

Chapter 2

Motion Along a Straight Line

First Edition

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Physics I

Physics for Technical and Engineering Students - Volume I

Mechanics, Fluids, Waves, and Thermodynamics

1 January 2026

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Preface

A translation stage moves a specimen into a beam, pauses while a reading is taken, and returns toward its starting position. A lift starts, cruises and stops. An object launched upward slows, reverses and passes below its release height. These motions differ physically, but the same questions organize their description: Where is the object? How quickly is its position changing? How quickly is that rate changing? What can a sequence of measurements establish about the motion between measurements?

This chapter develops *kinematics*, the quantitative description of motion, for a single straight line. It follows *Measurement, Units, and Physical Modelling*: the distinction between a physical model and a measurement remains important. A smooth position function is a model; recorded positions are data. Neither a plausible-looking curve nor a large number of decimal places establishes that the model is adequate.

The chapter introduces the calculus it needs through physical questions. A derivative becomes a local rate by taking the limit of slopes. An integral becomes an accumulated change by adding signed contributions over time. These ideas are then applied to constant-acceleration phases, vertical motion and general motion programs. The final enrichment section asks how differentiation behaves when the input consists of imperfect measurements rather than an exact function.

The Guideline

Choose a reference frame, a positive direction, an origin and a clock. Identify the interval and the model that applies on it. Then translate between words, a motion diagram, graphs and equations. Check the signs, units, initial conditions and domain before interpreting a numerical result.

Pedagogical boundary

This chapter describes motion before explaining its causes. It does not use force diagrams, Newton's laws or drag models as prerequisites. Formal vector algebra comes next in *Vectors*, followed by multidimensional kinematics. Numerical differentiation and its advanced error analysis are optional enrichment for the core route through mechanics.

Conventions and notation

The coordinate x is a signed position along a chosen fixed axis. Throughout the one-dimensional exposition, v and a denote the *signed components* of velocity and acceleration along that axis; they are not magnitudes. Speed is $|v| \geq 0$. For vertical motion, y increases upward unless explicitly stated otherwise, and g is a positive acceleration magnitude. Thus $a_y = -g$ in the ideal near-Earth free-fall model.

Symbol	Meaning	SI unit
x, y	Position coordinates	m
t, τ	Clock time; elapsed time from a stated event	s
Δx	Final position minus initial position	m
ℓ	Distance travelled along the actual path	m
$\bar{v}, v$	Interval-average and instantaneous velocity component	m s^{-1}
$\bar{a}, a$	Interval-average and instantaneous acceleration component	m s^{-2}
H	Vertical height difference in a launch or drop	m
h	Numerical time spacing	s
u_x, u_v	Standard uncertainty of position or velocity	same as quantity
ϵ_i	Additive error in reading i	m
α	Cubic coefficient in the constructed position model	m s^{-3}

Subscripts 0, i and f mean initial, initial endpoint and final endpoint when attached to physical quantities. An integer subscript on a data array labels a sample. Every time interval in an average has $t_f > t_i$. The notation dx/dt is introduced in the chapter, not assumed from earlier mechanics. A prime on a function denotes differentiation with respect to its displayed argument.

The metre and second are the SI units used for length and time [4]. Conversion factors and stipulated mathematical model parameters are not experimental uncertainty statements. Unless a problem states otherwise, numerical vertical-motion examples use $g = 9.81 \text{ m s}^{-2}$ as a rounded teaching approximation, not as an exact universal or locally measured value. Local gravitational acceleration varies with location [5].

A formula such as $x = -1 + 4t - t^2$ is used only when explicitly accompanied by the convention that the numerical values of t and x are in seconds and metres. Otherwise, coefficients carry their units: for example, $x = (-1 \text{ m}) + (4 \text{ m s}^{-1})t - (1 \text{ m s}^{-2})t^2$. Extra digits in a numerical-method comparison indicate computational resolution, not extra measurement accuracy.

Note

The core requires algebra, elementary graphs and a willingness to work through the introductory rate and accumulation arguments. The advanced section introduces Taylor expansions, variance and a small least-squares calculation where needed. Its results are not hidden prerequisites for the core problems.

Chapter 2

Motion Along a Straight Line

Motion is not fully described by a destination. Two stages can begin and end at the same positions while taking different times, travelling different distances and reversing at different places. A useful description must connect *position to time*, not merely list endpoints. Once that connection is made, its slope gives velocity and the change in slope gives acceleration. Conversely, accumulating velocity reconstructs position when an initial position is supplied.

