A characterization of the single-crossing domain

Robert Bredereck, Jiehua Chen, and Gerhard J. Woeginger

Abstract

We characterize single-crossing preference profiles in terms of two forbidden substructures, one of which contains three voters and six (not necessarily distinct) alternatives, and one of which contains four voters and four (not necessarily distinct) alternatives. We also provide an efficient way to decide whether a preference profile is single-crossing.

1 Introduction

Restricted domains of preferences. Single-peaked and single-crossing preferences have become standard domain restrictions in many economic models. Preferences are *single-peaked* if there exists a linear ordering of the alternatives such that any voter's preference relation along this ordering is either always strictly increasing, always strictly decreasing, or first strictly increasing and then strictly decreasing. Preferences are *single-crossing* if there exists a linear ordering of the voters such that for any pair of alternatives along this ordering, there is a single spot where the voters switch from preferring one alternative above the other one. In many situations, these assumptions guarantee the existence of a strategy-proof collective choice rule, or the existence of a Condorcet winner, or the existence of an equilibrium.

Single-peaked preferences go back to the work of Black [5] and have been studied extensively over the years. Single-peakedness implies a number of nice properties, as for instance non-manipulability (Moulin [19]) and transitivity of the majority rule (Inada [14]). Single-crossing preferences go back to the seminal paper of Roberts [20] on income taxation. Grandmont [12], Rothstein [21], and Gans & Smart [11] analyze various aspects of the majority rule under single-crossing preferences. Furthermore, single-crossing preferences play a role in the areas of income redistribution (Meltzer & Richard [18]), coalition formation (Demange [8]; Kung [15]), local public goods and stratification (Westhoff [24]; Epple & Platt [9]), and in the choice of constitutional voting rules (Barberà & Jackson [3]). Saporiti & Tohmé [23] study single-crossing preferences in the context of strategic voting and the median choice rule, and Saporiti [22] investigates them in the context of strategy proof social choice functions. Barberà & Moreno [4] develop the concept of top monotonicity as a common generalization of single-peakedness and single-crossingness (and of several other domain restrictions).

Forbidden substructures. Sometimes mathematical structures allow characterizations through forbidden substructures. For example, Kuratowski's theorem [16] characterizes planar graphs in terms of forbidden subgraphs: a graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$. For another example, Hoffman, Kolen & Sakarovitch [13] characterize totally-balanced 0-1-matrices in terms of certain forbidden submatrices. In a similar spirit, Lekkerkerker & Boland [17] characterize interval-graphs through five (infinite) families of forbidden induced subgraphs.

In the area of social choice, a beautiful result by Ballester & Haeringer [2] characterizes single-peaked preference profiles in terms of two forbidden substructures. The first forbidden substructure concerns three voters and three alternatives, where each of the voter ranks another one of the alternatives worst. The second forbidden substructure concerns two voters and four alternatives, where (sloppily speaking) both voters rank the first three alternatives in opposite ways with the second alternative in the middle, but prefer the fourth alternative to the second one.

Contribution of this paper. Inspired by the approach and by the results of Ballester & Haeringer [2], we present a forbidden substructure characterization of single-crossing preference profiles. One of our forbidden substructures consists of three voters and six alternatives (as described in Example 2.4) and the other one consists of four voters and four alternatives (as described in Example 2.5). We stress that the (six respectively four) alternatives in these forbidden substructures are not necessarily distinct: the substructures only partially specify the preferences of the involved voters; hence by identifying and collapsing some of the involved alternatives we can easily generate a number of smaller forbidden substructures (which of course are just special cases of our larger forbidden substructures). Finally, we will discuss the close relation of single-crossing preference profiles to *consecutive ones matrices*. A 0-1-matrix has the consecutive ones property if its columns can be permuted such that the 1-values in each row are consecutive. We hope that our results will turn out useful for future research on single-crossing profiles.

In Section 2 we summarize the basic definitions and provide some examples. In Section 3 we formulate and prove our main result (Theorem 3.1). In Section 4 we discuss the tightness of our characterization, and we argue that there does not exist a characterization that works with smaller forbidden substructures. Finally in Section 5 we show how to recognize the single-crossing property in polynomial time by using the connection to consecutive ones matrices.

