_2)x + (c_1+c_2)] = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ -b_1-b_2 & c_1+c_2 \end{bmatrix} =$$
$$= \begin{bmatrix} a_1 & b_1 \\ -b_1 & c_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ -b_2 & c_2 \end{bmatrix} = f(a_1x^2+b_1x+c_1) + f(a_2x^2+b_2x+c_2)$$

Zad. 2 $\varphi: \mathbb{R}^4 \to \mathbb{R}^3$ $\varphi(x,y,z,t) = (x+2y+z-t,\ x+2z+t,\ 2x+y+3t)$

a) $\operatorname{Im}\varphi = ?$ $\operatorname{Ker}\varphi = ?$ + ich bazy i wymiary

b) własności φ

- $\varphi(x,y,z,t) = x(1,1,2) + y(2,0,1) + z(1,2,0) + t(-1,1,3)$

$\operatorname{Im}\varphi = \operatorname{lin}\{(1,1,2), (2,0,1), (1,2,0), (-1,1,3)\}$ ← generatory, ale nie baza

$\operatorname{Im}\varphi \subset \mathbb{R}^3 \Rightarrow \dim \operatorname{Im}\varphi \leq 3$

$$r\begin{bmatrix} 1&1&2 \\ 2&0&1 \\ 1&2&0 \\ -1&1&3 \end{bmatrix} \underset{\substack{w_2-2w_1 \\ w_3-w_1 \\ w_4+w_1}}{=} r\begin{bmatrix} 1&1&2 \\ 0&-2&0 \\ 0&1&-2 \\ 0&2&5 \end{bmatrix} \underset{w_4+w_2}{\overset{w_2\cdot(-\frac{1}{2})}{=}} r\begin{bmatrix} 1&1&2 \\ 0&1&0 \\ 0&1&-2 \\ 0&0&5 \end{bmatrix} \underset{w_3-w_2}{=} r\begin{bmatrix} 1&1&2 \\ 0&1&0 \\ 0&0&-2 \\ \cancel{0}&\cancel{0}&\cancel{5} \end{bmatrix} = 3$$

$\dim \operatorname{Im}\varphi = 3$

Baza $\{(1,1,2), (0,1,0), (0,0,1)\}$

$\dim \operatorname{Im}\varphi = \dim \mathbb{R}^3$

φ – EPIMORFIZM

- $\dim \operatorname{Ker}\varphi = \dim \mathbb{R}^4 - \dim \operatorname{Im}\varphi = 4 - 3 = 1$

$\operatorname{Ker}\varphi = ?$ $\varphi(x,y,z,t) = (0,0,0) \iff \begin{cases} x+2y+z-t=0 \\ x+2z+t=0 \\ 2x+y+3t=0 \end{cases} \iff \begin{bmatrix} 1&2&1&-1 \\ 1&0&2&1 \\ 2&1&0&3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{cccc|c} 1&2&1&-1&0 \\ 1&0&2&1&0 \\ 2&1&0&3&0 \end{array}\right] \xrightarrow[w_3-2w_1]{w_2-w_1} \left[\begin{array}{cccc|c} 1&2&1&-1&0 \\ 0&-2&1&2&0 \\ 0&-3&-2&5&0 \end{array}\right] \xrightarrow{w_2-w_3} \left[\begin{array}{cccc|c} 1&2&1&-1&0 \\ 0&1&3&-3&0 \\ 0&-3&-2&5&0 \end{array}\right] \xrightarrow{w_3+3w_2} \left[\begin{array}{cccc|c} 1&2&1&-1&0 \\ 0&1&3&-3&0 \\ 0&0&7&-4&0 \end{array}\right]$$

$$\xrightarrow{w_3\cdot\frac{1}{7}} \left[\begin{array}{cccc|c} 1&2&1&-1&0 \\ 0&1&3&-3&0 \\ 0&0&1&-\frac{4}{7}&0 \end{array}\right] \xrightarrow[w_1-w_3]{w_2-3w_3} \left[\begin{array}{cccc|c} 1&2&0&-\frac{3}{7}&0 \\ 0&1&0&-\frac{9}{7}&0 \\ 0&0&1&-\frac{4}{7}&0 \end{array}\right] \xrightarrow{w_1-2w_2} \left[\begin{array}{cccc|c} 1&0&0&\frac{15}{7}&0 \\ 0&1&0&-\frac{9}{7}&0 \\ 0&0&1&-\frac{4}{7}&0 \end{array}\right]$$

