Algebra - Zadanie domowe nr 6

Zad. 1 Diagonalizacja. Baza.

$\varphi : \mathbb{R}^3 \to \mathbb{R}^3$

$\varphi(x,y,z) = (x+2y-2z,\ x+3z,\ x+3y)$

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 2 & -2 \\ 1 & -\lambda & 3 \\ 1 & 3 & -\lambda \end{vmatrix} = \lambda^2(1-\lambda) - 6 + 6 - 2\lambda - 9(1-\lambda) + 2\lambda = (1-\lambda)(\lambda^2-9)$$

$\operatorname{Spec}(\varphi) = \{-3, 1, 3\}$ trzy proste $\Rightarrow \varphi$ - diagonalizowalny

- $\lambda_1 = 1$

$(A-\lambda_1)X = 0$

$(A-I)X = 0 \iff \begin{bmatrix} 0 & 2 & -2 \\ 1 & -1 & 3 \\ 1 & 3 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 2 & -2 \\ 1 & -1 & 3 \\ 1 & 3 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 3 \\ 1 & 3 & -1 \\ 0 & 2 & -2 \end{bmatrix} \xrightarrow{W_2-W_1} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 4 & -4 \\ 0 & 2 & -2 \end{bmatrix} \xrightarrow[W_2/:4]{W_3-W_2} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{W_1+W_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\begin{cases} a+2c=0 \\ b-c=0 \end{cases}$ $\quad c=t$, $\ b=c=t$, $\ a=-2c=-2t$

$E_1 = \{(-2t, t, t) : t \in \mathbb{R}\} = \lim_{\mathbb{R}}\{(-2,1,1)\}$

- $\lambda_2 = 3$ $\quad (A-3I)X = 0$

$$\begin{bmatrix} -2 & 2 & -2 \\ 1 & -3 & 3 \\ 1 & 3 & -3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 & -2 \\ 1 & -3 & 3 \\ 1 & 3 & -3 \end{bmatrix} \xrightarrow{W_1:(-2)} \begin{bmatrix} 1 & -1 & 1 \\ 1 & -3 & 3 \\ 1 & 3 & -3 \end{bmatrix} \xrightarrow[W_3-W_1]{W_2-W_1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 2 \\ 0 & 4 & -4 \end{bmatrix} \xrightarrow{W_2:(-2)} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{W_1+W_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\begin{cases} a=0 \\ b-c=0 \end{cases}$ $\quad b=c$

$E_3 = \{(0, b, t), t \in \mathbb{R}\} = \lim_{\mathbb{R}}\{(0,1,1)\}$

- $\lambda_3 = -3$ $\quad (A+3I)X = 0$

$$\begin{bmatrix} 4 & 2 & -2 \\ 1 & 3 & 3 \\ 1 & 3 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 & -2 \\ 1 & 3 & 3 \\ 1 & 3 & 3 \end{bmatrix} \xrightarrow[W_1/2]{W_3-W_2} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 3 \\ 2 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{W_2-W_1\cdot 2} \begin{bmatrix} 1 & 3 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & 0 \end{bmatrix}$$

$\begin{cases} a+3b+3c=0 \\ -5b-7c=0 \end{cases}$ $\quad b = -\frac{7}{5}c$, $\ c=t$

$a = -3b-3c = \frac{21}{5}c - 3c = \frac{6}{5}c$

$E_{-3} = \{(\frac{6}{5}t, -\frac{7}{5}t, t), t \in \mathbb{R}\} = \lim_{\mathbb{R}}\{\frac{1}{5}(6, -7, 5)\}$

baza

$\mathcal{B} = \{(-2,1,1), (0,1,1), \frac{1}{5}(6,-7,5)\}$

$$M_{\mathcal{B}\mathcal{B}}(\varphi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Zadanie domowe – Diagonalizacja

Zad. 1 b) $\varphi \in End(\mathbb{C}^3)$, $\varphi(x_1,x_2,x_3) = (ix_1 - x_1 + 2x_2 + x_3,\ 2ix_2 + 3x_3,\ ix_3 - x_3)$

$A = M_\varphi(B_k, B_k)$ $\quad B_k = ((1,0,0),(0,1,0),(0,0,1))$

$$A = \begin{bmatrix} i-1 & 2 & 1 \\ 0 & 2i & 3 \\ 0 & 0 & i-1 \end{bmatrix}$$

$\chi_A(t) = \det(A - tI) = (2i-t)(i-1-t)^2$ $\quad Spec(A) = \{\lambda_1 = -1+i,\ \lambda_2 = 2i\}$

