ALGEBRA - Zadania domowe nr 5 - Odwzorowania liniowe

Zad. 1 Czy f jest liniowe?

a) $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (7x+y+7,\ 7x-y)$

NIE

$L = f(\alpha(x,y)) = f(\alpha x, \alpha y) = (7\alpha x + \alpha y + 7,\ 7\alpha x - \alpha y)$

$P = \alpha \cdot f(x,y) = \alpha(7x+y+7,\ 7x-y) = (7\alpha x + \alpha y + \alpha 7,\ 7\alpha x - \alpha y)$

$L \neq P$

b) $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x+y+z$ TAK

$\forall (x_1,y_1,z_1), (x_2,y_2,z_2) \in \mathbb{R}^3 \quad \forall \alpha \in \mathbb{R}$

$\alpha \cdot f(x,y,z) = \alpha \cdot (x+y+z) = \alpha x + \alpha y + \alpha z = f(\alpha x, \alpha y, \alpha z) = f(\alpha \cdot (x,y,z))$

$f(x_1,y_1,z_1) + f(x_2,y_2,z_2) = (x_1+y_1+z_1) + (x_2+y_2+z_2) = (x_1+x_2) + (y_1+y_2) + (z_1+z_2) =$

$= f(x_1+x_2, y_1+y_2, z_1+z_2) = f[(x_1,y_1,z_1) + (x_2,y_2,z_2)]$

c) $f: M_3(\mathbb{C}) \to \mathbb{C}$ $f(A) = \det A$ NIE

$L = 2 \cdot f(I_3) = 2 \cdot \det(I_3) = 2 \cdot 1$

$P = f(2I_3) = \det \begin{bmatrix} 2&0&0 \\ 0&2&0 \\ 0&0&2 \end{bmatrix} = 2^3$

$L \neq P$

d) $f: \mathbb{R}_2[x] \to M_2(\mathbb{R})$, $f(ax^2+bx+c) = \begin{bmatrix} a & b \\ -b & c \end{bmatrix}$ TAK

$\forall\ a_1x^2+b_1x+c_1 \in \mathbb{R}_2[x]$

$\forall\ a_2x^2+b_2x+c_2 \in \mathbb{R}_2[x]$

$\forall\ \alpha \in \mathbb{R}$

$\alpha \cdot f(a_1x^2+b_1x+c_1) = \alpha \begin{bmatrix} a_1 & b_1 \\ -b_1 & c_1 \end{bmatrix} = \begin{bmatrix} \alpha a_1 & \alpha b_1 \\ -\alpha b_1 & \alpha c_1 \end{bmatrix} = f(\alpha a_1x^2 + \alpha b_1x + \alpha c_1)$

$= f[\alpha \cdot (a_1x^2+b_1x+c_1)]$

$f[(a_1x^2+b_1x+c_1) + (a_2x^2+b_2x+c_2)] = f[(a_1+a_2)x^2 + (b_1+b_2)x + (c_1+c_2)] = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ -b_1-b_2 & c_1+c_2 \end{bmatrix} =$

$= \begin{bmatrix} a_1 & b_1 \\ -b_1 & c_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ -b_2 & c_2 \end{bmatrix} = f(a_1x^2+b_1x+c_1) + f(a_2x^2+b_2x+c_2)$

Zad. 2 $\varphi: \mathbb{R}^4 \to \mathbb{R}^3$ $\varphi(x,y,z,t) = (x+2y+z-t,\ x+2x+t,\ 2x+y+3t)$

a) $\operatorname{Im}\varphi = ?$ $\ker\varphi = ?$ $\oplus$ idą bazy i wymiary

b) Liniowość φ

- $\varphi(x,y,z,t) = x(1,1,2) + y(2,0,1) + z(1,2,0) + t(-1,1,3)$

$\operatorname{Im}\varphi = \operatorname{lin}\{(1,1,2), (2,0,1), (1,2,0), (-1,1,3)\}$ ← generatory, ale nie baza

$\operatorname{Im}\varphi \subset \mathbb{R}^3 \Rightarrow \dim \operatorname{Im}\varphi \leq 3$

$r\begin{bmatrix} 1&1&2 \\ 2&0&1 \\ 1&2&0 \\ -1&1&3 \end{bmatrix} \underset{\substack{W_3-W_1 \\ W_4+W_1}}{\overset{W_2-2W_1}{=}} r\begin{bmatrix} 1&1&2 \\ 0&-2&0 \\ 0&1&-2 \\ 0&2&5 \end{bmatrix} \overset{W_2\cdot(-\frac{1}{2})}{\underset{W_4+W_2}{=}} r\begin{bmatrix} 1&1&2 \\ 0&1&0 \\ 0&1&-2 \\ 0&0&5 \end{bmatrix} \underset{W_3-W_2}{=} r\begin{bmatrix} 1&1&2 \\ 0&1&0 \\ 0&0&-2 \\ 0&0&5 \end{bmatrix} = 3$

$\dim \operatorname{Im}\varphi = 3$

Baza $\{(1,1,2), (0,1,0), (0,0,1)\}$

$\dim \operatorname{Im}\varphi = \dim \mathbb{R}^3$

φ - EPIMORFIZM

- $\dim \ker\varphi = \dim \mathbb{R}^4 - \dim \operatorname{Im}\varphi = 4-3 = 1$