$\operatorname{Ker}\varphi = \{(-\frac{15}{7}t,\ \frac{9}{7}t,\ \frac{4}{7}t) ;\ t \in \mathbb{R}\} = \operatorname{lin}\{(-15,9,4)\}$ baza $\{(-15,9,4)\}$

$\operatorname{Ker}\varphi \neq \{(0,0,0,0)\}$

φ nie jest MONOMORFIZMEM

Zad. 3

$f: \mathbb{R}_2[x] \to \mathbb{R}_1[x]$ liniowe

$f(p)(x) = (3-x)\cdot p''(x) + 4p'(x)$

a) $A = ?$ $\quad A = M_f(\mathcal{B}, C)$ $\quad \mathcal{B} = (1, x, x^2)$ baza $\mathbb{R}_2[x]$
$C = (1, x)$ baza $\mathbb{R}_1[x]$

$f(1) = (3-x)\cdot 0 + 4\cdot 0 = 0 = [0,0]_C$

$f(x) = (3-x)\cdot 0 + 4\cdot 1 = 4 = [4,0]_C$

$f(x^2) = (3-x)\cdot 2 + 4\cdot 2x = 6 - 2x + 8x = 6 - 6x = [6,-6]_C$

$A = \begin{bmatrix} 0 & 4 & 6 \\ 0 & 0 & 6 \end{bmatrix}$

b) $\ker f = ?$, $\operatorname{Im} f = ?$ ⊕ ich bazy i wymiary

c) własności f

- $p \in \mathbb{R}_2[x]$ $\quad p(x) = ax^2 + bx + c$ $\quad p'(x) = 2ax + b$ $\quad p''(x) = 2a$

$f(p)(x) = (3-x)\cdot 2a + 4\cdot(2ax+b) = 6a - 2ax + 8ax + 4b = 6ax + (6a+4b) = 0$

$\Leftrightarrow \begin{cases} 6a = 0 \\ 6a + 4b = 0 \end{cases} \Leftrightarrow a = b = 0 \Rightarrow \ker f = \{p \in \mathbb{R}_2[x] : p(x) = c;\ c \in \mathbb{R}\}$

wielomiany stałe

$\ker f = \operatorname{lin}\{1\}$ BAZA $\quad \dim \ker f = 1$

f nie jest monomorfizmem

- $\dim \operatorname{Im} f = \dim \mathbb{R}_2[x] - \dim \ker f = 3 - 1 = 2$

lub $\quad \dim \operatorname{Im} f = r\begin{bmatrix} 0 & 4 & 6 \\ 0 & 0 & 6 \end{bmatrix} = 2$

$\operatorname{Im} f \subseteq \mathbb{R}_1[x] \ \wedge\ \dim \operatorname{Im} f = \dim \mathbb{R}_1[x] = 2 \Longrightarrow \operatorname{Im} f = \mathbb{R}_1[x]$

baza $(1, x)$

f – epimorfizm

Zad. 4

U, V – przestrzenie $\mathbb{R}$-liniowe

$\varphi: U \to V$ liniowe

$A = \begin{bmatrix} 1 & 3 & 4 & 1 & 1 \\ 2 & 1 & 1 & 0 & 1 \\ 4 & 7 & 9 & 2 & 3 \end{bmatrix}$ $\quad \dim \ker \varphi = ?$

$A \in M_{3\times 5}(\mathbb{R}) \Rightarrow \dim U = 5$, $\dim V = 3$

$\dim \ker \varphi = \dim U - \dim \operatorname{Im} \varphi = 5 - r(A)$

$r(A) \underset{\substack{w_2 - 2w_1 \\ w_3 - 4w_1}}{=\!=} r\begin{bmatrix} 1 & 3 & 4 & 1 & 1 \\ 0 & -5 & -7 & -2 & -1 \\ 0 & -5 & -7 & -2 & -1 \end{bmatrix} = 2 \Longrightarrow \dim \ker \varphi = 5 - 2 = 3$

Zad. 5

$f: \mathbb{R}^3 \to \mathbb{R}^2$ $\quad f(1,1,1) = (2,2)$, $f(1,0,1) = (1,0)$, $f(0,1,1) = (1,1)$

a) wzór f ?

