Algebra - Zadanie domowe nr 6

Zad. 1 Diagonalizacja. Baza.

$\varphi : \mathbb{R}^3 \to \mathbb{R}^3$

$\varphi(x,y,z) = (x+2y-2z,\ x+3z,\ x+3y)$

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 2 & -2 \\ 1 & -\lambda & 3 \\ 1 & 3 & -\lambda \end{vmatrix} = \lambda^2(1-\lambda) - 6 + 6 - 2\lambda - 9(1-\lambda) + 2\lambda = (1-\lambda)(\lambda^2-9)$$

$\mathrm{Spec}(\varphi) = \{-3, 1, 3\}$ widmo proste $\Rightarrow \varphi$ - diagonalizowalny

- $\lambda_1 = 1$

$(A-\lambda_1)X = 0$

$$(A-I)X = 0 \iff \begin{bmatrix} 0 & 2 & -2 \\ 1 & -1 & 3 \\ 1 & 3 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & -2 \\ 1 & -1 & 3 \\ 1 & 3 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 3 \\ 1 & 3 & -1 \\ 0 & 2 & -2 \end{bmatrix} \xrightarrow[w_2-w_1]{} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 4 & -4 \\ 0 & 2 & -2 \end{bmatrix} \xrightarrow[\substack{w_3-w_2 \\ w_2/:4}]{} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[w_1+w_2]{} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} a + 2c = 0 \\ b - c = 0 \end{cases}$$

$c = t$

$b = c = t$

$a = -2c = -2t$

$E_1 = \{(-2t, t, t) : t \in \mathbb{R}\} = \mathrm{lin}_{\mathbb{R}}\{(-2,1,1)\}$

- $\lambda_2 = 3$

$(A-3I)X = 0$

$$\begin{bmatrix} -2 & 2 & -2 \\ 1 & -3 & 3 \\ 1 & 3 & -3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 & -2 \\ 1 & -3 & 3 \\ 1 & 3 & -3 \end{bmatrix} \xrightarrow[w_1:(-2)]{} \begin{bmatrix} 1 & -1 & 1 \\ 1 & -3 & 3 \\ 1 & 3 & -3 \end{bmatrix} \xrightarrow[\substack{w_2-w_1 \\ w_3-w_1}]{} \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 2 \\ 0 & 4 & -4 \end{bmatrix} \xrightarrow[w_2:(-2)]{} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[w_1+w_2]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} a = 0 \\ b - c = 0 \end{cases} \quad b = c$$

$E_3 = \{(0, t, t), t \in \mathbb{R}\} = \mathrm{lin}_{\mathbb{R}}\{(0,1,1)\}$

- $\lambda_3 = -3$

$(A+3I)X = 0$

$$\begin{bmatrix} 4 & 2 & -2 \\ 1 & 3 & 3 \\ 1 & 3 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 & -2 \\ 1 & 3 & 3 \\ 1 & 3 & 3 \end{bmatrix} \xrightarrow[\substack{w_3-w_2 \\ w_1/2}]{} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 3 \\ 2 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[w_2-w_1\cdot 2]{} \begin{bmatrix} 1 & 3 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} a + 3b + 3c = 0 \\ -5b - 7c = 0 \end{cases}$$

$a = -3b - 3c = \frac{21}{5}c - 3c = \frac{6}{5}c$

$b = -\frac{7}{5}c$

$c = t$

$E_{-3} = \{(\frac{6}{5}t, -\frac{7}{5}t, t), t \in \mathbb{R}\}$

$\mathrm{lin}_{\mathbb{R}}\{\frac{1}{5}(6, -7, 5)\}$

baza

$\mathcal{B} = \{(-2,1,1), (0,1,1), \frac{1}{5}(6,-7,5)\}$

$$M_{\mathcal{B}\mathcal{B}}(\varphi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Algebra – Zadanie domowe nr 5

Diagonalizacja endomorfizmu

Zad. 2 $\varphi: \mathbb{R}_2[x] \to \mathbb{R}_2[x]$ $\forall p \in \mathbb{R}_2[x]$ $\varphi(p)(x) = 3xp''(x) + (x^2-1)p(1) + x \cdot p(2)$

$B_k = (1, x, x)$ baza $\mathbb{R}_2[x]$

$\varphi(1) = 3x \cdot 0 + (x^2-1) \cdot 1 + x \cdot 1 = x^2+x-1 = [-1,1,1]_{B_k}$

$\varphi(x) = 3x \cdot 0 + (x^2-1) \cdot 1 + x \cdot 2 = x^2+2x-1 = [-1,2,1]_{B_k}$

