Coordination via Polling in Plurality Voting Games under Inertia

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Abstract

We discuss a new model for strategic voting in plurality elections under uncertainty. In particular, we introduce the concept of *inertia* to capture players' uncertainty about poll accuracy. We use a sequence of pre-election polls as a source of partial information. Under some behavioural assumptions, we show how this sequence can help agents to coordinate on an equilibrium outcome. We study the model analytically under some special distributions of inertia, and present some simulation results for more general distributions. Some special cases of our model yield a voting rule closely related to the Instant Runoff voting rule and give insight into the political science principle known as Duverger's law. Our results show that the type of equilibrium and the speed of convergence to equilibrium depend strongly on the distribution of inertia and the preferences of agents.

1 Introduction

Voting as a preference aggregation method is widely used in human society and artificially designed systems of software agents. A large amount of recent research has considered the situation where a single individual or a small coalition attempts to manipulate an election result in its favour, assuming the remaining agents are naive (that is, always vote sincerely). Such an assumption on agent behaviour can be justified if the goal is to prove computational hardness results. However, if we wish to understand how voting rules function under fully strategic behaviour, we need to study a game-theoretic model of strategic manipulation.

The plurality rule is the most widely used voting rule, despite substantial criticism from social choice theorists. One point in its favour is its simplicity and space-efficiency: an agent needs only report a single alternative instead of submitting a full preference order, a list of utilities, or a binary approval vector, as is the case with most other rules. However, even such a simple rule can become complicated when strategic voting behaviour is considered. In this paper, we study plurality voting under the assumption that all agents act strategically, as a starting point for a study of further classes of rules.

Voting games notoriously have many equilibria, and agents often cannot coordinate on a particular equilibrium outcome. Hence, voting games are hard to understand. The lack of publicly known information can exacerbate the lack of coordination of agents. A commonly used device that addresses the coordination issue, especially for plurality elections, is to use publicly announced pre-election polls. Such polls, which amount to an approximate simulation of an election with the same agents and alternatives, increase the commonly known information among agents and may influence their strategic behaviour. However, the beliefs of agents regarding the accuracy of these results can be different. This is a key point in the present paper, and we introduce the concept of inertia to describe these differences in beliefs.

Several authors from the political science and economics disciplines have discussed the influence of pre-election polls in plurality elections, both empirically and theoretically. The key topic of interest is what is called "Duverger's law", a general political science principle stating that plurality voting tends to lead to two-party competition [13]. More recently some papers have appeared that study equilibria in plurality voting games from a more algorithmic viewpoint (e.g. [6], [1]). Most of the models that have been used, with a few exceptions (e.g. [3], [6]), concern static equilibria, classifying them as "duvergerian" or "non-duvergerian", and fail to discuss the dynamic process of converging to equilibria via the use of polls. There are several important differences between our work and existing literature. One of the differences is related to the different amount of information and strategic behaviour of agents. The other extra feature considered in the present paper is agent-dependent beliefs about the reliability of this information.

1.1 Our Contribution

We present a model for plurality elections that allows for heterogeneous agents. We introduce the concept of an agent's *inertia*, which is that agent's perception of the accuracy of the poll result. This perception is the result of each agent's belief about such sources of error as coverage bias, miscounting, roundoff error, and noise in the announcement of results. This concept is rather general and seems realistic enough to be used for both human society and for designed systems of autonomous agents. This article focuses on the plurality rule, places some restrictions on agent behaviour, and considers some particular distributions of inertia. We present some numerical and analytic results on convergence to equilibria, both duvergerian and non-duvergerian. For example, a duvergerian equilibrium often occurs when all agents have the same value of inertia.

2 Game Model

We have a set of agents each of whose set of allowable actions is to vote for a single alternative (not necessarily their most desirable alternative). Abstention is not allowed. Each agent has a total order on the set of alternatives (indifference is not allowed) but as the voting rule is plurality, they vote for one alternative. Agents participate in a sequence of pre-election polls before the real election. In our model, these polls include all agents and alternatives in real election, not just a random sample. The information that these polls reveal does not have any effect on the agents' sincere preference order. In fact, we are interested in the strategic voting effect of polls rather than the so-called bandwagon or underdog effects considered in some papers [5]. In those papers, agents do not have a fixed preference order and their preference for an alternative is influenced by the popularity of that alternative.

