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Chapter 1

Measurement, Units, and Physical Modelling

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Preface

A physical calculation begins before the first number is entered into a calculator. We have to decide what object is being described, which properties matter, what has actually been measured, and what a successful prediction would mean. This chapter develops that beginning. Its central example is an ordinary machine spacer: a short cylindrical part whose mass is initially estimated from an incomplete drawing. A missing bore turns an apparently simple arithmetic task into a question about evidence and models.

The aim is not to collect conversion tricks. It is to learn how to make a quantitative statement that another person can understand, reproduce and test. Units identify the scale of a quantity. Dimensions constrain equations. Estimates expose implausible results. A measurement record distinguishes an instrument indication from an inferred property. Graphs and residuals reveal where a model succeeds and where it needs to change. These habits remain useful when the objects become moving particles, rotating machines, fluids, waves or thermal systems.

The core assumes arithmetic and elementary algebra, not completed mechanics or calculus. Short mathematical explanations accompany powers, ratios, geometry, averages and straight-line graphs. A separate advanced section introduces local derivatives, covariance and dimensionless experimental design. Its problems are explicitly marked; none is a hidden prerequisite for the core pathway.

This opening chapter belongs to *Physics for Technical and Engineering Students - Volume I: Mechanics, Fluids, Waves, and Thermodynamics*. The explanations, teaching datasets, worked examples and figures have been composed for this chapter. The three supplied textbooks inform the comparative reading; current SI definitions and metrological terminology are checked against primary references. Source notes and a bibliography distinguish those roles.

The Guideline

Before asking “Which formula should I use?”, ask “What quantity do I need, what model connects it to the information available, and what observation could show that the model is inadequate?” Carry units through the reasoning and give the conclusion with its conditions, not just its digits.

Pedagogical boundary

The word *force* appears in a dimensional audit, with its dimensions supplied. We do not yet derive a law of motion or a theory of fluid resistance. Likewise, a calibration curve is used to study prediction, not to introduce electronics or full statistical regression. Those subjects require additional physics and mathematics.

Conventions and notation

A quantity symbol is italic; a unit symbol is upright. Thus m may denote mass, whereas m denotes the metre. Square brackets denote dimensions: $[V] = \text{L}^3$. The upright symbols $\text{M}, \text{L}, \text{T}$ denote mass, length and time dimensions; the italic L is a particular length. Numerical table headings such as L/mm mean that the entries are the numerical values of L in millimetres.

Symbol	Meaning in this chapter	Typical unit
m, V, ρ	Mass, material volume, uniform or stated average density	$\text{kg}, \text{m}^3, \text{kg m}^{-3}$
R_o, R_i, L	Outer radius, bore radius, axial length of a spacer	m
A	Cross-sectional area (not an instrument label in equations)	m^2
x, y	Specified input and output quantities	Stated locally
a, b, c_2	Linear coefficient, offset, quadratic coefficient	Stated locally
$\bar{x}, s_x, N$	Sample mean, sample standard deviation, reading count	Unit of x , unit of x , 1
u_x	Standard uncertainty associated with an estimate of x	Unit of x
$\delta x, \Delta x$	A perturbation; a stated finite change	Unit of x
r_j	Data-minus-reference or data-minus-model residual	Output unit
r_{oi}	Correlation coefficient for the two radius inputs	1
β, λ, Π_m	Dimensionless radius ratio, aspect ratio, mass group	1

An interval $[x_{\min}, x_{\max}]$ is a stated allowed range, not automatically a probability statement. The notation $x \pm u_x$ is used only after u_x has been identified as a standard uncertainty. A tolerance, an instrument's display increment, the spread of repeated readings and the standard uncertainty of their mean are different objects. They must not be exchanged merely because they all have units of length.

Exact equalities describe definitions, algebra and stipulated ideal models. The sign $\simeq$ or $\approx$ marks approximations or rounded evaluations. Intermediate calculations retain extra digits; final reporting reflects the quality of the inputs. Unless otherwise stated, geometry is Euclidean, lengths are positive, and the spacer is a uniform cylinder with a concentric circular through-bore. Its material occupies $0 \leq R_i < R_o$ and $L > 0$.

All numerical inspection and calibration data are **constructed teaching data**. Stated material densities are model inputs, not certified properties of an identified alloy. References to later chapters use their topics rather than invented section numbers.

Chapter 1

Measurement, Units, and Physical Modelling

Part I — Measurement and Mathematical Description

1.1 Physical questions and models

A drawing labels the outside radius and length of a small spacer. A technician wants to estimate the mass of a batch before it is weighed. A calculation based on a solid cylinder gives a plausible-looking answer. The actual part, however, has a bore that is missing from the drawing. A calculator cannot discover that omission: it will evaluate the wrong volume with just as many digits as the right one.

This is a useful first physics problem because it separates several tasks that are easily confused. Choosing a model is not the same as evaluating it. Measuring a length is not the same as knowing its exact value. Agreement with one result is not proof of every assumption. A better investigation combines a representation of the object, a quantitative prediction, and an observation capable of challenging it.

Learning goals

After the core pathway, you should be able to explain and test a geometric mass model; convert products and powers of units without changing the physical result; audit dimensions without mistaking consistency for proof; construct defensible estimates and bounds; distinguish resolution, precision, calibration and uncertainty; and compare calibration models using labelled graphs and independent test points. The advanced pathway adds covariance propagation and dimensionless experimental design.

1.1.1 A question must identify a quantity

“How large is this part?” is not yet a precise measurement question. It could refer to its outer diameter, its bore diameter, its length, the volume enclosed by its outer surface, or the volume of material that remains after machining. Each can be useful, but they answer different questions. A quantitative task begins by naming the quantity and defining the object and conditions to which it belongs.

For the spacer we ask for the mass of the dry, clean part alone, excluding packaging and any loose material in the bore. We describe its dimensions at a common, stable laboratory condition; temperature effects are not calculated here. The system boundary follows the material, including the inner wall of the bore. Air inside the hole is not counted as part of the spacer. A drawing should make that decision visible.

Definition 1.1: Physical model and measurand

A **physical model** is an explicit representation of a system together with assumptions and relationships used to explain or predict selected quantities. The **measurand** is the quantity intended to be measured, specified sufficiently clearly for the intended task. An instrument reading becomes a measurement result only through a procedure and interpretation.[5]

The model is deliberately selective. We neglect surface roughness and chamfers at first. We retain the bore because it removes a substantial volume. Whether an omission is acceptable depends on the observable and the required accuracy. A scratch may be irrelevant for estimating batch mass but important for another purpose. “Simple” should mean that unnecessary detail is omitted, not that a convenient formula is used regardless of the object.

1.1.2 From a material boundary to a mass model

For a material of uniform density, each equal volume contains the same mass. If a volume V_0 has mass m_0 , then two such volumes have mass $2m_0$, and subdividing a volume preserves the ratio. This motivates the uniform-density relation

$$m = \rho V, \quad \rho = \frac{m}{V}. \quad (1.1)$$

The ratio m/V can also define an *average* density for a nonuniform object, but that does not mean every small region has that density. Using a density measured on one sample to predict another object’s mass adds the assumption that the material is sufficiently comparable. Defining $\rho = m/V$ using the very mass one then “predicts” would be circular: it would only return the original input.

The cross-sectional area of a circle of radius R is πR^2 . A cylinder of constant cross-section is built by extending that area through an axial length L , giving area times length. Removing a concentric cylindrical bore therefore leaves

$$A = \pi R_o^2 - \pi R_i^2 = \pi(R_o^2 - R_i^2), \quad (1.2)$$

$$V = \pi L(R_o^2 - R_i^2), \quad m = \rho \pi L(R_o^2 - R_i^2). \quad (1.3)$$

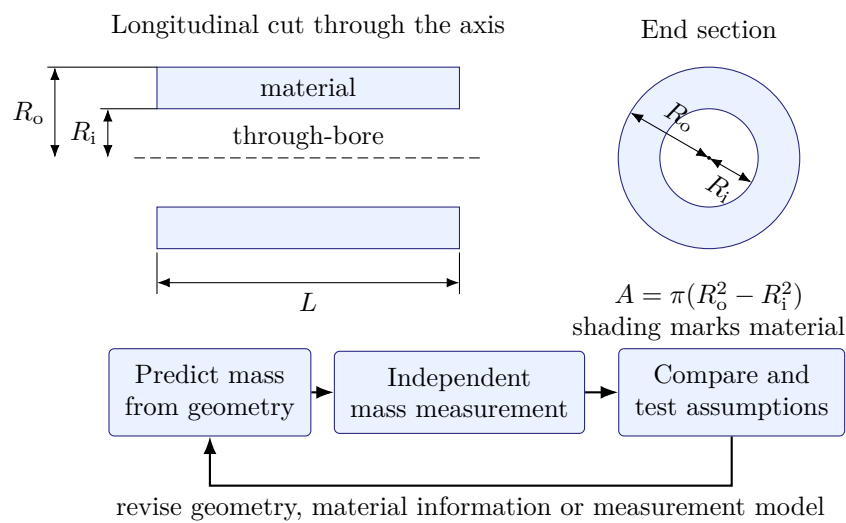


Figure 1.1: F01-01. A sectional cutaway of the spacer and the reasoning cycle. The axis is dashed; shaded regions are material, not the air in the bore. Radii and axial length are different dimensions. Geometry is schematic, not to the numerical scale of an example. The lower arrows show steps in reasoning, not a physical flow of matter.

1.1.3 Check the model before evaluating it

Equation (1.3) has three useful limiting checks. At $R_i = 0$ it becomes the solid-cylinder formula. Increasing R_i at fixed R_o, L, ρ removes material, so it decreases the mass. As R_i approaches R_o from below, the remaining volume approaches zero. A negative predicted mass means that the input geometry is inadmissible or that a sign has been mishandled; it does not describe a negative-mass spacer.

Another check uses the factorisation

$$R_o^2 - R_i^2 = (R_o - R_i)(R_o + R_i). \quad (1.4)$$

The first factor is the radial wall thickness; the second is twice the mean radius. Both are lengths. Their product is an area factor, as it must be. This form will later reveal why a thin wall can make a mass estimate sensitive to dimensional measurements.

Example 1.1: A machine spacer with an unrecorded bore

An incomplete drawing is used to model a spacer as solid. A subsequent inspection finds $R_i = 2R_o/3$. Assume that the bore extends through the full length and that the density is uniform. What fraction of the solid-model mass remains? What would an independent weighing test?

Represent. Figure 1.1 identifies the missing volume. The outer cylinder and the removed cylinder have the same length and density, so a ratio avoids unnecessary numerical inputs.

Relate and solve.

$$\frac{m_{\text{hollow}}}{m_{\text{solid}}} = \frac{\rho\pi L(R_o^2 - R_i^2)}{\rho\pi LR_o^2} = 1 - \left(\frac{R_i}{R_o}\right)^2 = 1 - \frac{4}{9} = \frac{5}{9}.$$

The hollow part contains about 55.6% of the solid-model mass. The omitted bore removes 44.4% of that solid-model mass. Relative to the actual hollow-model mass, the solid prediction is too large by

$$\frac{m_{\text{solid}} - m_{\text{hollow}}}{m_{\text{hollow}}} = \frac{4/9}{5/9} = 0.80.$$

Thus “an 80% overestimate” and “44.4% of the solid volume was removed” have different denominators and are consistent.

Test and interpret. The ratio lies between zero and one and recovers one when the bore vanishes. An independently calibrated balance can test the predicted mass, provided the geometry and density were not themselves inferred from that same weighing. Disagreement does not uniquely identify the bore: an incorrect density, a blind rather than through-bore, a zero offset or attached material could also matter. Inspection plus weighing is more informative than either alone.

Method

A reusable problem-solving sequence. Model the system and state assumptions. Represent its geometry or data. Relate the desired quantity to known quantities symbolically. Solve with consistent units. Test dimensions, signs, limits and an independent estimate. Interpret the result in words, including what remains unknown. These steps are a way to organise reasoning, not a requirement to repeat six headings in every short calculation.

Guided checks

Check. Could a solid cylinder with a smaller density have the same mass as the hollow part?

Answer. Yes. With the same outer dimensions it would need density $5\rho/9$. A mass alone does not distinguish these hypotheses. An observation of the bore or independent material information breaks the ambiguity.

1.1.4 Make the observation independent of the prediction

The word “test” identifies a relationship between two chains of reasoning. In the prediction chain, inspected dimensions and independently justified density information enter a geometric calculation. In the measurement chain, an instrument indication enters a separate measurement model, which may include a tare or calibration correction. These chains meet when their results are compared. They are not independent merely because one is written in a notebook and the other appears on a computer screen. If the density was chosen to reproduce the same balance reading, that reading cannot then validate the chosen density.

Independence here is an experimental design question. Shared inputs may still connect the chains: the same calibration reference, for example, can affect both results. Such connections should be recorded rather than assumed absent. For the core interval comparison, we need only state the allowed ranges in both chains and ask whether they can agree. A statistical interpretation requires more information and is developed in the advanced pathway.

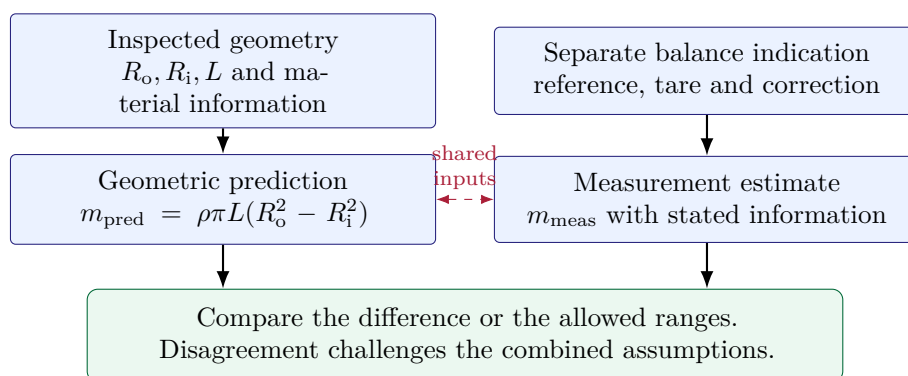


Figure 1.2: Two routes to a mass comparison. Arrows show information dependencies, not motion of the part. A separate balance observation is valuable because it can challenge the geometric prediction; a density inferred from that same observation would reconnect the two routes. The diagram asserts no numerical uncertainty or probabilistic independence.

Example 1.2: One observation, two competing explanations

A part has stipulated outer radius 12.0 mm and length 30.0 mm. A separate weighing is assumed to place its mass in the allowed interval $[58.3, 59.3]$ g. Compare two ideal models: a uniform through-bored part with $R_i = 8.0$ mm and density 7.80 g cm^{-3} , and a solid part with a different density. What is excluded, what remains possible, and which observation distinguishes the survivors?

Prediction. The through-bored model gives $m_H = 18.72\pi \text{ g} \simeq 58.81 \text{ g}$. A solid part at the same 7.80 g cm^{-3} density would give $m_S = 33.696\pi \text{ g} \simeq 105.86 \text{ g}$, outside the stated weighing range. That combination of geometry and density is excluded by this constructed comparison, assuming the weighing range and dimensions are valid.

Alternative inference. A solid model can instead reproduce 58.81 g if its density is

$$\rho_S = \frac{m_H}{\pi L R_o^2} = \rho_H \left[1 - \left(\frac{R_i}{R_o} \right)^2 \right] = \frac{5}{9} (7.80) \text{ g cm}^{-3} = 4.333 \dots \text{ g cm}^{-3}.$$

This is an inferred density, not a certified identification of a material. A single mass permits both models. Direct inspection of a through-bore would distinguish the specified geometries; independent material information would constrain the densities. Either observation adds information that the mass alone lacks.

Check and interpretation. Both candidate masses lie in the stipulated interval. Agreement permits a model but does not uniquely establish it. Selecting the second density because it makes the mass agree and then reporting that agreement as a successful prediction would count the same information twice.

The ambiguity has a useful graphical form. For fixed outer dimensions, all mass predictions depend on the product $\rho(1 - \beta^2)$, where $\beta = R_i/R_o$. Many different pairs of density and bore ratio can produce one measured mass. Inverting a model is possible without making that inversion unique.

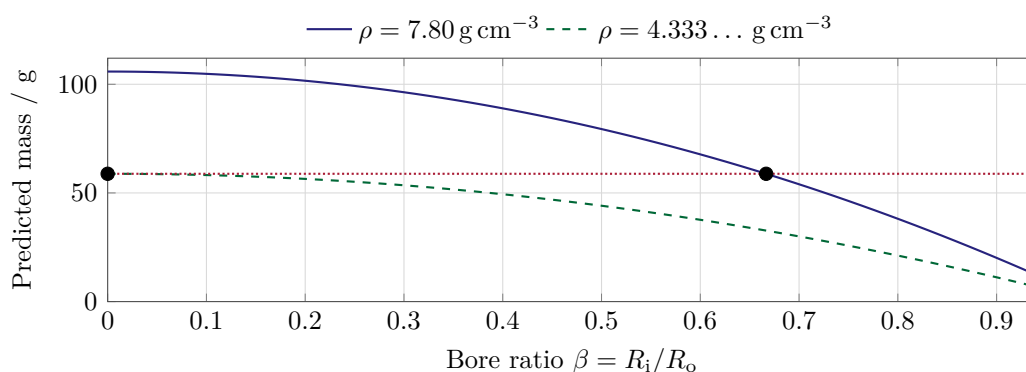


Figure 1.3: Different model inputs can give one mass. The dotted horizontal line is the central comparison value $18.72\pi \text{ g}$, and black points mark the two models in the preceding example. Both curves use $R_o = 12.0 \text{ mm}$ and $L = 30.0 \text{ mm}$. The curves are model predictions, not measured trends; density does not change along either curve.

1.2 SI quantities, units and standards

A length does not become a different physical length when its numerical value changes from 30.0 to 0.0300. The complete statements are $L = 30.0 \text{ mm}$ and $L = 0.0300 \text{ m}$. A number without its unit discards information. The same distinction applies inside a spreadsheet, a program and a laboratory notebook.

1.2.1 The value of a quantity is a number times a unit

For a chosen unit u , write $q = \{q\}_u u$, where $\{q\}_u$ is the numerical value. Changing u changes the number, not q . If a new length unit is one thousand times larger, a given length has a numerical value one thousand times smaller. Areas and volumes require the corresponding square and cube of that factor.

A *quantity equation*, such as $m = \rho V$, is independent of the units used, provided they are combined consistently. A *numerical-value equation*, such as

$$\frac{m}{\text{g}} = \frac{\rho}{\text{g cm}^{-3}} \frac{V}{\text{cm}^3},$$

is written for specified units. Inserting V/mm^3 on its right-hand side without changing the conversion factor would be a different, incorrect equation.

1.2.2 Base and derived units

The International System of Units, SI, retains seven base quantities and units. For this volume, the most frequently used at first are length, mass and time. The others are included to establish a common language for later physics. Table 1.1 lists the unit names and symbols.[4]

Table 1.1: SI base quantities and units. A unit symbol is not an abbreviation and is not pluralised.

Base quantity	Unit name	Symbol
Time	second	s
Length	metre	m
Mass	kilogram	kg
Electric current	ampere	A
Thermodynamic temperature	kelvin	K
Amount of substance	mole	mol
Luminous intensity	candela	cd

Derived units combine base units algebraically. Area has unit m^2 , volume m^3 and density kg m^{-3} . A speed, meaning distance travelled per elapsed time in a simple average, has unit m s^{-1} . The newton, N, is the name for kg m s^{-2} , and the joule, J, for $\text{kg m}^2 \text{s}^{-2}$. Their physical roles are developed later; their unit expressions can already be used to check an equation.

1.2.3 Seven defining constants and practical realisation

The SI fixes the following numerical values exactly. Their displayed digits are definitions, not the outcome of a measurement with a vanishingly small experimental error. The units in the final column belong to one connected system; a constant does not define an isolated unit independently of the others. The values and terminology follow the BIPM account of the SI [4].

Defining quantity	Exact numerical value	Unit
Cs-133 hyperfine transition frequency, $\Delta\nu_{\text{Cs}}$	9 192 631 770	Hz
Vacuum light speed, c	299 792 458	m s^{-1}
Planck constant, h	$6.626\,070\,15 \times 10^{-34}$	J s
Elementary charge, e	$1.602\,176\,634 \times 10^{-19}$	C
Boltzmann constant, k_{B}	$1.380\,649 \times 10^{-23}$	J K^{-1}
Avogadro constant, N_{A}	$6.022\,140\,76 \times 10^{23}$	mol^{-1}
Specified luminous efficacy, K_{cd}	683	lm W^{-1}

Table 1.2: The seven exact SI defining constants. $\Delta\nu_{\text{Cs}}$ refers to the unperturbed ground-state hyperfine transition of caesium-133. K_{cd} applies to monochromatic radiation at 540×10^{12} Hz. The familiar base units are recovered from these constants and their relationships.

The exactness of a definition does not make a clock, ruler, balance or thermometer exact. Realising a unit requires an experiment and an uncertainty budget. The former platinum–iridium kilogram is an historical standard; the present kilogram is defined through h . Likewise, a more accurate experiment improves the realisation of the metre rather than changing the exact assigned value of c .

Two details prevent later confusion. The degree Celsius is an SI derived unit: the numerical temperatures obey $t/^{\circ}\text{C} = T/\text{K} - 273.15$, while a temperature interval has the same numerical value in kelvins and degrees Celsius. Plane angle has SI dimension one, but is a specific kind of quantity;

retaining rad identifies that meaning. In formulas such as $\sin \theta$, use the angle's numerical value in radians, not its numerical value in degrees.

Table 1.3: Exact, measured and stipulated quantities must be distinguished. The parentheses in the value of G identify its standard uncertainty in the final digits.

Quantity or relation	Value	Status
Millimetre	$1 \text{ mm} = 10^{-3} \text{ m}$	Exact unit relation
Speed of light in vacuum	$299\,792\,458 \text{ m s}^{-1}$	Exact SI defining value
Planck constant	$6.626\,070\,15 \times 10^{-34} \text{ J s}$	Exact SI defining value
Gravitational constant	$G = 6.674\,30(15) \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$	Measured; 2022 CODATA recommendation[8]
Spacer density used below	7.80 g cm^{-3}	Stipulated teaching input; no material certification
Instrument reference below	20.000 mm , $u_{\text{ref}} = 0.010 \text{ mm}$	Constructed reference with a stated standard uncertainty

The exactness of a conversion factor does not remove the uncertainty of a measured length. Likewise, using many exact digits of a constant cannot compensate for an incorrectly specified material boundary.

1.2.4 Prefixes, powers and cancellation

SI prefixes denote exact decimal factors. Their capitalisation matters: m is milli, M is mega. Prefixes attach to unit symbols, so $(\text{mm})^2 = (10^{-3}\text{m})^2 = 10^{-6}\text{m}^2$. The prefix is part of the unit being squared. The current list extends from quecto to quetta.[4]

Table 1.4: SI decimal prefixes. The factors belong to the entire prefixed unit before a power is taken.

Factor	Name	Symbol	Factor	Name	Symbol
10^1	deca	da	10^{-1}	deci	d
10^2	hecto	h	10^{-2}	centi	c
10^3	kilo	k	10^{-3}	milli	m
10^6	mega	M	10^{-6}	micro	μ
10^9	giga	G	10^{-9}	nano	n
10^{12}	tera	T	10^{-12}	pico	p
10^{15}	peta	P	10^{-15}	femto	f
10^{18}	exa	E	10^{-18}	atto	a
10^{21}	zetta	Z	10^{-21}	zepto	z
10^{24}	yotta	Y	10^{-24}	yocto	y
10^{27}	ronna	R	10^{-27}	ronto	r
10^{30}	quetta	Q	10^{-30}	quecto	q

A conversion is multiplication by one in a useful form. For example,

$$35 \text{ mm} \left(\frac{1 \text{ m}}{1000 \text{ mm}} \right) = 0.035 \text{ m},$$

For density, both conversion factors are needed:

$$1 \text{ g cm}^{-3} = \frac{10^{-3} \text{ kg}}{(10^{-2} \text{ m})^3} = 10^3 \text{ kg m}^{-3}. \quad (1.5)$$

The factors in a density conversion act in opposite directions: converting a gram to kilograms decreases the numerator's number, but converting a cubic centimetre to cubic metres decreases the

denominator's number even more. This is why a density near 8 g cm^{-3} corresponds to thousands, not thousandths, of kilograms per cubic metre.

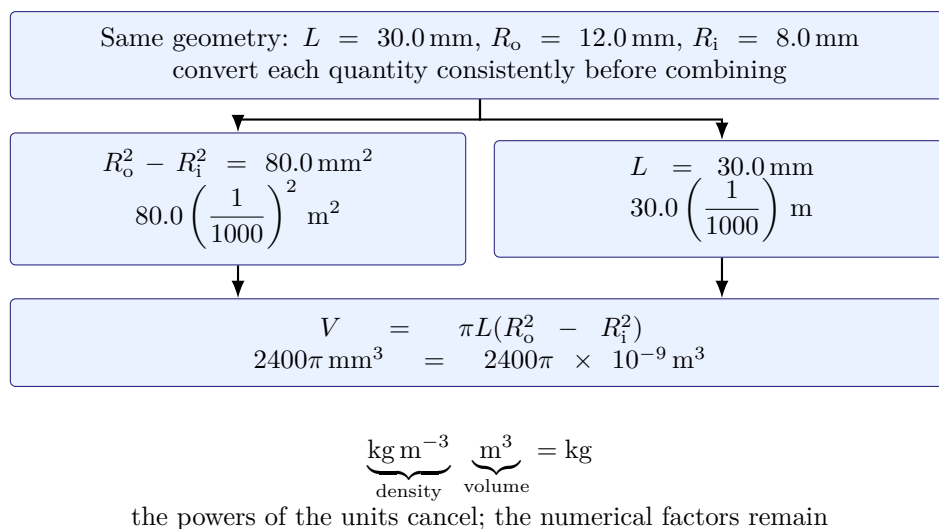


Figure 1.4: F01-02. The length factor occurs once in L and twice in the squared radii. Thus the numerical volume changes by 10^{-9} from mm^3 to m^3 . Boxes describe the same stipulated spacer, not different objects.

Example 1.3: Mixed-unit dimensional inspection

Use $R_o = 12.0 \text{ mm}$, $R_i = 8.0 \text{ mm}$, $L = 30.0 \text{ mm}$ and a stipulated uniform density $\rho = 7.80 \text{ g cm}^{-3}$. Find the spacer mass by two unit routes. These nominal values are model inputs; reporting more digits will not make them measured constants.

Geometry first. $R_o^2 - R_i^2 = (144 - 64) \text{ mm}^2 = 80 \text{ mm}^2$, so

$$V = \pi(30.0)(80) \text{ mm}^3 = 2400\pi \text{ mm}^3 \simeq 7539.82 \text{ mm}^3.$$

Route 1: centimetres and grams. Since $1 \text{ cm}^3 = 1000 \text{ mm}^3$, $V = 2.4\pi \text{ cm}^3 \simeq 7.53982 \text{ cm}^3$. Therefore

$$m = (7.80 \text{ g cm}^{-3})(2.4\pi \text{ cm}^3) = 18.72\pi \text{ g} \simeq 58.81 \text{ g}.$$

Route 2: SI base units. The density is 7800 kg m^{-3} , and the dimensions are 0.0120 , 0.0080 and 0.0300 m . Thus

$$m = (7800)\pi(0.0300)(0.0120^2 - 0.0080^2) \text{ kg} \simeq 0.05881 \text{ kg}.$$

Test. These are the same mass. The solid-model value is $33.696\pi \text{ g} \simeq 105.86 \text{ g}$, and $58.81/105.86 \simeq 5/9$, agreeing with Example 1.1. The approximate calculation $8 \times 7.5 \text{ g} \approx 60 \text{ g}$ checks the scale independently.

Interpretation. A useful nominal prediction is about 59 g . A justified uncertainty or allowed range requires more information, developed below. The displayed guard digits support comparison of methods, not a claim of 0.01 g measurement accuracy.