Learning goals

After the core, you should be able to define a one-dimensional observation frame; distinguish displacement from distance and velocity from speed; obtain instantaneous rates from slopes; diagnose speeding up and slowing down from signed quantities; derive and apply constant-acceleration equations with their initial conditions; select physical roots in vertical-motion problems; and reconstruct motion from signed areas. The supporting data section adds numerical estimates and checks against a known model.

2.1 Position, displacement and the observation frame

2.1.1 What must be specified before assigning a position?

A position measurement compares an object with a reference. For a carriage on a straight rail, attach an axis to the laboratory, choose one marked point as $x = 0$, and choose a positive direction. Record the coordinate of the same feature of the carriage on every observation. A point-particle description can refer to that marker even though the actual carriage has finite size. It is adequate when the question concerns translation rather than rotation, deformation or the extent of the carriage.

The observation frame includes the reference objects relative to which positions are specified. The carriage can move relative to the laboratory and remain at rest relative to a sensor bolted to it. These are not contradictory descriptions. In this chapter the chosen axes do not move or rotate relative to the stated laboratory. Transformations between moving frames are deferred; changing a stationary origin is already enough to show which quantities depend on our labels.

A negative coordinate does not mean negative distance travelled, nor does it mean motion in the negative direction. It says only that the object lies on the negative side of the chosen origin. Motion requires at least two times. An object can be at $x = -0.20$ m while moving toward increasing x , toward decreasing x , or not moving at all.

Choose a clock origin separately. Let $t = 0$ identify a reproducible event, such as the start of a recorded motion, not necessarily the moment an instrument is switched on. Positions and times must refer to the same events. A time stamp without an associated position, or a position with an unknown delay, cannot determine a reliable rate.

Definition 2.1: Displacement and distance

For initial and final events on the same axis,

$$\Delta x = x_f - x_i. \quad (2.1)$$

Displacement is signed and depends only on the endpoints. Distance travelled ℓ is the non-negative length of the actual route. For a sequence of monotone straight-line legs,

$$\ell = \sum_j |x_{j+1} - x_j|. \quad (2.2)$$

Every reversal must be included among the leg endpoints for this sum to give the full distance.

If the origin is shifted to the point with old coordinate b , then $x' = x - b$. The displacement becomes $(x_f - b) - (x_i - b) = \Delta x$: the origin cancels. Reversing the positive direction instead gives $x' = -x$ and $\Delta x' = -\Delta x$. Distances are unchanged in both cases. A consistent coordinate convention changes the description, not the physical motion.

Example 2.1: Translation-stage return motion

A stage marker is at -0.20 m at $t = 0$, at $+0.60$ m at 2.0 s, and at $+0.10$ m at 4.0 s. It moves monotonically between each pair of successive observations, reversing at the middle observation. Find the displacement and distance over the full interval. Then move the origin to the middle position.

Model and strategy. The laboratory axis is fixed, positive to the right. There are two monotone legs. Add their signed changes for displacement and their magnitudes for distance.

$$\Delta x_1 = 0.60 - (-0.20) = +0.80 \text{ m},$$

$$\Delta x_2 = 0.10 - 0.60 = -0.50 \text{ m},$$

$$\Delta x = +0.80 - 0.50 = +0.30 \text{ m},$$

$$\ell = 0.80 + 0.50 = 1.30 \text{ m}.$$

Independent endpoint check. Direct subtraction gives $0.10 - (-0.20) = +0.30$ m. The distance is larger because part of the travel is undone in the signed sum.

New origin. With $b = 0.60$ m, the three coordinates are -0.80 , 0 and -0.50 m. The same calculations still give $\Delta x = +0.30$ m and $\ell = 1.30$ m. All final coordinates can be non-positive even though the net displacement is positive. Figure 2.1 separates the two travel legs from their net change.

2.1.2 What sparse observations cannot establish

The monotonicity assumption in Example 2.1 is essential. Three positions alone do not exclude an extra excursion and return between two time stamps. Without information about such excursions, the sum in Eq. (2.2) is a *lower bound* on the distance, not necessarily its exact value. A smooth curve through the samples is one possible interpolation, not an observation of everything that happened between them.

For any route on a line, $\ell \geq |\Delta x|$. Equality holds for monotone motion, including intervals of rest. If the object returns exactly to its starting position, $\Delta x = 0$ while ℓ can be positive. A distance statement therefore answers a different question from an endpoint-position statement. That distinction will reappear when signed areas are used to reconstruct motion.