$k_1 = 2$, $k_2 = 1$

$\lambda_1 = -1+i$ $\quad E_{\lambda_1} = ?$

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = (A - \lambda_1 I)\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1+i & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \Rightarrow \begin{matrix} 2x_2 + x_3 = 0 \\ (1+i)x_2 + 3x_3 = 0 \end{matrix} \Rightarrow x_3 = -2x_2$$

$(1+i)x_2 - 6x_2 = 0$

$x_2 = x_3 = 0$

$E_{-1+i} = \{(x_1,0,0);\ x_1 \in \mathbb{C}\} = lin_{\mathbb{C}}\{(1,0,0)\}$ $\quad \dim E_{-1+i} = 1 \neq k_1 = 2$

Macierz nie jest diagonalizowalna

$\lambda_2 = 2i$ $\quad E_{2i} = ?$

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = (A - 2iI)\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix} -1-i & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & -1-i \end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \Rightarrow \begin{matrix} x_3 = 0 \\ 2x_2 - (1+i)x_1 = 0 \end{matrix}$$

$E_{2i} = \{(x_1, -\frac{1}{2}(1+i)x_1, 0);\ x_1 \in \mathbb{C}\} = lin_{\mathbb{C}}\{(1, -\frac{1}{2} - \frac{1}{2}i, 0)\}$ $\quad \dim E_{2i} = 1$

$= lin_{\mathbb{C}}\{(2, -1-i, 0)\}$

c) Zadanie rozwiązane na wykładzie – rozdział 8.2

Zad. 4c) $A = \begin{bmatrix} p & 2 & -1 \\ 1 & p & -1 \\ 0 & 0 & 2 \end{bmatrix} \in M_3(\mathbb{R})$ Dla jakich $p \in \mathbb{R}$ A jest diagonalizowalna?

$$\det(A - \lambda I) = \begin{vmatrix} p-\lambda & 2 & -1 \\ 1 & p-\lambda & -1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)\cdot(-1)^6 \begin{vmatrix} p-\lambda & 2 \\ 1 & p-\lambda \end{vmatrix} = (2-\lambda)\left[(p-\lambda)^2 - 2\right] =$$

$$= (2-\lambda)(p-\lambda-\sqrt{2})(p-\lambda+\sqrt{2})$$

$Spec(A) = \{2,\ p-\sqrt{2},\ p+\sqrt{2}\}$

① $p+\sqrt{2} \neq p-\sqrt{2}$ $\quad \forall p \in \mathbb{R}$ $\quad p - \sqrt{2} \neq 2$ $\quad p + \sqrt{2} \neq 2$ $\Rightarrow p \in \mathbb{R} \setminus \{2-\sqrt{2}, 2+\sqrt{2}\}$

$p \neq 2+\sqrt{2}$ $\quad p \neq 2-\sqrt{2}$

Widmo proste

A-diagonalizowalna

② $p = 2-\sqrt{2}$ $\quad A = \begin{bmatrix} 2-\sqrt{2} & 2 & -1 \\ 1 & 2-\sqrt{2} & -1 \\ 0 & 0 & 2 \end{bmatrix}$ $\quad \lambda_1 = 2$ $\quad k_1 = 2$

$\lambda_2 = 2-2\sqrt{2}$ $\quad k_2 = 1$

$$\dim E_{\lambda_1} = 3 - r(A - 2I) = 3 - r\begin{bmatrix} -\sqrt{2} & 2 & -1 \\ 1 & -\sqrt{2} & -1 \\ 0 & 0 & 0 \end{bmatrix} = 3 - 2 \neq 1 \neq k_1$$ nie diagonalizowalna

③ $p = 2+\sqrt{2}$ $\quad A = \begin{bmatrix} 2+\sqrt{2} & 2 & -1 \\ 1 & 2+\sqrt{2} & -1 \\ 0 & 0 & 2 \end{bmatrix}$ $\quad \lambda_1 = 2$ $\quad k_1 = 2$

$\lambda_2 = 2+2\sqrt{2}$ $\quad k_2 = 1$

$$\dim E_{\lambda_1} = 3 - r(A - 2I) = 3 - r\begin{bmatrix} \sqrt{2} & 2 & -1 \\ 1 & \sqrt{2} & -1 \\ 0 & 0 & 0 \end{bmatrix} = 3 - 2 = 1 \neq k_1$$ nie diagonalizowalna

Algebra – Zadanie domowe nr 5

Diagonalizacja endomorfizmu

Zad. 2 $\varphi: \mathbb{R}_2[x] \to \mathbb{R}_2[x]$ $\quad \forall p \in \mathbb{R}_2[x] \quad \varphi(p)(x) = 3xp''(x) + (x^2-1)p(1) + x\cdot p(2)$