1.2.5 Writing units so that they can be read unambiguously

Use a space between a number and its unit: 12.0 mm , not a fused identifier. Write kg m^{-3} or kg/m^3 , not an ambiguous chain of slashes. Unit symbols do not take a plural ending: 5 kg , not “kgs”. A symbol such as N begins with a capital letter even though the unit name *newton* does not. The kilogram is the base mass unit, but prefixes for mass attach to the gram: $1 \text{ mg} = 10^{-6} \text{ kg}$, not a compound prefix attached to kg .^[4]

A percentage is a dimensionless ratio times 100%. An angle in radians is also dimensionless, being an arc length divided by a radius; its quantity type remains an angle. “Dimensionless” does not mean “meaningless”. Do not mix degrees and radians inside a formula whose argument is specified in radians. The distinction will matter in waves and rotation.

Caution

Unit consistency and numerical precision solve different problems. A result can have correct units and the wrong magnitude because a prefix was lost. It can have the correct magnitude but unjustified decimal places. It can even have correct units and arithmetic while using the wrong physical model.

1.2.6 Equations for numbers require a declared unit convention

The symbol m denotes a mass; m/g denotes its numerical value in grams. This distinction makes a spreadsheet formula auditable. For the spacer, if the density is entered in g cm^{-3} and every length in millimetres, the numerical-value equation is

$$\frac{m}{\text{g}} = 10^{-3} \pi \frac{\rho}{\text{g cm}^{-3}} \frac{L}{\text{mm}} \left[\left(\frac{R_o}{\text{mm}} \right)^2 - \left(\frac{R_i}{\text{mm}} \right)^2 \right]. \quad (1.6)$$

Every ratio on the right is dimensionless. The 10^{-3} accounts for mm^3 to cm^3 ; it belongs to this numerical convention. It is absent from the unit-independent quantity equation $m = \rho \pi L (R_o^2 - R_i^2)$. Keeping both equations visible explains where an apparent “extra factor” comes from and prevents a unit conversion from being mistaken for new physics.

Example 1.4: A coating specification with squared units

A rectangular panel has stipulated dimensions 25.0 cm by 40.0 cm. A uniform coating adds an areal mass $\sigma = 85.0 \text{ g m}^{-2}$ on one face. Estimate the coating mass and audit a proposed answer of 85 kg. Ignore edge coating and thickness variation; these are explicit model assumptions.

Quantity model. Areal mass is mass per coated area, so $m_c = \sigma A = \sigma ab$. Converting lengths before multiplying gives $A = (0.250 \text{ m})(0.400 \text{ m}) = 0.100 \text{ m}^2$ and

$$m_c = (85.0 \text{ g m}^{-2})(0.100 \text{ m}^2) = 8.50 \text{ g}.$$

Alternatively, $A = 1000 \text{ cm}^2$ and $1 \text{ cm}^2 = 10^{-4} \text{ m}^2$, giving the same 0.100 m^2 . The numerical-value equation for centimetre inputs is

$$\frac{m_c}{\text{g}} = 10^{-4} \frac{\sigma}{\text{g m}^{-2}} \frac{a}{\text{cm}} \frac{b}{\text{cm}}.$$

Repair the proposal. Multiplying 85.0 by 1000 while discarding the units gives 85000, but its original unit product is $\text{g cm}^2 \text{ m}^{-2}$, not grams. Converting the area supplies 10^{-4} . The proposed 85 kg is 10^4 times the correct model mass.

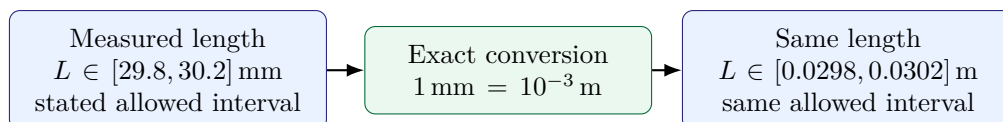
Independent scale check. The area is one tenth of a square metre, so it should receive one tenth of 85 g. The estimate is 8.50 g under the uniform one-face model. No uncertainty was supplied, so the three significant figures describe the stipulated inputs; they do not constitute a coating inspection certificate.

1.2.7 Exact relations do not make measured inputs exact

An integer count, an exact unit conversion and a measured geometric dimension have different statuses. In the statement “100 spacers”, 100 can be an exact count if all parts were counted and the

set is unambiguous. The factor $1 \text{ cm} = 10 \text{ mm}$ is exact by the prefix definitions. A caliper reading of 12.0 mm is neither made exact by its decimal notation nor made less uncertain by expressing it as 0.0120 m .

The constant π is an exact mathematical number, although a finite decimal approximation to it is rounded. In practice, retaining more digits of π is useful until its numerical rounding is negligible for the problem. Thereafter the density, geometry and measurement model control the quality of the prediction. Exact algebra preserves information; it cannot create information that was absent from the inputs.



$$\begin{aligned} &\text{Relative allowed half-width on either side:} \\ &0.2/30.0 = 0.0002/0.0300 = 1/150 \end{aligned}$$

Figure 1.5: Exact conversion scales the numerical value and its allowed half-width together. The brackets denote a stipulated allowed interval, not a standard uncertainty. The relative half-width is unchanged.

1.3 Dimensions and scaling

Dimensions describe the kind of quantity, whereas units describe a chosen scale for its value. Millimetres, metres and kilometres all have dimension L . A density in g cm^{-3} and the same density in kg m^{-3} both have dimension ML^{-3} . Dimensional analysis deliberately ignores the choice of scale to examine the structure of an equation.[1, 2, 3]

1.3.1 Dimensions obey algebra

For multiplication, add exponents of like base dimensions; for division, subtract them. For a power, multiply every exponent by that power. In the spacer model,

$$[\rho\pi L(R_o^2 - R_i^2)] = (ML^{-3})(1)(L)(L^2) = M.$$

The subtraction is permitted because both squared radii have the same dimension. By contrast, $R_o^2 - R_i$ is not an area: it tries to subtract a length from an area. No later multiplication by a constant number repairs that mismatch.

Every term added in an equation must have the same dimensions. A dimensionless numerical coefficient, such as π , cannot supply a missing length or time. A coefficient with units can do so, but its dimensions must be stated rather than silently chosen after the equation fails.

Definition 1.2: Dimensional homogeneity

An equation is **dimensionally homogeneous** when both sides have the same dimensions and every addition or subtraction combines like-dimensional quantities. Arguments of standard functions such as $\log$ and $\exp$ must be dimensionless. This is a necessary condition for a physical quantity equation, not a sufficient test of its truth.

For example, $\log(L/L_0)$ is meaningful for positive lengths L, L_0 , whereas $\log L$ without a reference scale is ambiguous. Its numerical result would change merely by expressing the same length in another unit. A straight-line numerical plot of $\log(L/\text{mm})$ is possible, but the chosen reference unit is then part of the plotted definition.

1.3.2 Finding exponents, and knowing what remains unknown

Suppose a proposed model for the *magnitude* of a resistance force contains density ρ , area A and a non-negative speed v . We supply $[F] = \text{MLT}^{-2}$; its mechanical interpretation comes later. Let

$$F = C\rho Av^q, \quad [C] = 1. \quad (1.7)$$

Since $[\rho] = \text{ML}^{-3}$, $[A] = \text{L}^2$ and $[v] = \text{LT}^{-1}$, the right-hand side has dimensions

$$\text{ML}^{-3+2+q}\text{T}^{-q} = \text{ML}^{q-1}\text{T}^{-q}.$$

Mass agrees automatically. Length requires $q - 1 = 1$, and time requires $-q = -2$. Both give $q = 2$. Checking both equations matters: one agreement alone could conceal an inconsistent input assumption.

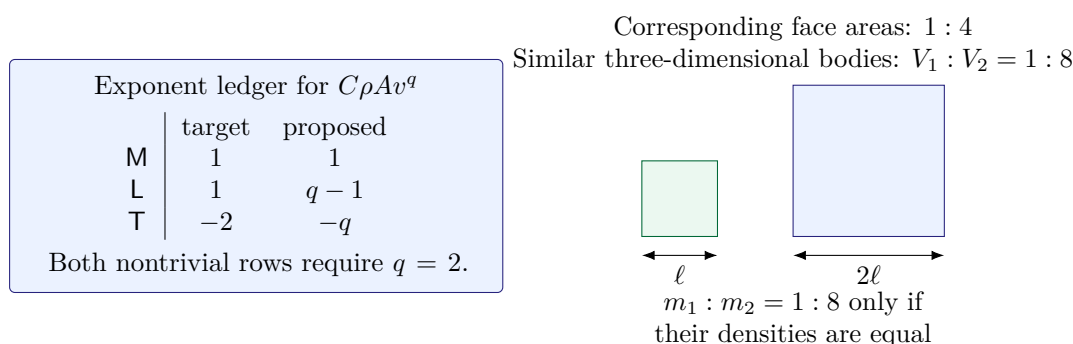


Figure 1.6: F01-03. Dimensional consistency fixes exponents within a stated model, while geometrical similarity fixes area and volume ratios. The right-hand squares represent corresponding faces, not complete mass distributions. Their side lengths are drawn in the stated $1 : 2$ ratio. Neither panel determines a force coefficient or a dynamical regime.

Example 1.5: Audit a proposed force model

A design note proposes $F = C\rho Av^q$ with dimensionless C . Determine q , predict the effect of doubling v within that model, and assess the claim that dimensions have proved the force law.

Solve. The independent exponent balances above give $q = 2$. Holding ρ, A, C fixed gives $F(2v)/F(v) = 4$ for $v > 0$. Density times area times speed squared has units $\text{kg m s}^{-2} = \text{N}$.

Test the claim. Both $F = 0.3\rho Av^2$ and $F = 0.9\rho Av^2$ are dimensionally valid. Dimensions do not choose between them. If another characteristic speed $v_0 > 0$ is relevant, the alternative

$$F = C_0\rho Av^2 \left[1 + \left(\frac{v}{v_0} \right)^2 \right]$$

is equally homogeneous but does not scale simply as v^2 over its entire domain. At $v = v_0$, doubling the speed changes the force by $4(5/2) = 10$, not four.

Interpretation. The exponent $q = 2$ follows from the specified monomial, its variable list and a dimensionless coefficient. The coefficient, completeness of that list, direction of the vector force and physical regime require evidence or further theory. This exercise is a dimensional audit, not a universal drag law.

1.3.3 Geometrical similarity is a conditional prediction

Two bodies are geometrically similar when every corresponding length is multiplied by the same positive factor s . A square face then has s^2 times the area, and a volume has s^3 times the volume.

The spacer provides a direct proof:

$$V' = \pi(sL)[(sR_o)^2 - (sR_i)^2] = s^3V, \quad \frac{m'}{m} = \frac{\rho'}{\rho}s^3. \quad (1.8)$$

Only when the densities are equal does mass scale as s^3 . A body's surface-area-to-volume ratio scales as s^{-1} under the same similarity assumption. This geometrical result can help identify scale effects, but a heat-transfer rate, a strength limit or a fluid force would require its own physical model.

Example 1.6: A scaled part that is not geometrically similar

The spacer's outer and inner radii are doubled while its length is unchanged. Its density remains the same. A student predicts eight times the mass because "all larger parts scale as length cubed." Repair the argument.

Geometry. The cross-sectional area is multiplied by four, but the axial length is not changed. Equation (1.3) therefore gives $m' = 4m$, or about 235 g for the nominal spacer.

Independent check. The new object can be compared with a fully doubled similar spacer, which would have mass $8m$ and twice the original length. Cutting that doubled part into two equal axial pieces gives $4m$ per piece. Similarity is a statement about *all* corresponding lengths; it is not implied by the word "larger".

1.3.4 Dimensions cannot identify every physical quantity

Quantities can share dimensions and still mean different things. Work and torque, for example, both have dimensions ML^2T^{-2} , but later chapters distinguish their definitions and roles. Likewise, a radius and a wall thickness both have dimension length, yet replacing one by the other changes the geometry. Dimensions catch mismatched kinds of quantities, not every mistaken choice within a kind.

A dimensionally correct equation may also violate a physical limit. The proposed spacer formula $m = \rho\pi L(R_o^2 + R_i^2)$ has mass dimensions, but predicts that enlarging an empty bore adds material. The minus sign is supplied by the material model, not by dimensional analysis. The most reliable tests are complementary: units, geometry, limiting cases and observations each reveal errors that the others may miss.

Guided checks

Check. Does doubling only the unit used for length change the physical mass by eight?

Answer. No. Changing units changes numerical values of all dimensional inputs consistently. Physically enlarging a similar object changes the object. Confusing these operations turns a conversion into a fictitious experiment.

1.3.5 Ratios, powers and logarithmic plots

A ratio answers a relative question only after its denominator is specified. If a bore removes the fraction $f = \beta^2$ of a solid cylinder, the fraction remaining is $1 - f$. The overestimate made by retaining the solid model, measured relative to the remaining mass, is $f/(1 - f)$. These are different dimensionless quantities. For $f = 4/9$, removal is $4/9$ of the solid mass while the overestimate is $4/5$ of the hollow mass. A percentage without its comparison quantity invites misunderstanding.

For a positive scale factor s , powers satisfy $s^a s^b = s^{a+b}$, $(s^a)^b = s^{ab}$ and $s^a / s^b = s^{a-b}$. These are algebraic rules, not physical scaling laws. The geometry determines whether a quantity contains one, two or three powers of a changing length. On a logarithmic axis, equal distances correspond to

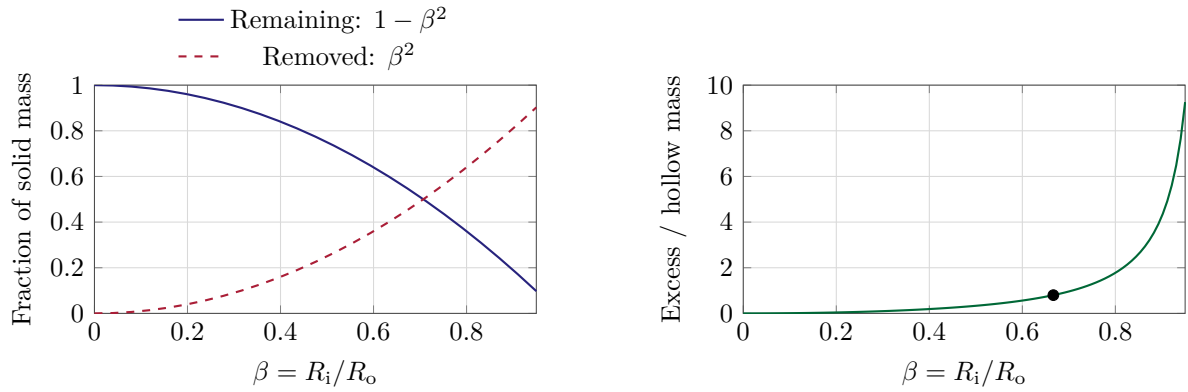


Figure 1.7: The same geometry supports different percentage statements. The left panel uses solid mass as denominator; the right uses hollow mass. The right-hand ratio diverges as $\beta \rightarrow 1$ because the remaining mass approaches zero, although the absolute removed mass remains finite. All curves assume fixed $\rho, L, R_o > 0$.

equal ratios. The interval from 1 to 10 is therefore the same plotted width as that from 10 to 100; it is not the same numerical difference.

If $Q/Q_0 = s^p$, taking base-ten logarithms gives

$$\log_{10}(Q/Q_0) = p \log_{10} s.$$

Both arguments are dimensionless. Thus a log-log plot turns an exact power law into a straight line whose slope is p . This identity does not prove the physical model: curved data, a changing coefficient or an offset can prevent a single power from describing the observations. Neither zero nor negative values can be placed on a conventional logarithmic axis.

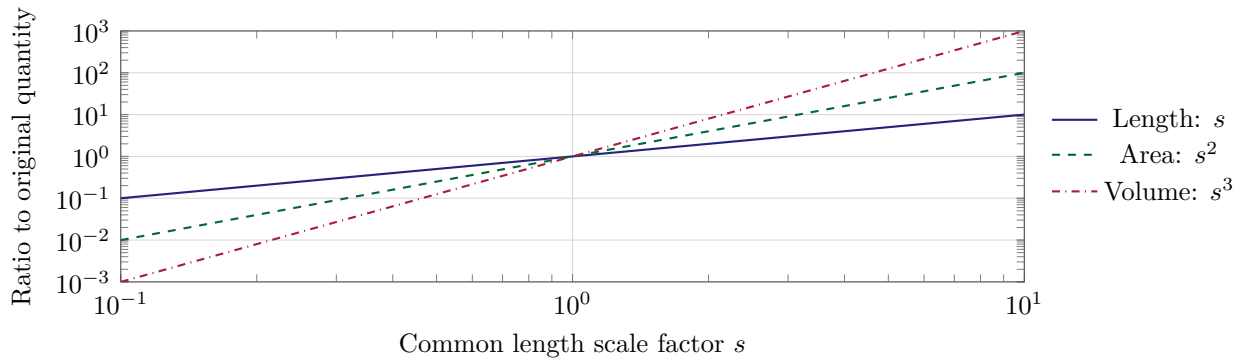


Figure 1.8: Similarity laws on logarithmic axes. Multiplying every corresponding length by ten multiplies area by 10^2 and volume by 10^3 . A scale reduction reverses these factors. Ratios and scale factors are positive and dimensionless; no mass law is asserted unless densities are also specified.

1.3.6 A dimensionally valid expression can hide an omitted scale

Suppose v_0 is a positive characteristic speed omitted from the first force proposal. Both $F/F_0 = z^2$ and $F/F_0 = z^2(1 + z^2)/2$, with $z = v/v_0$ and $F_0 = \rho A v_0^2$, are homogeneous and agree at $z = 1$. Their agreement there does not establish the same prediction at other speeds. Figure 1.9 deliberately compares formulas, not experimental drag data. It shows what dimensional analysis cannot decide.

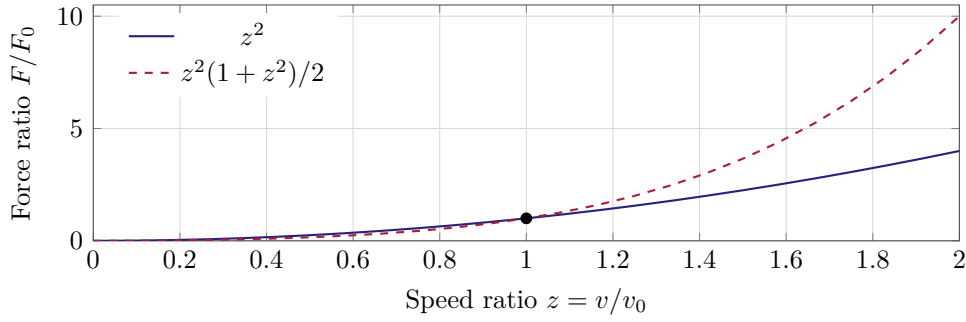


Figure 1.9: Homogeneity leaves dimensionless functional freedom. The black point is agreement at one stipulated speed. At twice that speed the models give 4 and 10 in the same force units. The normalisation uses $\rho, A, v_0 > 0$; these illustrative functions have not been established as physical drag laws.

Example 1.7: Doubling a spacer while retaining its wall thickness

A redesign doubles the spacer's outer radius and axial length, but keeps its radial wall thickness $t = R_o - R_i = 4.0$ mm. The original radii are 12.0 and 8.0 mm. Density is unchanged. Determine the mass ratio and compare it with full geometrical similarity.

Represent. The new outer radius is 24.0 mm. Keeping $t = 4.0$ mm requires the new inner radius to be 20.0 mm, rather than 16.0 mm. The revised cross-section is therefore not similar to the original.

Solve from geometry. For general $s > 0$, $R'_o = sR_o$, $L' = sL$, and $R'_i = sR_o - t$. The domain requires $sR_o \geq t$ so that the inner radius is non-negative. The exact ratio is

$$\frac{m'}{m} = s \frac{(sR_o)^2 - (sR_o - t)^2}{R_o^2 - (R_o - t)^2} = s \frac{2sR_o t - t^2}{2R_o t - t^2} = 1.2s^2 - 0.2s.$$

At $s = 2$, $m'/m = 4.4$. The nominal original mass 58.81 g therefore becomes approximately 258.8 g. Full similarity would instead double the wall thickness and give $8m \simeq 470.5$ g.

Checks. At $s = 1$, both ratios recover one. Directly, $2(24^2 - 20^2)/(12^2 - 8^2) = 2(176)/80 = 4.4$. For large s the fixed-wall model is approximately proportional to s^2 , because its cross-sectional material area grows nearly in proportion to radius while its length also grows. This asymptote is a geometric statement; no strength or thermal performance has been inferred.

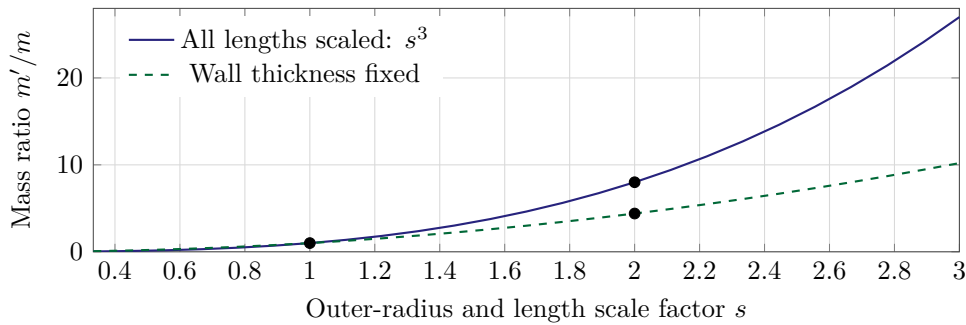


Figure 1.10: Mass scaling depends on the design constraint. Both models scale outer radius and length by s at unchanged density. One also scales wall thickness, while the other keeps $t = 4.0$ mm with original $R_o = 12.0$ mm. The fixed-wall domain is $s \geq 1/3$; below it the algebra would give a negative inner radius and would no longer describe this model.

1.4 Estimation and orders of magnitude

An estimate is a calculation in which the desired accuracy and the available information are deliberately matched. It is not a careless exact calculation. Before using a detailed model, we should know whether its answer ought to be closer to a gram, a kilogram or a tonne. Such a check often finds a unit error faster than repeating a long calculation.[1, 2, 3]

1.4.1 Powers of ten as a working language

Scientific notation writes a nonzero magnitude as $a \times 10^n$, with $1 \leq a < 10$ and integer n . Multiplication adds exponents, division subtracts them:

$$(3 \times 10^{-2})(4 \times 10^{-3}) = 12 \times 10^{-5} = 1.2 \times 10^{-4}.$$

When adding quantities, first express them at a common scale. For example, $3.0 \times 10^{-2} + 4.0 \times 10^{-3} = 3.4 \times 10^{-2}$, not 7.0×10^{-5} . The latter mistake applies the rule for multiplication to addition.

An “order of magnitude” normally means a power-of-ten scale, not a number of decimal places. In this chapter an order estimate usually names the decade and an approximate coefficient: “a few times 10^{-2} kg” is clearer than rounding the mass to a single power of ten. When a *nearest power of ten* is explicitly requested, use closeness in multiplicative factor. The boundary between 10^n and 10^{n+1} is $\sqrt{10} 10^n \simeq 3.16 10^n$, because that number differs from either by the same factor $\sqrt{10}$.

For the spacer, the outside cylinder has an area of order $(10 \text{ mm})^2 \pi$, a length of a few tens of millimetres, and a volume of order $10^4 \text{ mm}^3 = 10 \text{ cm}^3$. A density of a few grams per cubic centimetre suggests a mass of tens of grams. The bore removes a substantial fraction. A result of $5.9 \times 10^4 \text{ g}$ or $5.9 \times 10^{-5} \text{ g}$ would therefore deserve an immediate conversion audit.

1.4.2 An estimate improves when its assumptions improve

Suppose a rectangular component is between 9 and 11 cm long, between 4 and 5 cm wide, and between 1 and 2 cm thick. For a solid rectangular model, positive dimensions make the smallest volume the product of the three lower endpoints and the largest volume the product of the upper endpoints:

$$36 \text{ cm}^3 \leq V \leq 110 \text{ cm}^3.$$

The range is wide mainly because the thickness is poorly specified. Calculating π to more digits would not help, and in this rectangular model π is not needed at all. Measuring thickness is more useful than refining arithmetic.

The same strategy applies to a hollow cylinder, but the inner radius enters with the opposite sign. A larger bore makes a smaller material volume. Before selecting endpoints, read the model as a physical statement: increase quantities that add material when seeking the maximum, and decrease the quantity that removes it.

1.4.3 Constructing physically admissible bounds

Let the density, length and radii each lie in stated intervals. Assume that every combination of their endpoints is allowed and that

$$\rho_{\min} > 0, \quad L_{\min} > 0, \quad 0 \leq R_{i,\min} \leq R_{i,\max} < R_{o,\min}.$$

The last condition keeps a positive wall thickness throughout the entire rectangular input domain. The exact bounds within this model are

$$m_{\min} = \rho_{\min} \pi L_{\min} (R_{o,\min}^2 - R_{i,\max}^2), \quad (1.9)$$

$$m_{\max} = \rho_{\max} \pi L_{\max} (R_{o,\max}^2 - R_{i,\min}^2). \quad (1.10)$$

These expressions follow from monotonicity, without calculus. Squaring preserves order for non-negative radii; multiplying positive ordered factors preserves order; subtracting the bore term reverses its contribution. If the input intervals overlap so that $R_i \geq R_o$ is allowed algebraically, the rectangular domain includes impossible spacers. It must be restricted before its mathematical extrema are interpreted physically.

Intervals may also be linked. A manufacturing drawing may constrain *wall thickness* and outer diameter together, rather than allowing both radii to vary independently. The corner combinations in Eqs. (1.9)–(1.10) could then be inaccessible. Bounds are statements about both a model and its allowed input set.

Example 1.8: A mass range before a part is weighed

For the nominal spacer of Example 1.3, use the following *stipulated allowed intervals*, not standard uncertainties:

$$\begin{aligned} R_o &\in [11.9, 12.1] \text{ mm}, & R_i &\in [7.8, 8.2] \text{ mm}, \\ L &\in [29.8, 30.2] \text{ mm}, & \rho &\in [7.7, 7.9] \text{ g cm}^{-3}. \end{aligned}$$

Find the allowed mass range, and decide which dimensional measurement would be most useful to improve first.

Check the geometry. The largest bore radius, 8.2 mm, is smaller than the smallest outer radius, 11.9 mm. All combinations remain admissible. Assume a uniform through-bored cylinder and independent interval choices in the sense of allowed combinations, not statistical independence.

Evaluate the endpoints. The factor 10^{-3} converts mm^3 to cm^3 :

$$\begin{aligned} m_{\min} &= 7.7\pi(29.8)(11.9^2 - 8.2^2)10^{-3} \text{ g} \simeq 53.6111 \text{ g}, \\ m_{\max} &= 7.9\pi(30.2)(12.1^2 - 7.8^2)10^{-3} \text{ g} \simeq 64.1365 \text{ g}. \end{aligned}$$

An outward-rounded enclosure at 0.1 g resolution is $[53.6, 64.2]$ g. Outward rounding avoids excluding an allowed endpoint. This is not a 68% or 95% interval: no probability distribution has been supplied.

Compare one input at a time. Hold the other three inputs at their nominal values and vary the indicated input to its endpoints. Relative to $m_0 \simeq 58.8106$ g, the changes are

input	smaller mass change	larger mass change
R_o	−2.9875%	+3.0125%
R_i	−4.0500%	+3.9500%
L	−0.6667%	+0.6667%
ρ	−1.2821%	+1.2821%

Interpretation. At these stated interval widths, the bore measurement has the largest individual influence, followed by the outer radius. The unequal positive and negative changes in a radius are due to squaring, not an error in the table. Improving length alone would leave most of the range unchanged. The ranking belongs to this problem: different interval widths or a thinner wall could change it.