We now discuss the assumptions in our model regarding the information and strategic behaviour of agents.

The information available for agents

The amount of information available to agents is a very important factor in their choice of strategy. The effect of poll information on the election result has been discussed in [12]. Complete information in plurality voting has been assumed in [8] and there is incomplete information in [11].

In the context of a repeated game, such as this sequence of polls under the plurality rule, in order to have complete information each agent would have to know how many agents of each *type* (sincere preference order) there are (this is usually called the *voting situation*). Even if this is unknown, we might expect to know the number of agents expressing each preference order in the previous poll. However, opinion polls for plurality will typically report only the number of agents ranking each alternative first, which we call the *scoreboard*. This lack of information on further preferences of other agents is crucial in the analysis below.

We use the concept of *inertia* to describe the reaction of agents toward the announced poll result. Agent coverage bias, miscounting or error and noise in announcing the result cause different values of uncertainty. This uncertainty brings about an inertia in agents. Each agent has an inertia value from the interval $[0, 1]$. An agent with inertia value of zero believes that the poll result is accurate. However, the poll result is meaningless to an agent with inertia value of one. In fact this agent does not consider the poll result in his decision making process. Other agents lie between these two extremes. Each agent's inertia value does not change during the sequence of polls. This seems reasonable because the set of participants in each poll does not change (it is always the entire set of agents), and the same system is used for counting and announcing the results in polls.

As far as we know this concept is new. The probability of miscounting has been discussed in [8], but is the same for all agents, whereas we have different values of inertia for different agents. The Poisson model of population uncertainty, in which there is uncertainty about the numbers of each type of agent, has been considered in [10]. In this paper agents have beliefs about these numbers that have been modelled as independent Poisson random variables. However, in our model, each agent just knows his own inertia and sincere preference order, and the scoreboard after each poll. This assumption makes sense for a system with no communication or coordination. This incomplete information influences the equilibrium result. Roughly speaking, it allows more alternatives to remain viable from the viewpoint of each agent.

The strategic behaviour of agents

The voting game described so far is still very general and allows for a wide range of outcomes. Voting games with more than two alternatives have many Nash equilibria and are not necessarily dominance solvable [2]. Eliminating dominated strategies is not sufficient to determine the result. Other refinements of equilibria such as strong and coalition-proof Nash equilibria do not always exist [7]. Some authors try to restrict the strategies of players by additional assumptions such as by assuming no voting for an alternative from another party [9].

In this paper, we assume agents have lexicographic preferences. Each agent infinitely prefers alternative x to alternative y, so he does not ignore any chance of winning of a more preferred alternative x [4]. Lexicographic preferences are not consistent with the idea of a cardinal utility function and probabilities are not relevant. Rather, they give a strong bias toward sincere voting which can still be overcome when an alternative is perceived to be a definite loser.

We also assume that each voter votes in each poll in the same way that he would if that poll were the actual election. One scenario in which this would occur is when voters do not know whether the current poll is the actual election. For example, the system designer may introduce this requirement. Thus voters will not attempt to vote strategically in the sense of misleading other voters, although they do vote strategically in the sense of playing their perceived best response. Note that the restricted information given by the scoreboard helps in this regard. For example, if bca voters could infer how many cab voters there were , they could vote for c in order that the cab voters do not abandon c, which might allow a to defeat b.

Therefore, agents vote for their most preferred alternative whom they perceive as having a non-zero chance of winning in further polls.

After each poll, each agent considers a set W of potential winners, consisting of all alternatives whom that agent perceives as having non-zero chance to win sometime in future. This set does not depend on the agents' preference order and only depends on the scoreboard and his inertia value. Agents update this set after the announced result of each poll. Agents start by voting sincerely in the first poll. Then, they update their votes according to their beliefs about potential winners during the sequence of polls. All these assumptions on behaviour are common knowledge as far as agents are concerned.