$\varphi(x^2) = 3x \cdot 2 + (x^2-1) \cdot 1 + x \cdot 4 = x^2+10x-1 = [-1,10,1]_{B_k}$

$$A = M_\varphi(B_k, B_k) = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 2 & 10 \\ 1 & 1 & 1 \end{bmatrix}$$

$$0 = \det(A - tI) = \begin{vmatrix} -1-t & -1 & -1 \\ 1 & 2-t & 10 \\ 1 & 1 & 1-t \end{vmatrix} \underset{\substack{w_1+w_2 \\ w_3-w_2}}{=} \begin{vmatrix} -t & 1-t & 9 \\ 1 & 2-t & 10 \\ 0 & -1+t & -9-t \end{vmatrix} \underset{w_1+w_3}{=} \begin{vmatrix} -t & 0 & -t \\ 1 & 2-t & 10 \\ 0 & t-1 & -9-t \end{vmatrix} =$$

$$= -t(2-t)(-9-t) - t(t-1) - 10(t-1)(-t) = t\left[(2-t)(9+t) - t + 1 + 10t - 10\right] = t\left[18 + 2t - \cancel{9t} - t^2 + \cancel{9t} - 9\right]$$

$$= t\left[-t^2 + 2t + 9\right]$$

$\Delta = 4 + 36 = 40$

$\sqrt{\Delta} = 2\sqrt{10}$

$\mathrm{Spec}(\varphi) = \{0, 1-\sqrt{10}, 1+\sqrt{10}\}$ widmo proste

$\Downarrow$

φ - diagonalizowalny

Zad. 3 Niech $\varphi \in \mathrm{End}(\mathbb{R}^3)$ t. że

$\varphi(1,0,0) = (1,0,0)$

$\varphi(1,1,0) = (-1,-1,0)$

$\varphi(1,1,1) = (0,0,0)$

Oblicz $\varphi^{100}(3,6,9)$.

$\mathrm{Spec}(\varphi) = \{1, -1, 0\}$

Baza wektorów własnych $C = \{c_1 = (1,0,0), c_2 = (1,1,0), c_3 = (1,1,1)\}$

B - baza kanoniczna

$$P = P_{B \to C} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{macierz przejścia}$$

$$D = M_{CC}(\varphi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

I SPOSÓB:

$A = M_{BB}(\varphi)$ $\quad A = P \cdot D \cdot P^{-1}$, $A^{100} = P \cdot D^{100} \cdot P^{-1}$

$P^{-1} = ?$

$$[P|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[w_2-w_3]{w_1-w_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{w_1-w_2} \Bigg[I \Bigg| \underbrace{\begin{array}{ccc} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array}}_{P^{-1}} \Bigg]$$

$\varphi^{100}(3,6,9) = ?$

$$A^{100} \cdot \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = P \cdot D^{100} \cdot P^{-1} \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{100} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -3 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} -6 \\ -3 \\ 0 \end{bmatrix}$$

$\varphi^{100}(3,6,9) = (-6,-3,0)$

II SPOSÓB

$(3,6,9) = [-3,-3,9]_C$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 9 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & -6 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 9 \end{array}\right] \to \left[I \,\middle|\, \begin{array}{c} -3 \\ -3 \\ 9 \end{array}\right]$$

$$D^{100} \cdot \begin{bmatrix} -3 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -3 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ 0 \end{bmatrix}$$

$\varphi^{100}(3,6,9) = [-3,-3,0]_C = -3(1,0,0) - 3(1,1,0) =$

$= (-6,-3,0)$

ALGEBRA – Zadania domowe nr 6 – Diagonalizacja

Zad. 4

a) $A = \begin{bmatrix} 1 & -5 \\ 1 & 1 \end{bmatrix} \in M_2(\mathbb{R})$

$\det(A - tI) = \begin{vmatrix} 1-t & -5 \\ 1 & 1-t \end{vmatrix} = (1-t)^2 + 5 > 0$

brak pierwiastków rzeczywistych

$Spec(A) = \emptyset$ $\quad$ A nie jest diagonalizowalna

b) $A = \begin{bmatrix} 7 & 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

A - trójkątna górna

$Spec(A) = \{ \lambda_1 = 7, \lambda_2 = 1, \lambda_3 = 2 \}$, $k_1 = 1$, $k_2 = 3$, $k_3 = 1$

$\dim E_{\lambda_1} = k_1 = 1$

$\dim E_{\lambda_3} = k_3 = 1$

Zaś $\dim E_{\lambda_2} = 5 - r(A - 1 \cdot I) = 5 - 4 = 1 \neq k_2 = 3$

NIE jest diagonalizowalna

$A - I = \begin{bmatrix} 6 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 3 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad r(A - I) = 4$

c) $A = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 \end{bmatrix}$ $\quad \lambda_1 = 0 \in Spec(A)$, $k_1 = 3$

$r(A) = 1$ widać!

$\dim E_{\lambda_1} = ?$

$\dim E_{\lambda_1} = 4 - r(A - 0 \cdot I) = 4 - r(A) = 4 - 1 = 3 = k_1$

TAK, diagonalizowalna

d) $A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & p & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

dla jakich p macierz jest diagonalizowalna?