1.4.4 Which missing observation would change the conclusion?

A prediction from geometry and density cannot identify a material unambiguously. A light reading could mean a larger hidden cavity, a lower density, an incomplete part or a balance problem. A useful new observation should discriminate among those possibilities. Inspecting whether the bore is through-going tests geometry. Measuring several orientations tests whether a circular model is

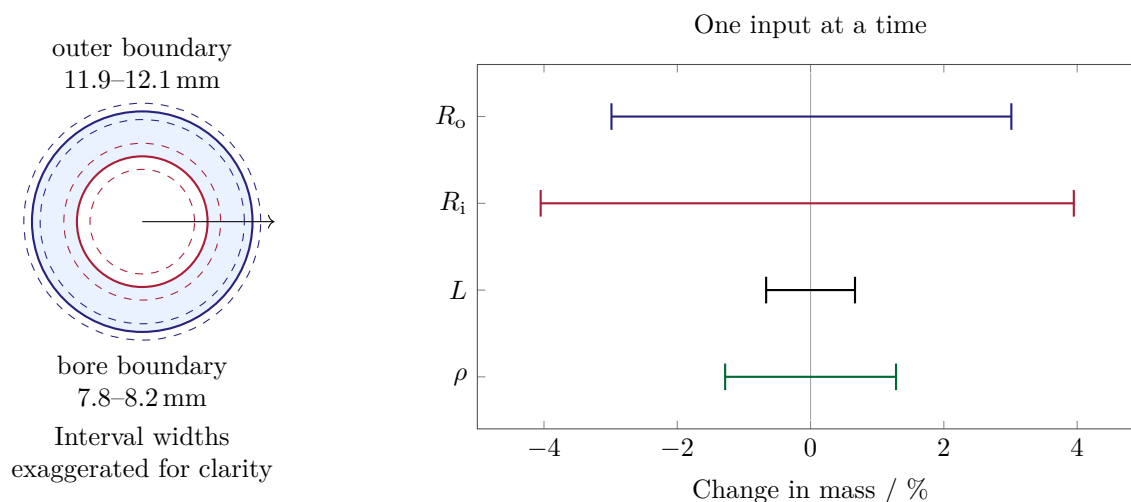


Figure 1.11: F01-04. Allowed radial intervals and a calculated sensitivity comparison for Example 1.8. Dashed circles are limiting boundaries, not probability contours. On the right, each segment joins the exact minimum and maximum change obtained by varying only the named input; the others remain nominal. These segments are not standard-error bars and must not be added as if they were independent variances.

adequate. Comparing with a calibrated mass reference tests the balance. Obtaining reliable material information tests the density assumption.

The nominal result and the interval also answer different questions. The nominal value tells where the chosen central inputs lead. The bounds identify all results allowed by a specified model and input set. Neither, by itself, gives the frequency of masses in a manufactured batch. A tolerance range is not a measured distribution of parts.

Guided checks

Check. A weighing gives 70 g while the stated model permits only 53.6–64.2 g. Should the answer be repaired by printing more digits of the nominal mass? **Answer.** No. After checking the weighing and units, reconsider the geometry, density and interval assumptions. Digits do not enlarge the model's allowed set.

1.4.5 A Fermi estimate is a chain of explicit assumptions

An estimate often starts before any input is measured precisely. Break the desired quantity into factors that are understandable and can be bounded. Then distinguish what is counted, what is approximately known and what is a model assumption. A rounded central calculation can identify a scale; a range calculation identifies the consequences of the input choices. Neither result automatically has a confidence level.

For a product of positive factors, multiplicative uncertainty is particularly transparent. A factor allowed to vary by a ratio of four can matter more than a second factor allowed to vary by only 10%. Improving a tightly known factor first may barely improve the final estimate. Correlations and linked choices can alter the extrema, so multiplying independent endpoint choices is justified only for a rectangular allowed input set.

Example 1.9: A first storage estimate for a space-instrument dataset

A proposed instrument is to store one uncompressed measurement per second during an observing day. For planning, stipulate between 80 and 120 channels, between 4 and 8 h of

observation, and 2 bytes per channel value. The exact time conversion is $1 \text{ h} = 3600 \text{ s}$. Estimate the raw payload, identify the dominant stated range, and say what this estimate omits. The byte here is stipulated as 8 bits, and decimal $1 \text{ MB} = 10^6 \text{ bytes}$ is used explicitly.

Model. If n_c is the number of channels, f the sampling frequency, τ the observing duration and b the bytes per value, then

$$B = n_c f \tau b.$$

The factors count stored values and storage per value. This is a payload accounting model, not an instrument performance law. We assume all channels are sampled at the stated rate throughout the interval and endpoint choices are allowed independently.

Central scale. With $n_c = 100$, $\tau = 6 \text{ h}$, $f = 1 \text{ s}^{-1}$ and $b = 2 \text{ bytes}$, $B_0 = 100(1)(21600)(2) = 4\,320\,000 \text{ bytes}$, or 4.32 MB . A quick check uses about 2×10^4 samples per channel and about 2×10^2 bytes per sample set, giving a few times 10^6 bytes.

Bounds. The allowed rectangular input set gives $B_{\min} = 80(4 \times 3600)(2) = 2.304 \text{ MB}$ and $B_{\max} = 120(8 \times 3600)(2) = 6.912 \text{ MB}$. An outward-rounded planning enclosure is $[2.3, 7.0] \text{ MB}$. These are allowable payload bounds under the stipulations, not a forecast distribution.

Priority and omissions. Channels contribute a maximum-to-minimum factor $120/80 = 1.5$; duration contributes $8/4 = 2$. Better observing-duration information removes the larger of these stated multiplicative ranges. Time tags, file headers, quality flags, packet framing, duplicated records and compression are omitted. Each requires its own accounting model before payload becomes a total storage requirement. A single unusually long observation could also invalidate the specified duration range.

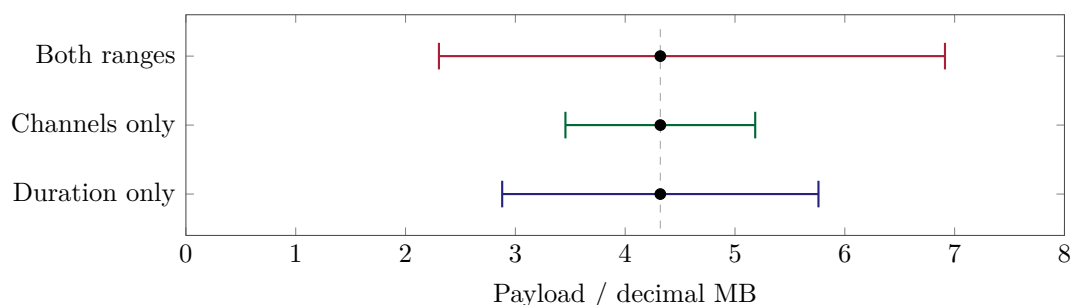


Figure 1.12: Which assumption controls this Fermi estimate? The lower two ranges vary one input while holding the other at its central value; the upper range varies both over the allowed rectangular set. Dots mark the stipulated central calculation, and segments are deterministic allowed ranges. Duration dominates the stated one-input ranges; omitted storage contributions are not included.

1.4.6 Use logarithmic midpoints for multiplicative questions

The arithmetic midpoint of positive bounds a and b is $(a + b)/2$; it is equally distant from the endpoints by subtraction. The geometric midpoint $\sqrt{ab}$ is equally distant by division, since $\sqrt{ab}/a = b/\sqrt{ab} = \sqrt{b/a}$. A logarithmic axis places this geometric midpoint halfway between the bounds. This is a choice of comparison measure, not an assertion that the midpoint is the most probable value.

For the exact payload bounds, the geometric midpoint is $\sqrt{2.304 \times 6.912} \text{ MB} \simeq 3.991 \text{ MB}$. It differs from the 4.32 MB product of stipulated central channel count and duration. There is no contradiction: a product of arithmetic input midpoints need not equal the geometric midpoint of the output range. Recording which central estimate was used makes the result reproducible.

1.5 Measurement quality and reporting

A digital instrument can show a stable last digit and still be wrong by an amount much larger than that digit. Repeating the reading may reproduce the same problem. Measurement quality therefore requires several distinct questions: how finely the instrument responds, how readings vary under stated conditions, how its indication relates to a reference, and what remains uncertain after corrections.[5, 6]

1.5.1 Resolution, precision, accuracy and calibration

Resolution concerns the smallest change in the measured quantity that causes a perceptible change in indication. In an ideal digital example, the display increment provides a useful resolution scale, but a real instrument's resolution can depend on operating conditions. **Precision** concerns agreement among repeated indications under specified conditions. A small spread indicates high precision for those conditions. **Accuracy** concerns closeness to the quantity's value; in practice it is assessed using suitable references and uncertainty information, not by counting displayed digits.[5]

Calibration establishes a relationship between reference values and indications, with their uncertainties. It need not physically adjust an instrument. An offset correction may be applied in the data analysis; an adjustment changes the device. Neither operation guarantees that all other influences have disappeared.[5]

For a simple additive indication model, let

$$x_j = x_* + b + \epsilon_j. \quad (1.11)$$

Here x_* is the model reference value of the quantity, b is a common indication offset and ϵ_j is a reading-dependent deviation. This is a model of how data could arise, not a claim that each deviation is known in a real experiment. Replacing b by $b + \Delta b$ shifts every reading by the same amount but leaves all pairwise differences unchanged. Thus repeatability cannot diagnose every common offset.

The terminology “random” and “systematic” describes different error behaviour, not two exclusive ways of evaluating uncertainty. Statistical analysis of repetitions is often called a Type A evaluation; other information, such as calibration or resolution information, can provide Type B evaluations. Both can contribute standard uncertainties. A known significant bias should be corrected when possible, and the uncertainty of that correction retained.[6]

1.5.2 Averages and residuals preserve different information

For N readings, the sample mean is

$$\bar{x} = \frac{1}{N} \sum_{j=1}^N x_j. \quad (1.12)$$

The summation sign instructs us to add the values indexed by $j = 1, 2, \dots, N$. It is a compact form of $(x_1 + x_2 + \dots + x_N)/N$. A residual relative to a reference is

$$r_j = x_j - x_{\text{ref}}. \quad (1.13)$$

Its sign matters. Positive residuals lie above the reference, and negative residuals below it. Residuals about the sample mean, $x_j - \bar{x}$, sum to zero by construction, even when every reading is biased. A graph centred on the sample mean can therefore hide precisely the offset one needs to investigate.

One measure of spread is the sample standard deviation,

$$s_x = \sqrt{\frac{\sum_{j=1}^N (x_j - \bar{x})^2}{N-1}}, \quad N > 1. \quad (1.14)$$

Squaring prevents positive and negative deviations from cancelling. The square root restores the unit of x . The divisor $N - 1$ is the conventional variance estimate for independent repetitions from a common distribution; one constraint has been used by estimating their mean. A full statistical derivation is not required to compute it, but the assumptions matter when it is interpreted.

For independent repetitions with the same mean and variance, the estimated repeatability contribution to the standard uncertainty of the mean is

$$u_{\bar{x},\text{rep}} = \frac{s_x}{\sqrt{N}}. \quad (1.15)$$

It is not the spread of individual readings and not necessarily the total measurement uncertainty. A common calibration error does not decrease as $1/\sqrt{N}$ as more readings are taken. Neither does drift disappear just because the data file becomes longer.[6]

Example 1.10: Two instruments measuring the same reference

A constructed reference has value $x_{\text{ref}} = 20.000$ mm and standard uncertainty $u_{\text{ref}} = 0.010$ mm. Two synthetic instruments produce the following readings:

j	1	2	3	4	5	6
A/mm	20.18	20.20	20.19	20.21	20.20	20.19
B/mm	19.9	20.1	20.0	20.0	19.9	20.1

Instrument A has display increment 0.01 mm and was constructed with offset +0.20 mm. Its additional deviations are $(-0.02, 0, -0.01, +0.01, 0, -0.01)$ mm. Instrument B has increment 0.10 mm and zero stipulated offset, with the displayed deviations about 20.0 mm. These are controlled data, not random draws or laboratory observations.

Calculate the means and spread.

$$\bar{x}_A = 20.195 \text{ mm}, \quad s_A \simeq 0.01049 \text{ mm}; \quad \bar{x}_B = 20.000 \text{ mm}, \quad s_B \simeq 0.08944 \text{ mm}.$$

The mean reference residuals are +0.195 mm and 0. A is more finely resolved and more tightly clustered, but its uncorrected indications are farther from the reference. B's six-reading mean happens to coincide with the reference; this does not prove that B is exact or universally more accurate.

Apply a correction. Subtracting the stipulated 0.20 mm from every A reading gives mean 19.995 mm. Its spread remains 0.01049 mm because differences from the mean have not changed. In a real calibration the offset itself would have uncertainty. The calculation of total uncertainty must include that information rather than pretending that repeated precision proves perfect calibration.

Test and interpret. All readings sit on their stated display grids. Figure 1.13 uses the same horizontal scale for both instruments. Comparing plots with separately rescaled horizontal axes could conceal the difference in spread or exaggerate it.

1.5.3 What an uncertainty statement does and does not say

A standard uncertainty is expressed as a standard deviation associated with an estimate. It can reflect incomplete knowledge of calibration, environmental corrections, geometric representation and repeatability. Its evaluation is not confined to scatter in repeated readings. An error is a difference from a reference or quantity value; an uncertainty characterises what is not known about the result. They are not synonyms.[6, 5]

If an ideal display rounds to the nearest 0.1 mm, its rounding error is within 0.05 mm under that idealisation. That is a bound for one mechanism. It does not establish a complete uncertainty

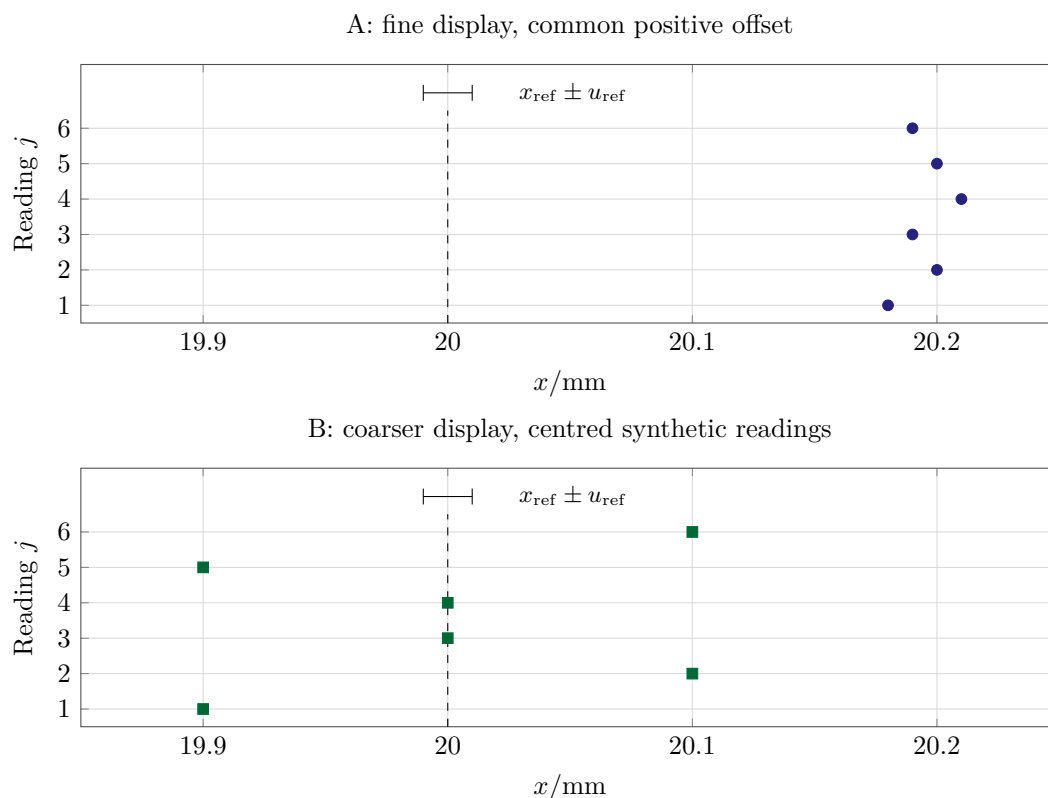


Figure 1.13: F01-05. Aligned dot plots of the constructed readings in Example 1.10; the vertical coordinate identifies a reading, not a second physical measurement. The dashed line is the same reference in both panels. The capped segment shows one stated *standard uncertainty of the reference*, not a tolerance band, a distribution of readings or a guaranteed coverage interval. The display increments, 0.01 mm for A and 0.10 mm for B, are separate from the observed spread.

budget, nor does it say that every value inside the interval is equally probable. An assumption of a uniform rounding distribution would supply additional probabilistic information; it is not supplied by the digit alone.

Similarly, reporting $x = (20.000 \pm 0.010)$ mm with $u_x = 0.010$ mm does not automatically promise 68% coverage. That interpretation requires an appropriate distribution and other qualifications. Expanded uncertainties use a stated coverage factor and a justified coverage interpretation. We do not assign one without the necessary information.[6]

1.5.4 Significant figures are a reporting convention

When a detailed uncertainty is unavailable, introductory calculations often report a product or quotient with no more significant figures than the least precise input, and an addition or subtraction to the coarsest decimal place after bringing the terms to common units. These are useful reporting conventions, not laws of probability.[2, 3]

Leading zeros locate the decimal point and are not significant figures: 0.00420 has three. A trailing zero after a decimal point can communicate a retained place: 4.20 mm is not the same precision claim as 4.2 mm. A bare integer such as 1200 mm is ambiguous without context; 1.20×10^3 mm or an explicit uncertainty removes much of that ambiguity. An exactly counted number of objects, or an exact unit conversion, does not limit a result's precision in the way a measured input does.

Example 1.11: Rounding after, not during, a calculation

A small plate has measured nominal sides $a = 12.34\text{ cm}$ and $b = 5.6\text{ cm}$. Using the usual significant-figure convention, report its rectangular area. Then consider the difference $d = 12.34\text{ cm} - 12.1\text{ cm}$.

Area. Retaining guard digits gives $ab = 69.104\text{ cm}^2$. The two-significant-figure convention gives 69 cm^2 , or $6.9 \times 10^1\text{ cm}^2$. This is not an uncertainty calculation: the actual uncertainty and correlations of the side measurements have not been supplied.

Difference. The unrounded difference is 0.24 cm . With the coarser input given to 0.1 cm , the conventional result is 0.2 cm . Subtraction has exposed a small difference, so retaining many relative digits would be misleading.

Check. Performing the same calculation in millimetres gives 2.4 mm before rounding and 2 mm after the same physical reporting rule. Unit conversion does not create more information. Rounding the raw inputs prematurely can change a small difference appreciably; preserve the original readings and round the reported result once.

1.5.5 A complete record makes another calculation possible

Record the quantity, units, raw indications, reference and calibration information, conditions, applied corrections, uncertainty meaning and model assumptions. Retain raw data rather than replacing them with rounded corrected values. A reader should be able to reconstruct both what the instrument displayed and how the final estimate was obtained.

For example, “ $L = 30.00$ ” omits the unit, the object, the method and any uncertainty meaning. A usable statement might instead read: “Axial length of this part, in millimetres; mean of six repositioned readings after the stated zero correction; u_L is the estimated standard uncertainty including the specified calibration contribution; temperature variation was not separately evaluated.” The details must be true of the actual procedure, not pasted into a report as decoration.

1.5.6 A digital increment can conceal a changing quantity

An ideal nearest-step digital indication with increment $\delta > 0$ can be modelled as

$$I(x) = \delta \text{ round}(x/\delta). \quad (1.16)$$

Here x and I have the same units. The exact convention at half-step boundaries must be specified in an implementation; away from those boundaries the model gives $|I - x| < \delta/2$. The display is a staircase: an entire interval of input values maps to one indication. Consequently, identical displayed readings do not establish an unchanged input.

Suppose the display increment is 0.01 mm and five underlying teaching-model lengths are 20.001 , 20.002 , 20.003 , 20.004 and 20.003 mm . Under Eq. (1.16), all five indicate 20.00 mm . The displayed sample spread is zero, although the underlying values span 0.003 mm . We do not know those hidden values in an actual measurement; they are stipulated here to demonstrate the information lost by quantization. Averaging five identical indications returns 20.00 mm and cannot reconstruct their unobserved digits.

1.5.7 Resolution, bias and repeatability act differently

Increasing the number of readings can improve the estimate of an average when their reading-dependent deviations are independent and comparable. It does not force a common calibration correction to become better known. Nor does it generally average an unresolved fixed rounding effect as if each reading had sampled a new independent rounding error. The sampling procedure, signal variation and instrument behaviour decide whether that model is appropriate.

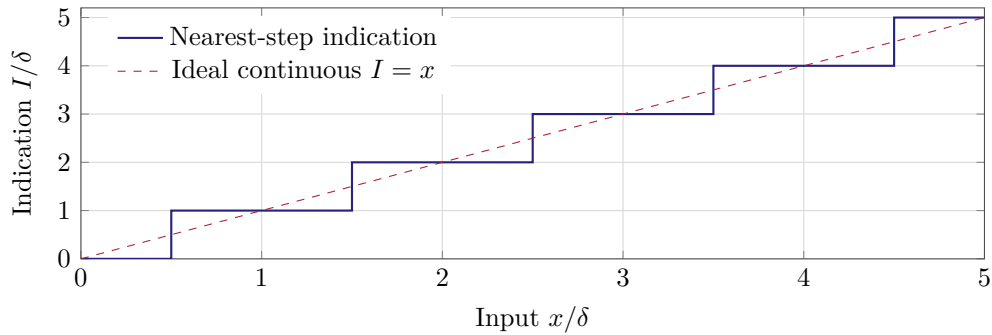


Figure 1.14: An ideal digital indication model, expressed in units of one display step. Horizontal sections represent intervals with unchanged display despite changing input. Vertical joining strokes make the staircase visible; they are not intermediate physical indications. The selection of a value exactly at a half-step boundary is an implementation convention and is not used in the numerical example.

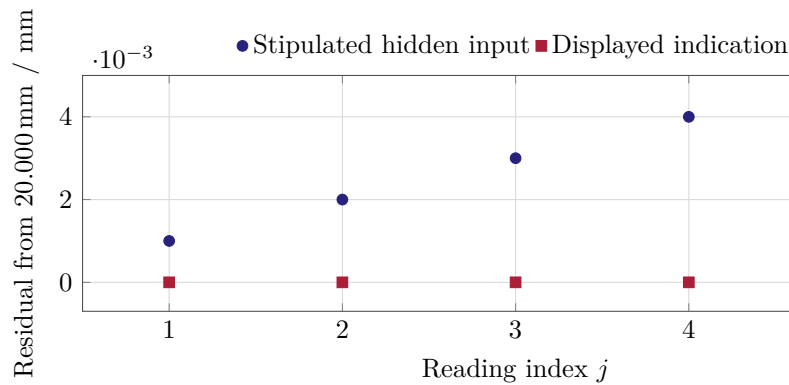


Figure 1.15: Zero spread in displayed readings need not mean zero input variation. All five stipulated values lie within the same nearest-step bin for $\delta = 0.01$ mm. Points are discrete constructed values; no line is drawn to imply an observed continuous trajectory. Hidden inputs are illustrative model values, not extra information available to a real observer.

For example, take an independent repeatability standard deviation $s = 0.020$ mm and a common correction standard uncertainty $u_b = 0.010$ mm. Under the explicitly independent-contribution model used in the computational investigation, the uncertainty of a corrected mean is $u(N) = \sqrt{s^2/N + u_b^2}$. The repeatability component falls with N , but the total approaches u_b . This formula is a stated model for standard uncertainties, unlike the deterministic half-step bound above.

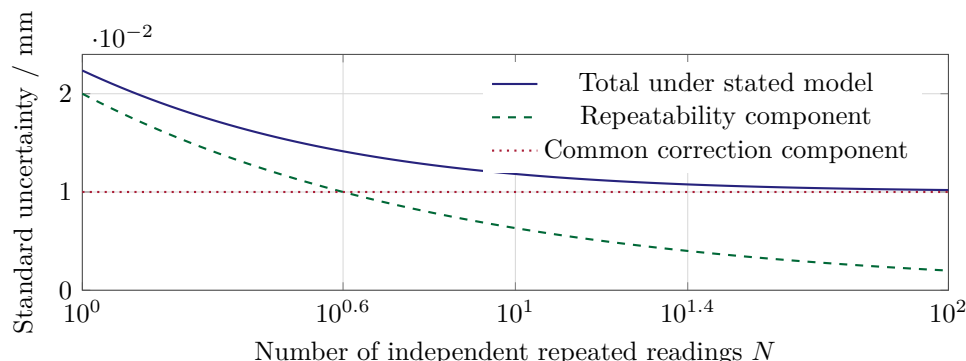


Figure 1.16: Repetition reduces one uncertainty component, not every component. The smooth curves show the algebraic dependence on N ; actual numbers of readings are integers. They assume independent repeatability deviations with fixed $s = 0.020$ mm and an independent common correction with $u_b = 0.010$ mm. No display-resolution contribution is added, and no claim is made that these values describe a real instrument.

1.5.8 Reporting is part of the measurement model

A complete result records the estimated quantity, unit, uncertainty or bound, its meaning and the conditions under which it applies. “ 20.00 ± 0.01 ” leaves both the quantity and unit unspecified and does not say whether 0.01 is a tolerance, a standard uncertainty or a rounding allowance. “Corrected outer diameter (20.00 ± 0.01) mm, with 0.01 mm a standard uncertainty under the stated calibration and repeatability model” makes the central claim interpretable, although the associated record must still explain the inputs and procedure.

Rounding should be done after the main calculation, retaining extra digits in intermediate steps. If an allowed interval is delivered, round its lower endpoint downward and its upper endpoint upward to preserve enclosure. If a standard uncertainty is rounded, the estimate is normally reported at a matching decimal place and the rounding convention recorded when it matters. A numerical formatting rule cannot rescue an incomplete uncertainty budget.

Unit labels also protect computational results. A table column headed L/mm contains numerical values in millimetres; a column headed merely L leaves the convention unstated. The same numerical array could otherwise be interpreted as metres or centimetres. Archive the raw readings and corrections separately so that another reader can reproduce the result, test a different correction, and distinguish information measured directly from information inferred through a model.

1.6 Graphs and model testing

A graph is a quantitative argument. Its axis definitions determine what the coordinates mean; its symbols distinguish information from assumptions; and its residuals show which discrepancies a large plotting scale can hide. A smooth line alone does not demonstrate a physical law.

1.6.1 Straight lines, slope and intercept

For the model

$$y = ax + b, \quad [a] = \frac{[y]}{[x]}, \quad [b] = [y], \quad (1.17)$$

the slope is the output change per input change, $a = (y_2 - y_1)/(x_2 - x_1)$, for distinct inputs on the line. The intercept is the model output at $x = 0$. It may represent an offset, but that interpretation requires a reason to extend the model to zero and a measurement procedure that makes zero meaningful. An intercept is not always an instrument defect.

The plotted line's angle on the page is not the numerical slope. Stretching an axis changes its angle without changing the physical data. Axes labelled x/mm and y/V have numerical slopes in volts per millimetre. Axes labelled x/m and y/V describe the same physical model with a different numerical slope.

1.6.2 Fit, residual and prediction are different operations

A calibration rule uses known inputs and observed outputs to select model parameters. A residual is the difference between a supplied output and its model prediction at the same input:

$$r_j = y_j - y_{\text{model}}(x_j). \quad (1.18)$$

A positive residual means the model underpredicts that output. Plotting residuals against input can reveal curvature; plotting them against acquisition order can reveal drift. Neither pattern by itself uniquely identifies its cause.

A model chosen using a dataset has already used information from those data. Its agreement with them tests descriptive adequacy, not independent predictive success. A stronger test sets aside other inputs or obtains new measurements, fixes the model before examining the outputs, and compares the predictions with those held-out values. In a teaching construction, we can illustrate this logic without pretending to have performed an experiment.