3 Game Dynamics

3.1 Notation

There is a set C of alternatives (we use index c for alternatives) which has m members, and a set V of players with n members (we use index ν for agents). We consider a sequence of K polls indexed by k, where the last poll is the election. However, agents are not aware of the value of K. Each agent has a sincere strict preference order on alternatives. There are $m!$ different preference orders (or types) which are indexed by t. We have plurality as our scoring rule in which each agent votes for only one alternative. Therefore, we can assume that the set of possible strategies for player ν is $S_\nu = C$. We use the following notations through the paper:

- $s_k(c)$: the normalized score of alternative c in poll k, namely the proportion of agents who have voted for c at poll k,
- $c_k(h)$: the alternative who has h-th highest score in poll k (e.g. $c_k(1)$ is the winner of poll k, note that we do not consider ties in this paper as this case occurs relatively rarely in large electorates),
- v_t: the number of agents with type (or preference order) t,
- $W_{\varepsilon,k}$: the set of potential winners from the view point of player with inertia value ϵ according to the result of poll k,
- $V_{c,k}$: the set of agents who vote for alternative c in poll k.

Definition 1 (**The concept of certain and doubtful**). Suppose that according to the poll result $s_k(i) < s_k(j)$. An agent with inertia ε is *certain* about this statement if

$$(1+\varepsilon)s_k(i) < (1-\varepsilon)s_k(j). \tag{1}$$

Otherwise, he is *doubtful*.

Note that this formula implies that if inertia of an agent is 0, then he will always be certain that j is ahead of i provided that such a result is reported. Also, Equation (1) implies that an agent with inertia equal to 1 will always be doubtful of any claimed scores.

The supporters of each alternative may be certain that the score of their favoured alternative is less than the winner, yet they might still consider that alternative as a potential winner and vote for him in the next poll. We study the concept of potential winner in the next section.

Example 1. Consider a 3 alternative election, and suppose the result of poll k is $s_k(c_k(1)) = 45\%$, $s_k(c_k(2)) = 30\%$ and $s_k(c_k(3)) = 25\%$. Any agent with inertia less than $\frac{1}{11}$ is certain that alternative 3 has fewer votes than alternative 2, but agents with inertia more than that are doubtful about this statement. In other words, those with $\varepsilon > \frac{1}{11}$ do not use this statement, while the others consider it in their strategic computations.

3.2 Set of potential winners

In the initial state ($k = 0$), an agent with inertia ε does not have any information about the number of supporters of each alternative. Therefore, he sees all alternatives as potential winners, $W_{\varepsilon,0} = C$, and he votes sincerely in the first poll. For the next poll, the agent votes for the most desirable alternative who can win in future (not necessarily the next poll) according to his interpretation of the poll result and the voting strategies of other agents (the strategy of agents is common knowledge).

Each agent's set of potential winners should satisfy some basic properties. The key necessary properties that we require are as follows. These are all common knowledge.

- non-emptiness: Any agent with any inertia value ε believes that there exists at least one agent with a positive chance of winning. W should clearly be nonempty for every voter, and contain the highest scoring candidate in the current poll.

- upward closure: if an agent with inertia ε believes that $c_k(x) \in W_{\varepsilon,k}$, then he believes $c_k(x-1) \in W_{\varepsilon,k}$. This seems reasonable: if an agent believes that some alternatives have a chance to win in future in the best case, then that agent also believes that all alternatives with higher current poll support also have a chance to win in future.

- overtaking: a possible winner must be able to overtake a higher scoring candidate who is also a possible winner. Overtaking the next higher scoring alternative is a necessary condition for winning, because the only chance an alternative has for attracting more support is that he improves his ranking position in the scoreboard. This is justified by the belief of agents about the upper closure of set of potential winners. For overtaking, alternative $c_k(x)$ needs extra support, and this support can only be obtained from the supporters of alternatives with a lower score than alternative $c_k(x)$. This is because agents who have already voted for higher scoring alternatives than $c_k(x)$ will change their votes to $c_k(x)$ if they perceive that their current choice does not have any chance to win. Upper closure of $W_{\varepsilon,k}$ would then lead to inconsistent beliefs.

  If $c_k(x)$ cannot overtake $c_k(x-1)$ in the next poll, in the most favourable case, then $x \notin W_{\varepsilon,k}$. We describe this case precisely in Proposition 1.

We first give an example to give the intuition behind our definitions.

