Proportional Representation as Resource Allocation: Approximability Results

Piotr Skowron[1] and **Piotr Faliszewski**[2] and **Arkadii Slinko**[3]

Abstract. We model Monroe's and Chamberlin and Courant's multiwinner voting systems as a certain resource allocation problem. We show that for many restricted variants of this problem, under standard complexity-theoretic assumptions there are no constant-factor approximation algorithms. Yet, we also show cases where good approximation algorithms exist (these variants correspond to optimizing total voter satisfaction under Borda scores, within Monroe's and Chamberlin and Courant's voting systems).

1 Introduction

Resource allocation is one of the most important issues in multiagent systems, equally important both to human societies and to artificial software agents [20]. For example, if there is a set of items (or a set of bundles of items) to distribute among agents then we may use one of many auction mechanisms (see, e.g., [15,20] for an introduction and a review, and numerous recent papers on auction theory for current results). Typically, in auctions if an agent obtains an item (a resource) then this agent has exclusive access to it. In this paper we consider resource allocation for items that can be shared, and we are interested in computing (approximately) optimal assignments (focusing on cases that reduce to multiwinner voting). We do not make any strategic considerations.

Let us explain our resource allocation problem through an example. Consider a company that wants to provide free sport classes to its employees. We have a set $N = \{1, \ldots, n\}$ of employees and a set $A = \{a_1, \ldots, a_m\}$ of classes. Naturally, not every class is equally appealing to each employee and, thus, each employee orders the classes from the most desirable one to the least desirable one. For example, the first employee might have preference order $a_1 \succ a_3 \succ \cdots \succ a_m$, meaning that for him or her a_1 is the most attractive class, a_3 is second, and so on, until a_m, which is least appealing. Further, each class a_i has some maximum capacity cap_{a_i}, that is, a maximum number of people that can comfortably participate, and a cost, denoted c_{a_i}, of opening the class (independent of the number of participants). The company wants to assign the employees to the classes so that it does not exceed its sport-classes budget B and the employees' satisfaction is maximal (or, equivalently, their dissatisfaction is minimal).

There are many ways to measure (dis)satisfaction. For example, we may measure an employee's dissatisfaction as the position of the class to which he or she was assigned in his or her preference order (and satisfaction as m less the voter's dissatisfaction). We may demand that, for example, the maximum dissatisfaction of an employee is as low as possible (minimal satisfaction is as high as possible; in economics this corresponds to egalitarian social welfare) or that the sum of dissatisfactions is minimal (the sum of satisfactions is maximal; this corresponds to the utilitarian approach in economics).

It turns out that our model generalizes two well-known multiwinner voting rules; namely, those of Monroe [13] and of Chamberlin and Courant [7]. Under both these rules voters from the set N submit preference orders regarding alternatives from the set A, and the goal is to select K candidates (the representatives) best representing the voters. For simplicity, let us assume that K divides $\|N\|$.[4] Under Monroe's rule we have to match each selected representative to $\frac{\|N\|}{K}$ voters so that each voter has a unique representative and so that the sum of voters' dissatisfactions is minimal (dissatisfaction is, again, measured by the position of the representative in the voter's preference order). Chamberlin and Courant's rule is similar except that there are no restrictions on the number of voters a given alternative represents (in this case it is better to think of the alternatives as political parties rather than particular politicians). It is easy to see that both methods are special cases of our setting: For example, for Monroe it suffices to set the "cost" of each alternative to be 1, to set the budget to be K, and to set the "capacity" of each alternative to be $\frac{\|N\|}{K}$. We can consider variants of these two systems using different measures of voter (dis)satisfaction, as indicated above (see also the works of Potthoff and Brams [17], Betzler et al. [3] and of Lu and Boutilier [12]).

It is known that both Monroe's method and Chamberlin and Courant's method are NP-hard to compute in essentially all nontrivial settings [3,12,18]. This holds even if various natural parameters of the election are low [3]. Notable exceptions include, e.g., the case where K is bounded by a fixed constant and the case where voter preferences are single-peaked [3].

Nonetheless, Lu and Boutilier [12]—starting from a very different motivation and context—propose to rectify the high computational complexity of Chamberlin and Courant's system by designing approximation algorithms. In particular, they show that if one focuses on the sum of voters' satisfactions, then there is a polynomial-time approximation algorithm with approximation ratio $(1 - \frac{1}{e}) \approx 0.63$ (i.e., their

[1] University of Warsaw, Warsaw, Poland
[2] AGH University of Science and Technology, Krakow, Poland
[3] University of Auckland, Auckland, New Zealand

[4] This assumption does not affect our results. Our algorithms maintain their quality without it. Yet, modeling Monroe's and Chamberlin and Courant's systems without it would be more tedious.

algorithm outputs an assignment that achieves no less than about 0.63 of optimal voter satisfaction). Unfortunately, total satisfaction is a tricky measure. For example, under standard Chamberlin and Courant's system, a $\frac{1}{2}$-approximation algorithm is allowed to match each voter to an alternative somewhere in the middle of this voter's preference order, even if there is a feasible solution that matches each voter to his or her most preferred candidate. On the other hand, it seems that a $\frac{1}{2}$-approximation focusing on total dissatisfaction would give results of very high quality.