2 Definitions, notations, and examples

Let $a_1, \ldots, a_m$ be m alternatives and let $V_1, \ldots, V_n$ be n voters. A *preference profile* specifies the *preference orderings* of the voters, where voter V_i ranks the alternatives according to a strict linear order $\succ_i$. For alternatives a and b, the relation $a \succ_i b$ means that voter V_i strictly prefers a to b.

An unordered pair of two distinct alternatives is called a *couple*. A subset $\mathcal{V}$ of the voters is *mixed* with respect to couple $\{a, b\}$, if $\mathcal{V}$ contains two voters one of which ranks a above b, whereas the other one ranks a below b. If $\mathcal{V}$ is not mixed with respect to $\{a, b\}$, then it is said to be *pure* with respect to $\{a, b\}$. Hence, an empty set of voters is pure with respect to any pair of alternatives. A couple $\{a, b\}$ *separates* two sets $\mathcal{V}_1$ and $\mathcal{V}_2$ of voters from each other, if no voter in $\mathcal{V}_1$ agrees with any voter in $\mathcal{V}_2$ on the relative ranking of a and b; in other words, sets $\mathcal{V}_1$ and $\mathcal{V}_2$ must both be pure with respect to $\{a, b\}$, and if both are non-empty then their union $\mathcal{V}_1 \cup \mathcal{V}_2$ is mixed.

An ordering of the voters is *single-crossing with respect to couple* $\{a, b\}$, if the ordered list of voters can be split into an initial piece and a final piece that are separated by $\{a, b\}$. An ordering of the voters is *single-crossing*, if it is single-crossing with respect to every possible couple. Finally a preference profile is single-crossing, if it allows a single-crossing ordering of the voters. It is easy to see that single-crossing is a monotone property of preference profiles:

Lemma 2.1 *Let $\mathcal{P}$ be a preference profile, and let $\mathcal{P}'$ result from $\mathcal{P}$ by removing some alternatives and/or voters. If $\mathcal{P}$ is single-crossing, then also $\mathcal{P}'$ is single-crossing.* $\square$

In the remaining part of this section we present several instructive examples of preference

V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9	V_{10}	V_{11}
1	1	1	1	5	5	5	5	5	5	5
2	2	2	5	1	1	1	4	4	4	4
3	3	5	2	2	2	4	1	1	3	3
4	5	3	3	3	4	2	2	3	1	2
5	4	4	4	4	3	3	3	2	2	1

Figure 1: A single-crossing preference profile with 11 voters and 5 alternatives.

profiles that are single-crossing (Section 2.1) respectively that are not single-crossing (Section 2.2).

2.1 Profiles from weak Bruhat orders

Let S_m denote the set of permutations of $1, \ldots, m$. We specify permutations $\pi \in S_m$ by listing the entries as $\pi = \langle \pi(1), \pi(2), \ldots, \pi(n) \rangle$. The *identity* permutation $\langle 1, 2, \ldots, m \rangle$ arranges the integers in increasing order, and the *order reversing* permutation $\langle m, m-1, \ldots, 2, 1 \rangle$ arranges them in decreasing order. A *descent* in π is a pair $(\pi(i), \pi(i+1))$ of consecutive entries with $\pi(i) > \pi(i+1)$. We write $\pi \lhd \rho$, if permutation π can be obtained from permutation ρ by a series of swaps, each of which interchanges the two elements of a descent.

The partially ordered set $(S_m, \lhd)$ is known as *weak Bruhat order*; see for instance Bóna[6]. The weak Bruhat order has the identity permutation as minimum element and the order reversing permutation as maximum element. Every maximal chain (that is: every maximal subset of pairwise comparable permutations) in the weak Bruhat order has length $\frac{1}{2}m(m-1)+1$ and contains the identity permutation and the order reversing permutation.

The following example illustrates the well-known connection between weak Bruhat orders and single-crossing preference profiles; we refer the reader to Abello [1] or Galambos & Reiner [10] for more information.