$\ker\varphi = ?$ $\varphi(x,y,z,t) = (0,0,0) \iff \begin{cases} x+2y+z-t=0 \\ x+2z+t=0 \\ 2x+y+3t=0 \end{cases} \iff \begin{bmatrix} 1&2&1&-1 \\ 1&0&2&1 \\ 2&1&0&3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{cccc|c} 1&2&1&-1&0 \\ 1&0&2&1&0 \\ 2&1&0&3&0 \end{array}\right] \xrightarrow[W_3-2W_1]{W_2-W_1} \left[\begin{array}{cccc|c} 1&2&1&-1&0 \\ 0&-2&1&2&0 \\ 0&-3&-2&5&0 \end{array}\right] \xrightarrow{W_2-W_3} \left[\begin{array}{cccc|c} 1&2&1&-1&0 \\ 0&1&3&-3&0 \\ 0&-3&-2&5&0 \end{array}\right] \xrightarrow{W_3+3W_2} \left[\begin{array}{cccc|c} 1&2&1&-1&0 \\ 0&1&3&-3&0 \\ 0&0&7&-4&0 \end{array}\right]$

$\xrightarrow{W_3\cdot\frac{1}{7}} \left[\begin{array}{cccc|c} 1&2&1&-1&0 \\ 0&1&3&-3&0 \\ 0&0&1&-\frac{4}{7}&0 \end{array}\right] \xrightarrow[W_1-W_3]{W_2-3W_3} \left[\begin{array}{cccc|c} 1&2&0&-\frac{3}{7}&0 \\ 0&1&0&-\frac{9}{7}&0 \\ 0&0&1&-\frac{4}{7}&0 \end{array}\right] \xrightarrow{W_1-2W_2} \left[\begin{array}{cccc|c} 1&0&0&\frac{15}{7}&0 \\ 0&1&0&-\frac{9}{7}&0 \\ 0&0&1&-\frac{4}{7}&0 \end{array}\right]$

$\ker\varphi = \{(-\frac{15}{7}t, \frac{9}{7}t, \frac{4}{7}t, t) : t \in \mathbb{R}\} = \operatorname{lin}\{(-15, 9, 4, 7)\}$ baza $\{(-15,9,4,7)\}$

$\ker\varphi \neq \{(0,0,0,0)\}$

φ nie jest MONOMORFIZMEM

Zad. 1 e) $f: C^1([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$ $\quad f(g)(x) = g'(x)e^x$

Dla $g_1, g_2 \in C^1([0,1], \mathbb{R})$, $\alpha \in \mathbb{R}$

$f(\alpha g)(x) = (\alpha g)'(x)e^x = \alpha \cdot g'(x)e^x = \alpha \cdot f(g)(x)$

$f(g_1+g_2)(x) = (g_1+g_2)'(x)e^x = \left(g_1'(x)+g_2'(x)\right)e^x = g_1'(x)e^x + g_2'(x)e^x = f(g_1)(x) + f(g_2)(x)$

TAK, liniowe

f) $f: \mathbb{R}[x] \to \mathbb{R}[x]$, $f(p)(x) = [p(x)]^2$

NIE JEST LINIOWE Niech $p(x) = x$ $\quad f(2p)(x) = [2x]^2 = 4x^2 \neq 2 \cdot f(p)(x) = 2 \cdot x^2$

g) $f: \mathbb{R}^2 \to \mathbb{R}_2[x]$, $f(a,b) = (a+b)x^2 + 2ax + a - b$

Niech $(a_1, b_1), (a_2, b_2) \in \mathbb{R}^2$, $\alpha \in \mathbb{R}$ $\quad f(\alpha(a,b)) = f(\alpha a, \alpha b) = (\alpha a + \alpha b)x^2 + 2(\alpha a)x + \alpha a - \alpha b =$

$= \alpha\left[(a+b)x^2 + 2ax + a - b\right] = \alpha \cdot f(a,b)$

$f((a_1,b_1)+(a_2,b_2)) = f(a_1+a_2, b_1+b_2) = (a_1+a_2+b_1+b_2)x^2 + 2(a_1+a_2)x + a_1+a_2-(b_1+b_2) =$

$= \left[(a_1+b_1)x^2 + 2a_1x + a_1 - b_1\right] + \left[(a_2+b_2)x^2 + 2a_2x + a_2 - b_2\right] = f(a_1,b_1) + f(a_2,b_2)$

tak, liniowe

h) $V = (\mathbb{C}, +, \mathbb{C}, \cdot)$ $\quad f: V \to V$ $\quad f(z) = \bar{z}$

Nie jest $\mathbb{C}$-liniowe $\quad \alpha = \beta + iy \in \mathbb{C}$ $\quad z = x + iy \in \mathbb{C}$ $\quad f(\alpha z) = \overline{\alpha z} = \bar{\alpha} \cdot \bar{z} \neq \alpha \cdot f(z) = \alpha \cdot \bar{z}$

$\alpha \neq \bar{\alpha}$

i) $\mathbb{C}^2 = (\mathbb{C}^2, +, \mathbb{C}, \cdot)$ $\quad f: \mathbb{C}^2 \to \mathbb{C}^2$, $f(z,w) = (w, 0)$ liniowe

