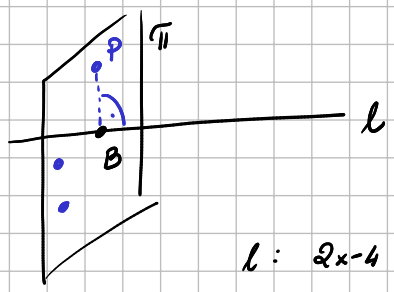
ALGEBRA - Zadanie domowe nr 3
Geometria analityczna

Zad. 1 Punkty, którym rzutem prostokątnym na $l: 2x-4 = 2-y = 2z+4$ jest $B=(3,0,-1)$.



$B \in l$, istotnie
$2\cdot 3-4=2 \quad 2-0=2 \quad 2\cdot(-1)+4=2$

$\pi \perp l$, $B \in \pi$ $\forall P \in \pi$ rzut P na l to B

$l: 2x-4 = 2-y = 2z+4$

$l: \frac{x-2}{\frac{1}{2}} = \frac{y-2}{-1} = \frac{z-(-2)}{\frac{1}{2}}$ $\quad \vec{a}=[1,-2,1] \parallel l \Rightarrow \vec{a} \perp \pi$

$\vec{a} \perp \pi \wedge B \in \pi$ $\quad \pi: 1\cdot(x-3)-2(y-0)+1\cdot(z+1)=0$
$\pi: x-2y+z-2=0$

Zad. 2

a) $\pi: \begin{cases} x=t \\ y=t-s \\ z=-1+2t+s \end{cases}$; $t,s \in \mathbb{R}$ Znaleźć równanie ogólne π.

$t=0, s=0 \Rightarrow P_0=(0,0,-1) \in \pi$ $\quad \vec{a}=[1,1,2] \parallel \pi$ $\quad \vec{b}=[0,-1,1] \parallel \pi$

$\vec{n} := \vec{a} \times \vec{b}$ $\quad \vec{n} \perp \pi$ $\quad \vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{vmatrix} = [3,-1,-1]$

$\pi: 3(x-0)-1\cdot(y-0)-1(z+1)=0$
$\pi: 3x-y-z-1=0$

b) $\pi: 3x-y-z-1=0$ Znaleźć parametryzację π.

Niech $y=\alpha$, $z=\beta$. Wówczas $3x-\alpha-\beta-1=0$, $x=\frac{1}{3}(\alpha+\beta+1)$

$\pi: \begin{cases} x=\frac{1}{3}(\alpha+\beta+1) \\ y=\alpha \\ z=\beta \end{cases}$; $\alpha,\beta \in \mathbb{R}$ ← Inna parametryzacja niż w podpunkcie a)

Zad. 3

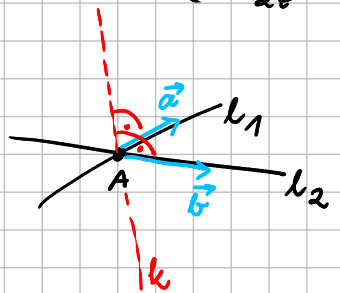
$l_1: x+4 = 2y+2 = z$ $\quad l_2: \begin{cases} x=-2s \\ y=-13+6s \\ z=0 \end{cases}$; $s \in \mathbb{R}$

$\{A\} = l_1 \cap l_2$
$k = ?$ t. że $k \perp l_1 \wedge k \perp l_2 \wedge A \in k$

$l_1: \frac{x-(-4)}{1} = \frac{y-(-1)}{\frac{1}{2}} = \frac{z-0}{1}$ $\quad \vec{a}=[2,1,2] \parallel l_1$ $\quad l_1: \begin{cases} x=-4+2t \\ y=-1+t \\ z=2t \end{cases}$; $t \in \mathbb{R}$

$\vec{b}=[-2,6,0] \parallel l_2$

$A=?$ $\quad \begin{cases} -4+2t=-2s \\ -1+t=-13+6s \\ 2t=0 \rightarrow t=0 \end{cases}$ $\quad \begin{matrix} -4=-2s \rightarrow s=2 \\ -1=13+6s \\ \text{sprzeczna} \end{matrix}$ $\quad A=(-4,-1,0)$



$\vec{n} := \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 2 \\ -2 & 6 & 0 \end{vmatrix} = [-12,-4,14] \parallel k$