Example 2.2 *Let $C = (\pi_1 \lhd \pi_2 \lhd \cdots \lhd \pi_n)$ be a maximal chain with $n = \frac{1}{2}m(m-1)+1$ permutations in the weak Bruhat order $(S_m, \lhd)$. We construct a profile by using $1, \ldots, m$ as alternatives, and by interpreting every permutation π as preference ordering $\pi(1) \succ \pi(2) \succ \ldots \succ \pi(n)$ over the alternatives. Voter V_i has preference ordering π_i. See Figure 1 for an illustration with $m = 5$ alternatives and $n = 11$ voters.*

The resulting profile is single-crossing: any two alternatives a and b start off in the right order in the identity permutation π_1, eventually are swapped into the wrong order, and then can never be swapped back again at later steps. Furthermore, the profile contains $n = \frac{1}{2}m(m-1)+1$ voters with pairwise distinct preference orderings. $\square$

If one starts the construction in Example 2.2 from arbitrary (not necessarily maximal!) chains in the weak Bruhat order, then one can generate this way every possible single-crossing preference profile (up to isomorphism). This is another well-known connection, which follows from the fact that $\pi \lhd \rho$ if and only if every inversion of permutation π also is an inversion of permutation ρ.

2.2 Some profiles that are not single-crossing

We next present three examples of profiles that are not single-crossing. The first example is due to Saporiti & Tohmé [23] and shows a profile that is single-peaked but fails to be single-crossing. The other two examples introduce two principal actors of this paper.

Example 2.3 *Consider four alternatives* $1, 2, 3, 4$ *and three voters* V_1*,* V_2*,* V_3 *with the following preference orders:*

$$\begin{aligned}
&\textit{Voter } V_1\textit{:} \quad 2 \succ_1 3 \succ_1 4 \succ_1 1\\
&\textit{Voter } V_2\textit{:} \quad 4 \succ_2 3 \succ_2 2 \succ_2 1\\
&\textit{Voter } V_3\textit{:} \quad 3 \succ_3 2 \succ_3 1 \succ_3 4
\end{aligned}$$

It can be verified that this profile is not single-crossing but single-peaked (with respect to the ordering $1 < 2 < 3 < 4$ *of alternatives, for instance).* $\square$

Example 2.4 (γ-Configuration)
A profile with three voters V_1*,* V_2*,* V_3 *and six (not necessarily distinct) alternatives* a, b, c, d, e, f *is a* γ*-configuration, if it satisfies the following:*

$$\begin{aligned}
&\textit{Voter } V_1\textit{:} \quad b \succ_1 a \quad \textit{and} \quad c \succ_1 d \quad \textit{and} \quad e \succ_1 f\\
&\textit{Voter } V_2\textit{:} \quad a \succ_2 b \quad \textit{and} \quad d \succ_2 c \quad \textit{and} \quad e \succ_2 f\\
&\textit{Voter } V_3\textit{:} \quad a \succ_3 b \quad \textit{and} \quad c \succ_3 d \quad \textit{and} \quad f \succ_3 e
\end{aligned}$$

This profile is not single-crossing, as none of the three voters can be arranged between the other two: the couple $\{a, b\}$ *prevents us from putting* V_1 *into the middle, the couple* $\{c, d\}$ *forbids voter* V_2 *in the middle, and the couple* $\{e, f\}$ *forbids* V_3 *in the middle.* $\square$

The observations stated in Example 2.4 provide a cheap proof that the profile in Example 2.3 is not single-crossing, as this profile contains a γ-configuration with $a = 3$, $b = c = 2$, $d = e = 4$, and $f = 1$.

Example 2.5 (δ-Configuration)
A profile with four voters V_1*,* V_2*,* V_3*,* V_4 *and four (not necessarily distinct) alternatives* a, b, c, d *is a* δ*-configuration, if it satisfies the following:*

$$\begin{aligned}
&\textit{Voter } V_1\textit{:} \quad a \succ_1 b \quad \textit{and} \quad c \succ_1 d\\
&\textit{Voter } V_2\textit{:} \quad a \succ_2 b \quad \textit{and} \quad d \succ_2 c\\
&\textit{Voter } V_3\textit{:} \quad b \succ_3 a \quad \textit{and} \quad c \succ_3 d\\
&\textit{Voter } V_4\textit{:} \quad b \succ_4 a \quad \textit{and} \quad d \succ_4 c
\end{aligned}$$

This profile is not single-crossing: the couple $\{a, b\}$ *forces us to place* V_1 *and* V_2 *next to each other, and to put* V_3 *and* V_4 *next to each other; the couple* $\{c, d\}$ *forces us to place* V_1 *and* V_3 *next to each other, and to put* V_2 *and* V_4 *next to each other. Then no voter can be put into the first position.* $\square$

3 A characterization through forbidden configurations

Examples 2.4 and 2.5 demonstrate that preference profiles that contain a γ-configuration or a δ-configuration cannot be single-crossing. It turns out that these two configurations are the only obstructions for the single-crossing property.