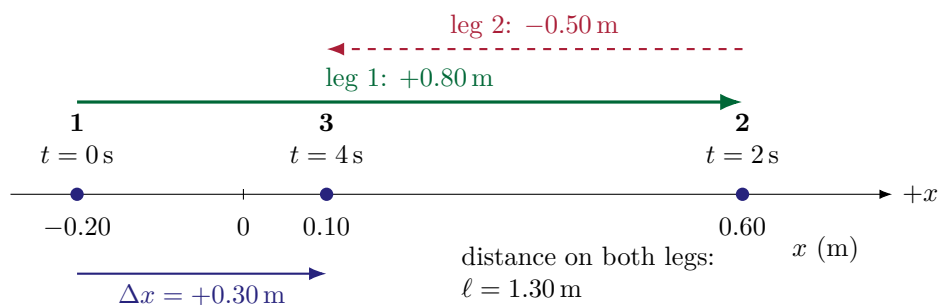


Figure 2.1: Three positions of one translation stage at equal 2 s time intervals, numbered chronologically. Horizontal distances and arrow lengths share the metre scale; vertical offsets merely separate annotations. The two travel legs are assumed monotone. A displacement arrow joins the initial and final positions, not every visited position. No arrow denotes a force.

Guided checks

Check. A marker goes from $x = 2$ m to 5 m and back to 2 m. After shifting the origin by 10 m, has its displacement become negative?

Answer. No. Both descriptions give zero displacement and 6 m distance for two monotone legs. Subtracting the same constant from every position cannot change an endpoint difference.

2.2 Average and instantaneous motion

2.2.1 A rate over an interval

Dividing a change in position by the elapsed time defines an interval-average velocity:

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x(t_f) - x(t_i)}{t_f - t_i}, \quad t_f > t_i. \quad (2.3)$$

This definition does *not* assume constant acceleration, constant velocity or monotone motion. Its sign follows the displacement. In the translation-stage example, the two leg-average velocities are $+0.40$ and -0.25 m s^{-1} , while the full-interval average is $0.30/4.0 = +0.075 \text{ m s}^{-1}$.

Average speed is a different quotient:

$$\text{average speed} = \frac{\ell}{\Delta t} \geq |\bar{v}|. \quad (2.4)$$

For that stage it is $1.30/4.0 = 0.325 \text{ m s}^{-1}$. Neither average says the instantaneous rate was constant. A short pause, a slow approach or a rapid excursion can be hidden by the same whole-interval average.

When a motion is divided into intervals of durations Δt_j , displacement addition gives

$$\bar{v} = \frac{\sum_j \bar{v}_j \Delta t_j}{\sum_j \Delta t_j}. \quad (2.5)$$

The arithmetic mean of two leg-average velocities works when the durations are equal, not in general. For instance, travelling at 0.20 m s^{-1} for 1 s and at 0.60 m s^{-1} for 3 s gives 2.0 m in 4 s, or 0.50 m s^{-1} , rather than 0.40 m s^{-1} . The weights represent time spent at each rate.

On a position–time graph, Eq. (2.3) is the slope of the straight *secant* joining the endpoint events. The vertical difference has units of metres, and the horizontal difference has units of seconds. Their ratio is a physical rate. A visually steep line can be made shallow by changing the drawing scales, but its numerically calculated slope and units remain the same.

2.2.2 A rate at one instant: the derivative

To describe the motion near a particular time t , compare $x(t)$ with $x(t+h)$ for progressively smaller nonzero intervals h . If the slopes approach a definite limit, define

$$v(t) = \lim_{h \rightarrow 0} \frac{x(t+h) - x(t)}{h} = \frac{dx}{dt}. \quad (2.6)$$

The limit is called the *derivative*. It is not obtained by setting $h = 0$ in the unsimplified fraction, which would give $0/0$. Instead, simplify or evaluate the quotient for nonzero h and determine its limiting value. Geometrically, the secant approaches the tangent at the selected event.

For $x(t) = x_0 + ut - ct^2$, with constant coefficients,

$$\begin{aligned} \frac{x(t+h) - x(t)}{h} &= \frac{uh - c[(t+h)^2 - t^2]}{h} \\ &= u - 2ct - ch \quad (h \neq 0). \end{aligned}$$

Thus $v(t) = u - 2ct$. The constant position x_0 disappears because a fixed shift of origin contributes no change. The coefficient u has units m s^{-1} , while c has units m s^{-2} , so both terms in v have the required velocity units.