$A \in k \Rightarrow k: \begin{cases} x=-4-12t \\ y=-1-4t \\ z=14t \end{cases}$; $t \in \mathbb{R}$

Algebra – Zad. domowe nr. 3 – Geometria analityczna

Zad. 4

$l_1: \frac{x-9}{8} = \frac{y-5}{3} = \frac{z-2}{1}$ $\quad l_2: \frac{3x+9}{4} = y+1 = \frac{24-3x}{7}$

$l_2: \frac{x+3}{\frac{4}{3}} = \frac{y+1}{1} = \frac{z-8}{-\frac{7}{3}}$

a) $l_1 \cap l_2 = \{P_0\}$ $\quad P_0 = ?$

$l_1: \begin{cases} x = 9+8t \\ y = 5+3t \\ z = 2+t \end{cases}$ $\quad t \in \mathbb{R}$

$l_2: \begin{cases} x = -3+\frac{4}{3}s \\ y = -1+s \\ z = 8-\frac{7}{3}s \end{cases}$

$\begin{cases} 9+8t = -3+\frac{4}{3}s \\ 5+3t = -1+s \\ 2+t = 8-\frac{7}{3}s \end{cases}$

$s = 6+3t$

$2+t = 8-\frac{7}{3}(6+3t) =$

$= 8-14-7t$

$8t = -8 \quad t = -1$

$s = 3$

$9+8\cdot(-1) = 1 = -3+\frac{4}{3}\cdot 3$ ok.

$P_0 = (1, 2, 1)$

b) $\pi = ?$ $\quad l_1, l_2 \subset \pi$

$\vec{a} = [8,3,1] \parallel l_1$, $\quad \vec{b} = [\frac{4}{3}, 1, -\frac{7}{3}] \parallel l_2$

równ. param.

$\pi: \begin{cases} x = 9+8t+\frac{4}{3}s \\ y = 5+3t+s \\ z = 2+t-\frac{7}{3}s \end{cases}$

$\vec{n} = \vec{a} \times \vec{b} \perp \pi$

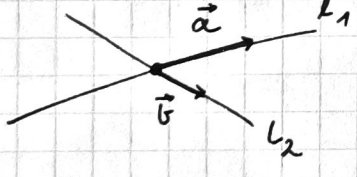
$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 8 & 3 & 1 \\ \frac{4}{3} & 1 & -\frac{7}{3} \end{vmatrix} = [-8, 20, 4]$

$\pi: \quad -8(x-9)+20(y-5)+4(z-2)=0$

$-2(x-0)+5(y-5)+z-2=0$

$-2x+5y+z-9=0$

c) l_3, l_4 dwusieczne kątów między l_1 i l_2



$\vec{v_3} = \frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}$ $\quad \vec{v_4} = \frac{\vec{a}}{|\vec{a}|} - \frac{\vec{b}}{|\vec{b}|}$



dwusieczne kątów to przekątne

$|\vec{a}| = \sqrt{64+9+1} = \sqrt{74}$

$|\vec{b}| = \sqrt{\frac{16}{9}+1+\frac{49}{9}} = \frac{1}{3}\sqrt{16+9+49} = \frac{\sqrt{74}}{3}$

$\vec{v_3} = \frac{1}{\sqrt{74}}\left([8,3,1] + 3[\frac{4}{3}, 1, -7]\right) =$

$= \frac{1}{\sqrt{74}} \cdot [12, 6, -20]$

$\vec{v_4} = \frac{1}{\sqrt{74}}[(8,3,1) - (4,3,-21)] = \frac{1}{\sqrt{74}}[4,0,22]$

$P_0 \in l_3, \; P_0 \in l_4$

$l_3: \begin{cases} x = 1+12t \\ y = 2+6t \\ z = 1-20t \end{cases}$ $\quad t \in \mathbb{R}$

$l_4: \begin{cases} x = 1+4s \\ y = 2 \\ z = 1+22s \end{cases}$ $\quad s \in \mathbb{R}$

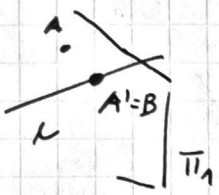
Zad. 5 $p = P_{\Delta ABC} = ?$ $\quad A = (1,2,3)$ $\quad l: \frac{3x-3}{} = \frac{6-y}{} = \frac{2z-8}{}$