Theorem 3.1 *A preference profile* $\mathcal{P}$ *is single-crossing if and only if* $\mathcal{P}$ *contains neither a* γ*-configuration nor a* δ*-configuration.*

The rest of this section is dedicated to the proof of Theorem 3.1. The (only if) part immediately follows from the monotonicity of the single-crossing property (Lemma 2.1) and from the observations stated in Examples 2.4 and 2.5.

For the (if) part, we first introduce some additional definitions and notations. An *ordered partition* $\langle X_1, \ldots, X_p \rangle$ of the voters $V_1, \ldots, V_n$ satisfies the following properties: every part X_i is non-empty, distinct parts are disjoint, and the union of all parts is the set of all voters. The *trivial* ordered partition has $p = 1$ and hence consists of a single part $\{V_1, \ldots, V_n\}$. We let $\{a_k, b_k\}$ with $1 \le k \le \frac{1}{2}m(m-1)$ be an enumeration of all the possible couples, and we define $\mathcal{C}_k$ as the set containing the first k couples in this enumeration.

Now let us prove the (if) part of the theorem. We consider some arbitrary preference profile $\mathcal{P}$ that neither contains a γ-configuration nor a δ-configuration. Our argument is algorithmic in nature. We start from the trivial partition $\mathcal{X}^{(0)}$ of the voters, and then step by step refine this partition while working through $\frac{1}{2}m(m-1)$ phases. The kth such phase generates an ordered partition $\mathcal{X}^{(k)} = \langle X_1^{(k)}, \ldots, X_p^{(k)} \rangle$ of the voters that satisfies the following two properties.

(i) For $1 \le j \le p-1$, the union of parts $X_1^{(k)}, \ldots, X_j^{(k)}$ is separated from the union of parts $X_{j+1}^{(k)}, \ldots, X_p^{(k)}$ by one of the couples in $\mathcal{C}_k$.

(ii) For every couple in $\mathcal{C}_k$, there is a j with $1 \le j \le p-1$ such that the couple separates the union of $X_1^{(k)}, \ldots, X_j^{(k)}$ from the union of $X_{j+1}^{(k)}, \ldots, X_p^{(k)}$.

Note that property (ii) implies that every part $X_j^{(k)}$ is pure with respect to every couple in $\mathcal{C}_k$. The following four lemmas summarize some useful combinatorial observations on the ordered partition $\mathcal{X}^{(k)}$ and how it relates to couple $\{a_{k+1}, b_{k+1}\}$.

Lemma 3.2 *At most one part in the ordered partition $\mathcal{X}^{(k)}$ is mixed with respect to couple $\{a_{k+1}, b_{k+1}\}$.*

Proof. Suppose for the sake of contradiction that the parts $X_s^{(k)}$ and $X_t^{(k)}$ with $1 \le s < t \le p$ both are mixed with respect to couple $\{a_{k+1}, b_{k+1}\}$. In other words, part $X_s^{(k)}$ contains a voter V_1' with $a_{k+1} \succ b_{k+1}$ and another voter V_2' with $b_{k+1} \succ a_{k+1}$, and part $X_t^{(k)}$ contains a voter V_3' with $a_{k+1} \succ b_{k+1}$ and another voter V_4' with $b_{k+1} \succ a_{k+1}$.