More generally, expanding $(t+h)^n$ shows that its term linear in h is $nt^{n-1}h$. After division by h , every higher power of h vanishes in the limit. For positive integers n ,

$$\frac{d}{dt}(t^n) = nt^{n-1}, \quad \frac{d}{dt}(Af + Bq) = Af' + Bq' \quad (2.7)$$

for constant A, B . These rules handle the polynomial models in the core. A physically useful derivative also requires a sensible model and time interval; differentiation cannot make an arbitrary fitted curve into a validated description.

Definition 2.2: Instantaneous velocity and speed

Velocity $v = dx/dt$ is the signed local rate of position change. Positive v means motion toward increasing x , and negative v toward decreasing x . Instantaneous speed is the magnitude $|v|$, which is non-negative and equals zero at rest.

Example 2.2: A polynomial path with a reversal

During $0 \leq t \leq 5$ s, a marker obeys

$$x(t) = (-1 \text{ m}) + (4 \text{ m s}^{-1})t - (1 \text{ m s}^{-2})t^2.$$

Find the reversal time, the final position, displacement, distance, whole-interval average velocity and average speed. Compare the average velocity with the instantaneous velocity at 1 s.

Strategy. Differentiate to locate sign changes of v . Split the distance calculation at an actual reversal rather than taking the absolute value of the final displacement.

$$v(t) = 4 - 2t \quad \text{in } \text{m s}^{-1} \text{ when } t \text{ is in s.}$$

The zero is at $t_* = 2$ s. Velocity is positive before that time and negative afterward, so this zero is a reversal. The positions are

$$x(0) = -1 \text{ m}, \quad x(2) = 3 \text{ m}, \quad x(5) = -6 \text{ m}.$$

Consequently,

$$\Delta x = -6 - (-1) = -5 \text{ m},$$

$$\ell = |3 - (-1)| + |-6 - 3| = 4 + 9 = 13 \text{ m},$$

$$\bar{v} = -5/5 = -1 \text{ m s}^{-1},$$

$$\text{average speed} = 13/5 = 2.6 \text{ m s}^{-1}.$$

At $t = 1 \text{ s}$, $x = 2 \text{ m}$ and $v = +2 \text{ m s}^{-1}$. Its positive instantaneous velocity is compatible with a negative whole-interval average because the later negative displacement is larger.

Graph check. The endpoint secant is $x = -1 - t$ in numerical SI coordinates and has slope -1 . The tangent at $t = 1$ is $x = 2 + 2(t - 1) = 2t$, with slope $+2$; both appear in Fig. 2.2. The maximum is 3 m at 2 s , agreeing with the sign change.

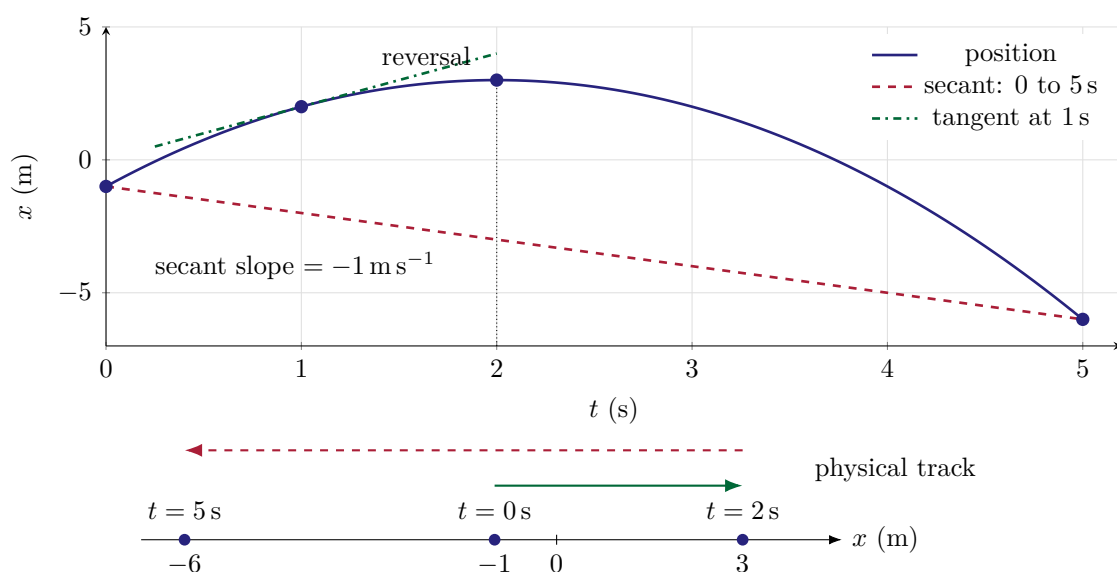


Figure 2.2: Calculated position $x = -1 + 4t - t^2$, with numerical values in SI units, for $0 \leq t \leq 5 \text{ s}$. The secant slope is the whole-interval average velocity; the tangent slope at 1 s is $+2 \text{ m s}^{-1}$. The small rail diagram below is the physical track, with a separate metre scale. Its vertically offset arrows distinguish the two travel legs; the graph above is not a bent physical path.