The goal of this paper is to provide an analysis of our resource allocation scenario, focusing on approximation algorithms for the special cases of Monroe's and Chamberlin and Courant's voting systems. We obtain the following results:

1. Monroe's and Chamberlin and Courant's systems are hard to approximate up to any constant factor for the case where we measure dissatisfaction, irrespective of whether we measure the total dissatisfaction or the dissatisfaction of the most dissatisfied voter (Theorems 1 and 2).
2. Monroe's and Chamberlin and Courant's systems are hard to approximate within any constant factor for the case where we measure satisfaction of the least satisfied voter (Theorems 3 and 4). However, there are good approximation algorithms for total satisfaction—for the Monroe's system we achieve approximation ratio arbitrarily close to 0.715 (and often a much better one; see Section 4). For Chamberlin and Courant's system we give a polynomial-time approximation scheme (Theorem 9).

Related work. Hardness of winner determination for multiwinner voting rules was studied by Procaccia, Rosenschein, and Zohar [18], by Lu and Boutilier [12] (who also gave the first approximation algorithm for Chamberlin and Courant's system), and by Betzler, Slinko and Uhlman [3]. Naturally, there is also a well-established line of work on winner-determination for single-winner voting rules, with results for, for example, Dodgson's rule [2,5,6,10], Ranked Pairs method [4], and many others.

In the context of resource allocation, our model resembles multi-unit resource allocation with single-unit demand [20, Chapter 11] (see also the work of Chevaleyre et al. [8] for a survey of the most fundamental issues in the multiagent resource allocation theory). The problem of multi-unit resource allocation is mostly addressed in the context of auctions (and so it is referred in the literature as multi-unit auctions); in contrast, we consider the problem of finding a solution maximizing the social welfare given the agents' preferences. More generally, our model is similar to resource allocation with sharable indivisible goods [1,8]. The most substantial difference is that we require each agent to be assigned to exactly one alternative. In the context of resource allocation with sharable items, it is often assumed that the agents' satisfaction is affected by the number of agents using the alternatives (the congestion on the alternatives; compare to congestion games [19]). Finally, it is worth mentioning that in the literature on resource allocation it is common to consider other criteria of optimality, such as envy-freeness [11], Pareto optimality, Nash equilibria [1], and others.

Our paper is very close in spirit (especially in terms of the motivation of the resource allocation problem) to the recent work of Darmann et al. [9].

2 Preliminaries

Alternatives, Profiles, Positional Scoring Functions. For each $n \in \mathbb{N}$, we take $[n]$ to mean $\{1, \dots, n\}$. We assume that there is a set $N = [n]$ of *agents* and a set $A = \{a_1, \dots a_m\}$ of *alternatives*. Each agent i has *weight* $w_i \in \mathbb{N}$, and each alternative a has *capacity* $\mathrm{cap}_a \in \mathbb{N}$ and cost $c_a \in \mathbb{N}$. The weight of an agent corresponds to its size (measured in some abstract way). An alternative's capacity gives the total weight of the agents that can be assigned to it, and its cost gives the price of selecting the alternative (the price is the same irrespective of the weight of the agents assigned to the alternative). Further, each agent i has a *preference order* $\succ_i$ over A, i.e., a strict linear order of the form $a_{\pi(1)} \succ_i a_{\pi(2)} \succ_i \cdots \succ_i a_{\pi(m)}$ for some permutation π of $[m]$. For an alternative a, by $\mathrm{pos}_i(a)$ we mean the position of a in i'th agent's preference order. For example, if a is the most preferred alternative for i then $\mathrm{pos}_i(a) = 1$, and if a is the most despised one then $\mathrm{pos}_i(a) = m$. A collection $V = (\succ_1, \dots, \succ_n)$ of agents' preference orders is called a *preference profile*. We write $\mathcal{L}(A)$ to denote the set of all possible preference orders over A. Thus, for preference profile V of n agents we have $V \in \mathcal{L}(A)^n$.

In our computational hardness proofs, we will often include subsets of alternatives in the descriptions of preference orders. For example, if A is the set of alternatives and B is some nonempty strict subset of A, then by saying that some agent has preference order of the form $B \succ A - B$, we mean that this agent ranks all the alternatives in B ahead of all the alternatives outside of B, and that the order in which this agent ranks alternatives within B and within $A - B$ is irrelevant (and, thus, one can assume any easily computable order).

A *positional scoring function* (PSF) is a function $\alpha^m : [m] \to \mathbb{N}$. A PSF α^m is an *increasing positional scoring function* (IPSF) if for each $i, j \in [m]$, if $i < j$ then $\alpha^m(i) < \alpha(j)$. Analogously, a PSF α^m is a *decreasing positional scoring function* (DPSF) if for each $i, j \in [m]$, if $i < j$ then $\alpha^m(i) > \alpha^m(j)$.