$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1 - 1 - 1 \neq 0 \Rightarrow ((1,1,1), (1,0,1), (0,1,1))$ pewna baza $\mathbb{R}^3$

$\mathcal{B}_e^3 = (e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1))$ baza kanoniczna $\mathbb{R}^3$

$f(x,y,z) = f(xe_1 + ye_2 + ze_3) \underset{f\text{-liniowe}}{=} x f(e_1) + y f(e_2) + z f(e_3)$ $\quad f(e_1) = ?,\ f(e_2) = ?,\ f(e_3) = ?$

$\begin{cases} f(1,1,1) = f(e_1) + f(e_2) + f(e_3) = (2,2) \\ f(1,0,1) = f(e_1) + f(e_3) = (1,0) \\ f(0,1,1) = f(e_2) + f(e_3) = (1,1) \end{cases} \Rightarrow \begin{array}{l} f(e_2) = (2,2) - (1,0) = (1,2) \\ f(e_1) = (2,2) - (1,1) = (1,1) \\ f(e_3) = (1,1) - f(e_2) = (1,1) - (1,2) = (0,-1) \end{array}$

$f(x,y,z) = x\cdot(1,1) + y(1,2) + z(0,-1) = (x+y,\ x + 2y - z)$

b) $\mathcal{B} = (b_1 = (3,1,1), b_2 = (5,1,6), b_3 = (4,-1,2))$ baza $\mathbb{R}^3$

$C = (c_1 = (-1,1), c_2 = (1,0))$ baza $\mathbb{R}^2$

$A' = M_f(\mathcal{B}, C) = ?$

$f(3,1,1) = (4,4) = 4c_1 + 8c_2 = [4,8]_C$

$f(5,1,6) = (6,1) = c_1 + 7c_2 = [1,7]_C$

$f(4,-1,2) = (3,0) = 3c_2 = [0,3]_C$

$A' = \begin{bmatrix} 4 & 1 & 0 \\ 8 & 7 & 3 \end{bmatrix}$

Zad. 5

c) $C' = (c_1' = (1,1),\ c_2' = (2,1))$ $\qquad P = P_{C \to C'} = ?$

$c_1' = (1,1) = (-1,1) + (2,0) = [1,2]_C$

$c_2' = (2,1) = (-1,1) + (3,0) = [1,3]_C$

$P = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

d) $v = [2,5]_C$ $\qquad v = [\alpha, \beta]_{C'} = ?$ $\qquad X = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ $\quad X' = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ $\qquad PX' = X$ $\qquad X' = P^{-1}X$

$P^{-1} = ?$ $\quad \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \xrightarrow[w_2 - 2w_1]{} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \xrightarrow[w_1 - w_2]{} \left[\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & -2 & 1 \end{array}\right]$, $\underbrace{\begin{matrix} 3 & -1 \\ -2 & 1 \end{matrix}}_{P^{-1}}$

$X' = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\qquad v = [1,1]_{C'}$

Zad. 6

$\varphi: \mathbb{R}^2 \to \mathbb{R}^3$ $\qquad A = M_\varphi(\mathcal{B}_e^2, \mathcal{B}_e^3) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$

$\mathcal{B} = (b_1 = (1,1),\ b_2 = (2,1))$ baza $\mathbb{R}^2$

$C = (c_1 = (1,2,0),\ c_2 = (2,3,0),\ c_3 = (0,0,1))$ baza $\mathbb{R}^3$ $\qquad A' = M_\varphi(\mathcal{B}, C) = ?$

$A' = Q^{-1} A P$ $\qquad P = P_{\mathcal{B}_e^2 \to \mathcal{B}} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ $\qquad Q = P_{\mathcal{B}_e^3 \to C} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$Q^{-1} \underset{\text{ćw}}{=} \begin{bmatrix} -3 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A' = \begin{bmatrix} -3 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -5 \\ 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -6 & -7 \\ 4 & 5 \\ 1 & 1 \end{bmatrix}$

Zad. 7

$\varphi: \mathbb{R}_1[x] \to \mathbb{R}_1[x]$ $\qquad \mathcal{B} = (1, x)$ $\qquad A = M_\varphi(\mathcal{B}, \mathcal{B}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$\mathcal{B}' = (1, -5x)$ $\qquad A' = M_\varphi(\mathcal{B}', \mathcal{B}') = ?$