Property (i) yields the existence of a couple $\{x, y\} \in \mathcal{C}_k$ that separates the union of parts $X_1^{(k)}, \ldots, X_s^{(k)}$ from the union of the parts $X_{s+1}^{(k)}, \ldots, X_p^{(k)}$. In particular, this couple separates $X_s^{(k)}$ from $X_t^{(k)}$. This implies that voters V_1' and V_2' agree on couple $\{x, y\}$ (say, with $x \succ y$), whereas voters V_3' and V_4' have the opposite ranking (say $y \succ x$). Then the four voters V_1', V_2', V_3', and V_4' together with the four alternatives a_{k+1}, b_{k+1}, x, and y form a δ-configuration; this yields the desired contradiction. $\square$

Lemma 3.3 *Consider s and t with $2 \le s < t \le p$. If some voter V_1' in part $X_1^{(k)}$ ranks $a_{k+1} \succ b_{k+1}$ and if some voter V_2' in part $X_s^{(k)}$ ranks $b_{k+1} \succ a_{k+1}$, then every voter V_3' in part $X_t^{(k)}$ ranks $b_{k+1} \succ a_{k+1}$.*

Proof. Suppose for the sake of contradiction that the voter V_3' ranks $a_{k+1} \succ b_{k+1}$. Then the couple $\{a_{k+1}, b_{k+1}\}$ separates V_2' from V_1' and V_3'. Property (i) yields a couple $\{x, y\} \in \mathcal{C}_k$ that separates $X_1^{(k)}$ from $X_s^{(k)} \cup X_t^{(k)}$; this couple separates V_1' from V_2' and V_3'. Property (i) yields also a couple $\{u, v\} \in \mathcal{C}_k$ that separates $X_t^{(k)}$ from $X_1^{(k)} \cup X_s^{(k)}$; this couple separates V_3' from V_1' and V_2'.

Then the three voters V_1', V_2', and V_3' together with the six alternatives a_{k+1}, b_{k+1}, x, y, u, and v form a γ-configuration; a contradiction. $\square$

The statement of the following lemma is symmetric to the statement of Lemma 3.3, and it can be proved by symmetric arguments.

Lemma 3.4 *Consider s and t with $1 \le s < t \le p-1$. If some voter V_2' in part $X_t^{(k)}$ ranks $a_{k+1} \succ b_{k+1}$ and some voter V_3' in part $X_p^{(k)}$ ranks $b_{k+1} \succ a_{k+1}$, then every voter V_1' in part $X_s^{(k)}$ ranks $a_{k+1} \succ b_{k+1}$.* □

Lemma 3.5 *There exists an index ℓ with $1 \le \ell \le p$ such that the couple $\{a_{k+1}, b_{k+1}\}$ separates the union of parts $X_1^{(k)}, \dots, X_{\ell-1}^{(k)}$ from the union of parts $X_{\ell+1}^{(k)}, \dots, X_p^{(k)}$.*

Proof. If $p = 1$ or if all voters in the profile agree on the relative ranking of a_{k+1} and b_{k+1}, the choice $\ell = 1$ works. Hence we assume that $p \ge 2$ and that there are two voters who disagree on the ranking of a_{k+1} and b_{k+1}. By Lemma 3.2 the parts $X_1^{(k)}$ and $X_p^{(k)}$ cannot both be mixed with respect to $\{a_{k+1}, b_{k+1}\}$.

If the first part $X_1^{(k)}$ is pure with respect to $\{a_{k+1}, b_{k+1}\}$, we pick an arbitrary voter V_1' from $X_1^{(k)}$. We choose ℓ as the smallest index for which $X_\ell^{(k)}$ contains some voter V_2' who ranks a_{k+1} versus b_{k+1} differently from voter V_1'. Then Lemma 3.3 yields that every voter V_3' in the parts $X_{\ell+1}^{(k)}, \dots, X_p^{(k)}$ must rank a_{k+1} versus b_{k+1} differently from voter V_1'. Hence the chosen index ℓ has all the desired properties, and this case is closed. In the remaining case the last part $X_p^{(k)}$ is pure with respect to $\{a_{k+1}, b_{k+1}\}$; this case can be settled in a symmetric fashion while using Lemma 3.4. □

Now let us finally describe how to construct the ordered partition $\mathcal{X}^{(k+1)}$ in the $(k+1)$th phase. Our starting point is the ordered partition $\mathcal{X}^{(k)}$, and we determine an index ℓ as defined in Lemma 3.5. If part $X_\ell^{(k)}$ is pure with respect to $\{a_{k+1}, b_{k+1}\}$, then we make the new partition $\mathcal{X}^{(k+1)}$ coincide with the old partition $\mathcal{X}^{(k)}$; properties (i) and (ii) are satisfied in $\mathcal{X}^{(k+1)}$. If part $X_\ell^{(k)}$ is mixed with respect to $\{a_{k+1}, b_{k+1}\}$, then we subdivide it into two parts Y and Z so that $\{a_{k+1}, b_{k+1}\}$ separates the union of parts $X_1^{(k)}, \dots, X_{\ell-1}^{(k)}, Y$ from the union of parts $Z, X_{\ell+1}^{(k)}, \dots, X_p^{(k)}$. Then the resulting partition