Example 1.12: Calibration over two measurement ranges

An idealised displacement instrument has input x in millimetres and output y in volts. The numerical data are constructed from

$$y_C = 0.10 \text{ V} + (0.20 \text{ V mm}^{-1})x + (0.020 \text{ V mm}^{-2})x^2.$$

Initially only the low-range pairs $(x/\text{mm}, y/\text{V}) = (0, 0.10), (1, 0.32), (2, 0.58)$ are supplied for calibration. Test a linear endpoint calibration and a quadratic description, then predict the held-out outputs at $x = 4$ and 6 mm .

Choose rules explicitly. The line is defined through the two endpoint pairs, not claimed to be a least-squares fit:

$$a_L = \frac{0.58 - 0.10}{2 - 0} \text{ V mm}^{-1} = 0.24 \text{ V mm}^{-1}, \quad b_L = 0.10 \text{ V}.$$

Thus $y_L = 0.10 \text{ V} + (0.24 \text{ V mm}^{-1})x$. Its low-range residuals are $(0, -0.02, 0) \text{ V}$.

Recover the curved rule. Put $y_C = b + ax + c_2x^2$. The point at zero gives $b = 0.10 \text{ V}$. The remaining points imply, in millimetre/volt numerical values,

$$a + c_2 = 0.22, \quad 2a + 4c_2 = 0.48.$$

Subtracting twice the first equation gives $2c_2 = 0.04$, so $c_2 = 0.020 \text{ V mm}^{-2}$ and $a = 0.20 \text{ V mm}^{-1}$. Three parameters can reproduce these three points exactly; that is not independent validation.

Predict before comparing.

x/mm	y_L/V	y_C/V	$y_{\text{held-out}}/\text{V}$
4	1.06	1.22	1.22
6	1.54	2.02	2.02

The held-out values were constructed using the stated quadratic generator and were not used to choose the two candidate rules. Linear residuals grow to $+0.16$ and $+0.48$ V. Quadratic residuals are zero by the construction, not by a discovery about a real sensor.

Interpretation. Suppose a comparison accepts discrepancies up to a stipulated reading bound 0.03 V, ignoring coefficient and input uncertainty for this illustrative test. The low-range discrepancy 0.02 V would not reject the line, but both held-out discrepancies would. A useful next experiment would extend the calibrated input range and test whether curvature persists under a changed acquisition order. It would not simply collect more points at the two endpoints already used.

1.6.3 Changing units without changing predictions

For the endpoint line, writing the input in metres changes the slope to 240 V m^{-1} , not 0.24 V m^{-1} . The same displacement $4 \text{ mm} = 0.004 \text{ m}$ then gives

$$0.10 \text{ V} + (240 \text{ V m}^{-1})(0.004 \text{ m}) = 1.06 \text{ V}.$$

The quadratic coefficients become 200 V m^{-1} and 20000 V m^{-2} . Their powers follow from the dimensions of the terms. Residuals in volts are unchanged. Expressing output in millivolts multiplies the numerical outputs, residuals, intercept and both coefficients by one thousand. The physical conclusion about model mismatch remains the same.

More generally, if numerical coordinates transform as $X' = k_x X$ and $Y' = k_y Y$, then a numerical equation $Y = AX + B$ becomes

$$Y' = \frac{k_y}{k_x} AX' + k_y B.$$

The numerical slope depends on both axis units. The physical relationship does not. This is a useful test of any computational implementation of a calibration.

Example 1.13: Using a calibration in reverse

The instrument gives a constructed output $y = 0.88$ V. Infer x using both fixed calibration rules of Example 1.12 for $x \geq 0$.

Linear inversion. $x_L = (0.88 - 0.10)/0.24 \text{ mm} = 3.25 \text{ mm}$.

Quadratic inversion. Subtract the intercept and divide the numerical equation by 0.02 to obtain $x^2 + 10x - 39 = 0$, with x here the numerical value in millimetres. Factorisation gives $(x - 3)(x + 13) = 0$. The allowed root is $x_C = 3.00 \text{ mm}$; the negative root is outside the stated input domain.

Check. Substitution into the original curves gives 0.88 V in both cases. Correct inversion of an inadequate calibration can still give the wrong displacement. The discrepancy 0.25 mm is a model-choice effect, not a failure of the algebra or an extra standard uncertainty inferred from these two numbers.

1.6.4 Residuals are clues, not unique diagnoses

A curved residual pattern could indicate an omitted nonlinear response, a distorted input scale or an unmodelled environmental effect. A time trend could indicate drift even when the plotted input dependence appears smooth, especially if inputs were always measured in increasing order. Interleaving a reference reading or changing the acquisition order helps distinguish these possibilities.

A proportional relation must pass through the origin: $y = ax$. A linear relation with an intercept need not. On logarithmic axes, a power law $y = Cx^p$ for positive quantities becomes a straight

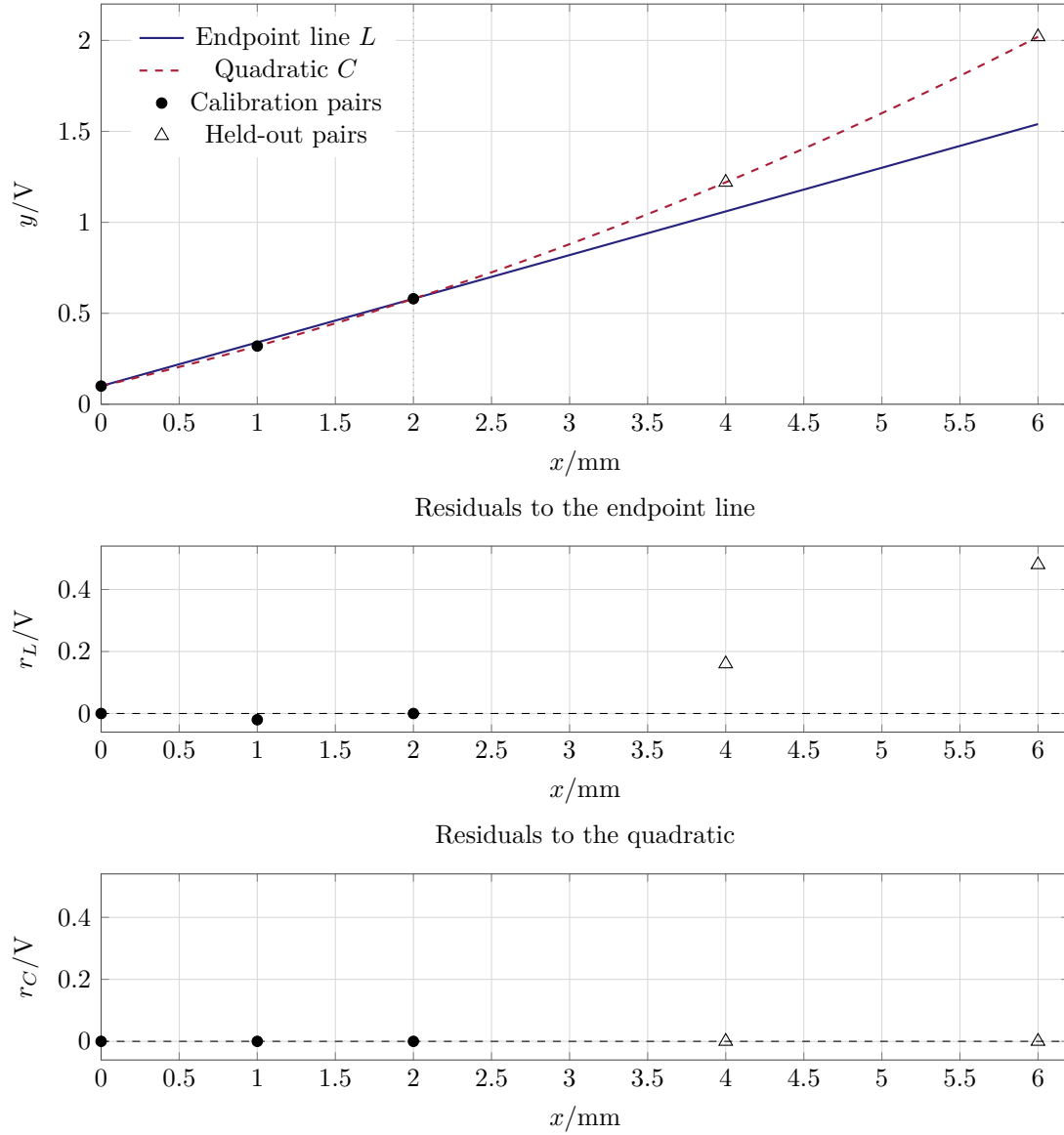


Figure 1.17: F01-06. Two calibration rules and their residuals for the constructed data of Example 1.12. The dotted line marks the upper end of the calibration range. Filled circles were used for calibration; open triangles were held out. Curves are analytical model evaluations, not hand-drawn trends. Both residual panels have the same vertical scale and explicit zero lines; positive residual means data above the model. No error bars or probabilistic coverage are implied.

line only after reference scales make the logarithms dimensionless. The line's slope estimates an exponent within that model; logarithmic plotting does not establish that other variables are absent. Full regression, parameter confidence regions and goodness-of-fit testing are deliberately deferred.

Key message

A good plot distinguishes data from models, fitted points from held-out points, and uncertainty statements from visual emphasis. Ask what was fixed before a comparison was made. A residual that is zero because a parameter was chosen to make it zero is not an independent test of the model.

1.6.5 Endpoint agreement does not identify a model

Two points determine a straight line *if a straight-line model has already been justified*. They do not establish that the physical response is linear. Write the endpoint rule from the earlier calibration as $\ell(x) = 0.10 \text{ V} + (0.24 \text{ V mm}^{-1})x$. Every member of the family

$$y_q(x) = \ell(x) + qx(x - 2 \text{ mm}) \quad (1.19)$$

passes through the same endpoints at $x = 0$ and 2 mm , whatever the coefficient q , whose unit is V mm^{-2} . The endpoints have made the added term zero and therefore contain no information about it. This failure to determine a parameter from the available observations is called *non-identifiability* for this experiment. It is not repaired by repeating those two exact endpoint conditions more accurately.

Figure 1.18 makes the ambiguity visible. The three curves coincide at both endpoints, yet differ between and beyond them. A new point at an input for which $x(x - 2 \text{ mm}) \neq 0$ can constrain the family. Choosing a very nearby endpoint gives weak leverage because that product is small. Choosing a faraway point gives more leverage on q , but also asks the assumed quadratic family to remain valid over a wider range.

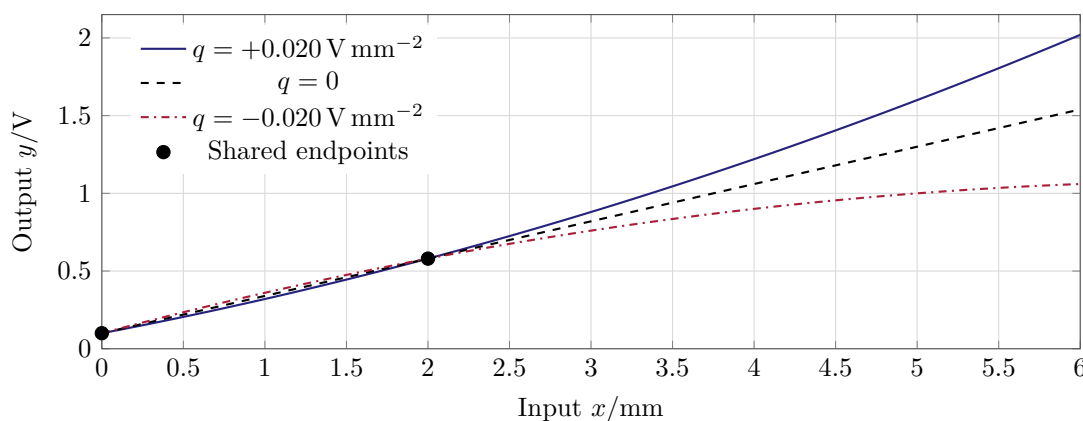


Figure 1.18: An analytical family left undetermined by two calibration endpoints. The input and output units are explicit; no observation uncertainties have been assigned. The quadratic term vanishes at both filled points, so agreement there cannot choose among these responses. Extrapolation amplifies the ambiguity.

Example 1.14: A new point constrains the curvature

Treat the two calibration endpoints as fixed for this teaching calculation. A separate constructed indication at $x = 4.000 \text{ mm}$ is $y = 1.220 \text{ V}$, with an allowed reading interval $[1.190, 1.250] \text{ V}$ and negligible stipulated input uncertainty. Infer q within the family above, then predict an output interval at $x = 6.000 \text{ mm}$. Compare it with a further observation $(2.020 \pm 0.030) \text{ V}$, where the stated $\pm$ is an allowed bound.

Inference. At 4 mm the line gives $0.10 + 0.24(4) = 1.06$ V, while $x(x - 2 \text{ mm}) = 8 \text{ mm}^2$. Hence

$$q = \frac{1.220 - 1.060}{8} \text{ V mm}^{-2} = 0.0200 \text{ V mm}^{-2}, \quad q \in [0.01625, 0.02375] \text{ V mm}^{-2}.$$

The bound is $0.030/8 = 0.00375 \text{ V mm}^{-2}$; it is not a fitted standard uncertainty.

Prediction and independent test. At 6 mm the linear contribution is 1.54 V and the coefficient multiplier is 24 mm^2 . The predicted central value is $1.54 + 24(0.0200) = 2.020$ V, with interval $[1.930, 2.110]$ V. The additional observation allows $[1.990, 2.050]$ V, which overlaps the prediction. These assumptions are not excluded by this comparison. The 4 mm indication is no longer a held-out test after it has been used to choose q ; the 6 mm indication remains independent of that choice.

Limits of the conclusion. The calculation tests a stipulated quadratic family with fixed endpoints. It does not prove that a cubic term, hysteresis, reference error or drift is absent. Endpoint uncertainty and input uncertainty would also need propagation in an actual calibration. Substitution of the central q into the original endpoint and test conditions reproduces all four central indications.

1.6.6 A residual pattern suggests the next experiment

Residuals should be inspected against both input and acquisition order. Figure 1.19 compares three deliberately isolated discrepancies against the same reference line: a constant added output, a slope difference, and a quadratic response. They produce different shapes here because all other effects have been excluded by construction. An actual pattern can have several causes. A temperature that changes systematically during an ordered input sweep, for example, can imitate an input dependence.

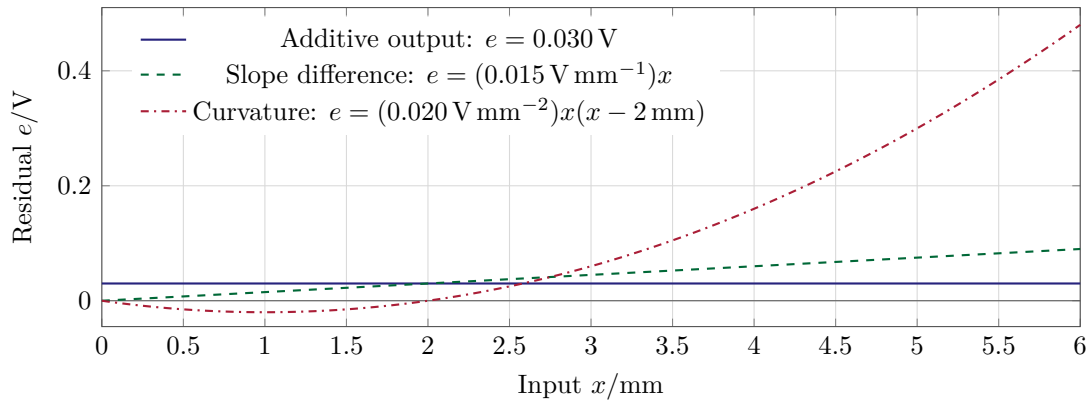


Figure 1.19: Analytical residual signatures for three constructed deviations from one fixed straight-line rule. Positive residual means output above the reference prediction. Line styles identify the curves in grayscale. These isolated models provide hypotheses to test; residual shape alone does not uniquely identify an instrument fault.

To separate time from input, record the order, revisit a reference input, and change the sweep order. Suppose seven readings are affected by the constructed drift $e_j = (0.006 \text{ V})(j - 4)$ for $j = 1, \dots, 7$. An ascending sweep assigns $x_j = (j - 1) \text{ mm}$ and makes the drift appear to be a slope error. A reordered sweep uses the same drift but assigns $x_j/\text{mm} = 0, 4, 1, 6, 2, 5, 3$. Figure 1.20 compares the two views. Reordering has not removed the drift; it has reduced the confounding between input and time. With noisy observations, revisiting the same reference input gives another direct way to probe drift.

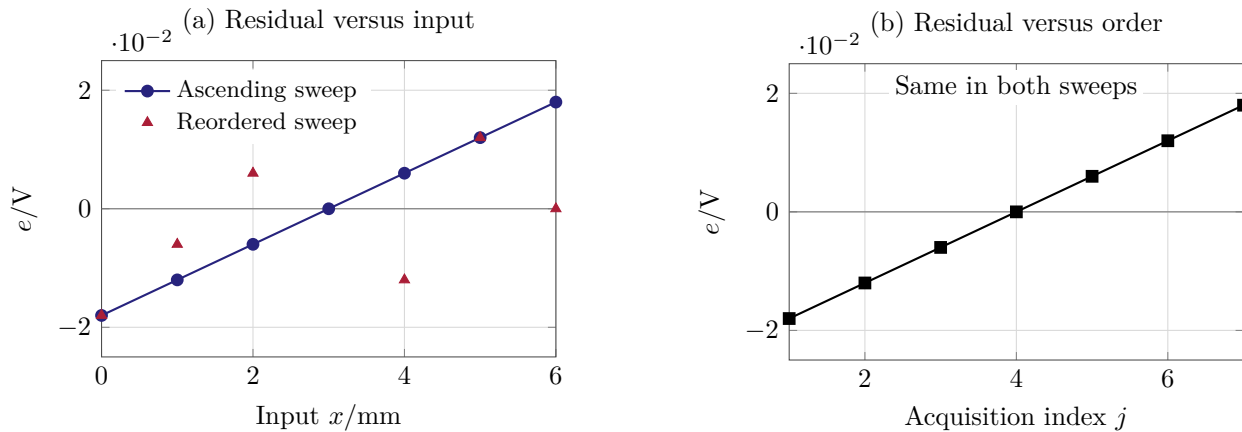


Figure 1.20: A deterministic drift can imitate input dependence when the experiment uses an ordered sweep. In (a), the input association changes when the seven assigned inputs are reordered; the overlapping first points correspond to the same first input. In (b), the drift with acquisition index remains identical. Both panels use the same residual scale; all values are constructed, with no statistical coverage attached.

1.7 Engineering inspection and a proposed measurement investigation

The spacer problem is an engineering application of elementary geometry, not a manufacturing acceptance certificate. A useful inspection report should distinguish a nominal drawing, a measured part, a model of its material and the decision being considered. We can calculate an expected mass range, identify influential dimensions and design an observation to challenge the model without claiming compliance with a product standard.

1.7.1 From a drawing to an inspection question

Suppose a batch is labelled with the nominal dimensions used above. Before predicting its total mass, ask whether the bore is through-going, whether dimensions denote radii or diameters, and whether the density describes the material or an average that includes pores. An exterior diameter entered as a radius produces a factor-of-four cross-sectional error if both radial dimensions are affected. A missing bore produces a different error. These mechanisms should not be hidden inside one vague allowance for “measurement error”.

If there are exactly N_p identical parts, the model predicts $M = N_p m$. For parts that differ, the correct relation is $M = \sum_j m_j$. Multiplying the mass of one inspected sample by N_p adds the assumption that it represents the batch. Repeatedly weighing the same part estimates repeatability; weighing different parts probes part-to-part variation as well. These are different sources of information.

The interval model permits a direct screening test. With the stipulated allowed range $[53.6, 64.2]$ g per part, a dry, unpackaged batch of 100 model parts has an enclosing mass range $[5.36, 6.42]$ kg. The factor 100 is an exact count in this exercise. Real weighing uncertainty, packing, contamination and sampling assumptions require separate treatment. A nominal batch mass near 5.88 kg is not itself a tolerance specification.

1.7.2 Proposed investigation: geometry versus an independent mass

The following investigation is a proposal, not a report of an experiment. It requires only a nonhazardous, clean object and suitable measuring instruments where available. No particular AGH apparatus is assumed. Choose a part with accessible geometry; do not force an instrument against sharp edges or damage its measuring faces.

First define the object and measurands. For a cylindrical spacer, record outer diameter D_o , bore diameter D_i and axial length L , with units. Inspect whether the hole is open at both ends and whether visible chamfers or rounded edges remove material. Measure diameters rather than treating a visual estimate of radius as an instrument indication, then use $R_o = D_o/2$ and $R_i = D_i/2$ with the exact factor two.

Before taking readings, record the instrument's display increment or scale division and perform an appropriate zero check. Inspect alignment: a tilted measurement can span a different distance from the desired diameter or axial length. Repeat measurements after removing and repositioning the instrument. Sample more than one orientation and axial location when possible. A reproducible orientation dependence may indicate noncircular geometry, not just random noise to be averaged away.

Keep the raw readings and their order. Record whether a correction was applied and where its value came from. Estimate a spread for repeated measurements made under comparable conditions. Keep variation between different directions or locations distinguishable from repeatability at one location; combining everything into a single standard deviation can hide model inadequacy.

Use independently justified density information to predict mass from geometry. If a balance is available, record its zero or tare procedure and a suitable calibration check, then weigh the clean part without packaging. This mass is useful as a test precisely because it was not used to select the density. If density is instead inferred from this mass and the same geometric volume, report it as an *inference*, not as an independent confirmation of $m = \rho V$.

1.7.3 A comparison needs an explicit criterion

For a simple interval-based report, construct one allowed interval for the model prediction and another for the weighing using stated bounds. Disjoint intervals show that the combined assumptions cannot explain the comparison. Overlap means only that the comparison has not excluded them; it does not prove every assumption. For standard uncertainties, use a stated model for the uncertainty of the difference, including any shared reference effects. The advanced section explains why adding uncertainty variances requires independence or explicit covariance.

An independent volume estimate by liquid displacement can be useful only for a compatible, nonreactive object and suitable equipment. Immersion, meniscus reading, trapped air and inaccessible cavities must be considered. A through-bore filled by liquid does not displace the same volume as a sealed air-filled cavity. The comparison is informative only after specifying which volume each method measures. No liquid displacement is required when the available object or setup makes it inappropriate.

Method

A useful investigation report answers four questions. What was directly read? What was inferred through a model? Which inputs were merely stipulated or taken from other information? Which additional observation would most reduce the uncertainty or discriminate between competing explanations? A report containing only a mean and a polished graph does not answer these questions.

For the nominal spacer's interval widths, better bore information is likely more valuable than finer length arithmetic. For a part with an uncertain material identity, density information could instead dominate. The priority follows from the actual sensitivity and uncertainty record, not from a universal rule that one kind of instrument is always the best investment.

1.8 Computational investigation: a correction is not new data

A small program should reproduce a transparent calculation before being trusted with a large dataset. We use instrument A from Example 1.10. The task is to preserve the raw readings, subtract a calibration offset, calculate the mean and repeatability spread, and check whether changing units or the correction changes the expected quantities.

1.8.1 State the uncertainty model before coding

For this teaching investigation, suppose a *separate calibration* supplies an offset estimate $b = 0.200$ mm with standard uncertainty $u_b = 0.010$ mm. Its uncertainty includes the reference contribution and is independent of the subsequent repeatability deviations. Adopt

$$\hat{x} = \bar{x} - b, \quad u_{\hat{x}} = \sqrt{\frac{s_x^2}{N} + u_b^2}. \quad (1.20)$$

This is a specified independent-contribution rule, not a full budget for an actual instrument. Its variance derivation appears in Section 1.10. Here the observed repeatability term is treated as the contribution from the discrete indication process; a second independent rounding term is not added to it without a distinct model. Other effects, such as geometry and temperature, are outside this constructed exercise.

The display increment is used to check the raw-data grid. It is not converted silently into a standard uncertainty. In particular, a sequence of identical displayed readings would give $s_x = 0$ but would not prove that the instrument or measurand had zero uncertainty. Unresolved rounding and calibration would then require additional information.

1.8.2 An executable calculation with explicit inputs

The following standard-library Python code uses millimetres throughout. Its numerical inputs are fully listed; it does not require a data file. The output uncertainty is meaningful only under the assumptions above.

```
from math import fsum, sqrt, isfinite

def analyse(readings, offset, u_offset, resolution):
    """All four inputs use the same length unit."""
    x = tuple(float(v) for v in readings)
    if len(x) < 2 or not all(isfinite(v) for v in x):
        raise ValueError("Need at least two finite readings")
    if not all(isfinite(v) for v in
               (offset, u_offset, resolution)):
        raise ValueError("Calibration inputs must be finite")
    if u_offset < 0 or resolution <= 0:
        raise ValueError("Invalid uncertainty or resolution")
    if any(abs(v/resolution - round(v/resolution)) > 1e-7
           for v in x):
        raise ValueError("Reading is off the stated display grid")
    n = len(x)
    mean = fsum(x)/n
    variance = fsum((v-mean)**2 for v in x)/(n-1)
    corrected = tuple(v-offset for v in x)
    return {"raw_mean": mean, "estimate": mean-offset,
            "s": sqrt(variance),
            "u": sqrt(variance/n + u_offset**2),
            "corrected": corrected}

raw = (20.18, 20.20, 20.19, 20.21, 20.20, 20.19)
```

```

result = analyse(raw, offset=0.200,
                 u_offset=0.010, resolution=0.01)
for key in ("raw_mean", "estimate", "s", "u"):
    print(key, f"{result[key]:.6f}", "mm")
# raw_mean 20.195000 mm; estimate 19.995000 mm
# s 0.010488 mm; u 0.010878 mm

```

For the illustrated uncertainty rule, a sensible report is $\hat{x} = (19.995 \pm 0.011)$ mm, with $u_{\hat{x}} = 0.011$ mm a standard uncertainty and with the model stated. The six printed code digits are for checking the computation; they are not six decimal places of experimentally justified precision.

1.8.3 Validate with a hand calculation and transformations

The raw sum is 121.17 mm, so the mean is 20.195 mm. Deviations from that mean are

$$(-0.015, 0.005, -0.005, 0.015, 0.005, -0.005) \text{ mm.}$$

Their squares sum to 0.00055 mm^2 . Dividing by five gives $s_x^2 = 0.00011 \text{ mm}^2$, and dividing this by six gives the repeatability variance of the mean. Adding $u_b^2 = 0.00010 \text{ mm}^2$ yields $u_{\hat{x}}^2 = 0.000118333 \dots \text{ mm}^2$. This direct decimal calculation checks the code independently of its accumulation order.

Now change the applied offset to 0.180 mm while keeping its assumed uncertainty fixed. The corrected estimate must rise by 0.020 mm to 20.015 mm; s_x and $u_{\hat{x}}$ must remain unchanged. Re-express every input, including resolution and offset uncertainty, in metres. All output lengths must become one thousand times smaller numerically, and every variance one million times smaller. The physical result is unchanged. Failure of either transformation test reveals an implementation or input-consistency problem.

