Problem on near-Skolem sequences

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The well known notions of Skolem sequence, hooked Skolem sequence, near-Skolem sequence and hooked near-Skolem sequence (cf.[2]) can be extended as follows.

Let $n \geq 3$, let $\{p, q\} \subset \{1, 2, \dots, n\}$, $N_{p,q} = \{1, 2, \dots, n\} \backslash \{p, q\}$. Similarly to [1], define a (p, q)*-near-Skolem sequence of order* n to be a sequence $S = (s_1, s_2, \dots, s_{2n-3})$ where $s_i \in N_{p,q} \cup \{0\}$ satisfies

(i) there is exactly one $k \in \{1, 2, \dots, 2n-3\}$ such that $s_k = 0$
(ii) for every $k \in N_{p,q}$ there are exactly two elements $s_i, s_j \in S$, $i < j$, such that $s_i = s_j = k$
(iii) if $s_i = s_j = k$ then $j - i = k$.

The integers p and q are *defects*. A (p, q)-near-Skolem sequence is *perfect* when $s_{2n-3} = 0$, and is *hooked* when $s_{2n-4} = 0$.

For example, $(1, 1, 7, 8, 3, 5, 2, 3, 2, 7, 5, 8, 0)$ is a perfect $(4, 6)$-near-Skolem sequence of order 8, and $(8, 6, 4, 7, 1, 1, 4, 6, 8, 3, 7, 0, 3)$ is a hooked $(2, 5)$-near-Skolem sequence of order 8.

Let $n \geq 7$. We propose the following conjecture.

Conjecture. Let $n \geq 7$.
(i) A perfect (p, q)-near-Skolem sequence of order n exists if and only if either $n \equiv 0, 1$ $(mod\ 4)$ and $p + q$ is even, or $n \equiv 2, 3$ $(mod\ 4)$ and $p + q$ is odd;
(ii) A hooked (p, q)-near-Skolem sequence of order n exists if and only if either $n \equiv 0, 1$ $(mod\ 4)$ and $p + q$ is odd, or $n \equiv 2, 3$ $(mod\ 4)$ and $p + q$ is even.

The condition $n \geq 7$ is necessary. I have verified the above conjecture for all n, $7 \leq n \leq 13$.

References

[1] N. Shalaby, *Skolem Sequences: Generalizations and Applications*, Ph.D. Thesis,
McMaster University 1991.

[2] N. Shalaby, *Skolem and Langford sequences* in: Handbook of Combinatorial Designs (Ed. C.J. Colbourn, J.H. Dinitz) CRC Press 2007 pp. 612–616.