Dla $\alpha \in \mathbb{C}$, $(z_1, w_1), (z_2, w_2) \in \mathbb{C}^2$ $\quad f(\alpha(z,w)) = f(\alpha z, \alpha w) = (\alpha w, 0) = \alpha(w, 0)$

$f((z_1,w_1)+(z_2,w_2)) = f(z_1+z_2, w_1+w_2) = (w_1+w_2, 0) = (w_1, 0) + (w_2, 0) = f(z_1,w_1) + f(z_2,w_2)$

j) $f: \mathbb{R}^2 \to \mathbb{C}$, $f(x,y) = x + iy$ $\quad \mathbb{C} = (\mathbb{C}, +, \mathbb{C}, \cdot)$

Dla $\alpha \in \mathbb{R}$ $\quad (x_1,y_1), (x_2,y_2) \in \mathbb{R}^2$ $\quad f(\alpha(x,y)) = f(\alpha x, \alpha y) = \alpha x + i \cdot \alpha y = \alpha(x+iy) = \alpha f(x,y)$

liniowe

$f((x_1,y_1)+(x_2,y_2)) = f(x_1+x_2, y_1+y_2) = x_1+x_2 + i(y_1+y_2) = x_1+iy_1 + x_2+iy_2 =$

$= f(x_1,y_1) + f(x_2,y_2)$

k) $V = (\mathbb{C}, +, \mathbb{C}, \cdot)$

$f: V \to V$, $f(z) = Re(z)$ $\quad$ Nie jest $\mathbb{C}$-liniowe

$z = x + iy$ $\quad f((1+i)z) = f((1+i)(x+iy)) = f(x+iy+xi-y) = x - y$

$(1+i)f(z) = (1+i)x$

l) $V = (\mathbb{C}, +, \mathbb{R}, \cdot)$ $\quad$ jest $\mathbb{R}$-liniowe

$\alpha \in \mathbb{R}$ $\quad z = x + iy \in \mathbb{C}$ $\quad f(\alpha \cdot z) = f(\alpha(x+iy)) = f(\alpha x + i\alpha y) = \alpha x = \alpha \cdot Re(z) = \alpha f(x+iy) = \alpha f(z)$

Zad. 3 $f: \mathbb{R}_2[x] \to \mathbb{R}_1[x]$ liniowe

$f(p)(x) = (3-x)\cdot p''(x) + 4p'(x)$

a) $A = ?$ $\quad A = M_f(\mathcal{B}, C)$ $\quad \mathcal{B} = (1, x, x^2)$ baza $\mathbb{R}_2[x]$, $C = (1, x)$ baza $\mathbb{R}_1[x]$

$f(1) = (3-x)\cdot 0 + 4\cdot 0 = 0 = [0,0]_C$

$f(x) = (3-x)\cdot 0 + 4\cdot 1 = 4 = [4,0]_C$

$f(x^2) = (3-x)\cdot 2 + 4\cdot 2x = 6 - 2x + 8x = 6 + 6x = [6,6]_C$

$$A = \begin{bmatrix} 0 & 4 & 6 \\ 0 & 0 & 6 \end{bmatrix}$$

b) $\ker f = ?$, $\operatorname{Im} f = ?$ + wiele bazy i wymiary

c) Własności f

- $p \in \mathbb{R}_2[x]$ $\quad p(x) = ax^2 + bx + c$ $\quad p'(x) = 2ax + b$ $\quad p''(x) = 2a$

$f(p)(x) = (3-x)\cdot 2a + 4(2ax + b) = 6a - 2ax + 8ax + 4b = 6ax + (6a + 4b) = 0$

$\Leftrightarrow \begin{cases} 6a = 0 \\ 6a + 4b = 0 \end{cases} \Leftrightarrow a = b = 0 \Rightarrow \ker f = \{p \in \mathbb{R}_2[x] : p(x) = c,\ c \in \mathbb{R}\}$ wielomiany stałe

$\ker f = \operatorname{lin}\{1\}$ ← BAZA $\quad \dim \ker f = 1$

f nie jest monomorfizmem

- $\dim \operatorname{Im} f = \dim \mathbb{R}_2[x] - \dim \ker f = 3 - 1 = 2$

lub $\dim \operatorname{Im} f = r\begin{bmatrix} 0 & 4 & 6 \\ 0 & 0 & 6 \end{bmatrix} = 2$

$\operatorname{Im} f \subseteq \mathbb{R}_1[x] \ \wedge \ \dim \operatorname{Im} f = \dim \mathbb{R}_1[x] = 2 \Rightarrow \operatorname{Im} f = \mathbb{R}_1[x]$

baza $(1, x)$

f – epimorfizm

Zad. 4 U, V – przestrzenie $\mathbb{R}$-liniowe

$\varphi: U \to V$ liniowe

$$A = \begin{bmatrix} 1 & 3 & 4 & 1 & 1 \\ 2 & 1 & 1 & 0 & 1 \\ 4 & 7 & 9 & 2 & 3 \end{bmatrix}$$

$\dim \ker \varphi = ?$

$A \in M_{3\times 5}(\mathbb{R}) \Rightarrow \dim U = 5$, $\dim V = 3$

$\dim \ker \varphi = \dim U - \dim \operatorname{Im}\varphi = 5 - r(A)$

$$r(A) \underset{\substack{W_2 - 2W_1 \\ W_3 - 4W_1}}{=} r\begin{bmatrix} 1 & 3 & 4 & 1 & 1 \\ 0 & -5 & -7 & -2 & -1 \\ 0 & -5 & -7 & -2 & -1 \end{bmatrix} = 2 \Rightarrow \dim \ker \varphi = 5 - 2 = 3$$

Zad. 5 $f: \mathbb{R}^3 \to \mathbb{R}^2$ $\quad f(1,1,1) = (2,2)$, $f(1,0,1) = (1,0)$, $f(0,1,1) = (1,1)$

a) wzór f ?