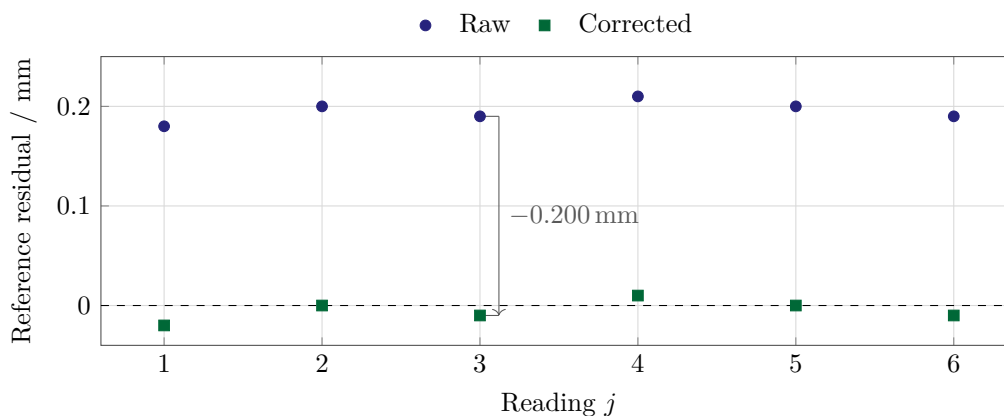


Figure 1.21: F01-09. Applying one common offset to the synthetic A readings translates their reference residuals without changing their spread. The arrow is a data transformation, not a measured trajectory. The reference is 20.000 mm; the zero line is explicit, and the dots are not connected to imply continuous time evolution.

A numerical investigation should report the raw inputs, correction rule, uncertainty assumptions, code output and checks. Copying the program's last line into a report is not a substitute for interpreting what it measures. The same distinction will recur when a numerical integrator produces a very precise answer to a poorly specified physical model.

1.8.4 How many repeated readings are useful?

An uncertainty budget can guide the amount of data collection. If σ is the standard deviation of independent repeatability contributions and u_b is the independent standard uncertainty of one

correction shared by all readings, the mean has

$$u(N) = \sqrt{\sigma^2/N + u_b^2}.$$

The curves introduced in Figure 1.16 approach the shared contribution rather than zero. The calculation below explores a different repeatability value and uses that limiting behaviour to plan an acquisition. Near the beginning, additional repetitions matter strongly; far into the flat part, improving the correction becomes more effective. This statement concerns the specified stationary model. Repetition during a changing temperature or a drifting instrument may not provide the assumed independent information.

Example 1.15: Plan repetitions against a shared uncertainty

Suppose a preliminary experiment supports $\sigma = 0.040$ mm, and an independent calibration supplies $u_b = 0.010$ mm. Determine the smallest integer N for which $u(N) \leq 0.015$ mm. Then consider a target 0.009 mm and an improved calibration with $u_b = 0.005$ mm.

Solve before counting. Squaring the first target and retaining units gives

$$\frac{0.001600}{N} + 0.000100 \leq 0.000225 \implies N \geq 12.8.$$

Thus $N = 13$ is the first acceptable integer. Direct checks give $u(12) = 0.015275$ mm and $u(13) = 0.014936$ mm.

Identify an unattainable target. For finite N , $u(N) > u_b = 0.010$ mm. The target 0.009 mm cannot be met by this calibration, however many readings are taken. The shared uncertainty must first be reduced.

Reallocate effort. With $u_b = 0.005$ mm, the original target gives $N \geq 0.001600/(0.000225 - 0.000025) = 8$. At $N = 8$, the total is exactly 0.015000 mm under the stated model. At $N = 13$ it is 0.012169 mm. The comparison supplies a quantitative reason to improve the calibration. It does not compare financial costs or establish that the preliminary estimate of σ is exact; those require further information.

1.9 Core summary and a working formula sheet

The opening mass estimate failed because its material boundary was incomplete. We repaired the geometry, not the calculator. Units then made the calculation transferable between numerical representations, dimensional analysis tested its structure, and interval reasoning showed what the allowed inputs could support. The instrument and calibration examples added another layer: a prediction is only as credible as the procedure connecting indications to physical quantities.

Summary

A model states what is being represented and what is omitted. A quantity value includes its unit. Dimensional homogeneity is necessary but cannot determine every coefficient, sign or regime. Similarity requires all relevant lengths to scale together. Bounds depend on an allowed input set; standard uncertainties depend on a stated uncertainty model. Repetition reveals some variability but does not remove a common calibration effect. Graphs and residuals test a model more effectively when fit points and held-out comparisons are distinguished.

Formula summary**Uniform, through-bored cylinder;** $\rho, L > 0, 0 \leq R_i < R_o$

$$A = \pi(R_o^2 - R_i^2), \quad V = AL, \quad m = \rho V, \quad \frac{m}{m_{\text{solid}}} = 1 - (R_i/R_o)^2.$$

Unit algebra

$$1 \text{ mm}^3 = 10^{-9} \text{ m}^3, \quad 1 \text{ g cm}^{-3} = 10^3 \text{ kg m}^{-3}.$$

Geometrically similar bodies, every length multiplied by $s > 0$

$$A' = s^2 A, \quad V' = s^3 V, \quad m'/m = (\rho'/\rho)s^3.$$

Data and a fixed calibration model

$$\bar{x} = \frac{\sum_j x_j}{N}, \quad s_x^2 = \frac{\sum_j (x_j - \bar{x})^2}{N-1}, \quad r_j = y_j - y_{\text{model}}(x_j).$$

A straight-line quantity equation

$$y = ax + b, \quad [a] = [y]/[x], \quad [b] = [y].$$

An intercept need not vanish; a proportional model requires $b = 0$. The standard uncertainty $s_x/\sqrt{N}$ describes only the repeatability contribution for independent, comparable readings, not an entire measurement budget.

Before accepting a result, ask whether its system boundary is explicit, its numerical units are consistent, its sign and limiting cases make sense, and its uncertainty claim matches the information supplied. Then ask whether the comparison used genuinely new information or merely returned data already used to construct the model. Core readers can proceed from here to the core exercises; the next section is optional enrichment.

1.10 Advanced Topics for Technical and Engineering Students

The core used exact geometry, allowed intervals and simple data summaries. The advanced pathway asks how incomplete knowledge of several inputs propagates through a nonlinear model, especially when those inputs share a calibration. It then asks how many independent dimensionless combinations are needed to organise an experiment. Both questions are developed through the same spacer rather than a new, unrelated physical system.

The additional mathematics is introduced below: local slopes and partial derivatives, variances and covariances, a quadratic form, and exponent balances for dimensionless groups. A derivative is used to approximate a small change, not assumed as prior mechanics knowledge. Matrix notation is a compact way to write sums; the two-input formulas remain usable without a separate linear-algebra course.

1.10.1 Local sensitivity: from a finite change to a derivative

For one input x , compare $f(x + \delta x)$ with $f(x)$. If the graph is smooth near x , a small change in the input gives approximately the tangent slope times the input change:

$$\delta y = f(x + \delta x) - f(x) \simeq f'(x) \delta x, \quad f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}. \quad (1.21)$$

The derivative's units are output units divided by input units. Its sign tells whether the output initially rises or falls. The limit defines the local slope; no finite experimental increment is literally zero.

For $f(x) = x^2$, direct algebra gives

$$(x + \delta x)^2 - x^2 = 2x\delta x + (\delta x)^2.$$

The first-order approximation keeps $2x\delta x$ and neglects the quadratic term. It is controlled when $|\delta x| \ll |x|$ and the neglected term is small compared with the accuracy required of the output. Near $x = 0$, the linear term vanishes and cannot describe the leading change.

For several inputs, a *partial derivative* varies one input while holding the others fixed. With $W = R_o^2 - R_i^2$,

$$\delta W = 2R_o\delta R_o - 2R_i\delta R_i + (\delta R_o)^2 - (\delta R_i)^2, \quad (1.22)$$

$$\delta W \simeq 2R_o\delta R_o - 2R_i\delta R_i. \quad (1.23)$$

Thus $\partial W / \partial R_o = 2R_o$ and $\partial W / \partial R_i = -2R_i$. Multiplying by π gives the derivatives of annular area. More generally, for $y = f(x_1, \dots, x_n)$,

$$\delta y \simeq \sum_i c_i \delta x_i, \quad c_i = \left. \frac{\partial f}{\partial x_i} \right|_{\text{input estimates}}. \quad (1.24)$$

The c_i are *sensitivity coefficients*. Their dimensions can differ because the inputs can be different kinds of quantities. Each product $c_i \delta x_i$ must have the same output unit.

For the complete spacer model, direct expansion to first order gives

$$\frac{\delta m}{m} \simeq \frac{\delta \rho}{\rho} + \frac{\delta L}{L} + \frac{2R_o\delta R_o - 2R_i\delta R_i}{R_o^2 - R_i^2}. \quad (1.25)$$

This is a signed local change, not yet an uncertainty formula. It predicts whether a particular perturbation increases or decreases the mass. Turning it into a variance requires information about how the input perturbations occur together.

1.10.2 Quantify the neglected finite-change term

The local approximation for $W = R_o^2 - R_i^2$ can be tested without assuming any probability distribution. Keep $R_i = 8$ mm fixed and vary $R_o = 12$ mm by a signed increment h . The exact change is $\Delta W = 2R_o h + h^2$, whereas the tangent prediction is $\Delta W_{\text{lin}} = 2R_o h$. For $h \neq 0$, the relative discrepancy in the *change* is

$$\frac{|\Delta W - \Delta W_{\text{lin}}|}{|\Delta W|} = \frac{|h|}{|2R_o + h|}. \quad (1.26)$$

This compares the approximation with the induced change, not with the whole original W . The two questions have different denominators. Figure 1.22 reveals a slight asymmetry: negative h reduces the denominator and therefore produces a larger relative discrepancy of the change at the same $|h|$. At $h = 0$ both changes vanish; the displayed relative-error curve uses its continuous limiting value zero.

For a physically admissible allowed interval $|h| \leq H < R_o - R_i$, the perturbed outer radius remains larger than the fixed inner radius throughout. This condition also ensures $H < 2R_o$. The largest discrepancy is then $H/(2R_o - H)$. Requiring it to be at most 1% gives $H \leq 2R_o(0.01)/(1.01)$. At $R_o = 12$ mm this is $H \leq 0.237624$ mm, safely below the physical limit $R_o - R_i = 4$ mm. This accuracy criterion makes the phrase “small change” operational for this particular output.

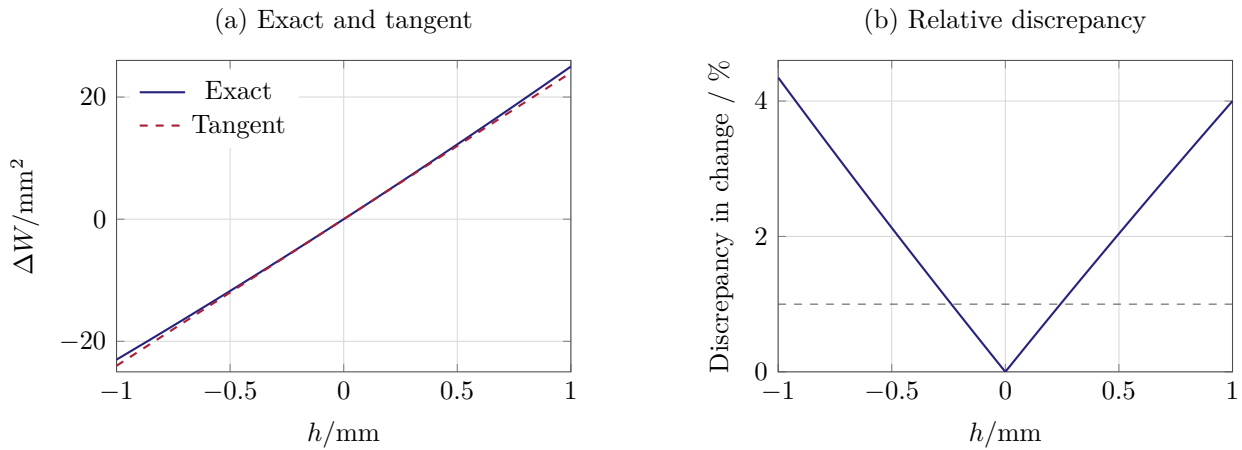


Figure 1.22: An independently calculable test of first-order linearisation. Panel (a) uses the exact polynomial and its tangent at the nominal radius; panel (b) compares the omitted term with the exact finite change. The curves use $R_o = 12$ mm and fixed $R_i = 8$ mm. The physical annulus remains admissible throughout the displayed interval.

1.10.3 From perturbations to standard uncertainties

To describe uncertainty, associate possible input deviations δx_i with a stated probability model about their estimates. An expectation $E[z]$ is a probability-weighted mean. For discrete possibilities it is simply $\sum_k p_k z_k$, where probabilities are non-negative and sum to one. A continuous distribution uses the corresponding weighted integral; we need only the moment rules here.

Assume the deviations are centred, $E[\delta x_i] = 0$. Define

$$u_i^2 = E[(\delta x_i)^2], \quad \text{Cov}(x_i, x_j) = E[\delta x_i \delta x_j]. \quad (1.27)$$

A covariance has units of x_i times units of x_j . It is positive when same-sign deviations tend to occur together, negative when opposite-sign deviations tend to occur together, and zero for independent centred deviations. Zero covariance does not in general prove independence.

Square Eq. (1.24) before taking its expectation. The cross-products survive unless the corresponding covariances vanish:

$$u_y^2 \simeq \text{E} \left[\left(\sum_i c_i \delta x_i \right)^2 \right] = \sum_i \sum_j c_i c_j \text{Cov}(x_i, x_j), \quad (1.28)$$

$$= \sum_i c_i^2 u_i^2 + 2 \sum_{i < j} c_i c_j \text{Cov}(x_i, x_j). \quad (1.29)$$

The factor two in the second form accounts for the two equal off-diagonal terms (i, j) and (j, i) . This is first-order propagation of standard uncertainty, not a rule for adding worst-case interval widths. It is consistent with the GUM treatment of correlated inputs.[6]

For independent inputs, the off-diagonal terms vanish. Then u_y is the square root of a sum of squared contributions. For a difference $y = x_1 - x_2$, this gives $u_y^2 = u_1^2 + u_2^2$, not $u_1^2 - u_2^2$. If the same uncertain offset affects both inputs, the covariance term can instead cancel it. The sign of a difference belongs in the sensitivity coefficients before the variance is formed.

1.10.4 Why a common calibration contribution does not average away

Let $x_j = \mu + b + \epsilon_j$, with independent centred repeatability deviations of variance σ^2 and an independent common uncertain offset of variance u_b^2 . Then

$$\text{Cov}(x_j, x_k) = \begin{cases} \sigma^2 + u_b^2, & j = k, \\ u_b^2, & j \neq k. \end{cases}$$

For the mean, every sensitivity coefficient is $1/N$. There are N repeatability contributions but N^2 contributions from the common offset in the double sum. Therefore

$$u_{\bar{x}}^2 = \frac{\sigma^2}{N} + u_b^2. \quad (1.30)$$

After applying an estimated offset correction, u_b represents the uncertainty remaining in that correction. Estimating σ by s_x recovers the rule used in the computational investigation. Multiplying the number of readings by four halves the repeatability contribution to u , but it leaves the common contribution unchanged.

The model also explains a frequent ambiguity in calibration. When the same reference contributes to both compared measurements, their uncertainties are correlated. Adding that reference uncertainty to each budget independently and then again to the difference can double-count it. The reliable procedure is to identify shared inputs and propagate from them explicitly.

1.10.5 Covariance must describe a possible set of inputs

For two positive input standard uncertainties, write

$$\text{Cov}(x_1, x_2) = r_{12} u_1 u_2, \quad -1 \leq r_{12} \leq 1. \quad (1.31)$$

The bound follows, for example, by requiring the variance of every linear combination of the two centred inputs to be non-negative. If we collect the covariances in a matrix Σ , this condition is

$$\mathbf{c}^T \Sigma \mathbf{c} \geq 0 \quad \text{for every real coefficient vector } \mathbf{c}.$$

Such a matrix is called *positive semidefinite*. For two inputs with non-negative diagonal entries, it requires

$$\det \Sigma = u_1^2 u_2^2 - \text{Cov}(x_1, x_2)^2 \geq 0. \quad (1.32)$$

For more inputs, pairwise correlation bounds alone are not enough; the complete matrix must satisfy the condition. A negative propagated variance indicates invalid inputs, an algebra or numerical problem, or a misapplied approximation. It is not a physical negative uncertainty to be replaced silently by zero.

1.10.6 Correlated radius measurements

With $W = R_o^2 - R_i^2$, Eq. (1.29) becomes

$$u_W^2 \simeq 4R_o^2 u_o^2 + 4R_i^2 u_i^2 - 8R_o R_i r_{oi} u_o u_i, \quad u_A = \pi u_W. \quad (1.33)$$

Here u_o, u_i are the radius standard uncertainties. The covariance term is negative for positive correlation because the two sensitivity coefficients have opposite signs. Increasing both radii together removes some of the otherwise independent change in their squared difference. This does not mean that positive correlation always reduces uncertainty: for a sum with positive sensitivities, it increases it.

Example 1.16: Correlated radius measurements

Use $R_o = 12.00$ mm, $R_i = 8.00$ mm, $u_o = 0.06$ mm, $u_i = 0.04$ mm and stipulated correlation $r_{oi} = 0.8$. These uncertainties are small relative to both radii and to the 4 mm wall thickness. Compare with zero correlation and a purely common multiplicative scale uncertainty.

Central geometry. $W = 80.00$ mm² and $A = 80\pi$ mm² $\simeq 251.327$ mm².

Covariance calculation. The covariance is $0.8(0.06)(0.04) = 0.00192$ mm², and

$$\Sigma = \begin{pmatrix} 0.0036 & 0.00192 \\ 0.00192 & 0.0016 \end{pmatrix} \text{ mm}^2.$$

Its determinant is 2.0736×10^{-6} mm⁴ > 0 . With sensitivities $(24, -16)$ mm,

$$\begin{aligned} u_W^2 &\simeq (24 \times 0.06)^2 + (16 \times 0.04)^2 - 2(24)(16)(0.00192) \text{ mm}^4 \\ &= 1.00864 \text{ mm}^4, \quad u_W \simeq 1.00431 \text{ mm}^2. \end{aligned}$$

Hence $u_A \simeq 3.1551$ mm² and $u_A/A \simeq 1.255\%$. A suitable rounded report is $A = (251.3 \pm 3.2)$ mm², with the uncertainty explicitly standard and first-order.

Zero-covariance comparison. Setting $r_{oi} = 0$ but keeping the same marginal uncertainties gives $u_W \simeq 1.57582$ mm² and relative uncertainty 1.970%. The difference is physical information about the assumed input relationship, not a choice of arithmetic convention.

Common-scale limit. If both radii are multiplied by the same uncertain factor $1 + \zeta$, with $u_\zeta = 0.005$, then $u_o = R_o u_\zeta$ and $u_i = R_i u_\zeta$, exactly the marginal values above, but now $r_{oi} = 1$. Direct geometry gives

$$W' = (1 + \zeta)^2 W, \quad \frac{u_W}{W} \simeq 2u_\zeta = 1.000\%.$$

The covariance formula gives $u_W = |1.44 - 0.64|$ mm² $= 0.80$ mm², agreeing with this independent scaling argument. Correlation does not remove the scale uncertainty; it changes how the two radius contributions combine.

The contour in Figure 1.23 can be constructed without calculating matrix eigenvectors. Parameterise it by $U = \cos \theta$, $V = r \cos \theta + \sqrt{1 - r^2} \sin \theta$. Substitution gives the displayed contour equation for $|r| < 1$. The greatest linear output change on that contour is

$$\max_{\theta} |c_o u_o U + c_i u_i V| = \sqrt{(c_o u_o + r c_i u_i)^2 + (1 - r^2)(c_i u_i)^2},$$

which is exactly the square root of the propagated variance. This is a geometrical interpretation of a covariance scale, not a probability assigned to the boundary.

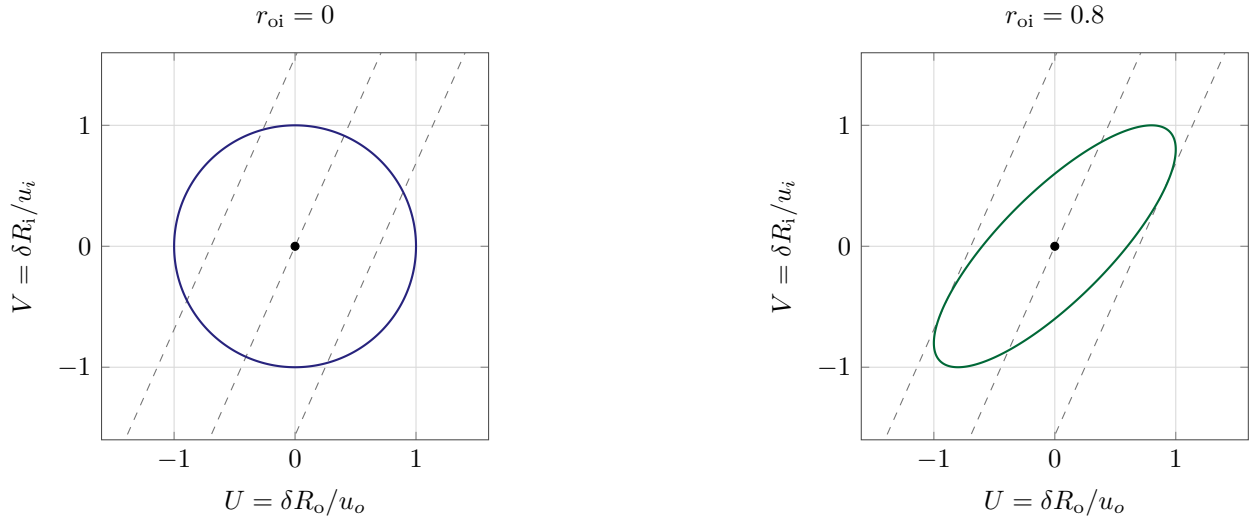


Figure 1.23: F01-07. Geometry of the two-input covariance model. Solid contours satisfy $(U^2 - 2rUV + V^2)/(1 - r^2) = 1$; they are covariance contours, not asserted confidence regions. Dashed lines are the local constant-output lines $\delta W/\text{mm}^2 = 1.44U - 0.64V$ at levels $-1, 0, +1$, with the zero line through the origin. Positive correlation aligns more variation in directions that partly cancel in W . Equal axis scales preserve the geometry. At perfect common-scale correlation the contour degenerates to a line, not an invertible ellipse.

1.10.7 Correlation changes the result continuously

The earlier correlated-radius example can be varied while holding both marginal standard uncertainties fixed. Its sensitivities are 24 and -16 mm, so the propagated expression reduces to

$$u_W(r) = \sqrt{2.4832 - 1.8432r} \text{ mm}^2, \quad -1 \leq r \leq 1.$$

Figure 1.24 shows why independence cannot be inserted as a convenience. The permitted correlation changes the output scale from 2.08 mm^2 at $r = -1$ to 0.80 mm^2 at $r = 1$, while the input marginal uncertainties remain unchanged. Perfect anticorrelation reinforces the two signed sensitivities; perfect positive correlation partly cancels them. At both endpoints the covariance matrix is singular but positive semidefinite, so propagation is still meaningful. An inverse covariance contour formula would require a separate limiting treatment there.

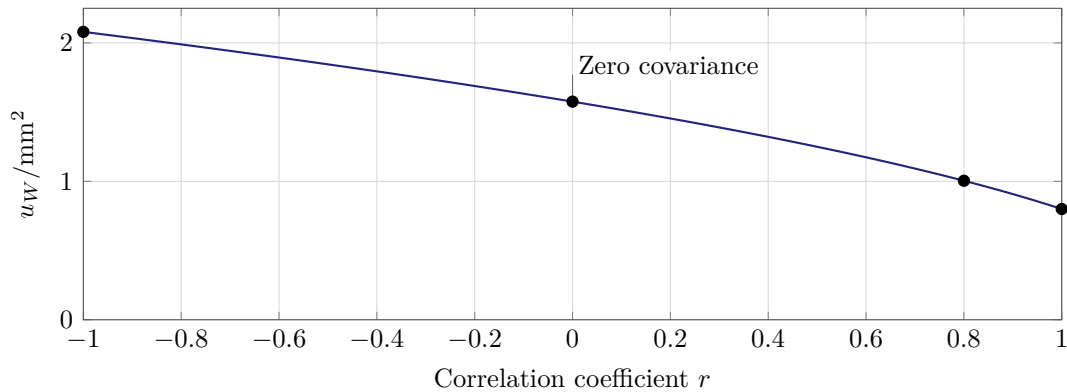


Figure 1.24: First-order annular-geometry uncertainty at fixed marginal input uncertainties, as only their correlation changes. The markers show $r = -1, 0, 0.8, 1$. This is a family of stipulated covariance models, not an empirical inference about a real measuring instrument. Positive correlation decreases this output uncertainty because the radius sensitivities have opposite signs.

1.10.8 Bounds and covariance describe different input information

An allowed input set and a covariance model answer different questions even if their numerical input scales happen to match. Consider the linear change $\delta W = 1.44U - 0.64V$ in mm^2 , with $U = \delta R_o / (0.06 \text{ mm})$ and $V = \delta R_i / (0.04 \text{ mm})$. If the two denominators are stipulated *bounds*, then $|U| \leq 1$ and $|V| \leq 1$ define a rectangle. The largest absolute first-order change on that rectangle is $1.44 + 0.64 = 2.08 \text{ mm}^2$, achieved at opposite corners.

If the same numerical denominators instead represent *standard uncertainties* with zero covariance, the propagated standard uncertainty is $\sqrt{1.44^2 + 0.64^2} = 1.57582 \text{ mm}^2$. The unit circle in Figure 1.25(b) geometrically displays that covariance scale: the largest linear change on that contour equals this value. It is not a boundary containing all possible inputs and is not automatically a confidence region. A probability model can include input deviations outside that circle. The comparison therefore does not rank one uncertainty method as universally more conservative. It displays the consequences of *different information*.

For finite bounded changes in the annulus, the exact expression also contains $(\delta R_o)^2 - (\delta R_i)^2$. Monotonicity allows an exact endpoint calculation. With the displayed numerical bounds, the maximum positive change is 2.0820 mm^2 and the most negative change is -2.0780 mm^2 . These differ slightly from the symmetric first-order bound. Retaining the distinction avoids using a standard uncertainty as though it were a guaranteed maximum error.

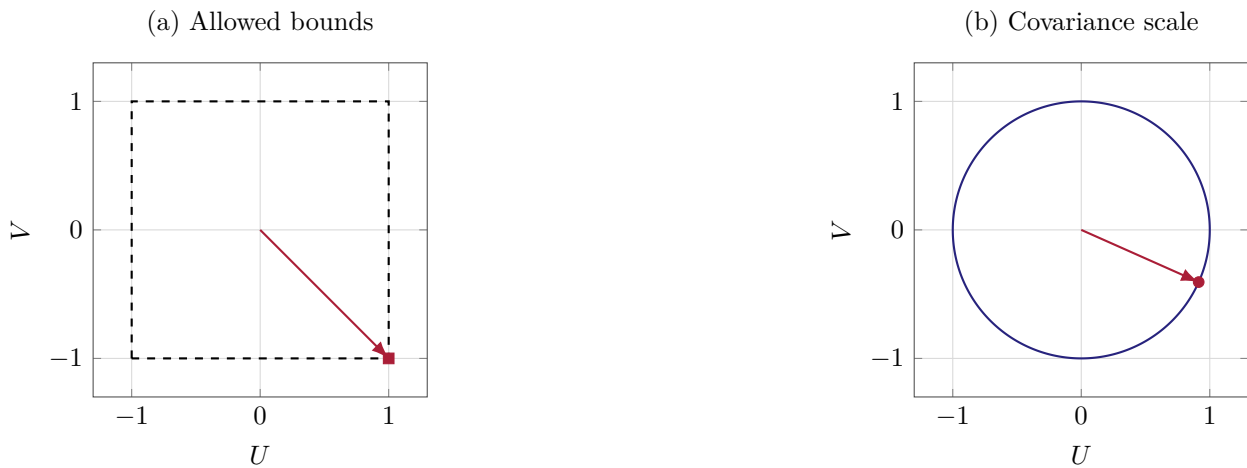


Figure 1.25: Two geometrical representations of different input information, using the same normalized coordinate scales. In (a), every point of a rectangular allowed set must be considered when calculating a first-order bound. In (b), the circle represents unit covariance scale for uncorrelated standardized deviations; the plotted extremum gives the propagated standard uncertainty. It neither declares bounded inputs nor assigns confidence coverage. Arrows locate the positive linear-output extrema in each construction.