$P = P_{\mathcal{B} \to \mathcal{B}'} = \begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix}$ $\qquad A' = P^{-1} A P$

$P^{-1} = -\frac{1}{5} \begin{bmatrix} -5 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{5} \end{bmatrix}$ $\qquad A' = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{5} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 0 & -5 \\ \frac{1}{5} & 0 \end{bmatrix}$

Zad. 8

$\varphi: \mathbb{R}^4 \to \mathbb{R}^3$

$\mathcal{B} = (b_1 = (1,1,1,1),\ b_2 = (1,1,1,0),\ b_3 = (1,1,0,0),\ b_3 = (1,0,0,0))$ baza $\mathbb{R}^4$

$C = (c_1 = (0,-1,0),\ c_2 = (1,0,0),\ c_3 = (1,1,1))$ baza $\mathbb{R}^3$

$A' = M_\varphi(\mathcal{B}, C) = \begin{bmatrix} 1 & -2 & -2 & -2 \\ 4 & 3 & 1 & 1 \\ 2 & 0 & 0 & 0 \end{bmatrix}$

a) $v = (5,0,1,0)$ $\qquad \varphi(v) = ?$

$v = [\alpha, \beta, \gamma, \delta]_{\mathcal{B}} = ?$ $\qquad P = P_{\mathcal{B}_e^4 \to \mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ $\qquad P \cdot \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \\ 0 \end{bmatrix} \iff \begin{cases} \alpha + \beta + \gamma + \delta = 5 \\ \alpha + \beta + \gamma = 0 \\ \alpha + \beta = 1 \\ \alpha = 0 \end{cases}$

$\Rightarrow \begin{array}{l} \alpha = 0 \\ \beta = 1 \\ \gamma = -\alpha - \beta = -1 \\ \delta = 5 - \alpha - \beta - \gamma = 5 - 0 - 1 + 1 = 5 \end{array}$ $\qquad v = [0, 1, -1, 5]_{\mathcal{B}}$

$A' \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} -10 \\ 7 \\ 0 \end{bmatrix}$ $\qquad \varphi(v) = [-10, 7, 0]_C = -10c_1 + 7c_2 = -10(0,-1,0) + 7(1,0,0) = \underline{\underline{(7, 10, 0)}}$

Zad. 8

b)
$$\begin{cases} \varphi(1,1,1,1) = [1,4,2]_C = (0,-1,0) + 4(1,0,0) + 2(1,1,1) = (6,1,2) & = \varphi(e_1)+\varphi(e_2)+\varphi(e_3)+\varphi(e_4) \\ \varphi(1,1,1,0) = [-2,3,0]_C = -2(0,-1,0) + 3(1,0,0) = (3,2,0) & = \varphi(e_1)+\varphi(e_2)+\varphi(e_3) \\ \varphi(1,1,0,0) = [-2,1,0]_C = -2(0,-1,0) + (1,0,0) = (1,2,0) & = \varphi(e_1)+\varphi(e_2) \\ \varphi(1,0,0,0) = [-2,1,0]_C = (1,2,0) & = \varphi(e_1) \end{cases}$$

$\Rightarrow \varphi(e_1) = (1,2,0)$, $\varphi(e_2) = (0,0,0)$, $\varphi(e_3) = (3,2,0) - (1,2,0) = (2,0,0)$

$\varphi(e_4) = (6,1,2) - (3,2,0) = (3,-1,2)$

$\varphi(x,y,z,t) = x\varphi(e_1) + y\varphi(e_2) + z\varphi(e_3) + t\varphi(e_4) =$
$= x(1,2,0) + y(0,0,0) + z(2,0,0) + t(3,-1,2) = (x+2z+3t,\ 2x-t,\ 2t)$

Zad. 9

$f : \mathbb{R}^3 \to \mathbb{R}^2$ $\quad f(x,y,z) = (x-y+z,\ 2y+z)$

$g : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad g(x,y) = (2x+y,\ x-y)$

$h := g^2 \circ f$

wzór h ?
$h(1,2,0) = ?$

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] \arrow[rrr, bend right=40, "g^2 \circ f"'] & \mathbb{R}^2 \arrow[r, "g"] & \mathbb{R}^2 \arrow[r, "g"] & \mathbb{R}^2 \end{tikzcd}$$

a)