$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1 - 1 - 1 \neq 0 \Rightarrow ((1,1,1), (1,0,1), (0,1,1))$ pewna baza $\mathbb{R}^3$

$\mathcal{B}_c^3 = (e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1))$ baza kanoniczna $\mathbb{R}^3$

$f(x,y,z) = f(xe_1 + ye_2 + ze_3) = x f(e_1) + y f(e_2) + z f(e_3)$ (f liniowe)

$f(e_1) = ?$, $f(e_2) = ?$, $f(e_3) = ?$

$$\begin{cases} f(1,1,1) = f(e_1) + f(e_2) + f(e_3) = (2,2) \\ f(1,0,1) = f(e_1) + f(e_3) = (1,0) \\ f(0,1,1) = f(e_2) + f(e_3) = (1,1) \end{cases} \Rightarrow \begin{aligned} f(e_2) &= (2,2) - (1,0) = (1,2) \\ f(e_1) &= (2,2) - (1,1) = (1,1) \\ f(e_3) &= (1,1) - f(e_2) = (1,1) - (1,2) = (0,-1) \end{aligned}$$

$f(x,y,z) = x\cdot(1,1) + y(1,2) + z(0,-1) = (x + y,\ x + 2y - z)$

b) $\mathcal{B} = (b_1 = (3,1,1), b_2 = (5,1,6), b_3 = (4,-1,2))$ baza $\mathbb{R}^3$

$C = (c_1 = (-1,1), c_2 = (1,0))$ baza $\mathbb{R}^2$

$A' = M_f(\mathcal{B}, C) = ?$

$f(3,1,1) = (4,4) = 4c_1 + 8c_2 = [4,8]_C$

$f(5,1,6) = (6,1) = c_1 + 7c_2 = [1,7]_C$

$f(4,-1,2) = (3,0) = 3c_2 = [0,3]_C$

$$A' = \begin{bmatrix} 4 & 1 & 0 \\ 8 & 7 & 3 \end{bmatrix}$$

Zad. 5

c) $C' = (c_1' = (1,1),\ c_2' = (2,1))$ $\quad P = P_{C \to C'} = ?$

$c_1' = (1,1) = (-1,1) + (2,0) = [1,2]_C$

$c_2' = (2,1) = (-1,1) + (3,0) = [1,3]_C$

$P = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

d) $v = [2,5]_C$ $\quad v = [\alpha, \beta]_{C'} = ?$ $\quad X = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ $\quad X' = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ $\quad PX' = X$ $\quad X' = P^{-1}X$

$P^{-1} = ?$ $\quad \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \xrightarrow{w_2 - 2w_1} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \xrightarrow{w_1 - w_2} \left[\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & \underbrace{-2}_{P^{-1}} & 1 \end{array}\right]$

$X' = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad v = [1,1]_{C'}$

Zad. 6

$\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ $\quad A = M_\varphi(B_c^2, B_c^3) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$

$\mathcal{B} = (b_1 = (1,1),\ b_2 = (2,1))$ baza $\mathbb{R}^2$

$C = (c_1 = (1,2,0),\ c_2 = (2,3,0),\ c_3 = (0,0,1))$ baza $\mathbb{R}^3$ $\quad A' = M_\varphi(\mathcal{B}, C) = ?$

$A' = Q^{-1} A P$ $\quad P = P_{B_c^2 \to \mathcal{B}} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ $\quad Q = P_{B_c^3 \to C} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$Q^{-1} \overset{\text{GW}}{=} \begin{bmatrix} -3 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad A' = \begin{bmatrix} -3 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -5 \\ 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -6 & -7 \\ 4 & 5 \\ 1 & 1 \end{bmatrix}$

Zad. 7 $\quad \varphi : \mathbb{R}_1[x] \to \mathbb{R}_1[x]$ $\quad \mathcal{B} = (1, x)$ $\quad A = M_\varphi(\mathcal{B}, \mathcal{B}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$\mathcal{B}' = (1, -5x)$ $\quad A' = M_\varphi(\mathcal{B}', \mathcal{B}') = ?$

$P = P_{\mathcal{B} \to \mathcal{B}'} = \begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix}$ $\quad A' = P^{-1} A P$

$P^{-1} = -\frac{1}{5} \begin{bmatrix} -5 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{5} \end{bmatrix}$ $\quad A' = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{5} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 0 & -5 \\ \frac{1}{5} & 0 \end{bmatrix}$