1.10.9 Complete mass uncertainty and the value of a new measurement

Assume now that density and length have standard uncertainties $u_\rho = 0.03 \text{ g cm}^{-3}$ and $u_L = 0.05 \text{ mm}$, independent of one another and of the correlated radii. Then

$$\left(\frac{u_m}{m}\right)^2 \simeq \left(\frac{u_\rho}{\rho}\right)^2 + \left(\frac{u_L}{L}\right)^2 + \left(\frac{u_W}{W}\right)^2. \quad (1.34)$$

For the correlated example, the three relative contributions are 0.003846, 0.001667 and 0.012554. Their quadrature gives $u_m \simeq 0.7784 \text{ g}$, or a first-order result $m = (58.81 \pm 0.78) \text{ g}$. This is not the earlier interval $[53.6, 64.2] \text{ g}$: the inputs and their uncertainty meanings differ.

The radius combination dominates this specified variance budget. Improving the density uncertainty by a factor two would reduce its variance contribution by four, but it would not remove the radius

contribution. Improving a shared scale calibration may be more useful than taking many extra repeated readings. A proposed improvement should name which uncertainty component it changes.

For an especially thin annulus, define $\beta = R_i/R_o$. If the two radii have equal relative standard uncertainty ε and zero covariance, Eq. (1.33) gives

$$\frac{u_A}{A} \simeq 2\varepsilon \frac{\sqrt{1+\beta^4}}{1-\beta^2}. \quad (1.35)$$

As $\beta \rightarrow 1$, the small difference of two large squared radii becomes relatively sensitive to independent input errors. This is a property of the input representation as well as the geometry. With a common multiplicative scale error, the relative sensitivity is instead 2ε at first order. With a common *additive* radius offset b , direct subtraction gives $\delta W \simeq 2(R_o - R_i)\delta b$, so $u_W/W \simeq 2u_b/(R_o + R_i)$. A shared scale and a shared zero offset are not interchangeable models.

1.10.10 When first-order propagation is insufficient

Small relative errors of each input are useful but not by themselves a complete validity test. Check the output's local sensitivity, cancellation, distance from a physical boundary, and the size of omitted nonlinear terms. A Gaussian input model that assigns appreciable probability to $R_i \geq R_o$ is inappropriate for a physical annulus unless that boundary is treated explicitly. The small uncertainties in Example 1.16 lie far from this issue.

Nonlinearity can affect the central estimate as well as its uncertainty. From Eq. (1.22), centred radius deviations give

$$E[W'] = (R_o^2 - R_i^2) + u_o^2 - u_i^2.$$

For the numerical example this shift is 0.0020 mm^2 , negligible compared with $u_W \simeq 1.00 \text{ mm}^2$ at the reporting precision. Its smallness has been checked, not assumed from the word “uncertainty”. More general nonlinear models can require higher-order terms or propagation of a fully specified input distribution; the current GUM amendment explicitly addresses this issue.[7]

Example 1.17: A zero local slope need not imply zero uncertainty

Consider the mathematical calibration model $Y = X^2$. The uncertain input has three equally probable values $-a, 0, +a$, with $a > 0$ and units of length. Its mean is zero. Compare first-order propagation at the mean with exact enumeration.

Linear prediction. The sensitivity $2E[X]$ is zero, so first-order propagation predicts zero output variance. It also uses $f(E[X]) = 0$ as the central output.

Exact discrete calculation. The outputs are $a^2, 0, a^2$. Their mean is $2a^2/3$, and the deviations from that mean are $a^2/3, -2a^2/3, a^2/3$. Therefore

$$\text{Var}(Y) = \frac{1}{3} \left(\frac{a^4}{9} + \frac{4a^4}{9} + \frac{a^4}{9} \right) = \frac{2a^4}{9}, \quad u_Y = \frac{\sqrt{2}}{3} a^2.$$

Interpretation. The first nonzero response is quadratic. A zero gradient has removed the leading *linear* term, not the physical variation. Exact enumeration supplies an independent check without Monte Carlo sampling, and every output has the correct area unit.

An allowed-interval calculation and a covariance calculation also have different domains. Exact interval endpoints can be preferable when only bounds are known and monotonicity is available. A covariance model uses more information about joint uncertainty but loses detail beyond the moments retained in a first-order calculation. Neither procedure is universally more conservative or more informative; the appropriate one follows from the available input knowledge and the question being asked.

1.10.11 Nondimensionalisation: separate size from shape

The spacer model appears to depend on five quantities: m, ρ, L, R_o, R_i . Only two independent base dimensions occur, mass and length. We can explicitly find combinations with no dimensions rather than relying on a named theorem alone.

For positive m, ρ, L, R_o, R_i , consider a monomial $m^a \rho^b L^c R_o^d R_i^e$. Its dimensions are

$$M^{a+b} L^{-3b+c+d+e}.$$

For a dimensionless combination, both exponents must vanish. Hence $b = -a$ and $d = -3a - c - e$. There are three freely selectable exponents, which can be represented by the three groups

$$\Pi_m = \frac{m}{\rho R_o^3}, \quad \lambda = \frac{L}{R_o}, \quad \beta = \frac{R_i}{R_o}. \quad (1.36)$$

The resulting geometric formula extends continuously to $R_i = 0$; arbitrary negative powers of R_i need not. Indeed, every monomial above can be written $\Pi_m^a \lambda^c \beta^e$. The two exponent constraints are independent: density carries a mass dimension while the radius supplies a length dimension. This is the rank-counting content of Buckingham-style dimensional reasoning for this controlled example.

Dimensions alone now permit a relation $\Pi_m = \Phi(\lambda, \beta)$ for an unspecified dimensionless function. Geometry and uniform density supply the actual function:

$$\Pi_m = \pi \lambda (1 - \beta^2), \quad \frac{m}{\rho \pi L R_o^2} = 1 - \beta^2, \quad \lambda > 0, \quad 0 \leq \beta < 1. \quad (1.37)$$

The second form removes the known axial and outer-size factors. It says that the fraction of the solid-model mass depends only on the radius ratio. Units cannot determine the function $1 - \beta^2$ by themselves; the subtraction of the bored volume does.

Example 1.18: A dimensionless comparison of two part sizes

Part A has the nominal dimensions $(R_o, R_i, L) = (12, 8, 30)$ mm and density 7.80 g cm^{-3} . Part B is geometrically similar with every length multiplied by 1.5 and the same density. Predict B's mass and the dimensionless groups for both.

Dimensional prediction. B has radii 18 and 12 mm and length 45 mm. Similarity gives $m_B = 1.5^3 m_A = 3.375 m_A \simeq 198.486 \text{ g}$.

Dimensionless prediction. Both parts have $\beta = 2/3$ and $\lambda = 2.5$, so

$$\Pi_m = \pi(2.5)(1 - 4/9) = \frac{25\pi}{18} \simeq 4.36332.$$

Their mass fractions relative to their own solid outer cylinders are both $5/9$.

Independent check. Direct substitution of B's dimensions into $m = \rho \pi L (R_o^2 - R_i^2)$ yields the same 198.486 g. Nondimensionalisation has identified which results should be invariant under a physical change of scale. It has not asserted that surface finish, density defects or measurement uncertainties also scale ideally.

1.10.12 Controlled experimental design and a thin-wall approximation

A useful experiment could compare several sizes at fixed β and λ , using independently justified material density. Equal Π_m would be consistent with the scale prediction. A second comparison could vary β at fixed λ , rather than changing every dimension at once. Plotting $m/(\rho \pi L R_o^2)$ against β tests the predicted shape dependence. Normalisation does not turn synthetic data into measurements, and using a density inferred separately from each same mass would make the test circular.

Dimensionless variables also quantify an approximation. Write the wall thickness as $t = R_o - R_i$. The exact area is

$$A = \pi t(2R_o - t). \quad (1.38)$$

When $t/R_o \ll 1$, neglecting t beside $2R_o$ gives $A_{\text{thin}} = 2\pi R_o t$. This overestimates the exact area. The relative overestimate is

$$\epsilon_{\text{thin}} = \frac{A_{\text{thin}} - A}{A} = \frac{t}{2R_o - t} = \frac{1 - \beta}{1 + \beta}. \quad (1.39)$$

The approximation's quality is controlled by a dimensionless shape parameter, not by whether the part looks small in millimetres. A micrometre-sized part with the same β has the same geometrical relative error.

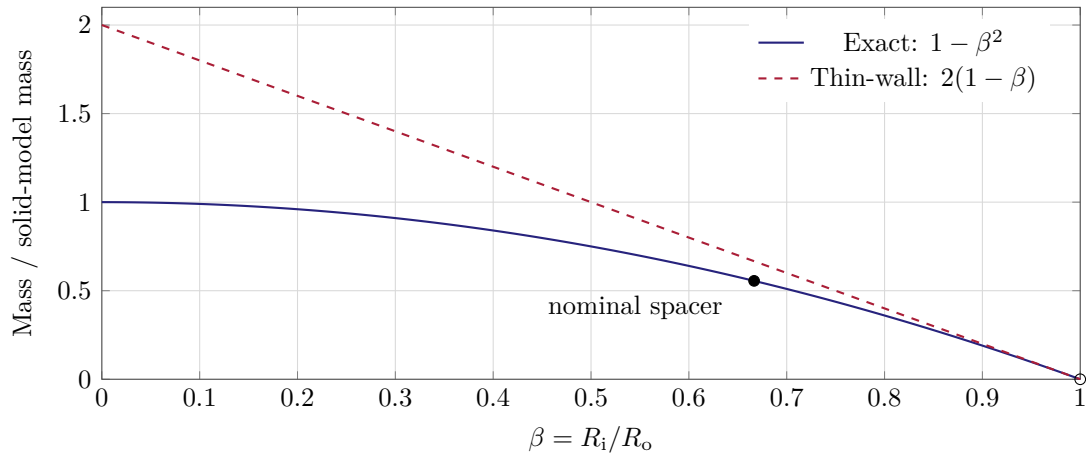


Figure 1.26: F01-08. Analytical dimensionless mass fraction and the outer-radius thin-wall approximation at the same density and outer geometry. The open endpoint at $\beta = 1$ is a zero-wall limit, not a physical spacer. The approximation is above the exact curve and becomes useful only near that limit. At $\beta = 2/3$ it overpredicts by 20% relative to the exact result.

Example 1.19: How thin is thin enough?

Require the thin-wall mass estimate to overpredict the exact uniform-cylinder mass by less than 1%. What restriction follows for the radius ratio? Does the nominal spacer satisfy it?

Solve the exact error criterion.

$$\frac{1 - \beta}{1 + \beta} < 0.01 \quad \Rightarrow \quad \beta > \frac{0.99}{1.01} \simeq 0.980198.$$

The wall must therefore be less than about 1.98% of the outer radius. The nominal spacer has $\beta = 2/3$ and $t/R_o = 1/3$, so it fails this criterion. Its overestimate is exactly 20%, even though its dimensions may be measured precisely.

Check an alternative representation. With the mean radius $R_m = (R_o + R_i)/2$, the identity $A = 2\pi R_m t$ is exact for a circular annulus. It is using the *outer* radius in place of R_m that introduces the thin-wall approximation. Choosing better variables can remove an approximation rather than merely reducing a numerical error.

1.10.13 Measure a sensitive difference directly when possible

Writing the annular geometry in terms of mean radius $R_m = (R_o + R_i)/2$ and wall thickness $t = R_o - R_i$ gives the *exact* identity

$$A = 2\pi R_m t. \quad (1.40)$$

Figure 1.27 identifies the dimensions. No thin-wall approximation was made: the term $-\pi t^2$ that appears when using the outer radius has been absorbed into $R_m = R_o - t/2$. This expression can suggest a more useful measurement procedure. However, merely changing variables in the same data cannot create additional precision. If R_m and t are inferred from the same two measured radii, their uncertainties and covariance must be transformed from those inputs.

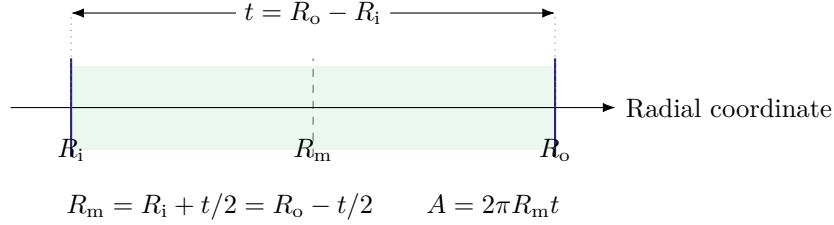


Figure 1.27: A magnified radial wall interval locates inner radius, mean radius and outer radius. The half-wall positions have equal spacing; the central axis and the rest of the annulus are omitted, so distances from the physical centre cannot be read from this sketch. The mean-radius formula for annular area is exact. The split dimensional marker adapts a construction by Alexandros Tsagkaropoulos, *Elliptical Orbit*, TikZ.net (CC BY-SA 4.0); the annular geometry and labels are new.

Example 1.20: Thin-wall inspection with two measurement procedures

A different part has $R_o = 12.00$ mm and $R_i = 11.80$ mm. Procedure A independently measures both radii with standard uncertainties 0.020 mm. Procedure B independently measures the mean radius with $u_{R_m} = 0.020$ mm and the thickness with $u_t = 0.0050$ mm. Compare first-order area uncertainties, and check the outer-radius thin-wall approximation. The independence assumptions describe separate stipulated procedures.

One geometry. Both procedures give $t = 0.200$ mm, $R_m = 11.90$ mm and $A = \pi(144 - 139.24) = 4.76\pi \text{ mm}^2 = 14.95398 \text{ mm}^2$. The alternative exact expression gives $2\pi(11.90)(0.200) = 4.76\pi \text{ mm}^2$.

Procedure A. With independent radii,

$$u_A = 2\pi\sqrt{(12.00 \times 0.020)^2 + (11.80 \times 0.020)^2} = 2.11489 \text{ mm}^2, \quad u_A/A = 14.1426\%.$$

The small wall thickness is obtained as a difference of two much larger inputs; $u_t = \sqrt{2}(0.020) = 0.02828$ mm already amounts to 14.14% of t .

Procedure B. Independent inputs in the exact product give

$$\frac{u_A}{A} \simeq \sqrt{(0.020/11.90)^2 + (0.0050/0.200)^2} = 0.0250564, \quad u_A = 0.374693 \text{ mm}^2.$$

The thickness has been measured with more useful information, so the relative uncertainty is now 2.506%. Rewriting Procedure A's same raw data as (R_m, t) would leave its propagated area uncertainty unchanged; it would not produce Procedure B's smaller u_t .

Approximation check. Using $A_{\text{thin}} = 2\pi R_o t$ gives $4.80\pi \text{ mm}^2 = 15.07964 \text{ mm}^2$, an overestimate $\pi t^2 = 0.125664 \text{ mm}^2$. Relative to the exact area the error is $0.04/4.76 = 0.84034\%$. This deterministic approximation error is different from either standard uncertainty. Using the mean-radius formula removes it exactly for this geometry.

1.10.14 A controlled dimensionless comparison needs two views

The three dimensionless groups tell us which conditions must be held fixed. Figure 1.28(a) shows that $\Pi_m = m/(\rho R_o^3)$ changes with both $\lambda = L/R_o$ and $\beta = R_i/R_o$. Comparing objects at unequal λ while attributing all mass change to the bore would therefore confound two geometrical effects. Dividing by the known factor $\pi\lambda$ produces the collapse in panel (b). This is a testable prediction of

a uniform through-bored cylinder, rather than a new law obtained from dimensions alone.

The four markers illustrate a constructed controlled series at fixed $\lambda = 2.5$: $(\beta, \Pi_m) = (0, 7.85398), (0.4, 6.59734), (2/3, 4.36332), (0.9, 1.49226)$. After normalisation their ordinates are 1, 0.84, 5/9, 0.19. Actual measurements would need uncertainty information and independently known density before a deviation could be interpreted. A density inferred from each same measured mass would force agreement and destroy the independence of the test.

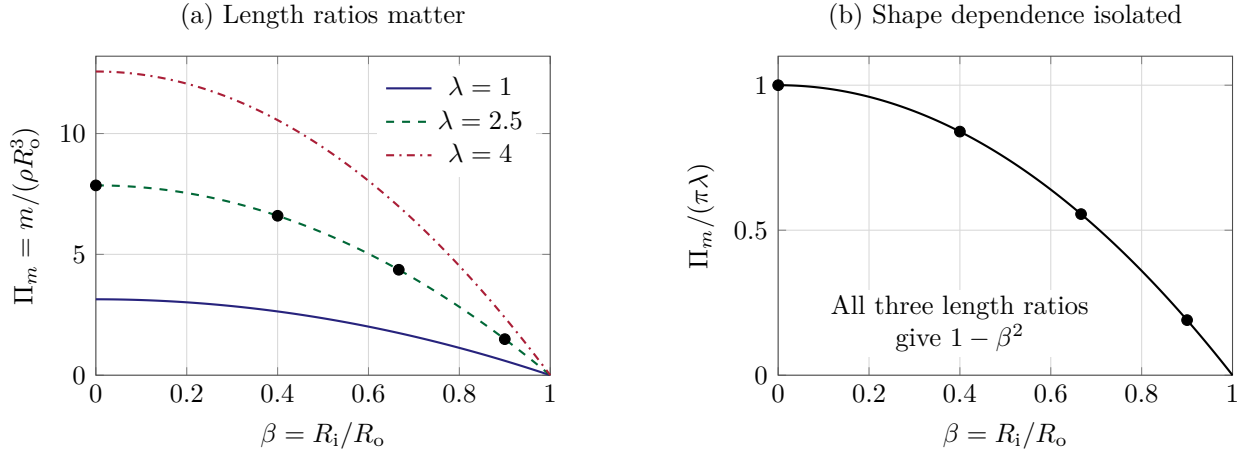


Figure 1.28: Analytical dimensionless mass families for a uniform annular cylinder. Panel (a) varies the length ratio; panel (b) removes its known multiplicative effect. Filled markers are calculated teaching values at $\lambda = 2.5$, not observations. The endpoint $\beta = 1$ is the zero-wall limiting geometry; a material part requires $\beta < 1$.

1.10.15 State an approximation tolerance in the shape variable

The relative error of the outer-radius thin-wall approximation becomes especially transparent if $\eta = t/R_o = 1 - \beta$. It is $\epsilon = \eta/(2 - \eta)$ for $0 < \eta \leq 1$. At $\eta = 0$ the exact area is zero, so the original relative-error ratio is undefined; the simplified expression has the continuous limiting value zero. A tolerance $\epsilon \leq \epsilon_0$ requires

$$\eta \leq \frac{2\epsilon_0}{1 + \epsilon_0}, \quad \beta \geq \frac{1 - \epsilon_0}{1 + \epsilon_0}. \quad (1.41)$$

Figure 1.29 resolves the region near a 1% tolerance. For a fixed $R_o = 12$ mm the allowed thickness is at most 0.237624 mm. For another size, the allowed absolute thickness changes proportionally, while the permitted ratio stays fixed. The nominal 4 mm wall of the opening spacer has $\eta = 1/3$ and a 20% relative overestimate, so it is outside the small plotted interval and outside this tolerance. The plot deliberately focuses on $\eta \leq 0.12$; the approximation error continues to rise beyond its right edge.

1.10.16 Advanced synthesis

The covariance calculation and the dimensionless calculation answer different questions. Covariance describes how uncertain inputs vary together near stated estimates. Nondimensionalisation identifies combinations that are invariant under unit changes and, under a model, under physical similarity transformations. Together they suggest a disciplined experiment: choose the quantity and dimensionless comparison, identify independent and shared inputs, quantify sensitivity, and check both nonlinear model error and measurement uncertainty.

A beautiful collapse of normalised data does not prove an omitted variable irrelevant outside the tested range. A small propagated standard uncertainty does not repair the wrong geometry. A well-designed chapter calculation makes those limitations visible, so that the next measurement has a clear purpose.

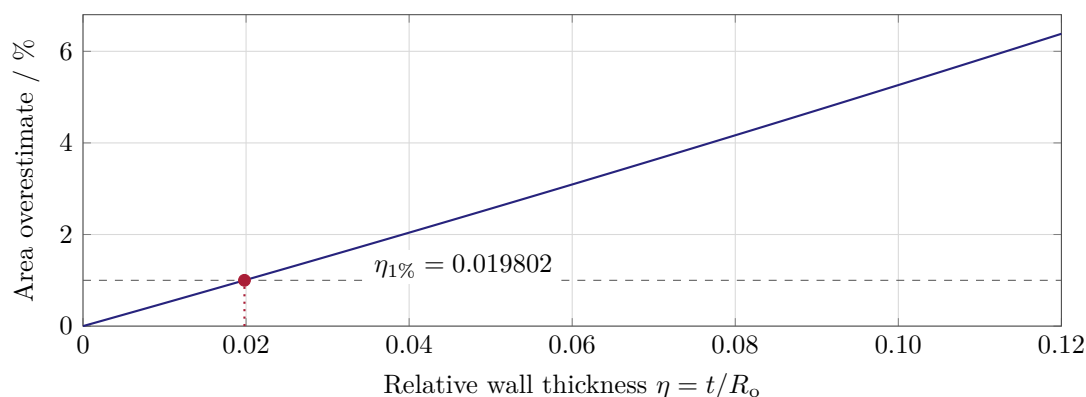


Figure 1.29: Exact relative error of $2\pi R_o t$ as an approximation to annular area, displayed near the thin-wall limit. The 1% boundary corresponds to $t/R_o = 0.019802$. The axis range is restricted to $0 \leq t/R_o \leq 0.12$ to make that boundary legible; its value at zero is the continuous limit because relative error for a zero-area wall is undefined. A material wall has $0 < t/R_o \leq 1$. Using the mean radius gives the exact area instead.

1.11 Exercises and problems

The problems use constructed teaching inputs, not specifications for an actual component or instrument. Unless stated otherwise, the spacer is a uniform, concentric, through-bored cylinder with $\rho > 0$, $L > 0$ and $0 \leq R_i < R_o$. Distinguish nominal values, allowed bounds and standard uncertainties. Retain intermediate digits and state the meaning of the final precision. Problems 31–40 and Problem 48 belong to the optional advanced pathway; the earlier problems require no derivatives or covariance calculation.

A. Models, units and physical interpretation

- 1. Repair a geometrical model.** A student proposes $m = \rho\pi L(R_o - R_i)^2$. It has the correct dimensions and becomes zero when the wall vanishes. Test its solid-cylinder limit, its response to a growing bore and its numerical prediction when $R_i = R_o/2$. Which tests does it pass despite being wrong? Explain the missing geometry using Figure 1.1.
- 2. One mass, two explanations.** A part's mass is 3/4 of the prediction for a solid cylinder made from a specified material. Give two distinct hypotheses: one with the specified density and a bore, and one with no bore but a different uniform density. Find the required radius ratio or density ratio. Identify an additional observation that discriminates between them.
- 3. Numbers are not quantities.** Two data files record the same length as 25.0 and 0.0250. Explain what information could make them consistent. Would changing the file's unit label without changing its stored numbers preserve the physical length? Give the factor of error for millimetres relabelled as metres.
- 4. Exact definition, uncertain realisation.** Explain why a more accurate laboratory determination of the speed of light does not change the SI definition of the metre. Does an exact definition make a length measured with a ruler exact? Contrast a defining constant such as c with the measured constant G .
- 5. Find a cubic conversion error.** A script obtains $V = 1250 \text{ mm}^3$ and converts it to 1.250 m^3 by dividing by 1000. With $\rho = 2.40 \text{ g cm}^{-3}$, find the correct mass using two coherent unit routes. By what factor is the script's converted volume wrong? Connect the diagnosis to Figure 1.4.
- 6. Prefixes under powers.** Convert $4.50 \mu\text{m}^2$ to m^2 , 0.0800 cm^{-1} to m^{-1} , and 1.25 mg mm^{-3}

to kg m^{-3} . State whether each exponent acts on the prefix as well as on the unit. Here μm^2 means $(\mu\text{m})^2$.

7. **Diameter is not radius.** A drawing specifies outer and bore diameters 24 mm and 16 mm, and length 30 mm. A calculation substitutes both diameters into the radius-based mass formula. Find the multiplicative error. Explain why relabelling the final answer's units cannot repair it.

B. Dimensions, scaling and estimates

8. **What does a dimensional result establish?** Suppose a model contains only a density ρ , area A and speed v , and proposes $F = C\rho^p A^q v^r$ with dimensionless C . Use $[F] = \text{MLT}^{-2}$ to determine p, q, r . Explain why this neither fixes C nor establishes that no other variable matters. Relate the conclusion to Figure 1.6.
9. **Audit every coefficient.** A recorded length is modelled as $x(t) = a + bt + ct^2$, where t is measured in seconds. Find the dimensions and SI units of a, b, c . Does dimensional homogeneity alone require the coefficient of t^2 to include a factor $1/2$?
10. **Dimensionless function arguments.** Let x and x_0 be positive lengths and τ a time. Diagnose $\ln x$, $\exp(-t/\tau)$ and $\sin(x + x_0)$. Give a dimensionally meaningful repair to each invalid expression. Explain why the choice of a reference scale is physical information, not merely typography.
11. **A change of units or a change of part?** In case A, every length is expressed in centimetres instead of millimetres. In case B, every physical length is made ten times larger while density is unchanged. Compare the mass in the two cases, and compare the numerical volumes before and after each change.
12. **Changing the constraint.** A bore is enlarged in a cylinder with fixed R_o , fixed material density and fixed total material volume. Derive the new length L' in terms of the old and new bore radii. What happens to L' as the new bore approaches R_o ? Explain why the fixed-length conclusion “a larger bore means less mass” does not apply to this different constraint.
13. **A justified order of magnitude.** For a preliminary estimate take $\pi \simeq 3$, $\rho \simeq 8 \text{ g cm}^{-3}$, $R_o \simeq 1.2 \text{ cm}$, $R_i \simeq 0.8 \text{ cm}$ and $L \simeq 3 \text{ cm}$. Estimate one part's mass and the mass of exactly 200 such parts. State two reasons the estimate is not a batch acceptance criterion. Use the chapter's nearest-power-of-ten convention for the one-part mass in grams.
14. **Bounds must describe an admissible object.** An input sheet gives $R_o \in [9.9, 10.1] \text{ mm}$ and $R_i \in [9.8, 10.2] \text{ mm}$. A program treats every endpoint combination as a physical annulus and returns a negative lower mass. Explain the error. What further geometrical restriction is necessary, and why can one not simply assert a positive lower bound or 95% coverage? Compare with the admissible intervals in Figure 1.11.
15. **An exact interval calculation.** For a different uniform spacer take independent allowed intervals $R_o \in [9.9, 10.1] \text{ mm}$, $R_i \in [5.8, 6.2] \text{ mm}$, $L \in [19.8, 20.2] \text{ mm}$ and $\rho \in [2.6, 2.8] \text{ g cm}^{-3}$. Find the exact model endpoints numerically and an enclosing mass interval rounded outward to 0.01 g. Explain why opposite choices of bore radius enter the two endpoints.

C. Measurement quality and graphical reasoning

16. **Read the meaning of a plus-minus sign.** A report says $x = (5.00 \pm 0.02) \text{ mm}$ without further explanation. Give three different meanings the 0.02 mm could have. Which additional

information is required before assigning a coverage probability?

17. **Translate residuals, not their spread.** Synthetic readings are (8.14, 8.16, 8.15, 8.15) mm for a reference 8.00 mm. A separate calibration prescribes subtraction of $b = 0.15$ mm. Find the raw and corrected means, both sets of reference residuals and the sample standard deviation. Explain why the corrected mean agreeing with the reference is not proof of zero uncertainty.
18. **Read an aligned plot.** Using Figure 1.13, identify which instrument has finer displayed steps, which has smaller repeatability spread, and which uncorrected mean is closer to the reference. Explain why plotting each dataset on an independently rescaled horizontal axis could mislead the reader.
19. **Rounding and cancellation.** Evaluate $(6.24 \text{ cm})(2.4 \text{ cm})$ and $6.24 \text{ cm} - 6.1 \text{ cm}$. Give unrounded values and values reported by the chapter's conventional significant-figure rules. Do these rules supply a probability distribution for either result?
20. **A mean can hide a time trend.** Two synthetic sequences, in millimetres, are

$$(10.00, 10.01, 10.02, 10.03, 10.04), \quad (10.02, 10.00, 10.04, 10.01, 10.03).$$

Compare their means and standard deviations. What information is preserved by a graph against reading order but lost by these two summary statistics? Suggest one check before interpreting either sequence as independent repeatability data.