$\mathcal{B}_k = (1, x, x)$ baza $\mathbb{R}_2[x]$

$\varphi(1) = 3x\cdot 0 + (x^2-1)\cdot 1 + x\cdot 1 = x^2+x-1 = [-1,1,1]_{\mathcal{B}_k}$

$\varphi(x) = 3x\cdot 0 + (x^2-1)\cdot 1 + x\cdot 2 = x^2+2x-1 = [-1,2,1]_{\mathcal{B}_k}$

$\varphi(x^2) = 3x\cdot 2 + (x^2-1)\cdot 1 + x\cdot 4 = x^2+10x-1 = [-1,10,1]_{\mathcal{B}_k}$

$$A = M_\varphi(\mathcal{B}_k, \mathcal{B}_k) = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 2 & 10 \\ 1 & 1 & 1 \end{bmatrix}$$

$$0 = \det(A - tI) = \begin{vmatrix} -1-t & -1 & -1 \\ 1 & 2-t & 10 \\ 1 & 1 & 1-t \end{vmatrix} \overset{\substack{W_1+W_2 \\ W_3-W_2}}{=} \begin{vmatrix} -t & 1-t & 9 \\ 1 & 2-t & 10 \\ 0 & -1+t & -9-t \end{vmatrix} \overset{W_1+W_3}{=} \begin{vmatrix} -t & 0 & -t \\ 1 & 2-t & 10 \\ 0 & t-1 & -9-t \end{vmatrix} =$$

$$= -t(2-t)(-9-t) - t(t-1) - 10(t-1)(-t) = t\left[(2-t)(9+t) - t + 1 + 10t - 10\right] = t\left[18 + 2t - 9t - t^2 + 9t - 9\right]$$

$$= t\left[-t^2 + 2t + 9\right]$$

$\Delta = 4 + 36 = 40$, $\sqrt{\Delta} = 2\sqrt{10}$

$Spec(\varphi) = \{0,\ 1-\sqrt{10},\ 1+\sqrt{10}\}$ trzy proste ⇓ φ – diagonalizowalny

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Zad. 3

a) Niech $\varphi \in End(\mathbb{R}^3)$ t.że $\varphi(1,0,0) = (1,0,0)$, $\varphi(1,1,0) = (-1,-1,0)$, $\varphi(1,1,1) = (0,0,0)$

Oblicz $\varphi^{100}(3,6,9)$.

$Spec(\varphi) = \{1, -1, 0\}$

Baza wektorów własnych $C = \{c_1 = (1,0,0),\ c_2 = (1,1,0),\ c_3 = (1,1,1)\}$

$\mathcal{B}$ – baza kanoniczna

$$P = P_{\mathcal{B}\to C} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$ macierz przejścia

$$D = M_{CC}(\varphi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

I sposób:

$A = M_{\mathcal{B}\mathcal{B}}(\varphi)$ $\quad A = P\cdot D\cdot P^{-1}$, $A^{100} = P\cdot D^{100}\cdot P^{-1}$

$P^{-1} = ?$ $\quad [P|I] = \left[\begin{array}{ccc|ccc} 1&1&1&1&0&0 \\ 0&1&1&0&1&0 \\ 0&0&1&0&0&1 \end{array}\right] \overset{W_1-W_3}{\underset{W_2-W_3}{\longrightarrow}} \left[\begin{array}{ccc|ccc} 1&1&0&1&0&-1 \\ 0&1&0&0&1&-1 \\ 0&0&1&0&0&1 \end{array}\right] \overset{W_1-W_2}{\longrightarrow} \left[I \,\middle|\, \underbrace{\begin{array}{ccc} 1&-1&0 \\ 0&1&-1 \\ 0&0&1 \end{array}}_{P^{-1}}\right]$

$\varphi^{100}(3,6,9) = ?$

$$A^{100}\cdot\begin{bmatrix}3\\6\\9\end{bmatrix} = P\cdot D^{100}\cdot P^{-1}\begin{bmatrix}3\\6\\9\end{bmatrix} = \begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&-1&0\\0&0&0\end{bmatrix}^{100}\begin{bmatrix}1&-1&0\\0&1&-1\\0&0&1\end{bmatrix}\begin{bmatrix}3\\6\\9\end{bmatrix}$$

$$\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\cdot\begin{bmatrix}-3\\-3\\9\end{bmatrix} = \begin{bmatrix}1&1&0\\0&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}-3\\-3\\9\end{bmatrix} = \begin{bmatrix}-6\\-3\\0\end{bmatrix}$$