Example 2. Consider scoreboard $(a, b, c, d) = (40\%, 29\%, 21\%, 10\%)$ and agent ν with $\varepsilon = 0$. Voter ν reasons as follows: for each agent with inertia ε, either alternative $d \in W_{\varepsilon,k}$ or not. If yes, then also alternatives $a, b, c \in W_{\sigma,k}$ (upward closure). The agents whose most desirable potential winner is alternative d have already voted for him, and the other agents prefer to vote for alternatives a, b or c in the next poll. Thus, the score of d cannot be increased and $d \notin W_{0,k}$. However, alternative $c \in W_{0,k}$ because it is possible that all supporters of alternative d switch to c, yielding scoreboard $(40\%, 29\%, 31\%, 0)$, and c can overtake alternative b, and in the next round all b-supporters may switch to alternative c, and he can overtake alternative a. Because of upward closure $b, a \in W_{0,k}$.

The basic properties above show that the currently highest-scoring alternative is always considered a potential winner by each agent. The necessary conditions do not define W uniquely. Because of lexicographic preferences, voters do not abandon candidates easily, and so it makes sense that W should be as large as possible. Of course if voters voted differently in the polls and the election (for example if they know that the next round is the election and have no other constraints on strategic action), W might be smaller. For example, a candidate may be able to win by successively attracting support from others, but the number of rounds remaining may not be enough for this to occur. We are ruling out this case by our assumptions on voter behaviour. For example, uncertainty about the time of the actual election allied to lexicographic preferences implies that W should be as large as possible. Thus we argue that the necessary conditions are sufficient.

We now show how to define the set of potential winners recursively starting from the top scoring alternative.

Definition 2. For $2 \leq i \leq m$, define condition $C_{ik\varepsilon}$ by

$$(1+\varepsilon)\sum_{h\geq i} s_k(c_k(h)) > (1-\varepsilon)s_k(c_k(i-1)). \qquad (C_{ik\varepsilon})$$

Proposition 1 (The conditions for being a potential winner). *After the announced result of poll k, $c_k(x) \in W_{\varepsilon,k}$ if and only if all conditions $C_{ik\varepsilon}$ for $2 \leq i \leq x$ hold. Algorithm 1 computes the set $W_{\varepsilon,k}$.*

Proof. Upward closure shows that the best chance of $c_k(x)$ overtaking $c_k(x-1)$ consists in attracting all supporters of agents currently voting for alternatives $c_k(h)$ with $h > x$, and retaining