21. **Slope with units.** A linear calibration gives 0.72 V at 3.0 mm and 1.52 V at 7.0 mm. Find its slope and intercept, predict the response at 5.0 mm, and rewrite the same physical calibration with the input length in metres. No uncertainty information is supplied; do not invent error bars.
22. **Do not reuse a test as its own validation.** A quadratic has been chosen to fit the three training points in Example 1.12 exactly. A student calls those zero residuals “three independent validations”. Repair the statement. Using the held-out point at 4 mm, compare the line and quadratic predictions. Rewrite their coefficients for a length input in metres and verify the same predictions.
23. **An inverse problem retains the calibration domain.** A response of 1.22 V is interpreted using $y_L = 0.10 \text{ V} + (0.24 \text{ V mm}^{-1})x$ and $y_C = 0.10 \text{ V} + (0.20 \text{ V mm}^{-1})x + (0.020 \text{ V mm}^{-2})x^2$. Find both inferred lengths and all algebraic roots of the quadratic. Which root is admissible on $0 \leq x \leq 6 \text{ mm}$? Why is the linear inference not equally supported by the held-out data?
24. **A new discrepancy.** At $x = 3.0 \text{ mm}$ a new constructed reading is 0.92 V. Calculate observed-minus-predicted residuals for the two models in Problem 23. If the allowed discrepancy is 0.03 V and all other inputs are stipulated exact for this test, does either model pass? Give two possible reasons for a discrepancy in a real experiment without claiming that the residual uniquely identifies one.

D. Engineering, measurement and computation

25. **Design a discriminating observation.** An outer-geometry model overpredicts a clean part's measured mass. Propose a short investigation that distinguishes an omitted through-bore from an incorrect density. State what is read directly, which quantity is inferred, and how to avoid using the same mass twice as both input and validation. Include zero and alignment checks where relevant. No specific apparatus availability is assumed.

26. **Screening is not certification.** Exactly 100 unpackaged model parts each obey the enclosing interval $[53.6, 64.2]$ g. An independent batch-mass bound is first $[6.60, 6.64]$ kg, and later, after a documented correction, $[6.30, 6.34]$ kg. Compare each with the model interval. What can and cannot be concluded from disjointness or overlap?
27. **A unit-invariance test for code.** Run the inline `analyse` function on instrument A's data, then divide every reading, the offset, its standard uncertainty and the display increment by 1000. State the predicted transformation of each returned quantity. Explain why converting only the readings is an invalid test.
28. **Input checks are part of a model.** Predict how the same function should respond to one reading, a nonfinite reading, a negative calibration uncertainty, a zero display increment, and the value 20.185 mm on a 0.01 mm display grid. What important physical problems can pass every one of these software checks?
29. **Revise a correction without erasing observations.** In the computational example, replace the offset 0.200 mm by 0.180 mm, retaining the same u_b . Find the new corrected mean, sample spread and combined standard uncertainty. Explain what changes in Figure 1.21, what remains unchanged, and which raw records must be preserved.
30. **A reproducible report.** Prepare a compact reporting template for a repeated length measurement. It must distinguish raw data, correction, repeatability, other uncertainty contributions and the physical model. State how the report changes when different positions on a part give reproducibly different dimensions. Evaluation is by completeness and correct interpretation, not by one preferred layout.

E. Advanced challenges

31. **Reject an impossible covariance.** A proposed two-length covariance matrix is

$$\Sigma = \begin{pmatrix} 0.040 & 0.030 \\ 0.030 & 0.010 \end{pmatrix} \text{ mm}^2.$$

Show that it is not positive semidefinite, both by its determinant and by finding a linear combination with negative formal variance. Explain why clipping that variance to zero is not a physical repair. Then state the allowed covariance range with those fixed diagonal entries.

32. **Correlation changes a subtraction.** For $W = R_o^2 - R_i^2$ use $R_o = 12$ mm, $R_i = 8$ mm, $u_o = 0.06$ mm and $u_i = 0.04$ mm. Calculate u_W for $r = -0.8, 0, +0.8, +1$. Which case corresponds to a single common fractional scale uncertainty of 0.5%? Explain the sign of the covariance contribution and why perfect correlation does not make u_W vanish.
33. **Two different shared errors.** The same two radii share a small additive zero error b with $u_b = 0.05$ mm, and have no other uncertainty in this exercise. Derive u_W/W at first order. Compare it with a common multiplicative scale uncertainty of 0.5%. Which shared error cancels exactly in $R_o - R_i$, and which leaves the ratio R_i/R_o unchanged?
34. **Choose an improvement using a variance budget.** Start from the mass budget following Example 1.16. Compare two ideal improvements: (A) halve u_ρ only; (B) halve both radius standard uncertainties while keeping their correlation coefficient 0.8. Keep central values and u_L unchanged. Calculate the new u_m in both cases. State why halving the marginal uncertainties while leaving their covariance unchanged would be inconsistent.

35. **Sensitivity and a physical boundary.** Let $R_o = 12$ mm, $R_i = 11.88$ mm and let the two independent radius uncertainties each be 0.5% of the corresponding radius. Find the first-order relative area uncertainty. Compare the wall thickness with the standard uncertainty of $R_o - R_i$. Explain why a small fractional radius uncertainty need not justify a Gaussian linear model for this thin annulus.
36. **Check a nonlinear result exactly.** Let X take $x_0 - a, x_0, x_0 + a$ with equal probabilities, where x_0, a have units of length and $a > 0$. For $Y = X^2$, derive the exact mean and variance. Compare the variance with first-order propagation about x_0 , including the limit $x_0 = 0$. Evaluate the ratio of the exact variance to the first-order variance when $x_0 = 2a$.
37. **A density change is not a new shape.** A new part is twice as large in every length as a reference spacer but has density 0.8ρ . Determine its mass ratio, its values of β and λ relative to the original, and its dimensionless mass Π_m relative to the original. Explain why a collapse in Π_m does not prove that the two materials have the same density.
38. **An approximation with an explicit tolerance.** Require the outer-radius thin-wall approximation to overestimate mass by less than 5%. Derive the exact lower limit on $\beta = R_i/R_o$. Test $\beta = 0.96$ and give its relative overestimate. Show how the mean-radius representation removes this geometrical approximation.
39. **A common reference cancels from a difference.** Two length estimates have central values 10.00 and 10.10 mm. Their independent repeatability standard uncertainties are 0.02 and 0.03 mm; both contain the same additive calibration contribution with standard uncertainty 0.05 mm. Find the standard uncertainty of the second minus the first. Compare it with the answer obtained by incorrectly treating the total uncertainties of the two estimates as independent.
40. **Design a dimensionless test, not a circular plot.** Propose one series of spacer measurements that varies overall size at fixed shape, and another that varies bore ratio at fixed L/R_o . Define axes, held-fixed quantities, independently supplied information and a residual or comparison criterion. Explain what goes wrong if a separate density is inferred from each very mass that is then plotted as a prediction. Identify a shared-input correlation that would matter in an uncertainty analysis of the normalised axes.

F. Further visual, engineering and investigation tasks

All datasets in this group are constructed teaching inputs. Problems 41–47 use the core methods; Problem 48 uses the optional shared-uncertainty treatment.

41. **Read residuals before accepting a calibration.** A frozen sensor model is $y = (0.20 \text{ V}) + (0.15 \text{ V mm}^{-1})x$. Figure 1.30 supplies independent test residuals, defined as observed minus predicted, at $x/\text{mm} = 0, 2, 4, 6, 8$; their exact values in volts are 0.010, -0.020 , 0.025, 0.040, 0.070. (a) Apply the stipulated acceptance criterion $|r| \leq 0.030 \text{ V}$ at each tested input. (b) Reconstruct the observed outputs at 6 and 8 mm. (c) Explain whether the plot establishes a single continuous acceptable input range, and propose a discriminating follow-up test.
42. **Separate a tare from a density.** A balance reports the mass of a small container plus a uniform material. Figure 1.31 has exact constructed pairs $(V/\text{cm}^3, m_{\text{read}}/\text{g}) = (0, 0.15), (1, 2.85), (2, 5.55), (3, 8.25), (4, 10.95)$. (a) Determine the slope, intercept and their physical meanings under $m_{\text{read}} = \rho V + m_c$. (b) Predict total and material-only masses at $V = 2.50 \text{ cm}^3$. (c) A student uses only the last pair and forces a line through the origin. Find the inferred density and its relative error. (d) Rewrite the correct relation with volume in m^3

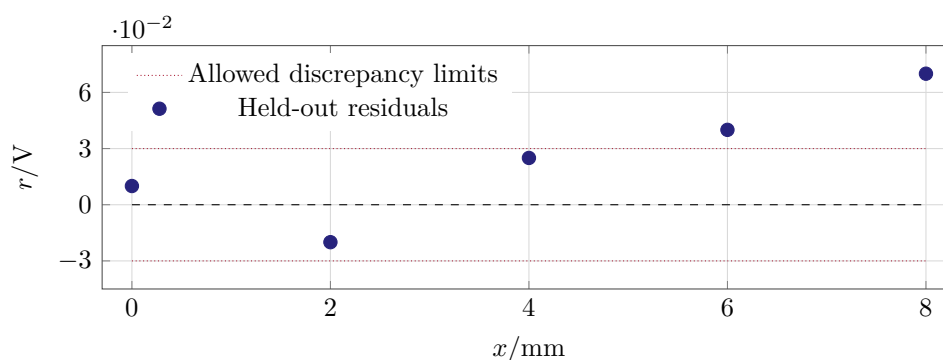


Figure 1.30: F01-P41. Constructed validation residuals for Problem 41. Dotted limits are a stipulated discrepancy criterion, not standard uncertainty or statistical coverage. Unconnected markers do not prescribe interpolation between tests.

and mass in kg; state what the agreement of all points does not establish about an actual material.

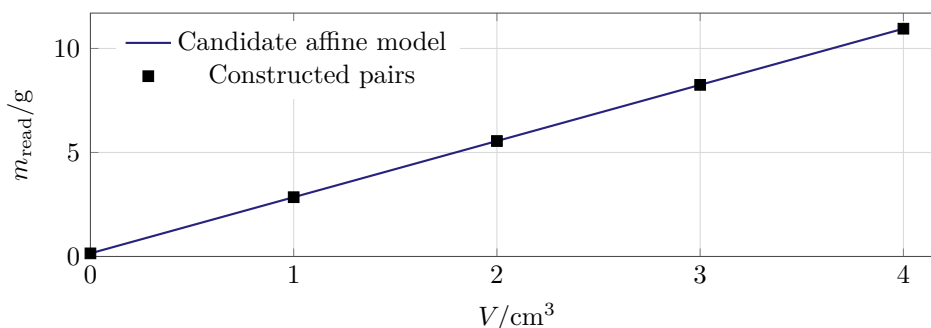


Figure 1.31: F01-P42. An affine mass–volume relation whose nonzero intercept matters to the density inference in Problem 42. The numerical pairs, including the zero-volume container reading, supply the information needed to distinguish slope from tare. No uncertainty information is supplied.

43. **An image measures a projection.** In the planar model of Figure 1.32, a straight segment of length d makes a known angle θ to the calibrated horizontal image axis. Its projected length is $z = d \cos \theta$. The constructed indication is $z = 19.32$ mm at $\theta = 15.0^\circ$. (a) Infer d . (b) Find the signed relative error if z is reported as the segment’s actual length. (c) Find the largest $|\theta| < 90^\circ$ for which this geometrical underestimate is at most 1%. (d) Explain why repeated identical image readings cannot remove this model error. Assume the projection relation, angle and scale are stipulated for the calculation; no uncertainty budget is given.

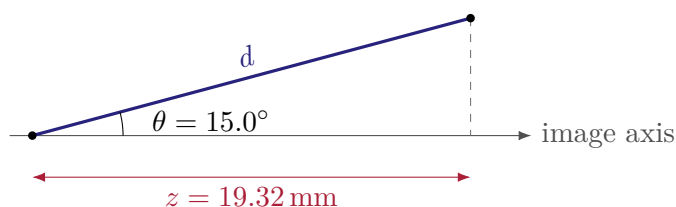


Figure 1.32: F01-P43. Planar projection geometry for Problem 43. The segment direction is drawn at the stated angle; the image scale is represented schematically. The vertical dashed line projects its endpoint onto the measurement axis. This is a defined projection model, not a claim about arbitrary camera perspective.

44. **Plan repeats around a common uncertainty floor.** Independent repeatability has standard deviation $s = 0.080$ mm and a separate common calibration has standard uncertainty $u_b = 0.010$ mm. Assume the planning model $u(N) = \sqrt{s^2/N + u_b^2}$ for integer $N \geq 1$, shown in Figure 1.33. (a) Calculate $u(16)$ and $u(64)$. (b) Find the least N giving $u(N) \leq 0.012$ mm.

- (c) Can repetitions alone reach 0.009 mm? (d) Describe what must be changed to improve the limiting uncertainty, and one experimental condition under which the curve would cease to be justified.

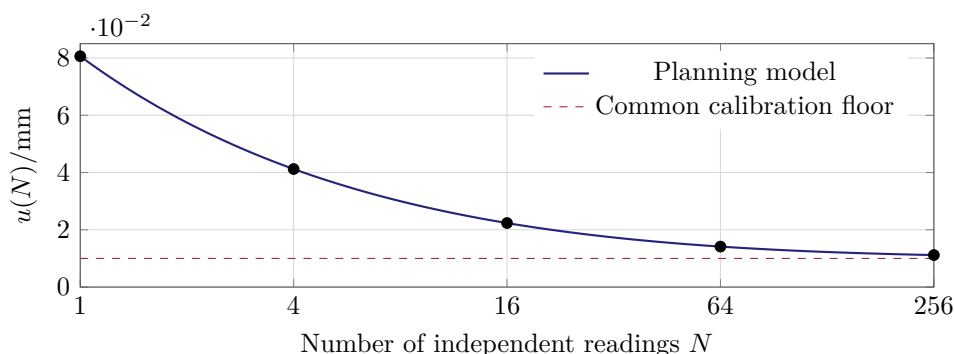


Figure 1.33: F01-P44. Analytical planning curve in Problem 44. The horizontal axis is logarithmic to show successive increases in the number of readings. Its continuous curve guides the eye; actual counts are integers. Increasing N suppresses independent repeatability but retains the same common calibration contribution.

45. **Audit a mixed-unit detector-window estimate.** A uniform rectangular detector window has width 80.0 mm, height 3.50 cm, thickness 250 μm and stipulated density 2.33 g cm^{-3} . Ignore coatings and frames in this model. (a) Find its volume and mass using one coherent unit system. (b) A script converts 250 μm to 0.250 cm. Diagnose the error and its effect on the mass. (c) The permitted total assembly mass is 2.00 g. What comparison is possible, and what information is missing before an assembly can be certified?
46. **Identical digits need an uncertainty model.** Four readings of a fixed length are all 7.20 mm on an ideal nearest-rounding display with increment $q = 0.01$ mm. A separate zero correction has already been applied. (a) Give the rounding interval and the sample standard deviation. (b) As an additional model, treat the unresolved rounding error of this one fixed indication as uniform on $[-q/2, q/2]$. Find its standard uncertainty. Should it be divided by $\sqrt{4}$ merely because the same digits were copied four times? (c) Combine it with an independent calibration standard uncertainty 0.004 mm and state a defensible report. Do not silently assume independent rounding noise in identical readings.
47. **Infer a parameter, then ask what was tested.** A uniform spacer has measured nominal mass 30.0 g, outer radius 10.0 mm, bore radius 6.0 mm and length 50.0 mm. (a) Infer its density under the stated hollow-cylinder model. (b) Repeat the inference if no bore were present but the outer radius and length were the same. (c) Does reproducing the input mass with each inferred density validate either geometry? State one independent observation that would decide between the models. No uncertainty information is supplied.
48. **Optional advanced investigation: spatial structure or repeatability?** Constructed length readings at locations P and Q, in millimetres, are (10.00, 10.02, 10.01, 10.01) and (10.20, 10.22, 10.21, 10.21). Each location has independent repeatability deviations; both estimates share the same additive calibration standard uncertainty 0.010 mm. (a) Calculate the two means, sample spreads and standard uncertainties of the means under this model. (b) Calculate Q minus P and its standard uncertainty. (c) Explain what is lost by reporting only the pooled mean. (d) Propose an acquisition sequence and two independent checks that distinguish stable spatial variation from drift or alignment effects; give an interpretation criterion, without claiming to have performed an experiment.

1.12 Complete solutions

Each solution addresses the full stated task. Bounds, standard uncertainties and model-discrepancy criteria retain their different meanings. In calculations, guard digits are shown to make the reasoning reproducible; the final reporting precision must reflect the stated inputs and uncertainty model.

Problem 1: dimensions and limits cannot establish the geometry

The proposed model has dimension $[\rho L(R_o - R_i)^2] = \text{M}$, gives the correct solid-cylinder value $\rho\pi LR_o^2$ at $R_i = 0$, decreases monotonically as the bore grows, and vanishes at $R_i = R_o$. These tests all pass. Nevertheless, at $R_i = R_o/2$ its predicted mass is $m_{\text{solid}}/4$, while the actual annular area in Figure 1.1 is $\pi R_o^2 - \pi(R_o/2)^2$, giving $3m_{\text{solid}}/4$. It underpredicts by a factor three. Thickness is not the radius of the material disk: the material surrounds a cavity. The correct factorisation is $(R_o - R_i)(R_o + R_i)$, which includes the circumference scale as well as thickness. Necessary consistency tests do not substitute for constructing the actual volume.

Problem 2: distinguish two hypotheses with another observation

At the specified density, $m/m_{\text{solid}} = 1 - (R_i/R_o)^2 = 3/4$, so $(R_i/R_o)^2 = 1/4$ and the nonnegative radius ratio is $1/2$. For an unbored cylinder of the same outer radius and length, $m/m_{\text{solid}} = \rho'/\rho$, so $\rho' = 3\rho/4$. Both hypotheses reproduce the one mass observation. Inspecting the bore, measuring material volume independently, or obtaining an independent density distinguishes them. A second calculation from the same mass supplies no new observation. The hypotheses assume uniform density and the same outer geometry; changes in length or other cavities would introduce further alternatives.

Problem 3: preserve the quantity, not only its stored number

The records are consistent if their units are respectively millimetres and metres: $25.0 \text{ mm} = 25.0 \times 10^{-3} \text{ m} = 0.0250 \text{ m}$. They also need to identify the same measurand and reading, rather than unrelated objects. A unit conversion changes both the numerical value and the unit label. Changing only the label from millimetres to metres states 25.0 m instead of 0.0250 m , an error factor 1000. The digit string alone does not identify a physical quantity or establish its precision.

Problem 4: exact definitions and imperfect realisations

The SI assigns the exact numerical value 299 792 458 to c in m s^{-1} . Together with the second, this defines the metre; a laboratory result inconsistent with that value tests the realisation or the measurement procedure rather than redefining the metre. A ruler reading still has limitations from calibration, graduation, alignment, temperature and interpretation of endpoints. Exact unit definitions do not make those operations exact. The gravitational constant G , by contrast, is an experimentally determined quantity with a stated uncertainty, rather than an exact SI defining constant. A better determination can revise its recommended numerical value and uncertainty.

Problem 5: cube the complete conversion factor

A millimetre is 10^{-3} m , hence

$$1250 \text{ mm}^3 = 1250(10^{-3} \text{ m})^3 = 1.250 \times 10^{-6} \text{ m}^3 = 1.250 \text{ cm}^3.$$

The centimetre route gives $m = (2.40 \text{ g cm}^{-3})(1.250 \text{ cm}^3) = 3.00 \text{ g}$. The SI route uses $\rho = 2400 \text{ kg m}^{-3}$, giving $m = 0.00300 \text{ kg}$, the same quantity. The script's 1.250 m^3 is 10^6 times the correct volume and would overpredict mass by the same factor. Two powers of the conversion were omitted; Figure 1.4 shows why a volume contains three length factors.

Problem 6: a prefix participates in the exponent

Using the stipulated interpretation of the first unit,

$$\begin{aligned} 4.50 \mu\text{m}^2 &= 4.50(10^{-6} \text{ m})^2 = 4.50 \times 10^{-12} \text{ m}^2, \\ 0.0800 \text{ cm}^{-1} &= 0.0800(10^{-2} \text{ m})^{-1} = 8.00 \text{ m}^{-1}, \\ 1.25 \text{ mg mm}^{-3} &= 1.25 \frac{10^{-6} \text{ kg}}{(10^{-3} \text{ m})^3} = 1.25 \times 10^3 \text{ kg m}^{-3}. \end{aligned}$$

In every powered unit the exponent acts on the full prefixed unit. A negative exponent inverts its conversion factor. The last result can be checked by noting that $1 \text{ mg mm}^{-3} = 1 \text{ g cm}^{-3}$, a plausible condensed-material density scale.

Problem 7: geometry cannot be repaired with a unit label

The correct radii are 12 mm and 8 mm, so $R_o^2 - R_i^2 = 80 \text{ mm}^2$. Substituting 24 and 16 mm gives $320 \text{ mm}^2 = 4W$. Density and length are unchanged, so predicted volume and mass are each four times too large. Both sets of inputs use millimetres; this is a misidentification of diameter as radius. Relabelling the output units cannot restore the missing factors 1/2 in the radial inputs. A diameter-based correct expression is $m = \rho\pi L(D_o^2 - D_i^2)/4$.

Problem 8: dimensional balance and its limits

The proposed right-hand dimensions are $\text{M}^p \text{L}^{-3p+2q+r} \text{T}^{-r}$. Matching the force dimensions gives $p = 1$, $r = 2$ and $-3 + 2q + 2 = 1$, so $q = 1$. Thus the restricted model is $F = C\rho A v^2$. All three independent balances are needed. A dimensionless constant does not appear in them, so C remains unknown. If a further speed v_0 matters, a dimensionally consistent model could instead contain a function of v/v_0 . The exponent construction in Figure 1.6 tests consistency with a chosen variable list, not the completeness of that list or agreement with experiment.

Problem 9: each added term has the same dimension

Because each term is added to a position, $[a] = \text{L}$, $[bt] = \text{L}$ and $[ct^2] = \text{L}$. Therefore $[b] = \text{LT}^{-1}$ and $[c] = \text{LT}^{-2}$, with SI units m , ms^{-1} and ms^{-2} . The coefficient named c here is a polynomial coefficient, not the speed of light. Dimensional homogeneity permits any dimensionless numerical factor, including 1/2. Such a factor arises from a separately specified physical definition or dynamical derivation; dimensions alone cannot require it.

Problem 10: functions require dimensionless arguments

A logarithm of a dimensional length changes meaning when its numerical unit changes, so replace $\ln x$ by $\ln(x/x_0)$ with a specified positive reference length. The exponential $\exp(-t/\tau)$ is valid when $\tau \neq 0$: its argument is a time ratio. The sum $x + x_0$ has units of length and cannot itself be a sine argument. One repair is $\sin[(x + x_0)/\ell]$, where ℓ is a specified nonzero length. Choosing ℓ determines how rapidly the sine changes with position. Different reference scales define physically different functions; inserting any convenient number without its meaning would only hide the problem.

Problem 11: numerical rescaling versus physical scaling

In case A, each length's numerical value is divided by ten when expressed in centimetres. The volume's numerical value is divided by 10^3 , while the density's numerical value in g cm^{-3} differs accordingly from one in g mm^{-3} . A coherent calculation gives the same physical volume and mass. In case B, the physical lengths become ten times larger at fixed density and shape; volume and mass grow by 10^3 in any unchanged unit system. Unit conversion changes the description. Enlarging the object changes the quantity. These cases can be distinguished by comparing complete quantities rather than unlabeled numbers.

Problem 12: enforce the volume constraint

At fixed material volume,

$$\pi L(R_o^2 - R_i^2) = \pi L'(R_o^2 - R_i'^2), \quad L' = L \frac{R_o^2 - R_i^2}{R_o^2 - R_i'^2}.$$

The ratio is dimensionless, giving length units to L' . For a larger admissible bore, $R_i' > R_i$, the denominator is smaller, so the length must increase and the mass ρV stays fixed. As $R_i' \rightarrow R_o^-$, the required length diverges. A real finite part cannot meet this constraint with exactly zero wall. The earlier decreasing-mass result held length fixed; it does not apply after changing that constraint.

Problem 13: an estimate and a decision have different inputs

In grams and centimetres the rough model gives

$$m \simeq 8(3)(3)(1.2^2 - 0.8^2) \text{ g} = 57.6 \text{ g}, \quad 200m \simeq 11.52 \text{ kg}.$$

Given the deliberately rounded inputs, sensible estimates are about 60 g per part and 12 kg per batch. Since $57.6 > \sqrt{10} \times 10 \simeq 31.6$, the chapter's nearest-power-of-ten order is 10^2 g. Changing the mass unit changes the numerical exponent, so retain the unit. The estimate supplies neither measured density/dimensions and their variation nor an acceptance tolerance. The exact count 200 introduces no counting uncertainty, but does not make the part masses exact.

Problem 14: reject invalid combinations before propagating bounds

The endpoint pair $R_o = 9.9 \text{ mm}$, $R_i = 10.2 \text{ mm}$ gives $R_o^2 - R_i^2 = -6.03 \text{ mm}^2$. It is outside the annulus model because a bore cannot exceed the outer radius. Restrict the joint allowed set to $0 \leq R_i < R_o$, or obtain tighter measurements that impose a positive minimum wall thickness. The given intervals overlap, so admissible radii can approach equality and the mass can approach zero. They supply no strictly positive lower mass. The comparison with Figure 1.11 is that its intervals are jointly admissible. Interval endpoints do not specify frequencies or probabilities; 95% coverage requires further information.

Problem 15: exact endpoints and outward rounding

Every input combination is admissible because $6.2 \text{ mm} < 9.9 \text{ mm}$ for the two radii. At fixed other inputs, mass grows with ρ, L, R_o and falls with R_i . Converting mm^3 to cm^3 therefore gives

$$m_{\min} = \frac{2.6\pi(19.8)(9.9^2 - 6.2^2)}{1000} \text{ g} = 9.63420784 \text{ g},$$

$$m_{\max} = \frac{2.8\pi(20.2)(10.1^2 - 5.8^2)}{1000} \text{ g} = 12.14856141 \text{ g}.$$

Round the lower end downward and upper end upward to obtain $[9.63, 12.15] \text{ g}$. The largest bore is chosen for the minimum because it removes the most material. These are allowed model bounds under the specified rectangular input set. No distribution or coverage probability is implied, and no unprovided correlation may be used to narrow the result.

Problem 16: define the meaning of the stated interval

The 0.02 mm could be a permitted deviation or tolerance bound; a standard uncertainty u_x ; or an expanded uncertainty $U = ku_x$ with a specified coverage factor. These meanings answer different questions. A bound needs its scope and assumptions; a standard uncertainty needs its evaluation model and is not a guaranteed range; expanded uncertainty needs the factor and coverage justification. Before assigning a probability, specify an applicable distribution or other supported coverage model, what contributions are included, and any relevant degrees of freedom. The displayed digits alone provide none of that information.