$\pi: -3x+2y-2z-29=0$

B – rzut prostokątny A na l

C – p. symetryczny do A wzgl. π

$l: \frac{x-1}{\frac{1}{3}} = \frac{y-6}{-1} = \frac{z-4}{\frac{1}{2}}$ $\quad \vec{v} = [\frac{1}{3}, -1, \frac{1}{2}] \parallel l$ $\quad l_1: \begin{cases} x = 1+\frac{1}{3}t \\ y = 6-t \\ z = 4+\frac{1}{2}t \end{cases}$



$\pi_1 \ni A$ $\quad \vec{v} \perp \pi_1$ $\quad \pi_1: \frac{1}{3}(x-1) - (y-2) + \frac{1}{2}(z-3) = 0$

$\pi_1 \perp l$

$\{B\} = \pi_1 \cap l$

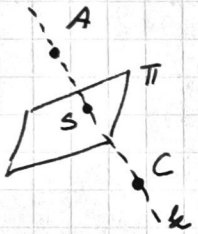
$2(x-1) - 6(y-2) + 3(z-3) = 0$

$2 \cdot \frac{1}{3}t - 6(4-t) + 3(1+\frac{1}{2}t) = 0$

$\frac{2}{3}t - 24 + 6t + 3 + \frac{3}{2}t = 0$

$\frac{4+36+9}{6}t = 21$ $\quad t = 21 \cdot \frac{6}{49_7} = \frac{18}{7}$

$B = (1+\frac{6}{7}, \frac{42-18}{7}, 4+\frac{9}{7}) = (\frac{13}{7}, \frac{24}{7}, \frac{37}{7})$



$k \perp \pi$

$A \in k$

$\vec{m} = [-3,2,-2] \perp \pi$

$\vec{m} \parallel k$

$k: \begin{cases} x = 1-3t \\ y = 2+2t \\ z = 3-2t \end{cases}; \quad t \in \mathbb{R}$

$\{S\} = \pi \cap k$

$-3(1-3t) + 2(2+2t) - 2(3-2t) - 29 = 0$

$17t = 34$

$t = 2$

$S = (-5, 6, -1)$

$C = T_{\vec{AS}}(S)$ $\quad \vec{AS} = [-6,4,-4]$ $\quad C = (-5-6, 6+4, -1-4)$

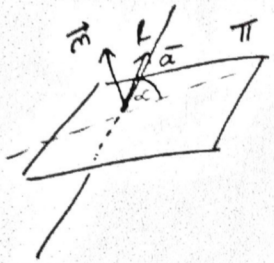
$C = (-11, 10, -5)$

$P_\Delta = \frac{1}{2}|\vec{AB} \times \vec{AC}|$ $\quad \vec{AB} = [\frac{6}{7}, \frac{10}{7}, \frac{16}{7}]$ $\quad \vec{AC} = [-12, 8, -8]$

$$\vec{AB} \times \vec{AC} = \frac{1}{7}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & 10 & 16 \\ -12 & 8 & -8 \end{vmatrix} = \frac{1}{7}[-208, -144, 168]$$

$$P_\Delta = \frac{1}{2} \cdot \frac{\sqrt{208^2 + 144^2 + 168^2}}{7^2} = \frac{1}{14}\sqrt{208^2 + 144^2 + 168^2}$$

Zad. 6 Oblicz miarę kąta między płaszczyzną $\pi: x - z = 0$ a prostą $l: \begin{cases} x = 3+2t \\ y = 1 \\ z = -2-3t \end{cases}; \quad t \in \mathbb{R}$.



$\vec{n} = (1,0,-1) \perp \pi$

$\vec{a} = (2,0,-3) \parallel l$

$\alpha = \sphericalangle(\pi, l) = \frac{\pi}{2} - \sphericalangle(\vec{n}, \vec{a})$ $\quad \alpha + \beta = \pi/2$

$\cos\beta = \cos(\frac{\pi}{2} - \alpha) = \sin\alpha = \frac{|\vec{n} \circ \vec{a}|}{|\vec{n}| \cdot |\vec{a}|} = \frac{|2+0+3|}{\sqrt{2} \cdot \sqrt{4+9}} = \frac{5}{\sqrt{26}}$