$Spec(A) = \{ \lambda_1 = 5, \lambda_2 = 3, \lambda_3 = 1 \}$, $k_1 = 2$, $k_2 = 1$, $k_3 = 1$

Na mocy tw. spektralnego, A diagonalizowalna $\iff \dim E_{\lambda_1} = 2$

$\dim E_{\lambda_1} = 4 - r(A - 5I) = 2 \iff r(A - 5I) = 4 - 2 = \underline{\underline{2}}$

$r(A - 5I) = r \begin{bmatrix} 0 & -2 & 6 & -1 \\ 0 & -2 & p & 0 \\ 0 & 0 & 0 & 4 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{-4} \end{bmatrix} \begin{matrix} \\ \\ /:4 \\ \\ \end{matrix} = r \begin{bmatrix} 0 & -2 & 6 & -1 \\ 0 & -2 & p & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = r \begin{bmatrix} -2 & 6 & -1 \\ -2 & p & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{w_2 - w_1}{=} r \begin{bmatrix} -2 & 6 & -1 \\ 0 & p-6 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$r(A - 5I) = \begin{cases} 3 & ; p \neq 6 \\ 2 & ; p = 6 \end{cases}$

A - diagonalizowalna $\iff p = 6$

Zad. 5 PRAWDA / FAŁSZ

$A = \begin{bmatrix} 7 & -3 & -1 & 0 \\ 0 & 4 & p^3 & 0 \\ 0 & 0 & 7 & \pi \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$A = M_\varphi(\mathcal{B}_e^4, \mathcal{B}_e^4)$

$k_1 = 2 \quad k_2 = 1 \quad k_3 = 1$

- $Spec(A) = \{ \lambda_1 = 7, \lambda_2 = 4, \lambda_3 = 1 \} \Rightarrow$ A diagonalizowalna $\iff \dim E_{\lambda_1} = 4 - r(A - 7I) = 2 \iff r(A - 7I) = 2$
- $\det A = 7 \cdot 4 \cdot 7 = 196 \neq 0 \Rightarrow r(A) = r(\varphi) = 4 \Rightarrow \varphi$ surjekcja $\Rightarrow \dim \ker\varphi = 4 - 4 = 0 \Rightarrow \varphi$ injekcja
- $\det\left(\frac{1}{\sqrt{7}} A^T\right) = \frac{1}{7^2} \cdot 196$

$A - 7I = \begin{bmatrix} 0 & -3 & -1 & 0 \\ 0 & -3 & p^3 & 0 \\ 0 & 0 & 0 & \pi \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$r(A - 7I) = r \begin{bmatrix} 0 & -3 & -1 & 0 \\ 0 & 0 & p^3 + 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{cases} 2 & ; p = -1 \\ 3 & ; p \neq -1 \end{cases}$

$\longrightarrow$ a) ~~FAŁSZ~~ PRAWDA $\quad$ b) FAŁSZ

c) PRAWDA

Ćwiczenia

ZAD.

$$A = \begin{bmatrix} p & 2 & -1 \\ 1 & p & -1 \\ 0 & 0 & 2 \end{bmatrix} \in M_3(\mathbb{R})$$

Dla jakich $p \in \mathbb{R}$ macierz A jest diagonalizowalna?

$$\chi_A(t) = \det(A - tI) = \begin{vmatrix} p-t & 2 & -1 \\ 1 & p-t & -1 \\ 0 & 0 & 2-t \end{vmatrix} = (2-t)\left[(p-t)^2 - 2\right]$$

$$= (2-t)(p-t-\sqrt{2})(p-t+\sqrt{2}) = -(t-2)(t-p+\sqrt{2})(t-p-\sqrt{2})$$

$\forall p \in \mathbb{R} \quad p-\sqrt{2} \neq p+\sqrt{2}$

(1)

$p+\sqrt{2} = 2 \Rightarrow p = 2-\sqrt{2}$

$p-\sqrt{2} = 2 \Rightarrow p = 2+\sqrt{2}$

$\Rightarrow p \neq 2 \pm \sqrt{2}$ wiemy prosto A-diagonalizowalna

(2) $p = 2-\sqrt{2}$

$$A = \begin{bmatrix} 2-\sqrt{2} & 2 & -1 \\ 1 & 2-\sqrt{2} & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

$\lambda_1 = 2 \quad k_1 = 2 \quad \dim E_{\lambda_1} = 3 - r(A - (2)I)$

$$= 3 - r\begin{bmatrix} -\sqrt{2} & 2 & -1 \\ 1 & -\sqrt{2} & -1 \\ 0 & 0 & 0 \end{bmatrix} = 3-2 = 1 \neq k_1 = 2$$

NIE DIAGONALIZOWALNA

(3) $p = 2+2\sqrt{2}$

$$A = \begin{bmatrix} 2+\sqrt{2} & 2 & -1 \\ 1 & 2+\sqrt{2} & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

$\lambda_1 = 2 \quad k_1 = 2 \quad \dim E_{\lambda_1} = 3 - r(A-2I) = 3 - r\begin{bmatrix} \sqrt{2} & 2 & -1 \\ 1 & \sqrt{2} & -1 \\ 0 & 0 & 0 \end{bmatrix} = 3-2 = 1 \neq k_1$

NIE DIAGONALIZOWALNA