$$\mathcal{X}^{(k+1)} \;=\; \langle X_1^{(k)},\, \dots,\, X_{\ell-1}^{(k)},\, Y,\, Z,\, X_{\ell+1}^{(k)},\, \dots,\, X_p^{(k)} \rangle$$

satisfies properties (i) and (ii) by construction.

We keep working like this and complete phase after phase, until in the very last phase $k = \frac{1}{2}m(m-1)$ we generate the final partition $\mathcal{X}^* = \langle X_1^*, \dots, X_q^* \rangle$. We construct an ordering π^* of the voters that lists the voters in every part X_j^* before all the voters in part X_{j+1}^* ($1 \le j \le q-1$). Property (ii) guarantees that every couple separates an initial piece of partition $\mathcal{X}^*$ from the complementary final piece, which implies that the ordering π^* for the voters in $\mathcal{P}$ is single-crossing. This completes the proof of Theorem 3.1.

We conclude this section with several comments on the above proof.

(1) Let $\langle X_1^{(k)}, \dots, X_p^{(k)} \rangle$ be the ordered partition determined in phase k, and consider an ordering σ of the voters that lists the voters in every part $X_j^{(k)}$ before all the voters in the succeeding part $X_{j+1}^{(k)}$. Let ordering σ^- list the voters in reverse order to σ. Then σ and σ^- are single-crossing with respect to all couples in $\mathcal{C}_k$. In fact, *any* ordering that is single-crossing with respect to all couples in $\mathcal{C}_k$ can be constructed in that fashion. This can be established by an inductive argument.

(2) By property (ii), every part X_j^* in the final partition $\mathcal{X}^*$ is pure with respect to every possible couple of alternatives. This means that all voters in part X_j^* have identical preference orderings, and that the ordering π^* is uniquely determined except for swapping voters with identical preference orderings.

(3) The preceding two comments imply the following. Let $\mathcal{P}$ be a preference profile in which distinct voters always have distinct preference orderings. If $\mathcal{P}$ is single-crossing, then there exist exactly two single-crossing orderings of the voters and these two orderings are mirror images of each other.

(4) By property (i), every two consecutive parts X_j^* and X_{j+1}^* must be separated by one of the couples. Since there are only $\frac{1}{2}m(m-1)$ distinct couples, there are at most $\frac{1}{2}m(m-1)+1$ parts in the final partition. This shows that a single-crossing preference profile contains at most $\frac{1}{2}m(m-1)+1$ voters with distinct preference orderings. (This bound of course is already known from the connection between single-crossing profiles and weak Bruhat orders as indicated in Section 2.1.)

4 The size of forbidden configurations

Throughout this short section, we speak of preference profiles with m alternatives and n voters as $m \times n$ configurations. Theorem 3.1 characterizes single-crossing preference profiles through certain forbidden 6×3 and 4×4 configurations. Are there perhaps other characterizations that work with smaller forbidden configurations? The following lemma shows that this is not the case, and hence our characterization uses the smallest possible forbidden configurations.

Lemma 4.1 *Every characterization of single-crossing preference profiles through forbidden configurations must forbid (a) some $m \times n$ configuration with $m \geq 6$ and $n \geq 3$ and (b) some $m \times n$ configuration with $m \geq 4$ and $n \geq 4$.*

Proof. Consider an arbitrary characterization of single-crossing profiles with forbidden configurations $F_1, \ldots, F_k$. Consider the following 6×3 configuration C.

Voter V_1: $b \succ_1 a \succ_1 c \succ_1 d \succ_1 e \succ_1 f$
Voter V_2: $a \succ_2 b \succ_2 d \succ_2 c \succ_2 e \succ_2 f$
Voter V_3: $a \succ_3 b \succ_3 c \succ_3 d \succ_3 f \succ_3 e$

This profile contains a γ-configuration and thus is not single-crossing. If we remove any alternative from C, the resulting 5×3 configuration is single-crossing and cannot be forbidden. And if we remove any voter from C, the resulting 6×2 configuration is again single-crossing and again cannot be forbidden. Hence the only possibility for correctly recognizing C as not single-crossing is by either forbidding C itself or by forbidding appropriate larger configurations that contain C. This proves (a). The proof of (b) is based on the following 4×4 configuration C' which contains a δ-configuration.