$\varphi^{100}(3,6,9) = (-6,-3,0)$

II sposób

$(3,6,9) = [-3,-3,9]_C$

$$\left[\begin{array}{ccc|c}1&1&1&3\\0&1&1&6\\0&0&1&9\end{array}\right] \to \left[\begin{array}{ccc|c}1&1&0&-6\\0&1&0&-3\\0&0&1&9\end{array}\right] \to \left[I\,\middle|\,\begin{array}{c}-3\\-3\\9\end{array}\right]$$

$$D^{100}\cdot\begin{bmatrix}-3\\-3\\9\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}-3\\-3\\9\end{bmatrix} = \begin{bmatrix}-3\\-3\\0\end{bmatrix}$$

$\varphi^{100}(3,6,9) = [-3,-3,0]_C = -3(1,0,0) - 3(1,1,0) =$

$= (-6,-3,0)$ ✓

ALGEBRA - Zadanie domowe nr 6

Zad. 3

b) $f: M_2(\mathbb{R}) \to M_2(\mathbb{R})$ $\quad f\left(\begin{bmatrix}1&1\\0&0\end{bmatrix}\right)=\begin{bmatrix}-1&-1\\0&0\end{bmatrix}$, $f\left(\begin{bmatrix}0&0\\1&1\end{bmatrix}\right)=\begin{bmatrix}0&0\\-1&-1\end{bmatrix}$

$\begin{bmatrix}1&0\\1&0\end{bmatrix} \in \ker f$ $\quad \begin{bmatrix}0&0\\0&1\end{bmatrix} \in \ker f$ $\quad f^{101}\left(\begin{bmatrix}1&1\\1&0\end{bmatrix}\right)=?$

$\lambda_1 = -1$ $\quad k_1 = 2$ $\quad B_1=\begin{bmatrix}1&1\\0&0\end{bmatrix}$ $\quad B_2=\begin{bmatrix}0&0\\1&1\end{bmatrix}$ w. własne odpowiadające $\lambda_1=-1$ są liniowo niezależne $\Rightarrow \dim E_{\lambda_1}=2$

$\lambda_2 = 0$ $\quad k_2 = 2$ $\quad B_3=\begin{bmatrix}1&0\\1&0\end{bmatrix}$ $\quad B_4=\begin{bmatrix}0&0\\0&-1\end{bmatrix}$ w. własne odp. $\lambda_2=0$ liniowo niezależne $\Rightarrow \dim E_{\lambda_2}=2$

Zatem $k_1=\dim E_{\lambda_1}$, $k_2=\dim E_{\lambda_2}$ $\quad k_1+k_2=\dim M_2(\mathbb{R})$

Czyli f jest diagonalizowalny

$A = PDP^{-1}$ $\quad D = M_f(\mathcal{B},\mathcal{B})$ $\quad \mathcal{B}=(B_1,B_2,B_3,B_4)$

$D=\begin{bmatrix}-1&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$ $\quad D^{101}=D$ $\quad F=\begin{bmatrix}1&1\\1&0\end{bmatrix}=[1,1,1,0]_{B_k}$

$B_k=(E_{11},E_{12},E_{21},E_{22})$

$D^{101}\begin{bmatrix}1\\1\\0\\1\end{bmatrix}=\begin{bmatrix}-1\\-1\\0\\1\end{bmatrix}$ $\quad F=B_1+B_2+B_4=[1,1,0,1]_{\mathcal{B}}$

$A^{101}(F)=-B_1-B_2=\begin{bmatrix}-1&-1\\-1&-1\end{bmatrix}$

c) $A=\begin{bmatrix}1&-1\\5&-3\end{bmatrix}\begin{bmatrix}-1&0\\0&2\end{bmatrix}\begin{bmatrix}1&-1\\5&-3\end{bmatrix}^{-1} \in M_2(\mathbb{R})$

P $\quad D$ (diagonalna) $\quad P^{-1}$

$\lambda_1=-1$ $\quad v_1=(1,5)$

$\lambda_2=2$ $\quad v_2=(-1,-3)$

$A^7\begin{bmatrix}-1\\-3\end{bmatrix}=2^7\cdot\begin{bmatrix}-1\\-3\end{bmatrix}$

$\varphi(v)=\lambda v$

$\varphi^2(v)=\varphi(\varphi(v))=\varphi(\lambda v)=\lambda\varphi(v)=\lambda(\lambda v)=\lambda^2 v$

$A^7\begin{bmatrix}0\\2\end{bmatrix}=A^7\left(\begin{bmatrix}1\\5\end{bmatrix}+\begin{bmatrix}-1\\-3\end{bmatrix}\right)$ $\quad$ itd...