2.2.3 The same position does not imply the same motion

The marker of Example 2.2 passes $x = 2 \text{ m}$ twice. Solving $-1 + 4t - t^2 = 2$ gives $(t - 1)(t - 3) = 0$: the events occur at 1 and 3 s . Their velocities are $+2$ and -2 m s^{-1} . One coordinate value is therefore not a complete state of motion. A position–time graph distinguishes the two events because their time coordinates differ.

A time shift $\tau = t - t_b$ changes the time labels but not elapsed intervals or the velocity at the same physical event. An origin shift $x' = x - b$ leaves v unchanged. Reversing the spatial axis gives $v' = -v$ but leaves speed unchanged. These consistency checks are simple ways to diagnose accidental sign changes.

Caution

A position–time graph is not a picture of the track. A downward-sloping graph means that position decreases as time increases; it does not mean the object physically moves down a sloping rail. Its ordinate gives position, while its slope gives velocity. Confusing height with slope is a different error from choosing the wrong positive direction.

2.3 Acceleration and motion diagrams

2.3.1 Change of velocity, not merely change of speed

The average acceleration over an interval is $\bar{a} = [v(t_f) - v(t_i)]/(t_f - t_i)$. The instantaneous acceleration is its limiting local value:

$$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}. \quad (2.8)$$

The second derivative notation means that position is differentiated twice with respect to time. It is not $(dx/dt)^2$. Acceleration has units $(\text{m s}^{-1})/\text{s} = \text{m s}^{-2}$ and dimensions LT^{-2} . A constant acceleration $+0.50 \text{ m s}^{-2}$ adds $+0.50 \text{ m s}^{-1}$ to the velocity during each second, regardless of its current sign.

For the polynomial in Example 2.2, differentiating $v = u - 2ct$ gives $a = -2c = -2 \text{ m s}^{-2}$. The acceleration is negative both before and after the reversal. The object first slows down while moving in the positive direction, then speeds up while moving in the negative direction. Negative acceleration is therefore not a synonym for slowing down.

Away from $v = 0$, speed can be differentiated separately. If $v > 0$, then $|v| = v$ and $d|v|/dt = a$. If $v < 0$, then $|v| = -v$ and $d|v|/dt = -a$. Hence

$$\frac{d|v|}{dt} = \text{sgn}(v) a \quad (v \neq 0). \quad (2.9)$$

Here $\text{sgn}(v)$ is $+1$ or -1 according to the sign of v . Velocity and acceleration of the same sign imply increasing speed; opposite signs imply decreasing speed. The sign test concerns the selected instant or interval, not a permanent label attached to the object.

v	a	va	Behaviour away from $v = 0$
+	+	+	positive-direction motion speeding up
+	−	−	positive-direction motion slowing down
−	+	−	negative-direction motion slowing down
−	−	+	negative-direction motion speeding up

Example 2.3: Classify four motion intervals

Motion A is the polynomial of Example 2.2. Motion B is its reflection, $x_B = -x_A$, on the same 0–5 s time interval. Classify the direction and speed change for each motion before and after 2 s. What happens precisely at that time?

Model. Reflection reverses all signed coordinates and their derivatives, so

$$v_A = 4 - 2t, \quad a_A = -2; \quad v_B = -4 + 2t, \quad a_B = +2$$

in numerical SI units. For $0 < t < 2$, A has $v_A > 0$, $a_A < 0$ and slows; B has $v_B < 0$, $a_B > 0$ and slows. For $2 < t < 5$, A has both quantities negative and speeds up; B has both positive and speeds up. Thus all four sign combinations occur without changing either constant acceleration.

At the common reversal. Both velocities vanish at 2 s, but neither acceleration does. For A, $|v_A| = 2|t - 2|$ in numerical SI units. Its one-sided slopes are -2 and $+2 \text{ m s}^{-2}$, so speed is not differentiable at that instant. Acceleration, defined from the *signed* velocity, remains -2 m s^{-2} . Motion B has the same speed history.