Zad. 8 $\quad \varphi : \mathbb{R}^4 \to \mathbb{R}^3$

$\mathcal{B} = (b_1 = (1,1,1,1),\ b_2 = (1,1,1,0),\ b_3 = (1,1,0,0),\ b_3 = (1,0,0,0))$ baza $\mathbb{R}^4$

$C = (c_1 = (0,-1,0),\ c_2 = (1,0,0),\ c_3 = (1,1,1))$ baza $\mathbb{R}^3$

$A' = M_\varphi(\mathcal{B}, C) = \begin{bmatrix} 1 & -2 & -2 & -2 \\ 4 & 3 & 1 & 1 \\ 2 & 0 & 0 & 0 \end{bmatrix}$

a) $v = (5,0,1,0)$ $\quad \varphi(v) = ?$

$v = [\alpha, \beta, \gamma, \delta]_{\mathcal{B}} = ?$ $\quad P = P_{B_c^4 \to \mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ $\quad P \cdot \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \\ 0 \end{bmatrix} \iff \begin{cases} \alpha + \beta + \gamma + \delta = 5 \\ \alpha + \beta + \gamma = 0 \\ \alpha + \beta = 1 \\ \alpha = 0 \end{cases}$

$\Rightarrow \alpha = 0,\ \beta = 1,\ \gamma = -\alpha - \beta = -1,\ \delta = 5 - \alpha - \beta - \gamma = 5 - 0 - 1 + 1 = 5$

$v = [0, 1, -1, 5]_{\mathcal{B}}$

$A' \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} -10 \\ 7 \\ 0 \end{bmatrix}$ $\quad \varphi(v) = [-10, 7, 0]_C = -10\, c_1 + 7\, c_2 = -10(0,-1,0) + 7(1,0,0) = \underline{(7, 10, 0)}$

Zad. 8

b) $\begin{cases} \varphi(1,1,1,1) = [1,4,2]_C = (0,-1,0) + 4(1,0,0) + 2(1,1,1) = (6,1,2) & = \varphi(e_1) + \varphi(e_2) + \varphi(e_3) + \varphi(e_4) \\ \varphi(1,1,1,0) = [-2,3,0]_C = -2(0,-1,0) + 3(1,0,0) = (3,2,0) & = \varphi(e_1) + \varphi(e_2) + \varphi(e_3) \\ \varphi(1,1,0,0) = [-2,1,0]_C = -2(0,-1,0) + (1,0,0) = (1,2,0) & = \varphi(e_1) + \varphi(e_2) \\ \varphi(1,0,0,0) = [-2,1,0]_C = (1,2,0) & = \varphi(e_1) \end{cases}$

$\Rightarrow \varphi(e_1) = (1,2,0)$, $\varphi(e_2) = (0,0,0)$, $\varphi(e_3) = (3,2,0) - (1,2,0) = (2,0,0)$

$\varphi(e_4) = (6,1,2) - (3,2,0) = (3,-1,2)$

$\varphi(x,y,z,t) = x\varphi(e_1) + y\varphi(e_2) + z\varphi(e_3) + t\varphi(e_4) =$

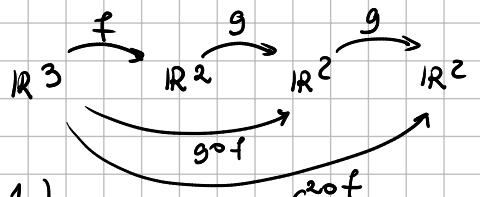
$= x(1,2,0) + y(0,0,0) + z(2,0,0) + t(3,-1,2) = (x+2z+3t, 2x-t, 2t)$

Zad. 9

$f: \mathbb{R}^3 \to \mathbb{R}^2$ $\quad f(x,y,z) = (x-y+z, 2y+z)$

$g: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad g(x,y) = (2x+y, x-y)$

$h := g^2 \circ f$ $\quad$ wzór h? $\quad h(1,2,0) = ?$



a)

- $A = M_f(\mathcal{B}_k^3, \mathcal{B}_k^2) = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ $\quad B = M_g(\mathcal{B}_k^2, \mathcal{B}_k^2) = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$

$C = M_h(\mathcal{B}_k^3, \mathcal{B}_k^2) = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}^2 \cdot \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -3 & 6 \\ 1 & 3 & 3 \end{bmatrix}$

$h: \mathbb{R}^3 \to \mathbb{R}^2$

$h(x,y,z) = ?$ $\quad C\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5x-3y+6z \\ x+3y+3z \end{bmatrix}$ $\quad h(x,y,z) = (5x-3y+6z, x+3y+3z)$

- $h(1,2,0) = ?$ $\quad C \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 & -3 & 6 \\ 1 & 3 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$ $\quad h(1,2,0) = (-1,7)$

Zad. 10 $\quad$ Prawda / fałsz

$\varphi: \mathbb{R}_3[x] \to \mathbb{R}_3[x]$ $\quad$ endomorfizm

a) Jeśli $\ker\varphi = \{0\}$, to φ jest izomorfizmem.