Voter V_1: $a \succ_1 b \succ_1 c \succ_1 d$
Voter V_2: $a \succ_2 b \succ_2 d \succ_2 c$
Voter V_3: $b \succ_3 a \succ_3 c \succ_3 d$
Voter V_4: $b \succ_4 a \succ_4 d \succ_4 c$

Since the argument is analogous to the one in (a), we omit the details. $\square$

5 Recognizing the single-crossing property

In this section, we sketch how to produce all (if any) single-crossing orderings of the voters by utilizing the PQ-tree algorithm as developed by Booth & Lueker [7]. The PQ-tree algorithm was designed to recognize, inter alia, *consecutive ones matrices.* A 0-1-matrix has the *consecutive ones property*, if its columns can be permuted such that the ones in each row are consecutive (and hence form an interval).

Hence let us consider an arbitrary preference profile $\mathcal{P}$, and let us transform it into a corresponding 0-1-matrix $M(\mathcal{P})$ in the following way. For each voter, the matrix $M(\mathcal{P})$

contains a corresponding column. For each ordered pair $\langle a, b\rangle$ of alternatives, matrix $M(\mathcal{P})$ has a corresponding row with value 1 at column j if voter j prefers alternative a to alternative b, and value 0 otherwise. For a preference profile with n voters and m alternatives, the resulting 0-1-matrix $M(\mathcal{P})$ has n columns and $m(m-1)$ rows. Example 5.1 illustrates this construction for a concrete profile with four voters and three alternatives.

Example 5.1 (A single-crossing profile and its 0-1-matrix representation) *Suppose that there are four voters V_1, V_2, V_3, and V_4 voting over three alternatives 1, 2, and 3. The preference orderings of the voters are as follows:*

Voter V_1: $3 \succ_1 1 \succ_1 2$
Voter V_2: $2 \succ_2 3 \succ_2 1$
Voter V_3: $2 \succ_3 1 \succ_3 3$
Voter V_4: $3 \succ_4 2 \succ_4 1$

Our construction yields the following 0-1-matrix corresponding to this profile.

	V_1	V_2	V_3	V_4
$\langle 1,2\rangle$	1	0	0	0
$\langle 2,1\rangle$	0	1	1	1
$\langle 1,3\rangle$	0	0	1	0
$\langle 3,1\rangle$	1	1	0	1
$\langle 2,3\rangle$	0	1	1	0
$\langle 3,2\rangle$	1	0	0	1

By applying the PQ-tree algorithm of Booth & Lueker [7], one can find all permutations of the columns with the consecutive ones property. One possible consecutive ones permutation of the columns is $\langle V_1, V_4, V_2, V_3\rangle$. As one can easily verify, this is also a single-crossing ordering of the voters in the original profile. □

Lemma 5.2 *A preference profile $\mathcal{P}$ is single-crossing if and only if the corresponding 0-1-matrix $M(\mathcal{P})$ has the consecutive ones property.*

Proof. An ordering of the voters is single-crossing for $\mathcal{P}$ if and only if this ordering permutes the columns of $M(\mathcal{P})$ so that the ones in each row are consecutive. □

The PQ-algorithm [7] solves the consecutive ones matrix problem in $O(x + y + z)$ time, where x and y are respectively the number of columns and rows, and z is the total number of 1s in the matrix. Hence, single-crossing profiles can be recognized in $O(m^2 + n + nm^2) = O(nm^2)$ time.

6 Conclusion

In this paper, we give an equivalent characterization of single-crossing preferences through two minimal forbidden substructures: γ- and δ-configurations. We demonstrate the close relation between single-crossing preferences and weak Bruhat orders. Futhermore, we can find all single-crossing orderings of a preference profile by transforming them into a binary matrix and asking whether this matrix has the consecutive ones property. This process needs subquadratic time and utilizes the consecutive ones matrix problem. Hence, searching for a direct and more efficient way of detecting the single-crossing property would be an interesting challenge.