$=A^7\begin{bmatrix}1\\5\end{bmatrix}+A^7\begin{bmatrix}-1\\-3\end{bmatrix}=(-1)^7\begin{bmatrix}1\\5\end{bmatrix}+2^7\begin{bmatrix}-1\\-3\end{bmatrix}=\begin{bmatrix}-1\\-5\end{bmatrix}+\begin{bmatrix}-2^7\\-3\cdot 2^7\end{bmatrix}$

ALGEBRA – Zadania domowe nr 6 – Diagonalizacja

Zad. 4

a) $A = \begin{bmatrix} 1 & -5 \\ 1 & 1 \end{bmatrix} \in M_2(\mathbb{R})$

$\det(A - tI) = \begin{vmatrix} 1-t & -5 \\ 1 & 1-t \end{vmatrix} = (1-t)^2 + 5 > 0$

brak pierwiastków rzeczywistych

$\mathrm{Spec}(A) = \emptyset$ A nie jest diagonalizowalna

Gdy $A = \begin{bmatrix} 1 & -5 \\ 1 & 1 \end{bmatrix} \in M_2(\mathbb{C})$

to $(1-t)^2 = -5 = \sqrt{5}^2 i^2$

$1 - t = \pm\sqrt{5}i$ $\quad t = 1 \pm \sqrt{5}i$

Widmo proste

istnieje baza $\mathbb{C}^2$ złożona z wektorów własnych

b) $A = \begin{bmatrix} 7 & 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

A - trójkątna górna

$\mathrm{Spec}(A) = \{\lambda_1 = 7, \lambda_2 = 1, \lambda_3 = 2\}$

$k_1 = 1 \quad k_2 = 3 \quad k_3 = 1$

$\dim E_{\lambda_1} = k_1 = 1$

$\dim E_{\lambda_3} = k_3 = 1$

Zał. $\dim E_{\lambda_2} = 5 - r(A - 1 \cdot I) = 5 - 4 = 1 \neq k_2 = 3$

NIE jest diagonalizowalna

$A - I = \begin{bmatrix} 6 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad r(A-1) = 4$

c) $A = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 \end{bmatrix}$

$\lambda_1 = 0 \in \mathrm{Spec}(A)$ $\quad k_1 = 3$

$\dim E_{\lambda_1} = ?$

$\dim E_{\lambda_1} = 4 - r(A - 0 \cdot 1) = 4 - r(A) = 4 - 1 = 3 = k_1$

TAK, diagonalizowalna

$r(A) = 1$ widać!

d) $A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & p & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

dla jakich p macierz jest diagonalizowalna?

$\mathrm{Spec}(A) = \{\lambda_1 = 5, \lambda_2 = 3, \lambda_3 = 1\}$

$k_1 = 2 \quad k_2 = 1 \quad k_3 = 1$

Na mocy tw. spektralnego, A diagonalizowalna $\Leftrightarrow \dim E_{\lambda_1} = 2$

$\dim E_{\lambda_1} = 4 - r(A - 5I) = 2 \Leftrightarrow r(A - 5I) = 4 - 2 = 2$

$$r(A-5I) = r\begin{bmatrix} 0 & -2 & 6 & -1 \\ 0 & -2 & p & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -4 \end{bmatrix} \overset{:4}{=} r\begin{bmatrix} 0 & -2 & 6 & -1 \\ 0 & -2 & p & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = r\begin{bmatrix} -2 & 6 & -1 \\ -2 & p & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{W_2 - W_1}{=} r\begin{bmatrix} -2 & 6 & -1 \\ 0 & p-6 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$r(A-5I) = \begin{cases} 3 & \text{dla } p \neq 6 \\ 2 & \text{dla } p = 6 \end{cases}$$

A - diagonalizowalna $\Leftrightarrow p = 6$

Zad. 5 PRAWDA / FAŁSZ

$A = \begin{bmatrix} 7 & -3 & -1 & 0 \\ 0 & 4 & p^3 & 0 \\ 0 & 0 & 7 & \pi \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad A = M_\varphi(\mathcal{B}_{\mathbb{R}^4}, \mathcal{B}_{\mathbb{R}^4}^4)$

- $\mathrm{Spec}(A) = \{\lambda_1 = 7, \lambda_2 = 4, \lambda_3 = 1\}$ $\quad k_1 = 2 \quad k_2 = 1 \quad k_3 = 1$ $\Rightarrow$ A diagonalizowalna $\Leftrightarrow \dim E_{\lambda_1} = 4 - r(A - 7I) = 2$ $\Leftrightarrow r(A - 7I) = 2$
- $\det A = 7 \cdot 4 \cdot 7 \cdot 1 = 196 \neq 0 \Rightarrow r(A) = r(\varphi) = 4 \Rightarrow \varphi$ surjekcja $\Rightarrow \dim \ker \varphi = 4 - 4 = 0 \Rightarrow \varphi$ injekcja
- $\det\left(\frac{1}{7}A^T\right) = \frac{1}{7^2} \cdot 196$