Intuitively, if β^m is an IPSF then $\beta^m(i)$ gives the *dissatisfaction* that an agent suffers from when assigned to an alternative that is ranked i'th on his or her preference order. Thus, we assume that for each IPSF β^m it holds that $\beta^m(1) = 0$ (an agent is not dissatisfied by his or her top alternative). Similarly, a DPSF γ^m measures an agent's satisfaction and we assume that for each DPSF γ^m it holds that $\gamma^m(m) = 0$.

We will often speak of families α of IPSFs (DPSFs) of the form $\{\alpha^m \mid m \in \mathbb{N}, \alpha^m \text{ is a PSF}\}$, where the following holds:

1. If we are dealing with IPSFs, then for each $m \in \mathbb{N}$ it holds that $(\forall i \in [m])[\alpha^{m+1}(i) = \alpha^m(i)]$.
2. If we are dealing with DPSFs, then for each $m \in \mathbb{N}$ it holds that $(\forall i \in [m])[\alpha^{m+1}(i+1) = \alpha^m(i)]$.

In other words, we build our families of IPSFs (DPSFs) by appending (prepending) values to functions with smaller domains. We assume that each function α^m from a family can be computed in polynomial time with respect to m. To simplify notation, we will refer to such families of IPSFs (DPSFs) as *normal* IPSFs (normal DPSFs).

We are particularly interested in normal IPSFs (normal DPSFs) corresponding to the Borda count method. That is, in the families of IPSFs $\alpha^m_{\mathrm{B,inc}}(i) = i - 1$ (in the families of DPSFs $\alpha^m_{\mathrm{B,dec}}(i) = m - i$).

Our Resource Allocation Problem. We consider a problem of finding function $\Phi : N \to A$ that assigns each agent to some alternative (we will call Φ an *assignment function*). We say that Φ is feasible if for each alternative a it holds that the total weight of the agents assigned to it does not exceed its capacity cap_a. Further, we define the cost of assignment Φ to be $\mathrm{cost}(\Phi) = \sum_{a:\Phi^{-1}(a)\neq\emptyset} c_a$.

Given an IPSF (DPSF) α^m, we consider two *dissatisfaction functions*, $\ell_1^\alpha(\Phi)$ and $\ell_\infty^\alpha(\Phi)$, (two *satisfaction functions*, $\ell_1^\alpha(\Phi)$ and $\min^\alpha(\Phi)$):

1. $\ell_1^\alpha(\Phi) = \sum_{i=1}^n \alpha(\mathrm{pos}_i(\Phi(i)))$.
2. $\ell_\infty^\alpha(\Phi) = \max_{i=1}^n \alpha(\mathrm{pos}_i(\Phi(i)))$ (or, $\min^\alpha(\Phi) = \min_{i=1}^n \alpha(\mathrm{pos}_i(\Phi(i))))$.

The former one measures agents' total dissatisfaction (satisfaction), whereas the latter one considers the most dissatisfied (the least satisfied) agent only. In welfare economics and multiagent resource allocation theory the two metrics correspond to, respectively, utilitarian and egalitarian social welfare. We define our resource allocation problem as follows.

Definition 1 *Let α be a normal IPSF. An instance of α-*ASSIGNMENT-INC *problem consists of a set of agents $N = [n]$, a set of alternatives $A = \{a_1, \dots a_m\}$, a preference profile V of the agents, a sequence $(w_1, \dots, w_n)$ of agents' weights, sequences $(\mathrm{cap}_{a_1}, \dots, \mathrm{cap}_{a_m})$ and $(c_{a_1}, \dots, c_{a_m})$ of alternatives' capacities and costs, respectively, and budget $B \in \mathbb{N}$. We ask for the assignment function Φ such that: (1) $\mathrm{cost}(\Phi) \le B$, (2) $\forall_{a\in A} \sum_{i:\Phi(i)=a} w_i \le cap_a$, and (3) $\ell_1^\alpha(\Phi)$ is minimized.*

In other words, in α-ASSIGNMENT-INC we ask for a feasible assignment that minimizes the total dissatisfaction of the agents without exceeding the budget.

Problem α-ASSIGNMENT-DEC is defined identically except that α is a normal DPSF and in the third condition we seek to maximize $\ell_1^\alpha(\Phi)$ (that is, in α-ASSIGNMENT-DEC our goal is to maximize total satisfaction). If we replace ℓ_1^α with ℓ_∞^α in α-ASSIGNMENT-INC then we obtain problem α-MINMAX-ASSIGNMENT-INC, where we seek to minimize the dissatisfaction of the most dissatisfied agent. If we replace ℓ_1^α with $\min^\alpha$ in α-ASSIGNMENT-DEC then we obtain problem α-MINMAX-ASSIGNMENT-DEC, where we seek to maximize the satisfaction of the least satisfied agent.

Focusing on either satisfaction or dissatisfaction is immaterial from the perspective of the optimal solution, but leads to very different approximation properties.