Algorithm 1 Function for constructing $W_{\varepsilon,k}$

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Require: k ≥ 1
  W_{ε,k} = {c_k(1)}
  for i = 2 to m do
    if Condition C_{ikε} holds then
      W_{ε,k} = W_{ε,k} ∪ {c_k(i)}
    else
      break
    end if
  end for
```

all current supporters. This yields condition $C_{xk\varepsilon}$, and so Algorithm 1 is clearly correct. Since overtaking of even higher alternatives must occur also, unrolling the loop in Algorithm 1 yields the result. □

Remark 1. In the majority case from the viewpoint of an agent with inertia value ε, in which

$$(1-\varepsilon)s_k(c_k(1)) > (1+\varepsilon)\sum_{c\neq c_{k(1)}} s_k(c),$$

alternative $c_k(2)$ and consequently all other alternatives except $c_k(1)$ do not have any chance to win in the future. Thus, $W_{\varepsilon,k} = \{c_k(1)\}$.

Example 3. Suppose the result of poll k is $s_k(a) = 55\%$, $s_k(b) = 30\%$ and $s_k(c) = 15\%$. According to Proposition 1,

$$W_{\varepsilon,k} = \begin{cases} \{a\} & 0 \leq \varepsilon \leq \frac{1}{11}; \\ \{a,b\} & \frac{1}{11} < \varepsilon \leq \frac{1}{3}; \\ \{a,b,c\} & \frac{1}{3} < \varepsilon \leq 1. \end{cases}$$

Therefore, we have 3 different sets for $W_{\varepsilon,k}$ based on the inertia value of agents. In the first inertia value interval, agents perceive the result of poll k as a majority case. Therefore, their set of potential winners is a singleton and they vote for a in poll $k+1$. In the second inertia value interval, they vote for a or b in poll $k+1$ based on their preference order. For example, an agent with preference order cab votes for a and an agent with preference order cba votes for b in poll $k+1$. In the third case where agents have high inertia, they do not care about the announced result of the poll. In fact, they believe each candidate to be viable and they just vote sincerely in poll $k+1$. An agent with inertia value of 1 always votes sincerely, regardless of the poll result.

4 Equilibrium Results for some special cases

4.1 Zero inertia

In the special case where inertia is identically zero for all agents, the set of potential winners is identical for all agents. We show that in this case the sequence of polls converges to a duvergerian equilibrium, i.e., a two party competition. Note that the inertia value is fixed in all polls and also we assume there is no majority case.

Theorem 1 (**duvergerian equilibrium**). *In a plurality voting game with common inertia value $\varepsilon = 0$, the polling sequence yields a duvergerian equilibrium in a non-majority case after at most $m-2$ polls.*

Proof. Let m be the number of alternatives and $\varepsilon = 0$. As agents have the same value of inertia, either all agents perceive the result as majority case or all of them perceive it as a non-majority case. As we explained before, in the majority case, agents vote for the highest scoring alternative (refer to Remark 1). In a non-majority case, we have $(s_k(c_k(1)) \leq \sum_{c \neq c_k(1)} s_k(c)$. According to Proposition 1, $c_k(2) \in W_{0,k}$, therefore, $\mid W_{0,k} \mid \geq 2$.

For all $\nu \in V_{c,k}$ for which $c \in C \setminus W_{0,k}$, ν changes his vote to his most desirable alternative in $W_{0,k}$. Thus, $s_{k+1}(c) = 0$, for each $c \in C \setminus W_{0,k}$. According to Proposition 1, $c_k(m) \notin W_{0,k}$. Therefore, in each poll, at least the last scored alternative is eliminated and after at most $m-2$ polls, we have a duvergerian equilibrium. □