$A - 7I = \begin{bmatrix} 0 & -3 & -1 & 0 \\ 0 & -3 & p^3 & 0 \\ 0 & 0 & 0 & \pi \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad r(A - 7I) = r\begin{bmatrix} 0 & -3 & -1 & 0 \\ 0 & 0 & p^3 + 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{cases} 2 & p = -1 \\ 3 & p \neq -1 \end{cases}$ $\longrightarrow$

a) ~~FAŁSZ~~ PRAWDA

b) FAŁSZ

c) PRAWDA

Zad. 11

a) $A \in M_3(\mathbb{R})$ $\quad \det(A - \lambda 1) = -\lambda^3 + \lambda + 7$ $\Rightarrow \det A \overset{\lambda = 0}{=} 7$ $\Rightarrow AX = 0$ Oznaczony PRAWDA

b) $P^{-1}AP = D \Rightarrow P^{-1}(A^6 + A^2 + 1)P = P^{-1}A^6P + P^{-1}A^2P + P^{-1}P = D^6 + D^2 + I$ ← diagonalna (suma diagonalnych) PRAWDA

c) $u, v \in \mathrm{Spec}(\varphi)$ $\quad u + v$ będzie wektorem własnym tylko jeśli u, v odpowiadają tej samej λ FAŁSZ

d) $X = v^T$ $\quad AX = \lambda X \Rightarrow (123A)X = 123(AX) = 123(\lambda X) = (123\lambda) \cdot X$ PRAWDA

e) $A \in M_{2\times 3}(\mathbb{R}) \Rightarrow A^TA \in M_3(\mathbb{R})$, A^TA jest symetryczna $\left((A^TA)^T = A^T(A^T)^T = A^TA\right) \Rightarrow$ rzeczywista i symetryczna jest ortogonalnie diag. PRAWDA

ALGEBRA — Zadanie domowe nr 6

Zad. 6 $\varphi \in End(\mathbb{R}_2[x])$, $\varphi(1)=4$, $\varphi(1+3x)=-2-6x$, $\varphi(x-x^2)=0$

a) $\varphi(1)=4$ $\quad \varphi(1+3x)=\varphi(1)+3\varphi(x)=-2-6x$

$\varphi(x)=\frac{1}{3}(-2-6x-\varphi(1))$

$=\frac{1}{3}(-2-6x-4)=$

$=-2x-2$

$\varphi(x-x^2)=\varphi(x)-\varphi(x^2)=0$

$\varphi(x)=\varphi(x^2)$

$\mathcal{B}=(1, x, x^2)$ $\quad M_\varphi(\mathcal{B},\mathcal{B})=\begin{bmatrix}4&-2&-2\\0&-2&-2\\0&0&0\end{bmatrix}$

b) $Spec(\varphi)=\{4,-2,0\}$ widmo proste $\Rightarrow$ φ-diagonalizowalny

c) $\mathcal{B}'=(v_1, v_2, v_3)$ gdzie

$\varphi(1)=4\cdot 1$ $\quad \lambda_1=4$ $\quad v_1=1$, $\varphi(1+3x)=-2(1+3x)$ $\quad \lambda_2=-2$ $\quad v_2=1+3x$, $\varphi(x-x^2)=0\cdot(x-x^2)=0$ $\quad \lambda_3=0$ $\quad v_3=x-x^2$

$P=\begin{bmatrix}1&1&0\\0&3&1\\0&0&-1\end{bmatrix}$ $\quad D=\begin{bmatrix}4&0&0\\0&-2&0\\0&0&0\end{bmatrix}$

Zad. 7

a) $A\in M_2(\mathbb{R})$ $\quad tr(A)=13$, $\det A=29$ $\quad A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$

$\det(A-\lambda I)=\begin{vmatrix}a-\lambda & b\\ c & d-\lambda\end{vmatrix}=(a-\lambda)(d-\lambda)-bc=\lambda^2-\underbrace{(a+d)}_{tr A}\lambda+\underbrace{(ad-bc)}_{\det A}=\lambda^2-13\lambda+29$

b) $A\in M_3(\mathbb{R})$, $\lambda_1=4$ $k_1=1$, $\lambda_2=3$ $k_2=2$ $\quad A=PDP^{-1}$