Check. Reflection changes v and a together, leaving va unchanged. It must therefore leave speeding up versus slowing down unchanged. The linked curves for A in Fig. 2.3 make that distinction visible.

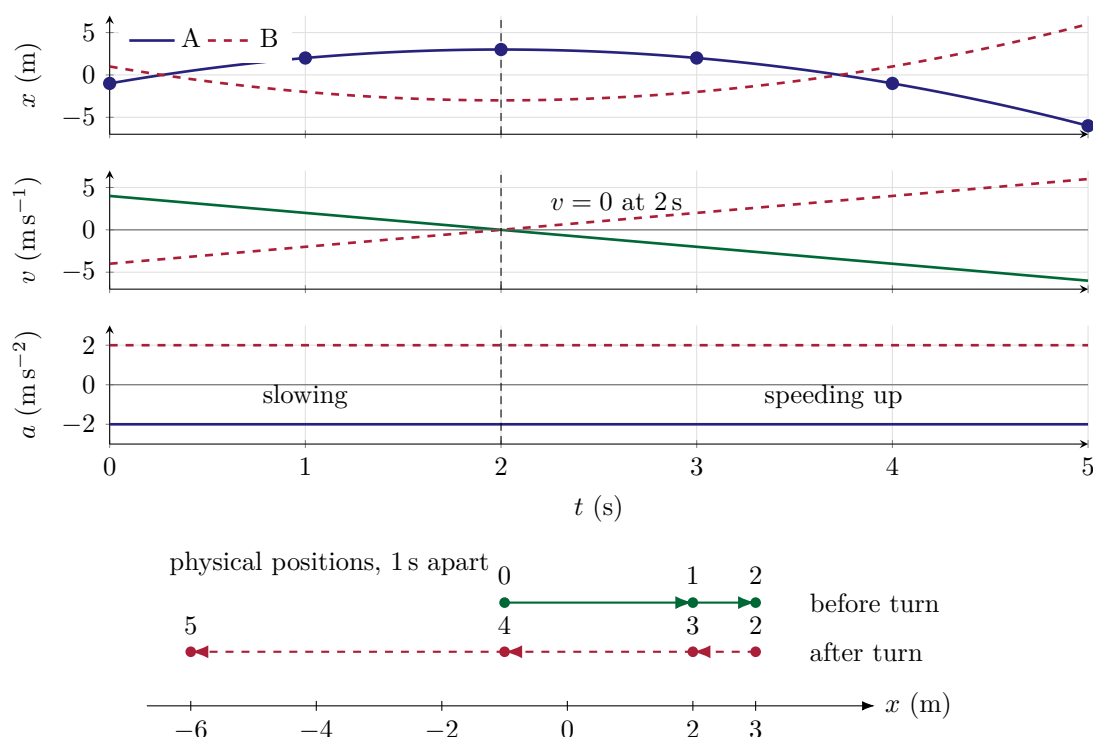


Figure 2.3: Matched analytical position, velocity and acceleration for motion A (solid) and reflected motion B (dashed). Dots on the position curve are one second apart, not equally spaced in position. All panels use the same time scale, and the vertical dashed line identifies the same event. Constant negative acceleration first reduces the positive velocity to zero, then makes the negative velocity more negative. The separate rail strips show physical directions and positions at the labelled times in seconds; their vertical separation is not motion. The two graph families explicitly show both acceleration signs; the rail strips represent A only.

2.3.2 Reading a motion diagram without guessing

A motion diagram places successive object positions on the physical track at specified times. With equal time spacing, large separations imply large *interval-average speeds* if each interval is monotone. Increasing separation in a given direction then suggests speeding up. Equal spatial separations do not imply equal velocities unless the time separations are also equal. A diagram without time information cannot establish acceleration.

For motion A, the one-second samples have coordinates $-1, 2, 3, 2, -1, -6 \text{ m}$. Reading them chronologically shows shrinking positive steps $+3, +1 \text{ m}$, then negative steps $-1, -3, -5 \text{ m}$. Repeated positions must be labelled by time or by sequence number: the points at $x = 2 \text{ m}$ and $x = -1 \text{ m}$ are visited twice. Drawing all dots without labels would hide that ordering. The graph in Fig. 2.3 retains it.

Three representations should agree. An increasing $x(t)$ corresponds to positive $v(t)$; a tangent that becomes less positive corresponds to negative $a(t)$; and a zero crossing of $v(t)$ can indicate a

reversal. On a velocity graph, the ordinate is velocity and the slope is acceleration. A high positive velocity need not imply positive acceleration, just as a large positive position need not imply positive velocity.