Remark 2. There is a connection with the voting method instant runoff (IRV). When $m = 3$, if inertia is identically zero then our assumptions mean that the plurality election is actually just IRV. For general inertia and general m, we could fix some $\beta > 0$ and require that the election system automatically deletes the alternative whose support becomes less than β for the next poll. If we assume that 2 alternatives do not reach this boundary β simultaneously, we again simulate IRV. However, our procedure is more general, as several alternatives may be eliminated at one step.

4.2 Constant non-zero inertia

Suppose that all agents have the same value of inertia θ, with $0 < \theta \leq 1$. Again note that the set of potential winners is identical for all agents at all times and the inertia value is fixed in all polls. This case is similar to the setup of Messner and Polborn [8] where the probability of miscounting is positive but small. Messner and Polborn introduce the concept of robust equilibrium and show that for plurality games with 3 alternatives, all such equilibria are duvergerian. However, in that paper, the value of θ is common knowledge between all agents, and this is not the case in our model. The behavioural assumptions of agents also differ. Paper [8] shows that duvergerian equilibrium happens in all robust equilibria of plurality games with 3 alternatives.

We consider a 3-alternative election with a large number of agents, with a fixed inertia value θ which is the same for all agents. W.l.o.g. we may assume that $s_1(c) < s_1(b) < s_1(a)$. We also assume there is no majority case (refer to Remark 1).

Proposition 2. *Let*

$$\theta' = \max\{\frac{s_1(a) - s_1(b) - s_1(c)}{s_1(a) + s_1(b) + s_1(c)}, \frac{s_1(b) - s_1(c)}{s_1(b) + s_1(c)}\}. \tag{2}$$

A c supporter with inertia $\theta \leq \theta'$ will change his vote to a or b in the second poll.

Proof. According to Proposition 1,

$$c \in W_{\theta,1} \Leftrightarrow \begin{cases} (1+\theta)(s_1(b) + s_1(c)) > (1-\theta)s_1(a) \\ (1+\theta)s_1(c) > (1-\theta)s_1(b) \end{cases}$$

Therefore, $c \in W_{\theta,1} \Leftrightarrow \theta > \theta'$, and $c \in C \setminus W_{\theta,1} \Leftrightarrow \theta \leq \theta'$. □

Theorem 2. *Consider a plurality voting game with $m = 3$, and fixed inertia value θ which is the same for all agents. Assuming a non-majority case, the polling sequence yields a duvergerian equilibrium after 1 poll if $\theta \leq \theta'$.*

Proof. Similar to previous case, as agents have the same value of inertia, either all agents perceive the result as majority case or all of them perceive it as a non-majority case. As we explained before, in the majority case, agents vote for the highest scoring alternative (refer to Remark 1). In a non-majority case, according to Proposition 2, as the inertia values of all agents are equal, c supporters abandon c immediately, and a duvergerian equilibrium is reached after one poll. □

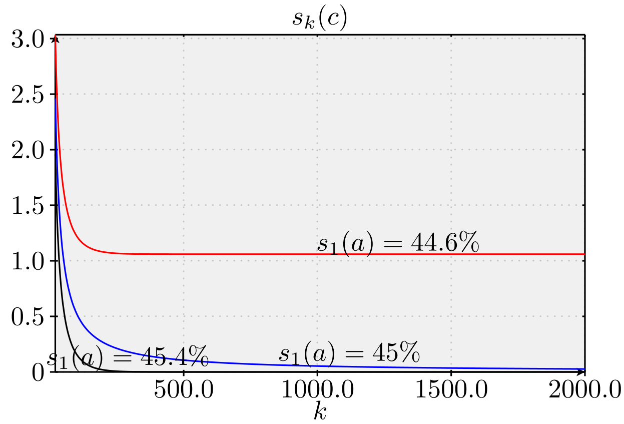

Figure 1: Score of the last alternative (c) as a function of k with uniform inertia distribution for three different cases where $V = (s_1(a), 35\%, 100\% - s_1(a) - 35\%, 5\%)$

Remark 3. Note that same constant non-zero inertia cases do not yield duvergerian equilibrium, depending on the value of θ. If $\theta > \theta'$, then every agent continues voting sincerely and the poll results will not change in the sequence.

Example 4. Consider plurality rule with 3 alternatives where the the scoreboard of the first poll is $(40\%, 35\%, 25\%)$. If the inertia value of all agents are θ and $\theta \leq \frac{1}{6}$, we have a duvergerian equilibrium.

4.3 Uniform distribution of inertia

We consider a 3-alternative election with a large number of agents, with a uniform inertia distribution on [0,1]. We describe the initial setup via a quadruple which is based on the first poll result $(s_1(a), s_1(b), s_1(c))$ and the true percentage v_6 of type cba agents (note this value is not known to any agent). W.l.o.g., we may assume that $s_1(c) < s_1(b) < s_1(a)$ and we approximate the discrete uniform distribution across agents by a continuous one for purposes of computation.