$\det A=\det D=\det\begin{bmatrix}4&0&0\\0&3&0\\0&0&3\end{bmatrix}=36$

$tr A=tr D=4+3+3=10$

c) $A\in M_3(\mathbb{R})$ $\quad tr(A)=2$

$\lambda_1=7$ $\quad v=(1,2,1)$, $w=(1,1,0)$

v, w - liniowo nieza leżne $\Rightarrow$ $k_1\geq 2$ $\quad \dim E_{\lambda_1}\geq 2$ $\quad \det A=\det D=7\cdot 7\cdot(-12)$

$\lambda_2+7+7=tr D=tr A=2 \Rightarrow \lambda_2=2-14=-12$

d) $B\in M_3(\mathbb{C})$ $\quad B=[b_{ij}]$ $b_{ij}\in\mathbb{R}$ diagonalizowalna, $tr B=2$, $v_1=(2,i,1)$ odpow. $\lambda_1=1+3i$

↳ wielomian charakt. ma współczynniki rzeczywiste

zatem $\lambda_2=\overline{\lambda_1}=1-3i$ to też wartość własna

odpowiada jej $v_2=(2,\overline{i},1)=(2,-i,1)$

v_1 i v_2 są liniowo niezależne bo odpow. różnym λ_1,λ_2

A diagonalizowalna

$\det A=\det D=?$

$tr A=tr D \iff 2=\lambda_1+\lambda_2+\lambda_3=1+3i+1-3i+\lambda_3=2+\lambda_3 \Rightarrow \lambda_3=0$

$\det A=\lambda_1\cdot\lambda_2\cdot\lambda_3=(1+3i)(1-3i)\cdot 0=0$

Zad. 8

φ_1 ↰ $\frac{\pi}{11}$ rotacja

φ_2 ↙ π rotacja

φ_3 symetria wzgl. $y=x$

φ_4 rzut prostokątny na Oy

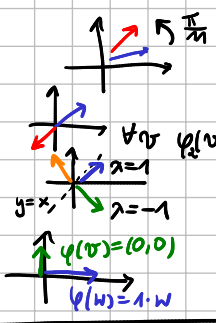
WIDMO

$W=\emptyset$

$\mathcal{U}=\{-1\}$

$S=\{1,-1\}$

$R=\{0,1\}$



$\varphi(v)=\lambda v$

$v, \lambda v$ ten sam kierunek

Zad. 9 $\varphi\in End(\mathbb{R}^7)$, $Spec(\varphi)=\{\lambda_1=0,\lambda_2=4\}$

a) $0\in Spec(\varphi) \Rightarrow$ φ nie injekcja PRAWDA

b) $\dim E_{\lambda_2}=6 \Rightarrow \dim E_{\lambda_2}=k_2 \wedge \dim E_{\lambda_1}=1=k_1$ PRAWDA

c) $\det M_\varphi=0$ bo $0\in Spec(\varphi)$ PRAWDA

! d) $0\in Spec(\varphi) \Rightarrow \dim\ker\varphi\geq 1 \Rightarrow \dim Im\varphi=7-\dim\ker\varphi\leq 6\neq\dim\mathbb{R}^7$ FAŁSZ

Zad. 10 $A\in M_3(\mathbb{R})$ osobliwa

I $AX=\begin{bmatrix}0\\0\end{bmatrix}$ jednorodny $\Rightarrow$ zb. rozw. to prz. liniowa PRAWDA

$\det A=0$ $\Rightarrow$ nie jest to prz. trywialna $\{(0,0)\}$

II. $0\in Spec(A)$ PRAWDA

III. $\exists B\in M_{2\times 1}(\mathbb{R})$ t.że $AX=B$ sprzeczny PRAWDA

$\det A=0 \Rightarrow r(A)\leq 1$ nowy schodek

Można tak dopisać B by $r(B)=1+r(A)$

ODP. B

Zadanie domowe – Diagonalizacja

Zad. 12 $a_1 = a_2 = 2$, $a_3 = 4$, $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$, $k = 3$

Wzór ogólny?

$$\begin{bmatrix} a_n \\ a_{n-1} \\ a_{n-2} \end{bmatrix} = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n-1} \\ a_{n-2} \\ a_{n-3} \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A}^{m-3} \cdot \begin{bmatrix} a_3 \\ a_2 \\ a_1 \end{bmatrix} \quad (n-k)$$

$$W_A(t) = \det(A - tI) = \begin{vmatrix} 2-t & 1 & -2 \\ 1 & -t & 0 \\ 0 & 1 & -t \end{vmatrix} = t^2(2-t) - 2 + t = -t^2(t-2) + (t-2) = (t-2)(1-t^2) = (t-2)(1-t)(1+t)$$