Clearly, each of our four ASSIGNMENT problems is NP-complete: Even without costs they reduce to the standard NP-complete PARTITION problem, where we ask if a set of integers (in our case these integers would be agents' weights) can be split evenly between two sets (in our case, two alternatives with the capacities equal to half of the total agent weight). However, in very many applications (for example, in the sport classes example from the introduction) it suffices to consider unit-weight agents and we focus on this case.

Our four problems can be viewed as generalizations of Monroe's [13] and Chamberlin and Courant's [7] multiwinner voting systems (see the introduction for their definitions). For Monroe's system, it suffices to set the budget $B = K$, the cost of each alternative to be 1, and the capacity of each alternative to be $\frac{\|N\|}{K}$ (for simplicity, throughout the paper we assume that K divides $\|N\|$). We will refer to such variants of our problems as MONROE-ASSIGNMENT variants. For Chamberlin and Courant's system, it suffices to take the same restrictions as for Monroe's system, except that each alternative has capacity equal to $\|N\|$. We will refer to such variants of our problems as CC-ASSIGNMENT variants.

Approximation Algorithms. For many normal IPSFs α (e.g., for Borda count), even the above-mentioned restricted versions of the original problem, namely, α-MONROE-ASSIGNMENT-INC, α-MINMAX-MONROE-ASSIGNMENT-INC, α-CC-ASSIGNMENT-INC, and α-MINMAX-CC-ASSIGNMENT-INC are NP-complete [3,18] (the same holds for the DEC variants). Thus, we seek approximate solutions.

Definition 2 *Let β be a real number such that $\beta \ge 1$ ($0 < \beta \le 1$) and let α be a normal IPSF (a normal DPSF). An algorithm is a β-approximation algorithm for α-*ASSIGNMENT-INC *problem (for α-*ASSIGNMENT-DEC *problem) if on each instance I it returns a feasible assignment Φ that meets the budget restriction and such that $\ell_1^\alpha(\Phi) \le \beta \cdot \mathrm{OPT}$ (and such that $\ell_1^\alpha(\Phi) \ge \beta \cdot \mathrm{OPT}$), where OPT is the optimal aggregated dissatisfaction (satisfaction) $\ell_1^\alpha(\Phi_{\mathrm{OPT}})$.*

We define β-approximation algorithms for the MINMAX variants analogously. For example, Lu and Boutilier [12] present a $(1 - \frac{1}{e})$-approximation algorithm for the case of CC-ASSIGNMENT-DEC.

Throughout this paper, we will consider each of the MONROE-ASSIGNMENT and CC-ASSIGNMENT variants of the problem and for each we will either prove inapproximability with respect to any constant β (under standard complexity-theoretic assumptions) or we will show an approximation algorithm. We use the following NP-complete problems.

Definition 3 *An instance I of* SET-COVER *consists of set $U = [n]$ (called the ground set), family $\mathcal{F} = \{F_1, F_2, \dots, F_m\}$ of subsets of U, and positive integer K. We ask if there exists a set $I \subseteq [m]$ such that $\|I\| \le K$ and $\bigcup_{i\in I} F_i = U$.* X3C *is a special case of* SET-COVER *where $\|U\|$ is divisible by 3, each member of $\mathcal{F}$ has exactly three elements, and $K = \frac{n}{3}$.*

X3C is NP-hard even if we assume that n is divisible by 2 and each member of U appears in at most 3 sets from $\mathcal{F}$.

3 Hardness of Approximation

In this section we present our inapproximability results for MONROE-ASSIGNMENT and CC-ASSIGNMENT variants of the resource allocation problem. In particular, we show that if we focus on voter dissatisfaction (i.e., on the INC variants) then for each $\beta > 1$, neither Monroe's nor Chamberlin and Courant's system has a polynomial-time β-approximation algorithm. Further, we show that analogous results hold if we focus on the satisfaction of the least satisfied voter.

Naturally, these inapproximability results carry over to more general settings. In particular, unless P = NP, there are no polynomial-time constant-factor approximation algorithms for the general resource allocation problem for the case where we focus on voter dissatisfaction. On the other hand, our results do not preclude good approximation algorithms for the case where we measure agents' total satisfaction.

Theorem 1 *For each normal IPSF α and each constant factor β, $\beta > 1$, there are no polynomial-time β-approximation algorithms for either of α-*MONROE-ASSIGNMENT-INC *and α-*MINMAX-MONROE-ASSIGNMENT-INC*, unless* P = NP.

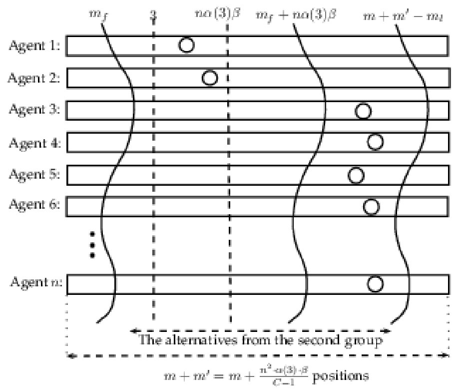

Figure 1. The alignment of the positions in the preference orders of the agents. The positions are numbered from left to right. The left wavy line shows the positions $m_f(\cdot)$, each no greater than 3. The right wavy line shows the positions $m_l(\cdot)$, each higher than $n \cdot \alpha(3) \cdot \beta$. The alternatives from A_2 (one such alternative is illustrated with a circle) are placed only between the peripheral wavy lines. Each alternative from A_2 is placed on the left from the middle wavy line exactly 2 times.