All c supporters who believe that c is a loser change their votes in favour of their second alternative. The percentage of type t agents (cab and cba) who vote in favour of alternative i (a and b respectively) in poll $k+1$ is denoted by $\alpha_{t,i,k}$. Note that the assumption of a common inertia distribution implies that for all k, $\alpha_{cab,a,k} = \alpha_{cba,b,k} \equiv \alpha_k$ and $\alpha_0 = 0$.

Proposition 3. *For a uniform distribution of inertia for all agents during the sequence of polls and initial result* $V = (s_1(a), s_1(b), s_1(c), v_6)$, *we have*

$$\alpha_k = \cfrac{1}{1 + \cfrac{2^k\left(\frac{s_1(c)-v_6}{s_1(b)+v_6}\right)^k (s_1(b)+v_6-2s_1(c))}{(s_1(b)-s_1(c))\left(-2^k\left(\frac{s_1(c)-v_6}{s_1(b)+v_6}\right)^k + \left(1-\frac{v_6}{s_1(c)}\right)^k\right)}} \tag{3}$$

Proof. According to the order of alternatives in the first poll and Proposition 1, a c supporter concludes that c is a loser and changes his vote if $(1+\varepsilon)s_k(c) < (1-\varepsilon)s_k(b)$.

Therefore, $\alpha_k = p\{\varepsilon < \frac{s_k(b)-s_k(c)}{s_k(b)+s_k(c)}\}$. The score of alternatives a, b and c in poll k is given by:

$$s_k(a) = s_1(a) + \alpha_{k-1}v_5 \qquad s_k(b) = s_1(b) + \alpha_{k-1}v_6 \tag{4}$$

$$s_k(c) = s_1(c) - \alpha_{k-1}v_6 - \alpha_{k-1}v_5 \tag{5}$$

Therefore,

$$\alpha_k = p\{\varepsilon < \frac{s_1(b) - s_1(c) + \alpha_{k-1}(s_1(c) + v_6)}{s_1(b) + s_1(b) - \alpha_{k-1}(s_1(c) - v_6)}\} \text{ for all } k \geq 1. \quad (6)$$

The stated solution formula for this recurrence is readily established by induction. □

Proposition 4. *The score of the last alternative in the first poll (which we denote by c) satisfies*

$$\lim_{k\to\infty} s_k(c) = \begin{cases} 0 & \textit{if } s_1(b) + v_6 \geq 2s_1(c) \\ \left(\frac{2s_1(c) - v_6 - s_1(b)}{s_1(c) - v_6}\right) s_1(c) & \textit{if } s_1(b) + v_6 < 2s_1(c) \end{cases} \quad (7)$$

Proof. The score of alternative c after $k+1$ polls is

$$s_{k+1}(c) = (1 - \alpha_k)s_1(c) \quad (8)$$

According to Proposition 3, if we converge k to infinity, we have

$$\lim_{k\to\infty} \alpha_k = \begin{cases} 1 & s_1(b) + v_6 \geq 2s_1(c); \\ \frac{s_1(b) - s_1(c)}{s_1(c) - v_6} & s_1(b) + v_6 < 2s_1(c). \end{cases}$$

The result follows immediately.

□

Remark 4. The convergence to zero is exponentially fast with the exponential rate decreasing as we approach the boundary between the two cases, and at the boundary it is subexponential. Figure 1 shows three special cases (the boundary case and 2 different cases in its neighbourhood).

Theorem 3. *In a plurality voting game with 3 alternatives and initial result $V = (s_1(a), s_1(b), s_1(c), v_6)$ and uniform distribution of inertia, the polling sequence yields a duvergerian equilibrium if and only if $s_1(b) + v_6 \geq 2s_1(c)$.*

Proof. Follows immediately from Proposition 4. □

Fig 1 illustrates this inequality when $v_6 = 5\%$ and $s_1(b) = 35\%$. For $s_1(a) \geq 45\%$, we have a duvergerian equilibrium.

4.4 Other distributions of inertia

The above results are for very special inertia distributions; explicit analysis of this type is not possible for general distributions. In this subsection, we investigate some different distributions via numerical simulations. Intuitively, we expect that distributions skewed to the left (with more agents of low inertia) will converge to the $\varepsilon \equiv 0$ case more quickly.

We consider the continuous triangular distribution $T(p)$ whose density function's graph has vertices at $(0, 0), (p, 2)$ and $(1, 0)$.