- $A = M_f(\mathcal{B}_k^3, \mathcal{B}_k^2) = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ $\qquad B = M_g(\mathcal{B}_k^2, \mathcal{B}_k^2) = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$

$C = M_h(\mathcal{B}_k^3, \mathcal{B}_k^2) = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}^2 \cdot \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -3 & 6 \\ 1 & 3 & 3 \end{bmatrix}$

$h : \mathbb{R}^3 \to \mathbb{R}^2$

$h(x,y,z) = ?$ $\qquad C\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5x - 3y + 6z \\ x + 3y + 3z \end{bmatrix}$ $\qquad h(x,y,z) = (5x-3y+6z,\ x+3y+3z)$

- $h(1,2,0) = ?$ $\qquad C \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 & -3 & 6 \\ 1 & 3 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$ $\qquad h(1,2,0) = (-1,7)$

Zad. 10 Prawda / fałsz

$\varphi : \mathbb{R}_3[x] \to \mathbb{R}_3[x]$ endomorfizm

a) Jeśli $\operatorname{Ker}\varphi = \{0\}$, to φ jest izomorfizmem.

PRAWDA $\quad \operatorname{Ker}\varphi = \{0\} \Rightarrow \varphi$ - monomorfizm

$\dim \operatorname{Im}\varphi = \dim \mathbb{R}_3[x] - \dim \operatorname{Ker}\varphi = 4 - 0 = 4 \Rightarrow \operatorname{Im}\varphi = \mathbb{R}_3[x]$

φ - epimorfizm

b) Jeśli $\operatorname{Ker}\varphi = \{0\}$ to $\{\varphi(1), \varphi(x), \varphi(x^2)\}$ jest liniowo niezależny

PRAWDA $\quad \operatorname{Ker}\varphi = \{0\} \Rightarrow \varphi$ - monomorfizm, więc przeprowadza układ liniowo niezależny $\{1, x, x^2\}$ w układ lin. niezależny

c) $A = M_\varphi(\mathcal{B}, \mathcal{B})$ $\qquad \mathcal{B} = (1, x, x^2, x^3)$

Czy kwadratowa stopnia 3 ? $\qquad$ FAŁSZ $\qquad A \in M_4(\mathbb{R})$

Zad. 9 - ciąg dalszy

b) $\varphi(1,3) - \varphi(1,1) = \varphi(0,2) = 2 \cdot \varphi(0,1)$
$(6,4) - (2,0) = (4,4)$
$\Rightarrow \varphi(0,1) = (2,2)$

$\varphi(1,0) = \varphi(1,1) - \varphi(0,1) = (2,0) - (2,2) = (0,-2)$

$D = M_\varphi(\mathcal{B}_k^2, \mathcal{B}_k^2) = \begin{bmatrix} 0 & 2 \\ -2 & 2 \end{bmatrix}$ $\qquad \det D = 4 \neq 0 \Rightarrow \varphi$ - izomorfizm

$\det B = -3 \neq 0 \Rightarrow g$ - izomorfizm

$D^{-1} = M_{\varphi^{-1}}(\mathcal{B}_k^2, \mathcal{B}_k^2)$
$B^{-1} = M_{g^{-1}}(\mathcal{B}_k^2, \mathcal{B}_k^2)$
$\Rightarrow M_{g^{-1} \circ \varphi^{-1}} = M_{g^{-1}} \cdot M_{\varphi^{-1}} = D^{-1}B^{-1} = (BD)^{-1}$

$g^{-1} \circ \varphi^{-1}(x,y) = ?$

$BD = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 6 \\ 2 & 0 \end{bmatrix}$ $\qquad \det(BD) = -12$ $\qquad (BD)^{-1} = \frac{-1}{12}\begin{bmatrix} 0 & -2 \\ -6 & -2 \end{bmatrix}^T = \frac{1}{12}\begin{bmatrix} 0 & 6 \\ 2 & 2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 0 & 3 \\ 1 & 1 \end{bmatrix}$

$\frac{1}{6}\begin{bmatrix} 0 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 3y \\ x+y \end{bmatrix}$

$g^{-1} \circ \varphi^{-1}(x,y) = (\frac{1}{2}y,\ \frac{1}{6}x + \frac{1}{6}y)$