Proof We give a proof for the case of α-Monroe-Assignment-Inc only. Let us fix a normal IPSF α and let us assume, for the sake of contradiction, that there is some constant β, $\beta > 1$, and a polynomial-time β-approximation algorithm $\mathcal{A}$ for α-Monroe-Assignment-Inc.

Let I be an instance of X3C with ground set $U = [n]$ and family $\mathcal{F} = \{F_1, \dots, F_m\}$ of 3-element subsets of U. W.l.o.g., we assume that n is divisible by 6 and that each member of U appears in at most 3 sets from $\mathcal{F}$.

Given I, we build instance I_M of α-Monroe-Assignment-Inc as follows. We set $N = U$ (that is, the elements of the ground set are the agents) and we set $A = A_1 \cup A_2$, where $A_1 = \{a_1, \dots, a_m\}$ is a set of alternatives corresponding to the sets from the family $\mathcal{F}$ and A_2, $\|A_2\| = \frac{n^2 \cdot \alpha(3) \cdot \beta}{2}$, is a set of dummy alternatives needed for our construction. We let $m' = \|A_2\|$ and we rename the alternatives in A_2 so that $A_2 = \{b_1, \dots, b_{m'}\}$. We set $K = \frac{n}{3}$.

We build agents' preference orders using the following algorithm. For each $j \in N$, set $M_f(j) = \{a_i \mid j \in F_i\}$ and $M_l = \{a_i \mid j \notin F_i\}$. Set $m_f(j) = \|M_f(j)\|$ and $m_l(j) = \|M_l(j)\|$; as the frequency of the elements from U is bounded by 3, $m_f(j) \leq 3$. For each agent j we set his or her preference order to be of the form $M_f(j) \succ_j A_2 \succ_j M_l(j)$, where the alternatives in $M_f(j)$ and $M_l(j)$ are ranked in an arbitrary way and the alternatives from A_2 are placed at positions $m_f(j) + 1, \dots, m_f(j) + m'$ in the way described below (see Figure 1 for a high-level illustration of the construction).

We place the alternatives from A_2 in the preference orders of the agents in such a way that for each alternative $b_i \in A_2$ there are at most two agents that rank b_i among their $n \cdot \alpha(3) \cdot \beta$ top alternatives. The following construction achieves this effect. If $(i + j) \bmod n < 2$, then alternative b_i is placed at one of the positions $m_f(j) + 1, \dots, m_f(j) + n \cdot \alpha(3) \cdot \beta$ in j's preference order. Otherwise, b_i is placed at a position with index higher than $m_f(j) + n \cdot \alpha(3) \cdot \beta$ (and, thus, at a position higher than $n \cdot \alpha(3) \cdot \beta$). This construction can be implemented because for each agent j there are exactly $m' \cdot \frac{2}{n} = n \cdot \alpha(3) \cdot \beta$ alternatives $b_{i_1}, b_{i_2}, b_{i_{n\alpha(3)\beta}}$ such that $(i_k + j) \bmod n < 2$.

Let Φ be an assignment computed by $\mathcal{A}$ on I_M. We will show that $\ell_1^{\alpha}(\Phi) \leq n \cdot \alpha(3) \cdot \beta$ if and only if I is a *yes*-instance.

($\Leftarrow$) If there exists a solution for I (i.e., an exact cover of U with $\frac{n}{3}$ sets from $\mathcal{F}$), then we can easily show an assignment in which each agent j is assigned to an alternative from the top $m_f(j)$ positions of his or her preference order (namely, one that assigns each agent j to the alternative $a_i \in A_1$ that corresponds to the set F_i, from the exact cover of U, that contains j). Thus, for the optimal assignment Φ_{OPT} it holds that $\ell_1^{\alpha}(\Phi_{\mathrm{OPT}}) \leq \alpha(3) \cdot n$. In consequence, $\mathcal{A}$ must return an assignment with the total dissatisfaction at most $n \cdot \alpha(3) \cdot \beta$.