Example 5 (**The effect of inertia distribution: Triangular vs. Uniform**). Consider the initial result $V = (s_1(a), s_1(b), s_1(c), v_6) = (45\%, 35\%, 20\%, 5\%)$. According to Theorem 3, we have a limiting duvergerian equilibrium for uniform inertia distribution. Numerical results in Figure 1 (the line for $s_1(a) = 45\%$) also confirm this result. When we change the inertia distribution to be triangular with apex 0.5, we have the result in Figure 2. As we see in Figure 1, the convergence is very slow but changing the inertia distribution to $T(0.5)$ accelerates the process.

Example 6 (**The effect of voting situation**). In Figure 2, we have 5% cba agents. Figure 3 shows the result of the same situation with 10% cba agents which leads to a faster convergence. Note that the voting situation is not known to agents.

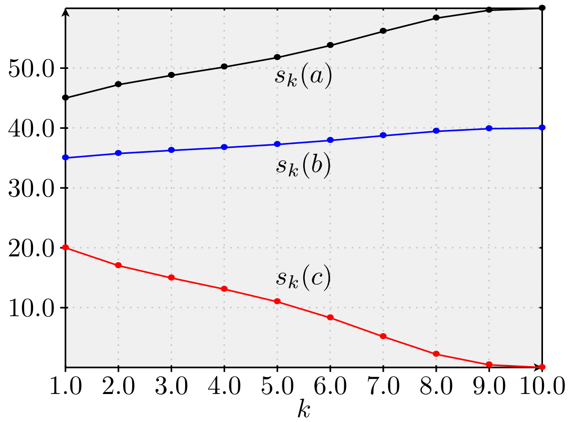

Figure 2: $V = (45\%, 35\%, 20\%, 5\%)$ and $T(0.5)$ inertia distribution

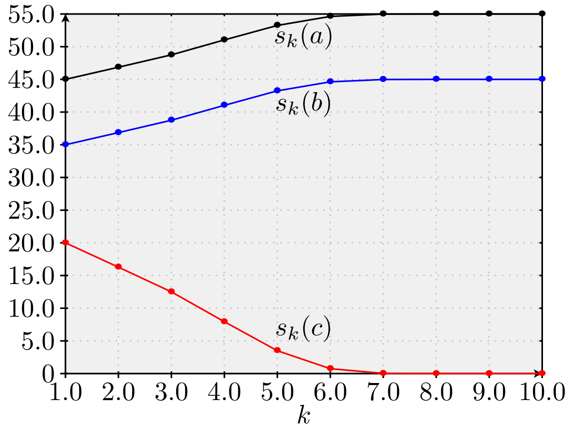

Figure 3: $V = (45\%, 35\%, 20\%, 10\%)$ and $T(0.5)$ inertia distribution

Example 7 (The effect of skewness of inertia distribution). Consider $V = (40\%, 35\%, 25\%, 10\%)$ with an inertia distribution of $T(0.5)$. This yields a non-duvergerian equilibrium, and it appears that the score of c converges to 22, as shown in Figure 4. However, the same voting situation with an inertia distribution $T(0.3)$ results in a duvergerian equilibrium as shown in Figure 5. In this case, more agents validate the poll result, and we have a duvergerian equilibrium after 10 polls.

5 Conclusion and Future Directions

In this paper we tried to study a repeated game with unknown number of rounds and incomplete information. The strategy of each player depends on his belief about the belief of other players. The sequence of opinion polls helps agents to coordinate on an equilibrium in an environment with some uncertainties about the accuracy of these polls. The amount of information available to agents has a critical role in influencing the strategic choices of agents. In this paper, we try to simplify the model with some assumptions about the strategy of players as a starting point for studying this game. Even in this simplified model, there are too many special cases that can happen depending on the inertia

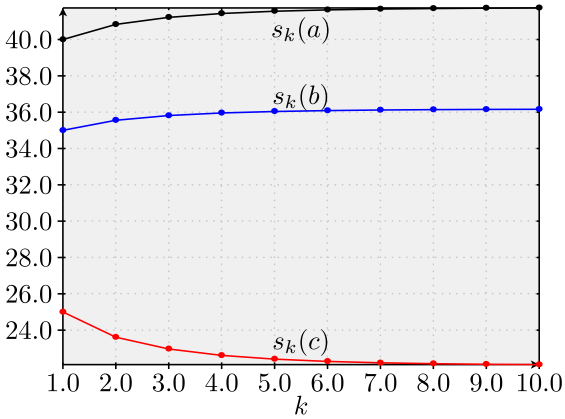

Figure 4: $V = (40\%, 35\%, 25\%, 10\%)$ and $T(0.5)$ inertia distribution

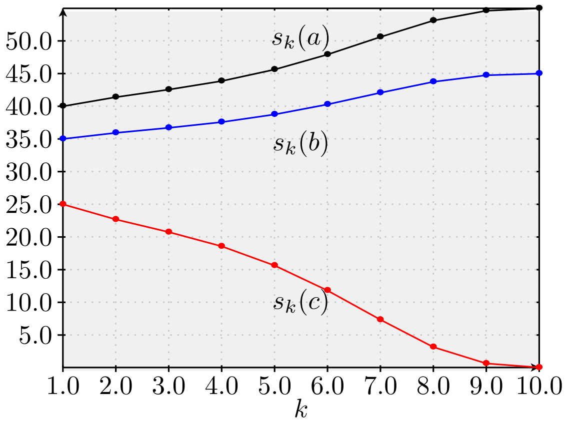

Figure 5: $V = (40\%, 35\%, 25\%, 10\%)$ and $T(0.3)$ inertia distribution

distribution or preference distribution of agents. We try to explain the model by some examples that give insight into different scenarios.

As a future direction, it is interesting to study how the strategy of agents will change if they have more information or in a more complicated model, each agent has different amounts of information. For example, some agents may have extra information than others regarding the inertia distribution of other agents or their preference order or the number of rounds ahead. Therefore, they may have different belief about the strategy of each agent.

Another interesting direction would be to to allow inertia to change from one poll to the next. For example, if random sampling is used instead of polling all voters, the sample size might vary between polls. More generally we want to explore the effect of inertia in other models with different behavioural assumptions for example, when voters use some some simple heuristic strategies. We expect to observe substantial differences in equilibrium outcomes when non-zero inertia is introduced into the model.

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