References

[1] J. Abello (1991). The weak Bruhat order of S_Σ, consistent sets, and Catalan numbers. *SIAM Journal on Discrete Mathematics 4*, 1–16

[2] M.A. Ballester and G. Haeringer (2011). A characterization of the single-peaked domain. *Social Choice and Welfare 36*, 305–322.

[3] S. Barberà and M.O. Jackson (2004). Choosing how to choose: Self-stable majority rules and constitutions. *Quarterly Journal of Economics 119*, 1011–1048.

[4] S. Barberà and B. Moreno (2011). Top monotonicity: A common root for single peakedness, single crossing and the median voter result. *Games and Economic Behavior 73*, 345–359.

[5] D. Black (1948). On the rationale of group decision-making. *Journal of Political Economics 56*, 23–34.

[6] M. Bóna (2004). *Combinatorics of Permutations.* Chapman and Hall/CRC.

[7] K.S. Booth and G. S. Lueker (1976). Testing for the consecutive ones property, interval graphs, and graph planarity using PQ-tree algorithms. *Journal of Computer and System Sciences 13*, 335–379.

[8] G. Demange (1994). Intermediate preferences and stable coalition structures. *Journal of Mathematical Economics 23*, 45–58.

[9] D. Epple and G.J. Platt (1998). Equilibrium and local redistribution in an urban economy when households differ in both preferences and incomes. *Journal of Urban Economics 43*, 23–51.

[10] A. Galambos and V. Reiner (2008). Acyclic sets of linear orders via the Bruhat orders. *Social Choice and Welfare 30*, 245–264.

[11] J.S. Gans and M. Smart (1996). Majority voting with single-crossing preferences. *Journal of Public Economics 59*, 219–237.

[12] J.-M. Grandmont (1978). Intermediate preferences and majority rule. *Econometrica 46*, 317–330.

[13] A.J. Hoffman, A.W.J. Kolen, and M. Sakarovitch (1985). Totally-balanced and greedy matrices. *SIAM Journal on Algebraic Discrete Methods 6*, 721–730.

[14] K. Inada (1969). The simple majority rule. *Econometrica 37*, 490–506.

[15] F.-C. Kung (2006). An algorithm for stable and equitable coalition structures with public goods. *Journal of Public Economic Theory 8*, 345–355.

[16] K. Kuratowski (1930). Sur le problème des courbes gauches en topologie. *Fundamenta Mathematicae 15*, 271–283.

[17] C. Lekkerkerker and D. Boland (1962). Representation of finite graphs by a set of intervals on the real line. *Fundamenta Mathematicae 51*, 45–64.

[18] A.H. Meltzer and S.F. Richard (1981). A rational theory of the size of government. *Journal of Political Economy 89*, 914–927.

[19] H. Moulin (1980). On strategy-proofness and single peakedness. *Public Choice 35*, 437–455.

[20] K.W.S. Roberts (1977). Voting over income tax schedules. *Journal of Public Economics 8*, 329–340.

[21] P. Rothstein (1990). Order restricted preferences and majority rule. *Social Choice and Welfare 7*, 331–342.

[22] A. Saporiti (2009). Strategy-proofness and single-crossing. *Theoretical Economics 4*, 127-163.

[23] A. Saporiti and F. Tohmé (2006). Single-crossing, strategic voting and the median choice rule. *Social Choice and Welfare 26*, 363–383.

[24] F. Westhoff (1977). Existence of equilibria in economies with a local public good. *Journal of Economic Theory 14*, 84–112.

Robert Bredereck
Institut für Eletrotechnik und Theoretische Informatik
Technische Universität Berlin
Ernst Reuter Platz 7, Sekr. TEL 5-1, 10587 Berlin, Germany
Email: `robert.bredereck@tu-berlin.de`

Jiehua Chen
Institut für Eletrotechnik und Theoretische Informatik
Technische Universität Berlin
Ernst Reuter Platz 7, Sekr. TEL 5-1, 10587 Berlin, Germany
Email: `jiehua.chen@tu-berlin.de`

Gerhard J. Woeginger
Department of Mathematics and Computer Science
TU Eindhoven
P.O. Box 513, 5600 MB Eindhoven, the Netherlands
Email: `gwoegi@win.tue.nl`