($\Rightarrow$) Let us now consider the opposite direction. We assume that $\mathcal{A}$ found an assignment Φ such that $\ell_1^{\alpha}(\Phi) \leq n \cdot \alpha(3) \cdot \beta$ and we will show that I is a *yes*-instance of X3C. Since we require each alternative to be assigned to either 0 or 3 agents, if some alternative b_i from A_2 were assigned to some 3 agents, at least one of them would rank him or her at a position worse than $n \cdot \alpha(3) \cdot \beta$. This would mean that $\ell_1^{\alpha}(\Phi) \geq n \cdot \alpha(3) \cdot \beta + 1$. Analogously, no agent j can be assigned to an alternative that is placed at one of the $m_l(j)$ bottom positions of j's preference order. Thus, only the alternatives in A_1 have agents assigned to them and, further, if agents x, y, z, are assigned to some $a_i \in A_1$, then it holds that $F_i = \{x, y, z\}$ (we will call each set F_i for which alternative a_i is assigned to some agents x, y, z *selected*). Since each agent is assigned to exactly one alternative, the selected sets are disjoint. Since the number of selected sets is $K = \frac{n}{3}$, it must be that the selected sets form an exact cover of U. So I is a *yes*-instance of X3C. □

Is Theorem 1 an artifact of our strict bound on the cost? This seems unlikely as there is also no β-γ-approximation algorithm that finds an assignment with the following properties: (1) the aggregated dissatisfaction $\ell_1^{\alpha}(\Phi)$ is at most β times higher than the optimal one, (2) the number of alternatives to which agents are assigned is at most γK and (3) each selected alternative, is assigned to no more than $\gamma \lceil \frac{n}{K} \rceil$ and no less than $\frac{1}{\gamma} \lceil \frac{n}{K} \rceil$ agents.

Result analogous to Theorem 1 holds for CC as well.

Theorem 2 *For each normal IPSF α and each constant factor β, $\beta > 1$, there are no polynomial-time β-approximation algorithms for either of α-*CC-Assignment-Inc *and α-*Minmax-CC-Assignment-Inc*, unless* P = NP.

The above results show that approximating the minimal dissatisfaction of agents is difficult. On the other hand, if we focus on agents' total satisfaction then constant-factor approximation exist (see [12] and the next section). Yet, the case of satisfying the least satisfied voter remains hard.

Theorem 3 *For each normal DPSF α (where each entry is coded in unary) and each constant factor β, $0 < \beta \leq 1$, there is no β-approximation algorithm for α-*Minmax-Monroe-Assignment-Dec *unless* P = NP.

Unfortunately, for the case of Minmax-CC-Assignment-Dec family of problems our inapproximability argument holds for the case of Borda DPSF only and we show a weaker collapse of W[2] to FPT. (See the book of Niedermeier [14] for an overview of parametrized complexity theory.)

Theorem 4 *Let $\alpha^m_{\mathrm{B,dec}}$ be the Borda DPSF ($\alpha^m_{\mathrm{B,dec}}(i) = m - i$). For each constant factor β, $0 < \beta \leq 1$, there is no β-approximation algorithm for $\alpha^m_{\mathrm{B,dec}}$-*Minmax-CC-Assignment-Dec *unless* FPT = W[2].

Monroe	Chamberlin-Courant
Maximizing total satisfaction (Borda scores)	
Randomized algorithms: (a) $0.715 - \epsilon$; (b) $\frac{1+\frac{K}{m}-\frac{K^2}{m^2-m}+\frac{K^3}{m^3-m^2}}{2} - \epsilon$ Deterministic algorithm: $1 - \frac{K-1}{2(m-1)} - \epsilon$	Deterministic algorithm: $1 - \frac{2\mathrm{w}(K)}{K}$, PTAS For general DPSFs, there is a $(1 - \frac{1}{e})$-approximation algorithm [12]
Minimizing total (minimal) dissatisfaction, maximizing minimal satisfaction	
Inapproximability: Theorems 1 and 3	Inapproximability: Theorems 2 and 4

Table 1. Summary of results for MONROE and CC variants.

4 Approximation Algorithms

We now turn to approximation algorithms for Monroe's and Chamberlin and Courant's rules. Indeed, if one focuses on agents' total satisfaction then it is possible to obtain high-quality approximation results. We show the first non-trivial approximation algorithms for Monroe's system and the first polynomial-time approximation scheme (PTAS) for Chamberlin-Courant's system. These results stand in a sharp contrast to those from the previous section, where we have shown that approximation is hard for essentially all remaining variants of the problem.

Hardness of α-MONROE/CC-ASSIGNMENT lays in selecting the alternatives to assign to agents. Given those, finding the optimal assignment is easy through network-flow arguments (this is implicit in the paper of Betzler et al. [3]).

Monroe's System. A natural iterative approach to solve $\alpha_{\mathrm{B,dec}}$-MONROE-ASSIGNMENT-DEC is to in each step pick some not-yet-assigned alternative a_i (using some criterion) and assign him or her to those $\lceil \frac{N}{K} \rceil$ agents that (a) are not assigned to any other alternative yet, and (b) whose satisfaction of being matched with a_i is maximal. This idea—implemented formally in Algorithm 1—works very well in many cases. (For each positive integer k, we let $H_k = \sum_{i=1}^{k} \frac{1}{i}$ be the k'th harmonic number. Recall that $H_k = \Theta(\log k)$.)