PRAWDA $\quad \ker\varphi = \{0\} \Rightarrow \varphi$ monomorfizm

$\dim \operatorname{Im}\varphi = \dim \mathbb{R}_3[x] - \dim \ker\varphi = 4 - 0 = 4 \Rightarrow \operatorname{Im}\varphi = \mathbb{R}_3[x]$

φ epimorfizm

b) Jeśli $\ker\varphi = \{0\}$ to $\{\varphi(1), \varphi(x), \varphi(x^2)\}$ jest liniowo niezależny

PRAWDA $\quad \ker\varphi = \{0\} \Rightarrow \varphi$ monomorfizm, więc przeprowadza układ liniowo niezależny $\{1, x, x^2\}$ w układ lin. niezależny

c) $A = M_\varphi(\mathcal{B}, \mathcal{B})$ $\quad \mathcal{B} = (1, x, x^2, x^3)$

Czy kwadratowa stopnia 3? $\quad$ FAŁSZ $\quad A \in M_4(\mathbb{R})$

Zad. 9 - ciąg dalszy

b) $\left.\begin{array}{l} \varphi(1,3) - \varphi(1,1) = \varphi(0,2) = 2\cdot\varphi(0,1) \\ (6,4) - (2,0) = (4,4) \end{array}\right\} \Rightarrow \varphi(0,1) = (2,2)$

$\varphi(1,0) = \varphi(1,1) - \varphi(0,1) = (2,0) - (2,2) = (0,-2)$

$D = M_\varphi(\mathcal{B}_k^2, \mathcal{B}_k^2) = \begin{bmatrix} 0 & 2 \\ -2 & 2 \end{bmatrix}$ $\quad \det D = 4 \neq 0 \Rightarrow \varphi$ izomorfizm

$\det B = -3 \neq 0 \Rightarrow g$ izomorfizm

$D^{-1} = M_{\varphi^{-1}}(\mathcal{B}_k^2, \mathcal{B}_k^2)$

$B^{-1} = M_{g^{-1}}(\mathcal{B}_k^2, \mathcal{B}_k^2)$ $\quad \Rightarrow M_{g^{-1}\circ\varphi^{-1}} = M_{g^{-1}} \cdot M_{\varphi^{-1}} = B^{-1} \cdot D^{-1} = (DB)^{-1}$

$DB = \begin{bmatrix} 0 & 2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & -4 \end{bmatrix}$ $\quad \det(DB) = -8 - 4 = -12 \neq 0$ $\quad (DB)^{-1} = \frac{-1}{12}\begin{bmatrix} -4 & 2 \\ 2 & 2 \end{bmatrix}^T = \frac{1}{6}\begin{bmatrix} 2 & -1 \\ -1 & -1 \end{bmatrix}$

$(DB)^{-1}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 2 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 2x-y \\ -x-y \end{bmatrix}$

$(g^{-1}\circ\varphi^{-1})(x,y) = \left(\frac{2x-y}{6}, \frac{-x-y}{6}\right)$

Zad. 10 $\varphi : \mathbb{R}_3[x] \to \mathbb{R}_3[x]$ liniowe

a) $\ker\varphi = \{0\} \iff \dim\ker\varphi = 0 \Rightarrow \dim \operatorname{Im}\varphi = \dim\mathbb{R}^3 - \dim\ker\varphi = 3-0=3 \Rightarrow$

$\Rightarrow \operatorname{Im}\varphi \subseteq \mathbb{R}^3 \wedge \dim\operatorname{Im}\varphi = 3 \Rightarrow \operatorname{Im}\varphi = \mathbb{R}^3 \Rightarrow \varphi$ - surjekcja

$\ker\varphi = \{0\} \Rightarrow \varphi$ - injekcja

Zatem φ - liniowe, bijektywne (izomorfizm) PRAWDA

b) $\ker(\varphi) = \{0\} \Rightarrow \varphi$ - injekcja

$\{1, x, x^2\}$ liniowo niezależne $\Rightarrow \{\varphi(1), \varphi(x), \varphi(x^2)\}$ liniowo niezależny PRAWDA / twierdzenie z wykładu

c) $\dim \mathbb{R}_3[x] = 4$, $\mathcal{B} = (1, x, x^2, x^3)$ baza $\Rightarrow M_\varphi(\mathcal{B},\mathcal{B}) \in M_4(\mathbb{R})$ FAŁSZ

Zad. 11

$A = \begin{bmatrix} \cos\frac{\pi}{8} & -\sin\frac{\pi}{8} \\ \sin\frac{\pi}{8} & \cos\frac{\pi}{8} \end{bmatrix}$

$\det A = \cos^2\frac{\pi}{8} + \sin^2\frac{\pi}{8} = 1 \neq 0$

izomorfizm

(monomorfizm + epimorfizm)

$B = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 0 & 2 \end{bmatrix}$

$f : V \to W$

$\dim V = 3$, $\dim W = 2$

$r(B) = 2 \Rightarrow \dim\operatorname{Im} f = 2 \Rightarrow$ epimorfizm

$\dim\ker\varphi = 3-2 = 1 \neq 0$

nie monomorfizm

$C = \begin{bmatrix} 1 & 4 \\ 2 & 7 \\ 2 & 9 \end{bmatrix}$

$f : V \to W$

$\dim V = 2$, $\dim W = 3$

$r(C) = 2$, $\dim\operatorname{Im} f = 2 \neq 3$

nie epimorfizm

$\dim\ker f = 2-2 = 0$

monomorfizm

$D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$f : V \to W$ $\dim V = 2$, $\dim W = 3$