$\operatorname{Spec}(A) = \{-1, 1, 2\}$ wielomian prosty $\Rightarrow$ A diagonalizowalna

$\lambda_1 = -1$: $(A+I)\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 & 1 & -2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} b+c=0 \\ a+b=0 \end{matrix}$ $E_{-1} = \{(-b, b, -b) : b \in \mathbb{R}\}$ $v_1 = (1, -1, 1)$

$\lambda_2 = 1$: $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = (A-I)\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow a = b = c$ $E_1 = \{(a, a, a) : a \in \mathbb{R}\}$ $v_2 = (1, 1, 1)$

$\lambda_3 = 2$: $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = (A-2I)\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 0 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow \begin{matrix} a-2b=0 \\ b-2c=0 \end{matrix}$ $E_2 = \{(4c, 2c, c) : c \in \mathbb{R}\}$ $v_3 = (4, 2, 1)$

$$\begin{bmatrix} a_n \\ a_{n-1} \\ a_{n-2} \end{bmatrix} = P \cdot D^{n-3} \cdot P^{-1} \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 1 & 4 \\ -1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$P^{-1} = ?$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 4 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 4 & 1 & 0 & 0 \\ 0 & 2 & 6 & 1 & 1 & 0 \\ 0 & 0 & -3 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -\frac{1}{3} & 0 & \frac{4}{3} \\ 0 & 2 & 0 & -1 & 1 & 2 \\ 0 & 0 & -3 & -1 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & 0 & \frac{4}{3} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 0 & 1 & \frac{1}{3} & 0 & -\frac{1}{3} \end{array}\right] \to \left[I_3 \,\middle|\, \underbrace{\begin{matrix} \frac{1}{6} & -\frac{1}{2} & \frac{4}{3} \\ -\frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{3} & 0 & -\frac{1}{3} \end{matrix}}_{P^{-1}} \right]$$

$$P^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & 6 \\ 2 & 0 & -2 \end{bmatrix}$$

$$\begin{bmatrix} a_n \\ a_{n-1} \\ a_{n-2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 4 \\ -1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} (-1)^{n-3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^{n-3} \end{bmatrix} \frac{1}{6} \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & 6 \\ 2 & 0 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} (-1)^{n-3} & 1 & 2^{n-1} \\ (-1)^{n-2} & 1 & 2^{n-2} \\ (-1)^{n-3} & 1 & 2^{n-3} \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$$

$$a_n = \frac{1}{6} \cdot \left[2 \cdot (-1)^{n-3} + 6 + 4 \cdot 2^{n-1}\right] = \frac{1}{3} \cdot (-1)^{n-3} + \frac{1}{3} \cdot 2^n + 1$$

Zad. 13

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 2 \end{bmatrix} \qquad A^5 - A^4 = ? \qquad B = \begin{bmatrix} i & 1 \\ 0 & i \end{bmatrix} \qquad B^{77} = ?$$

$$W_A(t) = \begin{vmatrix} 2-t & 3 & 1 \\ 0 & -t & 0 \\ 0 & -1 & 2-t \end{vmatrix} = -t(2-t)^2 = -t(4 - 4t + t^2) = -t^3 + 4t^2 - 4t \quad \Rightarrow / -A^3 + 4A^2 - 4A = 0$$

$$A^5 - A^4 = 24A^2 - 32A$$

$$(t^5 - t^4) : (t^3 - 4t^2 + 4t) = t^2 + 3t + 8$$
$$t^5 - 4t^4 + 4t^3$$
$$3t^4 - 4t^3$$
$$3t^4 - 12t^3 + 12t^2$$
$$8t^3 - 12t^2$$
$$8t^3 - 32t^2 + 32t$$
$$20t^2 - 32t$$

$B^2 - 2iB - I = 0 \;/\cdot B^{-1}$ $\det B \neq 0 \Rightarrow \exists B^{-1}$ policzyć...

$B - 2iI - B^{-1} = 0$

$B^{-1} = B - 2iI$

$W_B(t) = (i-t)^2 \Rightarrow (iI - B)^2 = 0 \quad -I - 2iB + B^2 = 0$

$$\begin{cases} t^{77} = q(t) \cdot (i-t)^2 + \underbrace{at+b}_{r(t)} \\ 77\,t^{76} = q'(t) \cdot (i-t)^2 + q(t) \cdot 2(i-t) \cdot (-1) + a \end{cases}$$

$t = i$:

$$\begin{cases} i^{77} = ai + b \\ 77\,i^{76} = a \end{cases}$$

$a = (i^4)^{19} \cdot 77 = 77$

$b = i^{77} - ai = i - 77i = -76i$

$B^{77} = 77B - 76iI = \ldots$ policzyć