Lemma 5 *Algorithm 1 is a* $(1 - \frac{K-1}{2(m-1)} - \frac{H_K}{K})$*-approximation algorithm for* $\alpha_{\mathrm{B,dec}}$*-MONROE-ASSIGNMENT-DEC that runs in polynomial time.*

Algorithm 1: The algorithm for MONROE-ASSIGNMENT.

Notation: $\Phi \leftarrow$ a map defining a partial assignment, iteratively built by the algorithm.
$\Phi^{\leftarrow} \leftarrow$ the set of agents for which the assignment is already defined.
$\Phi^{\rightarrow} \leftarrow$ the set of alternatives already used in the assignment.
if $K \leq 2$ **then**
  return the solution given by the algorithm of Betzler et al. [3].
$\Phi = \{\}$
for $i \leftarrow 1$ **to** K **do**
  $score \leftarrow \{\}$, $bests \leftarrow \{\}$
  foreach $a_i \in A \setminus \Phi^{\rightarrow}$ **do**
    $agents \leftarrow$ sort $N \setminus \Phi^{\leftarrow}$ so that $j \prec k$ in $agents$ $\implies pos_j(a_i) \leq pos_k(a_i)$
    $bests[a_i] \leftarrow$ chose first $\lceil \frac{N}{K} \rceil$ elements from $agents$
    $score[a_i] \leftarrow \sum_{j \in bests}(m - pos_j(a_i))$
  $a_{best} \leftarrow \mathrm{argmax}_{a \in A \setminus \Phi^{\rightarrow}} score[a]$
  foreach $j \in bests[a_{best}]$ **do**
    $\Phi[j] \leftarrow a_{best}$

Proof Our algorithm computes an optimal solution for $K \leq 2$. Thus we assume $K \geq 3$. Let us consider the situation in the algorithm after the i'th iteration of the outer loop (we have $i = 0$ if no iteration has been executed yet). So far, the algorithm has picked i alternatives and assigned them to $i\frac{n}{K}$ agents (recall that for simplicity we assume that K divides n evenly). Hence, each agent has $\lceil \frac{m-i}{K-i} \rceil$ unassigned alternatives among his or her $i + \lceil \frac{m-i}{K-i} \rceil$ top-ranked alternatives. By pigeonhole principle, this means that there is an unassigned alternative a_ℓ who is ranked among top $i + \lceil \frac{m-i}{K-i} \rceil$ positions by at least $\frac{n}{K}$ agents. To see this, note that there are $(n - i\frac{n}{K})\lceil \frac{m-i}{K-i} \rceil$ slots for unassigned alternatives among the top $i + \lceil \frac{m-i}{K-i} \rceil$ positions in the preference orders of unassigned agents, and that there are $m - i$ unassigned alternatives. As a result, there must be an alternative a_ℓ for whom the number of agents that rank him or her among the top $i + \lceil \frac{m-i}{K-i} \rceil$ positions is at least: $\frac{1}{m-i}\left((n - i\frac{n}{K})\lceil \frac{m-i}{K-i} \rceil\right) \geq \frac{n}{m-i}\left(\frac{K-i}{K}\right)\left(\frac{m-i}{K-i}\right) = \frac{n}{K}$. In consequence, the $\lceil \frac{n}{K} \rceil$ agents assigned in the next step of the algorithm will have the total satisfaction at least $\lceil \frac{n}{K} \rceil \cdot (m - i - \lceil \frac{m-i}{K-i} \rceil)$. Thus, summing over the K iterations, the total satisfaction guaranteed by the assignment Φ computed by Algorithm 1 is at least the following value (see the comment below for the fourth inequality; for the last inequality we assume $K \geq 3$): $\ell_1^{\alpha_b}(\Phi) \geq \sum_{i=0}^{K-1} \frac{n}{K} \cdot \left(m - i - \lceil \frac{m-i}{K-i} \rceil\right) \geq \sum_{i=0}^{K-1} \frac{n}{K} \cdot \left(m - i - \frac{m-i}{K-i} - 1\right) = \sum_{i=1}^{K} \frac{n}{K} \cdot \left(m - i - \frac{m-1}{K-i+1} + \frac{i-2}{K-i+1}\right) = \frac{n}{K}\left(\frac{K(2m-K-1)}{2} - (m-1)H_K + K(H_K - 1) - H_K\right) > (m-1)n\left(1 - \frac{K-1}{2(m-1)} - \frac{H_K}{K}\right)$. The fourth equality holds because $K(H_K - 1) - H_K = \sum_{i=1}^{K}\left(\frac{K}{i} - 1\right) - H_K = \sum_{i=1}^{K}\left(\frac{K}{K-i+1} - 1\right) - H_K = \sum_{i=1}^{K} \frac{i-1}{K-i+1} - H_K = \sum_{i=1}^{K} \frac{i-2}{K-i+1}$. If each agent were assigned to his or her top alternative, the total satisfaction would be equal to $(m-1)n$. Thus we get that $\frac{\ell_1^{\alpha_{\mathrm{B,dec}}}(\Phi)}{\mathrm{OPT}} \leq 1 - \frac{K-1}{2(m-1)} - \frac{H_K}{K}$. □

In the above proof we measure the quality of our assignment against a perhaps-impossible solution, where each agent is assigned to his or her top alternative. Thus for relatively large m and K, and small $\frac{K}{m}$ ratio, the algorithm can achieve a close-to-ideal solution irrespective of the voters' preference orders. This is an argument in favor of Monroe's system.