$\alpha = \arcsin\frac{5}{\sqrt{26}}$

ALGEBRA – Zadanie domowe nr 3

Zad. 7

a) $A=(2,3,2)$, $B=(4,3,-1)$, $C=(3,5,p^2+1)$, $D=(4,2,1)$

$p=?$ A,B,C,D współpłaszczyznowe

$\vec{AB}=[2,0,-3]$ $\vec{AC}=[1,2,p^2-1]$, $\vec{AD}=[2,-1,-1]$

$$[\vec{AB},\vec{AC},\vec{AD}]=\begin{vmatrix} 2 & 0 & -3 \\ 1 & 2 & p^2-1 \\ 2 & -1 & -1 \end{vmatrix}=$$

$$=-4+3+12+2(p^2-1)=2p^2+9\neq 0 \quad \forall p\in\mathbb{R}$$

NIGDY nie leżą w jednej płaszczyźnie (dla żadnego p)

b) $\vec{c}=[1,4,3]$ $\vec{b}$ – rzut prostokątny $\vec{c}$ na $Oy \Rightarrow \vec{b}=[0,4,0]$, $|\vec{b}|=4$

$\vec{a}$ wersor $\Rightarrow |\vec{a}|=1$

$$8\vec{a}-\vec{b}\perp\vec{b} \iff 0=(8\vec{a}-\vec{b})\circ\vec{b}=8\vec{a}\circ\vec{b}-\vec{b}\circ\vec{b}=8|\vec{a}||\vec{b}|\cos\sphericalangle(\vec{a},\vec{b})-|\vec{b}|^2=$$

$$=8\cdot 1\cdot 4\cdot\cos\sphericalangle(\vec{a},\vec{b})-16 \Rightarrow \cos\sphericalangle(\vec{a},\vec{b})=\frac{16}{32}=\frac{1}{2}\Rightarrow\sphericalangle(\vec{a},\vec{b})=\frac{\pi}{3}$$

Zad. 8

$$l:\begin{cases} x=-s \\ y=1+(1+a)s \\ z=2-2as \end{cases} s\in\mathbb{R} \qquad k:\begin{cases} x=t \\ y=1-t \\ z=at \end{cases} t\in\mathbb{R}$$

a – parametr

$\vec{u}=[-1,1+a,-2a]\parallel l$ $\quad A=(0,1,2)\in l$

$\vec{v}=[1,-1,a]\parallel k$ $\quad B=(0,1,0)\in k$

$\vec{AB}=[0,0,-2]$

$$[\vec{u},\vec{v},\vec{AB}]=\begin{vmatrix} -1 & 1+a & -2a \\ 1 & -1 & a \\ 0 & 0 & 2 \end{vmatrix}=2\cdot(-1)^6\begin{vmatrix} -1 & 1+a \\ 1 & -1 \end{vmatrix}=2(1-1-a)=-2a\neq 0 \iff a\neq 0$$

$a\neq 0$ → proste skośne

$a=0$ → proste leżą w jednej płaszczyźnie

a) $k\parallel l$? $a=0$ $\vec{u}=\vec{v}$ $k\parallel l$ PRAWDA

b) k,l skośne $\Rightarrow a=0$ FAŁSZ

c) $\forall a<2$ $k\cap l\neq\emptyset$ PRAWDA → gdy $a=0$ $k\parallel l$ czy $k=l$? $B=(0,1,0)\in k$ ale $B\notin l$

→ gdy $a\neq 0$ skośne

$a=0$ $\begin{cases} 0=s \\ 1=1 \\ 0=2 \end{cases}$ ↯ tzn. $k\cap l=\emptyset$

Zad. 9

a) $|\vec{u}\times\vec{v}|=|\vec{u}|\cdot|\vec{v}|\cdot\underbrace{|\sin\sphericalangle(\vec{u},\vec{v})|}_{\leq 1}\leq|\vec{u}|\cdot|\vec{v}|$ PRAWDA

b) $$l:\begin{cases} x+y-2z-1=0 \\ x+3y-z-4=0 \end{cases} \qquad \vec{v}=[-2,-14,-2]$$

$\vec{a}\parallel l$ $\quad \vec{a}=[1,1,-2]\times[1,3,-1]=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -2 \\ 1 & 3 & -1 \end{vmatrix}=[5,-1,2]$

$\vec{a}\circ\vec{v}=[-2,-14,-2]\circ[5,-1,2]$

$=-10+14-4=0$

tzn. $\vec{a}\perp\vec{v}$

PRAWDA