Problem 17: a correction translates the data without changing spread

The raw mean is 8.15 mm and raw reference residuals are (0.14, 0.16, 0.15, 0.15) mm. Subtracting $b = 0.15$ mm gives readings (7.99, 8.01, 8.00, 8.00) mm, mean 8.00 mm and reference residuals (−0.01, 0.01, 0, 0) mm. Deviations about either corresponding mean are identical, so

$$s = \sqrt{\frac{0.01^2 + 0.01^2}{4 - 1}} \text{ mm} = 0.00816497 \text{ mm}.$$

The corrected mean agrees exactly with the central reference value in this constructed calculation. That equality neither removes scatter nor supplies the missing uncertainties of the calibration and reference. It is evidence about central values, not proof of zero measurement uncertainty.

Problem 18: compare the plots on one physical scale

Instrument A's displayed increment is 0.01 mm compared with B's 0.10 mm, so A has finer displayed steps. Its sample spread is also smaller: $s_A = 0.0104881$ mm versus $s_B = 0.0894427$ mm. But the uncorrected means are 20.195 and 20.000 mm against the same 20.000 mm reference, so B's mean is closer. Figure 1.13 keeps the horizontal scales aligned. Independently zooming around each dataset can make their displayed spreads appear equal and hide A's offset. Finer resolution and smaller scatter do not themselves establish greater accuracy; the reference and calibration uncertainties still matter.

Problem 19: round once according to the relevant operation

Multiplication gives $6.24(2.4) = 14.976 \text{ cm}^2$ before rounding. The conventional limit is two significant figures, giving 15 cm^2 or $1.5 \times 10^1 \text{ cm}^2$. Subtraction gives $6.24 - 6.1 = 0.14 \text{ cm}$; the coarser decimal place is tenths of a centimetre, so report 0.1 cm. Cancellation explains why the small difference retains few relative digits. These rules summarise reporting precision; they do not give a probability distribution, standard uncertainty or confidence interval. Those require the measurement information and dependence of the inputs.

Problem 20: preserve acquisition order

Both lists contain the same five numbers, so their common mean is $50.10/5 = 10.02$ mm. Their squared deviations sum to 0.0010 mm^2 , giving

$$s = \sqrt{0.0010/4} \text{ mm} = 0.0158114 \text{ mm}.$$

A graph against reading order distinguishes the first sequence's steady increase of 0.01 mm per reading from the second's reordered pattern. Mean and spread are invariant under that permutation and therefore cannot identify a time trend. Check a stable reference before and after the sequence, record temperature/contact conditions, or repeat in a changed acquisition order. Even the reordered sequence alone does not prove independence; that is a modelling assumption supported by the procedure and further evidence.

Problem 21: slopes transform with the input unit

The slope is

$$a = \frac{1.52 - 0.72}{7.0 - 3.0} \text{ V mm}^{-1} = 0.20 \text{ V mm}^{-1}, \quad b = 0.72 - 0.20(3.0) = 0.12 \text{ V}.$$

Thus $y(5.0 \text{ mm}) = 1.12 \text{ V}$. With metre input the same calibration is $y = 0.12 \text{ V} + (200 \text{ V m}^{-1})x$. Substitution of $x = 0.0050 \text{ m}$ reproduces 1.12 V. The two supplied points determine the chosen line exactly, but provide no scatter, coefficient uncertainty, uncertainty bars or proof of validity at other inputs.

Problem 22: interpolation is not independent validation

The three training points determined the three quadratic coefficients. Their zero residuals are interpolation, not three independent predictive tests. At the held-out input 4 mm, the frozen models give $y_L = 0.10 + 0.24(4) = 1.06$ V and $y_C = 0.10 + 0.20(4) + 0.020(4)^2 = 1.22$ V. Against the held-out 1.22 V, observed-minus-predicted residuals are +0.16 V and zero. Under the stipulated 0.03 V criterion, the line fails this test and the quadratic passes.

For metre input, the line slope is 240 V m^{-1} , and the quadratic coefficients are 200 V m^{-1} and 20000 V m^{-2} . At 0.004 m they give $0.10 + 240(0.004) = 1.06$ V and $0.10 + 200(0.004) + 20000(0.004)^2 = 1.22$ V. Unit invariance checks the algebra; one constructed held-out agreement does not prove universal physical correctness.

Problem 23: retain the calibration domain after inversion

The line gives $x_L = (1.22 - 0.10)/0.24 \text{ mm} = 14/3 \text{ mm} = 4.66667 \text{ mm}$. For the quadratic use the dimensionless numerical variable $z = x/(1 \text{ mm})$:

$$0.020z^2 + 0.20z - 1.12 = 0, \quad z^2 + 10z - 56 = (z - 4)(z + 14) = 0.$$

The algebraic roots are $x = 4 \text{ mm}$ and -14 mm . Only the first lies within $0 \leq x \leq 6 \text{ mm}$; substituting it recovers 1.22 V. The line's inferred input also lies in that broad interval, but its extrapolation beyond the original 0–2 mm training range failed the held-out test. Algebraic admissibility and predictive support are distinct requirements.

Problem 24: neither candidate explains this new reading

At 3.0 mm, the line predicts $0.10 + 0.24(3) = 0.82$ V, while the quadratic predicts $0.10 + 0.20(3) + 0.020(3)^2 = 0.88$ V. The corresponding observed-minus-predicted residuals are $0.92 - 0.82 = +0.10$ V and $0.92 - 0.88 = +0.04$ V. Both magnitudes exceed 0.03 V, so neither passes the stipulated exact-input comparison. In a real experiment, an offset drift, an incorrect input reading, or an inadequate response model could each contribute. A residual identifies a mismatch under assumptions, not a unique cause; independent checks are needed to distinguish them.

Problem 25: make the next observation discriminating

One workable investigation first records a separately calibrated mass, then inspects whether the bore passes through the full part. Measure outer and inner dimensions and length with zero checks and repeated orientations to test alignment, circularity and contact. Obtain density from an independently characterised material sample or reliable material information, with its applicable uncertainty. Dimensions and balance indications are direct observations; material volume and predicted mass are inferred through the chosen geometry and density model.

A detected through-bore supports correcting the volume; a verified solid part with independently lower density supports the second explanation. Either, neither or both effects may be present. Using the observed mass to fit density and then calling the reconstructed same mass a validation is circular. A satisfactory design makes the observations independent, declares model limits and names how each alternative would be distinguished.

Problem 26: interval overlap is a consistency test

The exact count gives $100[53.6, 64.2] \text{ g} = [5.36, 6.42] \text{ kg}$ for unpackaged parts. The first weighing interval $[6.60, 6.64] \text{ kg}$ is disjoint: its lowest value exceeds the highest model value by 0.18 kg. Under the stipulated exact count and absence of packaging, at least one geometry, density, part-bound or weighing assumption is inconsistent with the batch; the comparison does not select which one. The documented corrected interval $[6.30, 6.34] \text{ kg}$ lies entirely within the model interval, so this test

no longer excludes the combined assumptions. Overlap does not certify every part, identify its material or validate the balance. The correction must have independent justification, rather than being chosen to enforce overlap.

Problem 27: convert every dimensional input

In millimetres the function returns $\bar{x} = 20.195$, $\hat{x} = 19.995$, $s = 0.0104881$ and $u = 0.0108781$, with corrected readings (19.98, 20.00, 19.99, 20.01, 20.00, 19.99). If every dimensional input is divided by 1000, those four scalars and every corrected reading are divided by 1000 and interpreted in metres. For example, the estimate becomes 0.019995 m and $u = 1.08781 \times 10^{-5}$ m. Internally, squared deviations and variances scale by 10^{-6} , and the count and grid ratios do not change.

Converting only readings leaves an offset in the wrong numerical unit and makes the grid test incompatible. The output can then be wrong or rejected. A unit-invariance test must preserve all complete quantities, including resolution and uncertainty.

Problem 28: distinguish numerical admissibility from physical adequacy

One reading fails the requirement $N \geq 2$ for this sample-variance calculation. A nonfinite reading fails the finite-input test. Negative u_b and zero resolution fail their sign tests. Finally, $20.185/0.01 = 2018.5$ is half a grid interval away from either allowed integer, far beyond the code's numerical tolerance, so this raw indication is rejected. Each case should raise a `ValueError`, rather than produce a plausible-looking result.

All checks can still pass for a misaligned instrument, a drifting zero, a wrong measurand, an inappropriate calibration domain, or a nonuniform part described by a single length. The checks enforce the function's stated numerical contract; physical validity requires further observations and model review. A grid-based input check is appropriate to raw indications, not to subsequently averaged or corrected quantities.

Problem 29: keep the raw evidence and revise the correction

The raw mean remains 20.195 mm. The revised estimate is $20.195 - 0.180 = 20.015$ mm, and corrected readings become (20.00, 20.02, 20.01, 20.03, 20.02, 20.01) mm. Subtracting a common value changes no deviation from the corresponding mean, so $s = \sqrt{0.00011} = 0.0104881$ mm. With the unchanged $u_b = 0.010$ mm,

$$u = \sqrt{0.00011/6 + 0.010^2} \text{ mm} = 0.0108781 \text{ mm}.$$

Every corrected reference residual in Figure 1.21 moves upward by 0.020 mm; raw residuals, reading order and spread remain unchanged. Preserve the raw readings, units, acquisition order, original calibration and the provenance/date of its revision. Correction history should accompany, rather than overwrite, observations.

Problem 30: an example reporting template and its rubric

A compact template is: “Measurand: [specified part, location and direction]; conditions: [temperature, alignment/contact procedure, acquisition order]; raw indications and units: [list]; display increment: [value]; calibration/correction: [source, sign, value, uncertainty and date]; model: [equation and assumptions]; raw mean: [value]; corrected estimate: [value]; repeatability: [N , s , $s/\sqrt{N}$ and independence assumptions]; other contributions: [values, meanings and shared dependencies]; combined uncertainty: [rule, value and meaning]; reporting/rounding: [result]; limitations and reconstruction steps: [record].”

Evaluation requires enough information to reproduce the result and to distinguish scatter from common uncertainty and allowed bounds. Consistent spatial differences require location-labelled

records and a model of geometry or an explicitly defined spatial average. Blindly averaging locations into one “length” can conceal the variation that matters. Accept alternative layouts meeting these criteria; an unsupported claim of complete uncertainty does not meet them.

Problem 31: repair the joint model, not the negative result

The determinant is $0.040(0.010) - 0.030^2 = -0.0005 \text{ mm}^4$, so the proposed matrix is not positive semidefinite. Directly, the coefficient vector $(1, -2)$ gives

$$\text{Var}(X_1 - 2X_2) = 0.040 + 4(0.010) - 4(0.030) = -0.040 \text{ mm}^2.$$

Variance is the expectation of a squared centred quantity and cannot be negative. Clipping this result to zero leaves an impossible covariance model and does not repair other combinations. With the given diagonal entries, the allowed covariance range is

$$-\sqrt{0.040(0.010)} \leq \text{Cov}(X_1, X_2) \leq \sqrt{0.040(0.010)}, \quad \text{or} \quad [-0.020, 0.020] \text{ mm}^2.$$

Equality gives a singular but possible perfect-correlation model. The proposed 0.030 mm^2 lies outside that range; revisit how the joint input information was obtained.

Problem 32: signed sensitivities control the covariance term

For $W = 80 \text{ mm}^2$, the signed sensitivity contributions are $2R_o u_o = 1.44 \text{ mm}^2$ and $-2R_i u_i = -0.64 \text{ mm}^2$. Thus

$$u_W^2 = (1.44^2 + 0.64^2 - 2r(1.44)(0.64)) \text{ mm}^4 = (2.4832 - 1.8432r) \text{ mm}^4.$$

For $r = -0.8, 0, 0.8, 1$, respectively, $u_W = 1.98941, 1.57582, 1.00431, 0.80000 \text{ mm}^2$. The negative cross term arises because increasing the outer radius adds area while increasing the inner radius removes it. For a shared fractional scale uncertainty 0.005 , the radii have these marginal uncertainties and $r = 1$. Since $W' = (1 + \varepsilon)^2 W$, first-order relative uncertainty is $2u_\varepsilon = 0.010$, reproducing $u_W = 0.8 \text{ mm}^2$. Perfect correlation does not cancel the common scale of a nonzero area.

Problem 33: shared additive and multiplicative modes differ

With the same additive deviation b in both radii,

$$(R_o + b)^2 - (R_i + b)^2 = W + 2(R_o - R_i)b.$$

The b^2 terms cancel exactly. Hence $u_W = 2(12 - 8)(0.05) = 0.40 \text{ mm}^2$ and $u_W/W = 0.40/80 = 0.005 = 0.5\%$. A common fractional scale uncertainty 0.5% instead gives $u_W/W \simeq 1.0\%$. The additive mode cancels exactly in $(R_o + b) - (R_i + b)$ but generally changes the ratio. A multiplicative mode cancels in $(1 + \varepsilon)R_i/[(1 + \varepsilon)R_o]$ but changes the wall thickness. Shared inputs must be represented by the appropriate physical mode, not by the word “correlated” alone.

Problem 34: reduce the component that dominates the budget

Use $m = 58.8106145 \text{ g}$, $\rho = 7.80 \text{ g cm}^{-3}$, $L = 30.00 \text{ mm}$ and the stated baseline radius variance $u_W^2 = 1.00864 \text{ mm}^4$. The independent relative density, length and radius variances are $(0.03/7.80)^2$, $(0.05/30.00)^2$ and $1.00864/80^2$. For A,

$$u_{m,A} = m \sqrt{(0.015/7.80)^2 + (0.05/30.00)^2 + 1.00864/80^2} = 0.753318 \text{ g}.$$

For B, halving both radius uncertainties at the same correlation divides their complete radius variance by four:

$$u_{m,B} = m \sqrt{(0.03/7.80)^2 + (0.05/30.00)^2 + 1.00864/(4 \cdot 80^2)} = 0.443896 \text{ g}.$$

B is more effective within this budget because radii dominate. Covariance equals $ru_o u_i$ and must also be quartered. Keeping its old value after halving both marginals would require $r = 3.2$, an impossible input model.

Problem 35: a nearby boundary compromises the linear model

Here $\beta = 0.99$, $W = 12^2 - 11.88^2 = 2.8656 \text{ mm}^2$, and independent uncertainties are $u_o = 0.0600 \text{ mm}$, $u_i = 0.0594 \text{ mm}$. First-order propagation gives

$$\frac{u_A}{A} = \frac{\sqrt{(24 \cdot 0.0600)^2 + (23.76 \cdot 0.0594)^2}}{2.8656} = 0.703624.$$

This is about 70%, despite the small marginal percentages. Thickness is $t = 0.120 \text{ mm}$, while $u_t = \sqrt{0.0600^2 + 0.0594^2} = 0.0844296 \text{ mm}$, so $t/u_t \simeq 1.42$. The forbidden $t \leq 0$ region is close on the uncertainty scale. A naive unbounded Gaussian radius model can assign substantial probability to impossible annuli. Linearisation and coverage therefore need scrutiny or a physically constrained joint model; the numerical first-order result is not a justified coverage statement.

Problem 36: exact enumeration tests the approximation

Write $X = x_0 + Z$ with $Z = -a, 0, a$ equally likely. Then $E Z = E Z^3 = 0$, $E Z^2 = 2a^2/3$ and $E Z^4 = 2a^4/3$. Since $Y = x_0^2 + 2x_0Z + Z^2$,

$$E Y = x_0^2 + \frac{2a^2}{3}, \quad \text{Var } Y = 4x_0^2 E Z^2 + E Z^4 - (E Z^2)^2 = \frac{8x_0^2 a^2}{3} + \frac{2a^4}{9}.$$

The covariance of Z and Z^2 vanishes because their relevant odd moments vanish. First-order propagation gives only $8x_0^2 a^2/3$. At $x_0 = 0$ it predicts zero although the exact variance is $2a^4/9 > 0$. At $x_0 = 2a$, the exact-to-linear variance ratio is $(98/9)/(96/9) = 49/48$. The mean shift remains $2a^2/3$ even when the variance approximation is close. Mean and variance have units of length squared and length to the fourth power respectively.

Problem 37: normalising density deliberately removes its effect

The volume scale is $2^3 = 8$, so the mass ratio is $0.8 \times 8 = 6.4$. Both shape ratios are unchanged: $\beta' = R'_i/R'_o = \beta$ and $\lambda' = L'/R'_o = \lambda$. In

$$\Pi_m = \frac{m}{\rho R_o^3} = \pi \lambda (1 - \beta^2),$$

the new numerator and denominator both acquire the factor 6.4, so $\Pi'_m = \Pi_m$. Equal dimensionless masses mean that the stipulated geometry model agrees after its density normalisation. They cannot establish equal material densities because density has explicitly been divided out.

Problem 38: solve the strict approximation criterion

For an admissible annulus the relative overestimate is $\epsilon = (1 - \beta)/(1 + \beta)$. Since $1 + \beta > 0$, requiring $\epsilon < 0.05$ gives $1 - \beta < 0.05(1 + \beta)$, or

$$\beta > \frac{0.95}{1.05} = \frac{19}{21} = 0.9047619.$$

Equality gives exactly 5% and is excluded by “less than”. At $\beta = 0.96$, $\epsilon = 0.04/1.96 = 1/49 = 0.0204082$, about 2.04%, so it passes. With $R_m = (R_o + R_i)/2$ and $t = R_o - R_i$, the identity $A = \pi t(R_o + R_i) = 2\pi R_m t$ is exact. It is replacing R_m by R_o , rather than the thickness variable itself, that introduces the approximation.

Problem 39: a common additive reference cancels

Represent estimates by $X_1 = 10.00 \text{ mm} + \epsilon_1 + B$ and $X_2 = 10.10 \text{ mm} + \epsilon_2 + B$. Their difference is $D = 0.10 \text{ mm} + \epsilon_2 - \epsilon_1$, so

$$u_D = \sqrt{0.02^2 + 0.03^2} \text{ mm} = 0.0360555 \text{ mm}.$$

Incorrectly treating total uncertainties as independent gives $\sqrt{0.02^2 + 0.05^2 + 0.03^2 + 0.05^2} = 0.0793725$ mm. The covariance formula subtracts $2u_B^2$ and reproduces the direct cancellation. The difference may sensibly be reported as (0.100 ± 0.036) mm with standard uncertainty under the stated model. Cancellation would not be exact for unequal reference sensitivities or a multiplicative common error; the shared-input model must describe the actual operation.

Problem 40: an exemplar dimensionless investigation

For the size series, choose a fixed admissible β and λ , and several R_o values; set $R_i = \beta R_o$, $L = \lambda R_o$ in the design. Measure the realised geometry and mass, supplying density independently. Plot $m/(\rho R_o^3)$ against R_o with model level $\pi\lambda(1 - \beta^2)$. A systematic trend would challenge the combined geometry, density, measurement and scaling assumptions. For the shape series, vary β at fixed λ and, if practical, fixed R_o ; plot $m/(\rho\pi L R_o^2)$ against β and compare with $1 - \beta^2$. For an explicit interval criterion, supply defensible allowed intervals for geometry, density and measured mass, propagate them through the model, and compare the resulting prediction interval with the independent measurement interval. Disjoint intervals reject the joint model and bound assumptions for that observation; overlap means they are not excluded, without proving the model. These allowed bounds are not assigned a coverage probability.

If each density is inferred as m/V from the same plotted mass, the normalised result is forced to agree by algebra; it is circular evidence. The same measured R_o enters β , λ and mass normalisation, making derived axes correlated. An adequate response specifies both series, independent inputs, comparison criteria and these dependencies. Accept alternative designs preserving those requirements; agreement, a trend or isolated discrepancies are all possible experimental outcomes.

Problem 41: discrete test decisions and reconstructed readings

(a) Compare magnitudes with the inclusive limit 0.030 V: inputs 0, 2, 4 mm pass; 6, 8 mm fail. The sign tells the direction of the discrepancy, whereas acceptance uses its magnitude. (b) The model outputs are $0.20 + 0.15(6) = 1.10$ V and $0.20 + 0.15(8) = 1.40$ V. Adding the plotted residuals gives observed 1.140 V and 1.470 V. (c) Unconnected points certify only the tested comparisons under the stipulated criterion. They do not establish that every intermediate input passes. The increasing high-range positive residuals suggest, but do not identify, model curvature or drift. Freeze the model, add intermediate/high-range tests, randomise their order and recheck the zero with an independent reference. Persistent input dependence supports a model limitation; acquisition-time dependence instead supports drift. Neither result is assumed in advance.

Problem 42: the slope is density and the intercept is container mass

(a) Any two pairs give $\rho = (10.95 - 0.15)/(4 - 0) = 2.70$ g cm⁻³ and $m_c = 0.15$ g. A zero-volume observation independently identifies the tare under this model. (b) At 2.50 cm³, material mass is $2.70(2.50) = 6.75$ g; total indicated mass is 6.90 g. (c) Forcing the last pair through the origin gives $\rho_{\text{wrong}} = 10.95/4 = 2.7375$ g cm⁻³, with relative error $(2.7375 - 2.70)/2.70 = 0.013889$, about +1.39%. It attributes container mass to material. (d) The coherent SI relation is

$$m_{\text{read}} = (2700 \text{ kg m}^{-3})V + 0.00015 \text{ kg}.$$

At $V = 2.50 \times 10^{-6} \text{ m}^3$ it gives 0.00690 kg. Exact agreement of constructed pairs checks this chosen relation, but cannot prove uniformity, calibration or uncertainty for an actual material.

Problem 43: correct the projection model before repeating readings

(a) The stipulated projection gives $d = z/\cos\theta = 19.32/\cos 15^\circ = 20.0015$ mm, nominally 20.00 mm under the input precision convention. (b) Reporting z as d gives signed relative error $(z - d)/d = \cos 15^\circ - 1 = -0.0340742$, an underestimate of about 3.41%. (c) For $|\theta| < 90^\circ$, the criterion is

$1 - \cos \theta \leq 0.01$, hence $|\theta| \leq \arccos(0.99) = 8.1096^\circ$. It includes equality because “at most” is inclusive. (d) Identical repeats reproduce the same projection, so they cannot turn z into d . Correct the geometry, align the segment, or measure the angle and its uncertainty. This solution assumes a planar orthogonal projection with a valid calibrated scale; perspective distortion would require a different model.

Problem 44: more repetitions cannot remove the common floor

(a) The two requested planning values are

$$u(16) = \sqrt{0.0064/16 + 0.0001} \text{ mm} = 0.0223607 \text{ mm},$$

$$u(64) = \sqrt{0.0064/64 + 0.0001} \text{ mm} = 0.0141421 \text{ mm}.$$

(b) Squaring the positive target and solving gives

$$\frac{0.0064}{N} \leq 0.012^2 - 0.010^2 = 0.000044, \quad N \geq 145.4545,$$

so the smallest integer is 146. It gives $u(146) = 0.0119931 \text{ mm}$; 145 gives 0.0120057 mm and fails. (c) Every finite N has $u(N) > 0.010 \text{ mm}$, and the limit is 0.010 mm , so 0.009 mm is unattainable by repetition alone. (d) Improve the common calibration contribution to lower the floor. A drifting reference, correlated repeatability or changing measurand would invalidate the assumed s^2/N term. Planning also treats s as known; estimating it from a short pilot adds uncertainty to the forecast.

Problem 45: convert the detector thickness before forming volume

(a) In centimetres, width is 8.00 cm , height 3.50 cm and thickness $250 \mu\text{m} = 0.0250 \text{ cm}$. Thus $V = 8.00(3.50)(0.0250) = 0.700 \text{ cm}^3$ and $m = 2.33(0.700) = 1.631 \text{ g}$, conventionally 1.63 g . (b) The script thickness is ten times too large, so its volume and mass are ten times too large: 7.00 cm^3 and 16.31 g . Checking thickness separately exposes the error. (c) The window-only nominal mass is below 2.00 g by 0.369 g . The total assembly includes omitted coating/frame masses, and the component tolerances, actual density and measurement uncertainties are absent. Therefore this model supports a preliminary mass budget, not certification of the complete assembly.

Problem 46: zero scatter is not zero uncertainty

(a) Ideal nearest rounding places the underlying fixed indication within $[7.195, 7.205] \text{ mm}$, with endpoint allocation depending on the tie rule; this enclosing interval is sufficient here. All deviations from the mean vanish, so the sample standard deviation is zero. (b) A uniform error interval with half-width a has standard uncertainty $a/\sqrt{3}$. Here $a = q/2$, so

$$u_q = \frac{q}{\sqrt{12}} = 0.00288675 \text{ mm}.$$

This is the root-mean-square deviation about zero: equal-length portions of the interval have equal probability, and their squared deviations average to $q^2/12$. The half-width $q/2$ is a bound, whereas $q/\sqrt{12}$ describes this stated distribution. *Optional calculus check:* $u_q^2 = q^{-1} \int_{-q/2}^{q/2} e^2 \text{ de} = q^2/12$. The unresolved error belongs to the same fixed indication, so copying its digits four times does not create four independent rounding errors or divide u_q by two. (c) Independent calibration and rounding contributions give $u = \sqrt{0.004^2 + 0.01^2/12} = 0.00493288 \text{ mm}$. One defensible report is $(7.2000 \pm 0.0049) \text{ mm}$ with standard uncertainty under these two explicit models. Other contributions were not evaluated. A uniform assumption is additional information, not implied by the display increment.

Problem 47: fitted density cannot validate the same mass

(a) The hollow volume is $\pi(50.0)(10.0^2 - 6.0^2)/1000 = 3.2\pi \text{ cm}^3 = 10.0531 \text{ cm}^3$. Hence $\rho = 30.0/(3.2\pi) = 2.98416 \text{ g cm}^{-3}$, conventionally 2.98 g cm^{-3} . (b) With no bore, volume is $5\pi \text{ cm}^3$ and fitted density is $30.0/(5\pi) = 1.90986 \text{ g cm}^{-3}$. (c) Substituting either inferred density into its own volume reproduces 30.0 g by construction; that is parameter inference, not validation. Inspect the through-bore or determine material density from a separate sample. The calculations are nominal and assume uniformity; no uncertainty or material identity follows from their numerical precision.

Problem 48: report location dependence and use the shared calibration correctly

(a) The location means are 10.010 and 10.210 mm . At either location the deviations are

$$(-0.01, 0.01, 0, 0) \text{ mm}, \quad s = \sqrt{0.0002/3} \text{ mm} = 0.00816497 \text{ mm}.$$

The repeatability uncertainty of either mean is $s/2 = 0.00408248 \text{ mm}$. Each total standard uncertainty is

$$u_{\bar{x}} = \sqrt{s^2/4 + 0.010^2} \text{ mm} = 0.0108012 \text{ mm}.$$

(b) The common additive calibration cancels in Q minus P, giving $D = 0.200 \text{ mm}$ and $u_D = \sqrt{2s^2/4} = 0.00577350 \text{ mm}$. Treating the two total uncertainties as independent would double-count the common term.

(c) The pooled mean 10.110 mm hides the reproducible location separation; it need not be the dimension at either location. (d) An exemplar sequence alternates P and Q, repeats in reversed order, and records times and orientations. Check a stable reference before, between and after acquisitions to diagnose drift; independently rotate/reposition the part and check alignment/contact to test geometry. One investigator-chosen screening rule is $|D| > 3u_D$. Here $3u_D = 0.0173205 \text{ mm}$, so the 0.200 mm separation is flagged for investigation. This threshold assigns no automatic coverage probability and does not identify a cause. If a location-linked separation persists under reversed order and repositioning while the independent reference checks meet their predeclared drift criterion, it supports spatial variation. A separation following acquisition time or orientation instead calls for revising that interpretation. A complete design specifies raw records, independent checks, uncertainty assumptions and decision criteria; actual outcomes remain to be observed.

1.13 Bridge to the mathematical description of motion

Measurement and modelling remain essential in mechanics. A moving object's position needs a coordinate system, units and reading times; an observed sequence is not automatically a valid trajectory model. Changing coordinates or units must preserve the physical prediction.

The next chapter, *Motion Along a Straight Line*, develops position, displacement, velocity and acceleration along one axis, introducing the required mathematics as it is used. Its core pathway builds on this chapter's algebra, units, graphs and model checks. The advanced covariance and derivative-based uncertainty treatment remains optional. Carry forward the habits of drawing the system, defining the observable, checking dimensions and limits, preserving evidence, and distinguishing a model's prediction from what a measurement actually establishes.

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