Betzler et al. [3] showed that for each fixed constant K, $\alpha_{\mathrm{B,dec}}$-MONROE-ASSIGNMENT-DEC can be solved in polynomial time. Thus, for small values of K for which the fraction $\frac{H_K}{K}$ affects the approximation guarantees of Algorithm 1 too much, we can use this polynomial-time algorithm to find an optimal solution. This means that we can essentially disregard the $\frac{H_K}{K}$ part of Algorithm 1's approximation ratio. In consequence, the quality of the solution produced by Algorithm 1 most strongly depends on the ratio $\frac{K-1}{m-1}$. In most cases we can expect it to be small. If it is not, we can use an algorithm that randomly samples K alternatives and matches them optimally to the agents.

Lemma 6 *A single sampling step of the randomized algorithm for* $\alpha_{\mathrm{B,dec}}$-Monroe-Assignment-Dec *achieves expected approximation ratio of* $\frac{1}{2}(1+\frac{K}{m}-\frac{K^2}{m^2-m}+\frac{K^3}{m^3-m^2})$. *Let* p_ϵ *denote the probability that the relative deviation between the obtained total satisfaction and the expected total satisfaction is higher than* ϵ*; for* $K \geq 8$ *we have* $p_\epsilon \leq \exp\left(-\frac{K\epsilon^2}{128}\right)$.

The threshold for $\frac{K}{m}$, where the randomized algorithm is (in expectation) better than the greedy algorithm is about 0.57. Combining the two algorithms, we get the next result.

Theorem 7 *For each fixed* ϵ*, there is an algorithm that provides a* $(0.715-\epsilon)$*-approximate solution for the problem* $\alpha_{\mathrm{B,dec}}$-Monroe-Assignment-Dec *with probability* λ*, in time polynomial with respect to the input size and* $-\log(1-\lambda)$.

Chamberlin and Courant's System. Let us now move on to the Chamberlin and Courant's system. It turns out that the additional freedom of this system allows us to design a polynomial-time approximation scheme for $\alpha_{\mathrm{B,dec}}$-CC-Assignement-Dec.

The idea of our method is to compute a certain value x and to greedily seek an assignment that (approximately) maximizes the number of agents assigned to their top-x alternatives (and match the remaining agents arbitrarily; recall that for nonnegative real numbers, Lambert's W function, $\mathrm{w}(x)$, is defined to be the solution of the equality $x = \mathrm{w}(x)e^{\mathrm{w}(x)}$.)

Lemma 8 *There is a polynomial-time* $(1-\frac{2\mathrm{w}(K)}{K})$*-approximation algorithm for* $\alpha_{\mathrm{B,dec}}$-CC-Assignement-Dec.

(Independently, Oren [16] gave a sampling-based algorithm with expected approximation ratio of $(1-\frac{1}{K+1})(1+\frac{1}{m})$.) Since for each $\epsilon > 0$ there is a value K_ϵ such that for each $K > K_\epsilon$ it holds that $\frac{2\mathrm{w}(K)}{K} < \epsilon$, and $\alpha_{\mathrm{B,dec}}$-CC-Assignment problem can be solved optimally in polynomial time for each fixed constant K (see the work of Betzler et al. [3]), there is a polynomial-time approximation scheme (PTAS) for $\alpha_{\mathrm{B,dec}}$-CC-Assignment (i.e., a family of algorithms such that for each fixed β, $0 < \beta < 1$, there is a polynomial-time β-approximation algorithm for $\alpha_{\mathrm{B,dec}}$-CC-Assignment).

Theorem 9 *There is a PTAS for* $\alpha_{\mathrm{B,dec}}$-CC-Assignment.

5 Conclusions

We have defined a certain (shared) resource allocation problem and have shown that it generalizes multiwinner voting rules of Monroe and of Chamberlin and Courant. Since these rules are hard to compute [3,12,18], we have investigated the possibility of computing approximate solutions. Our results are summarized in Table 1. Except for the case of maximizing total voter satisfaction, both rules turned out to be hard to approximate. However, for the the case of maximizing total voter satisfaction, we have obtained the first nontrivial approximation algorithms for Monroe's rule (our randomized algorithm obtains approximation ratios arbitrarily close to 0.715) and the first PTAS for Chamberlin and Courant's rule. Natural open problems include seeking a PTAS for Monroe's system and empirical evaluation of our algorithms.

Acknowledgements The authors were supported in part by AGH Univ. grant 11.11.120.865, by the Foundation for Polish Science's Homing/Powroty program, and by EU's Human Capital Program "National PhD Programme in Mathematical Sciences" carried out at the University of Warsaw.

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