2.3.3 Zero derivatives require interpretation

A zero velocity is a necessary condition for a smooth interior reversal, but it is not sufficient. Consider $x(t) = \beta t^3$ for $-1 \text{ s} \leq t \leq 1 \text{ s}$ and $\beta > 0$. Then $v = 3\beta t^2$ is positive on both sides of $t = 0$ and zero only at that instant. Position continues increasing; the object does not reverse. Check the sign on each side, not just the root.

Similarly, $a = 0$ at one instant says only that $v(t)$ has a horizontal tangent there. It does not by itself mean zero velocity, a finite period of constant velocity, or an extremum of velocity. Constant velocity throughout an interval requires $a = 0$ throughout that interval under the differentiable model. These distinctions prevent graph rules from being applied outside their assumptions.

Guided checks

Check. At a selected event, $v = -0.30 \text{ m s}^{-1}$ and $a = +0.10 \text{ m s}^{-2}$. Is the object moving toward positive x ?

Answer. No. It is moving toward negative x and slowing down. If the same acceleration persists, it reaches zero velocity after 3 s; continuing the model beyond that event makes it move toward positive x . A model intended only to describe braking must end or change at the stop.

2.3.4 How a family of secants becomes one tangent

A single secant measures an interval rate, while a tangent measures a local rate. Comparing several secants makes the distinction visible and also gives a numerical test of the limiting operation. Consider the exact teaching model

$$x(t) = (0.20 \text{ m}) + (0.40 \text{ m s}^{-1})t + (0.30 \text{ m s}^{-2})t^2, \quad 0 \leq t \leq 2 \text{ s}. \quad (2.10)$$

Select the event $t_0 = 1 \text{ s}$, where $x(t_0) = 0.90 \text{ m}$. An interval h can be positive or negative, provided $t_0 + h$ remains inside the model domain. Negative h compares the selected event with an earlier event; it does not describe negative elapsed time in the actual journey. Because both numerator and denominator change their ordering consistently, the quotient remains a velocity.

Expanding the polynomial before taking the limit gives

$$\begin{aligned} q(h) &= \frac{x(t_0 + h) - x(t_0)}{h} \\ &= 0.40 \text{ m s}^{-1} + (0.60 \text{ m s}^{-2})t_0 + (0.30 \text{ m s}^{-2})h \\ &= 1.00 \text{ m s}^{-1} + (0.30 \text{ m s}^{-2})h, \quad h \neq 0. \end{aligned} \quad (2.11)$$

The value at $h = 0$ in the last expression is the continuous extension of the simplified quotient, not the evaluation of its original $0/0$ form. Thus the tangent velocity is $v(t_0) = 1.00 \text{ m s}^{-1}$.

Example 2.4: Approach the event from both sides

Evaluate $q(h)$ for $h = -1.00, -0.50, -0.10, +0.10, +0.50, +1.00 \text{ s}$. Determine the sign of each error relative to the tangent velocity, and explain whether a secant through a later event overestimates or underestimates it.

Solution. Substitution into Eq. (2.11) gives

h (s)	-1.00	-0.50	-0.10	+0.10	+0.50	+1.00
$q(h)$ (m s ⁻¹)	0.70	0.85	0.97	1.03	1.15	1.30
$q - v(t_0)$ (m s ⁻¹)	-0.30	-0.15	-0.03	+0.03	+0.15	+0.30

The motion has positive acceleration. A later interval includes velocities greater than the velocity at its initial event, so its average exceeds that initial velocity. An earlier interval has the opposite error. Both sides converge to the same value. As a direct check, $x(2) = 2.20$ m gives $[2.20 - 0.90]/1 = 1.30$ m s⁻¹, while $x(0) = 0.20$ m gives $[0.20 - 0.90]/(-1) = 0.70$ m s⁻¹.

What the graph can establish. The secant lines all pass through the selected event, but their slopes differ. Their apparent angles depend on the scales of both axes. Reading coordinates and units, rather than measuring an angle on the printed page, identifies the physical velocities.