$\forall v \in V \ \varphi(v) = 0_V$

$\ker f = V$ $\operatorname{Im} f = \{0_V\}$

ani monomorfizm

ani epimorfizm

$E = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $r(E) = r\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} = 2$

$W_2 - 4W_1$, $W_3 - 7W_1$

$f : V \to W$ $\dim V = \dim W = 3$

$r(E) = 2 \Rightarrow \dim\operatorname{Im} f = 2 \neq 3$ nie epimorfizm

$\dim\ker f = 3-2 = 1 \neq 0$ nie monomorfizm

Zad. 12

a) $f : \mathbb{R}_3[x] \to \mathbb{R}_3[x]$ $\mathcal{B} = (1, x, x^2, x^3)$

$f(p)(x) = x \cdot p''(x) - 2x\,p'(x) + p(x)$

$b_1 = 1$ $f(b_1) \overset{(x)}{=} 0 - 0 + 1 = 1$

$b_2 = x$ $f(b_2) \overset{(x)}{=} 0 - 2x \cdot 1 + x = -x$

$b_3 = x^2$ $f(b_3) \overset{(x)}{=} x \cdot 2 - 2x \cdot 2x + x^2 = 2x - 3x^2$

$b_4 = x^3$ $f(b_4) \overset{(x)}{=} x \cdot 6x - 2x \cdot 3x^2 + x^3 = 6x^2 - 5x^3$

$M_f(\mathcal{B},\mathcal{B}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -3 & 6 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

b) $f : \mathbb{R}_3[x] \to \mathbb{R}_3[x]$, $f(p)(x) = x^2 p'''(x) + p'(x+1) + p(-2x)$ $\mathcal{B} = (1, x, x^2, x^3)$

$b_1 = 1$ $f(b_1)(x) = 0 + 0 + 1 = 1$

$b_2 = x$ $f(b_2)(x) = 0 + 1 + (-2x) = 1 - 2x$

$b_3 = x^2$ $f(b_3)(x) = x^2 \cdot 0 + 2(x+1) + 4x^2 = 2 + 2x + 4x^2$

$b_4 = x^3$ $f(b_4)(x) = x^2 \cdot 6 + 3(x+1)^2 + (-2x)^3 = 6x^2 + 3(x^2 + 2x + 1) - 8x^3 = 3 + 6x + 9x^2 - 8x^3$

$M_f(\mathcal{B},\mathcal{B}) = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & -2 & 2 & 6 \\ 0 & 0 & 4 & 9 \\ 0 & 0 & 0 & -8 \end{bmatrix}$

d) $f : \mathbb{R}_2[x] \to M_2(\mathbb{R})$, $f(c + bx + ax^2) = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ $\mathcal{B} = (1, x, x^2)$

$b_1 = 1$ $f(b_1) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $b_2 = x$ $f(b_2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $b_3 = x^2$ $f(b_3) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$M_\varphi(\mathcal{B},\mathcal{C}) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

c) $\varphi : \mathbb{R}^2 \to \mathbb{R}_2[x]$ $\varphi(a,b) = (a+2b)x^2 + (a-2b)x + a$ $\mathcal{B} = (1, x, x^2)$

$\varphi(1,0) = x^2 + x + 1$

$\varphi(0,1) = 2x^2 - 2x$

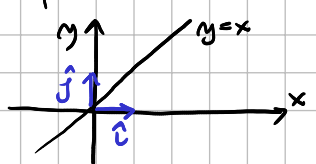
$M_\varphi(\mathcal{B}_k^2, \mathcal{B}) = \begin{bmatrix} 1 & 0 \\ 1 & -2 \\ 1 & 2 \end{bmatrix}$

c) $f : \mathbb{R}_2[x] \to \mathbb{R}^2$, $f(p) = (p(1), p'(1))$ $\mathcal{B} = (1, x, x^2)$

$b_1 = 1$ $f(b_1) = (1, 0)$, $b_2 = x$ $f(b_2) = (1, 1)$, $b_3 = x^2$ $f(b_3) = (1, 2)$

$M_f(\mathcal{B}, \mathcal{B}_k^2) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

f) $f : \mathbb{R}^2 \to \mathbb{R}^2$ odbicie względem $y = x$



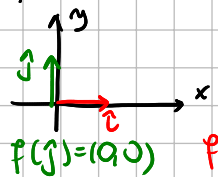
$\varphi(i) = j$ $\varphi(1,0) = (0,1)$

$\varphi(j) = i$ $\varphi(0,1) = (1,0)$

$M_f(\mathcal{B}_k^2, \mathcal{B}_k^2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

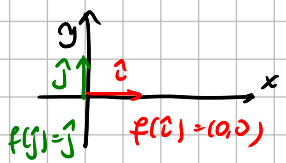
Zad. 12

g) $f: \mathbb{R}^2 \to \mathbb{R}^2$ f - rzut prostokątny na oś Ox



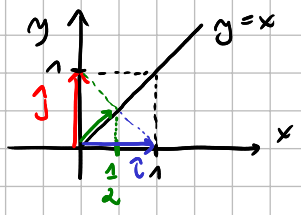
$M_f(B_k^2, B_k^2) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

h) $f: \mathbb{R}^2 \to \mathbb{R}^2$, rzut prostokątny na oś Oy



$M_f(B_k^2, B_k^2) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

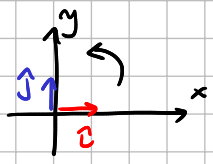
i) $f: \mathbb{R}^2 \to \mathbb{R}^2$, rzut prostokątny na $y=x$



$f(j) = (\frac{1}{2}, \frac{1}{2})$
$f(i) = (\frac{1}{2}, \frac{1}{2})$

$M_f(B_k^2, B_k^2) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

j) $f: \mathbb{R}^2 \to \mathbb{R}^2$, obrót $\circlearrowleft \frac{\pi}{2}$



$f(i) = j$ $f(1,0) = (0,1)$ $M_f(B_k^2, B_k^2) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$f(j) = -i$ $f(0,1) = (-1,0)$ $f(x,y) = (-y, x)$

Zad. 13 a) $A \in M_{6\times 10}(\mathbb{R})$

$A = M_\varphi$ $\dim \ker \varphi = 4 \Rightarrow \dim \operatorname{Im} \varphi = r(A) = 10 - 4 = 6$

$\varphi: \mathbb{R}^{10} \to \mathbb{R}^6$

PRAWDA

$r(A) = 6$ $AX = B$ $U = [A|B] \in M_{6\times 11}(\mathbb{R})$

$r(U) = r(A) = 6$ $\exists$ rozw.

b) $\exists\, \varphi: \mathbb{R}^3 \to \mathbb{R}^3$ liniowe t.że $\forall (x,y,z) \neq (0,0,0)$ $\varphi(x,y,z) \neq (x,y,z)$ $\wedge$ $M_\varphi(B,B) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 4 & 3 & 1 \end{bmatrix}$

$B = (b_1, b_2, b_3)$ $\varphi(b_3) = 0 \cdot b_1 + 0 \cdot b_2 + 1 \cdot b_3 = b_3$

$b_3 \neq (0,0,0)$

FAŁSZ

c) $f: \mathbb{R}[x] \to \mathbb{R}[x]$
$f(p)(x) = p'(x)$ NIE JEST MONOMORFIZMEM, bo $f(1) = f(100) = 0$ FAŁSZ
WIĘC nie jest izomorfizmem

d) $\varphi: \mathbb{R}^8 \to \mathbb{R}^5$ liniowe $\dim \ker \varphi = 8 - \dim \operatorname{Im} \varphi \geq 8 - 5 = 3$ FAŁSZ

$\operatorname{Im} \varphi \subseteq \mathbb{R}^5 \Rightarrow \dim \operatorname{Im} \varphi \leq 5$

e) $g: \mathbb{R}_2[x] \to \mathbb{R}^3$
$g(p)(x) = [p(1), p'(1), p''(1)]$ czy izomorfizm

$B = (1, x, x^2)$

$b_1 = 1$ $g(b_1) = (1,0,0)$
$b_2 = x$ $g(b_2) = (1,1,0)$
$b_3 = x^2$ $g(b_3) = (1,2,2)$

$M_g(B, B_k^2) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

$\det M_g(B, B_k^2) = 2 \neq 0 \Rightarrow g$ - izomorfizm

Zad. 14 $u = (1,0,11)$, $v = (-1,5,7)$, $w = (0,0,3) \in \mathbb{R}^3$ $\varphi: \mathbb{R}^3 \to \mathbb{R}^3$, $\varphi(u) = \varphi(v) = (0,0,0)$, $\varphi(w) = 2w$

u, v - liniowo niezależne, $u, v \in \ker \varphi \Rightarrow \dim \ker \varphi \geq 2$

$\dim \ker \varphi = 3 \iff \dim \operatorname{Im} \varphi = 0$ zał $w \neq (0,0,0) \wedge w \in \operatorname{Im}(\varphi)$ $\not\downarrow$: $\dim \operatorname{Im} \varphi \geq 1$

$\dim \ker \varphi + \dim \operatorname{Im} \varphi = 3 \Rightarrow \dim \ker \varphi = 2$, $\dim \operatorname{Im} \varphi = 1$

Zad. 15 $A = M_f(B_k^2, B_k^2)$ $f: \mathbb{R}^2 \to \mathbb{R}^2$ t.że $\operatorname{Im} f$ - prosta $y = 3x$, $\ker f$ - prosta $y = -2x$

$f(x,y) = (ax + by, cx + dz)$

$A = M_f(B_k^2, B_k^2) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\operatorname{Im} f = \{(x, 3x); x \in \mathbb{R}\} = \operatorname{lin}\{(1,3)\}$

tzn. $f(i), f(j) \in \operatorname{lin}\{(1,3)\} \Rightarrow c = 3a$, $d = 3b$, $A = \begin{bmatrix} a & b \\ 3a & 3b \end{bmatrix}$

$\ker f = \{(x, -2x), x \in \mathbb{R}\} = \operatorname{lin}\{(1,-2)\}$ tzn. $A \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} a & b \\ 3a & 3b \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow a + 2b = 0 \Rightarrow a = -2b$

$A = \begin{bmatrix} -2b & b \\ -6b & 3b \end{bmatrix}$ $Np.\ A = \begin{bmatrix} -2 & 1 \\ -6 & 3 \end{bmatrix}$