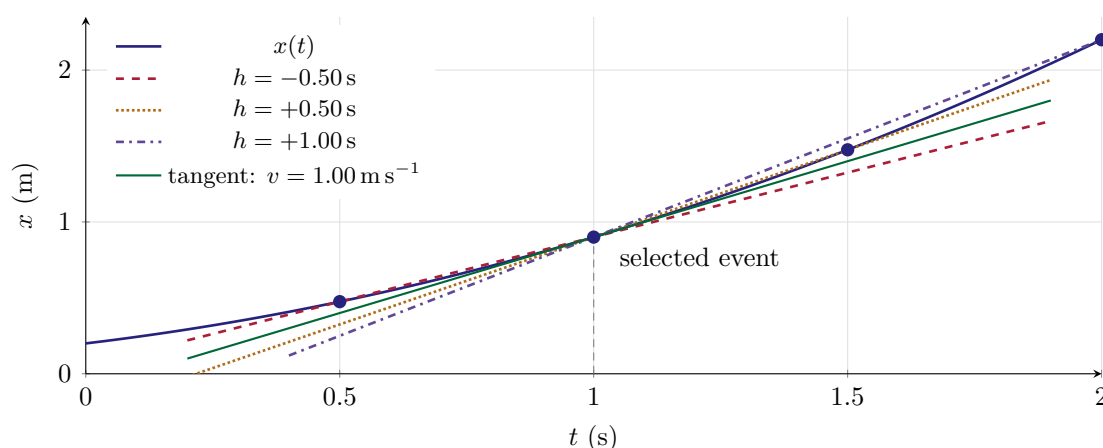


Figure 2.4: A quantitative secant family for Eq. (2.10). All lines meet at (1.00 s, 0.90 m). Each dashed or dotted line also meets a second point of the position curve. The tangent has the limit slope 1.00 m s⁻¹; shortening either signed interval brings its secant toward that tangent.

2.3.5 Equal-time dots for motion in either direction

Motion dots display sampled positions on the physical line. They can help diagnose speed changes, provided the time separation is stated. The visual rule must also respect direction: dots becoming farther apart can represent either increasingly positive velocity or increasingly negative velocity.

Two ideal motions on the same 0 to 4 m rail illustrate this explicitly:

$$\begin{aligned} x_+(t) &= (0.25 \text{ m s}^{-2})t^2, & v_+(t) &= (0.50 \text{ m s}^{-2})t, & a_+ &= +0.50 \text{ m s}^{-2}, \\ x_-(t) &= 4.00 \text{ m} - (0.25 \text{ m s}^{-2})t^2, & v_-(t) &= -(0.50 \text{ m s}^{-2})t, & a_- &= -0.50 \text{ m s}^{-2}, \end{aligned} \quad (2.12)$$

for $0 \leq t \leq 4$ s. The trajectories satisfy $x_- = 4.00 \text{ m} - x_+$, so every displacement and velocity is reflected, while every speed is preserved. Reflection also reverses the acceleration component. This is a change in physical direction within the stated fixed axis; a coordinate reversal would instead relabel a given motion.

Example 2.5: Extract the rates encoded by the dots

Find the interval-average velocities between the five dots for each motion. Compare them with the instantaneous velocities at the four interval midpoints. State precisely which conclusion depends on knowing the analytical model rather than merely seeing five dots.

Solution. For x_+ the positions are 0, 0.25, 1.00, 2.25, 4.00 m. Dividing consecutive differences by 1 s gives +0.25, +0.75, +1.25, +1.75 m s⁻¹. For x_- the order is 4.00, 3.75, 3.00, 1.75, 0 m and

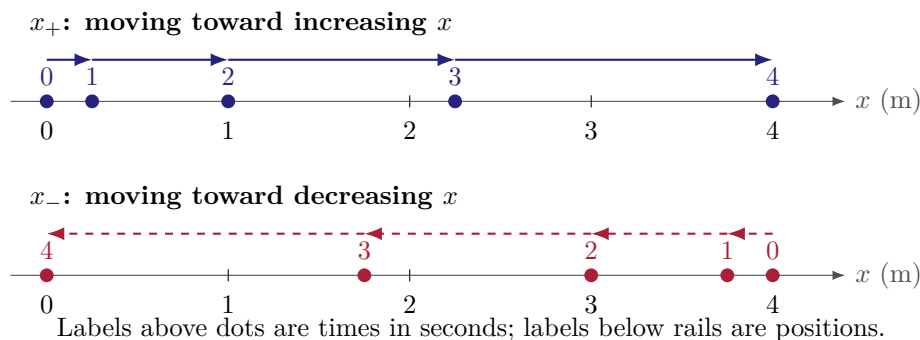


Figure 2.5: Calculated positions at $t = 0, 1, 2, 3, 4$ s for Eq. (2.12), on identical metre scales. The labels above dots are times in seconds. Arrow lengths indicate successive displacements over one second. They are not instantaneous velocity vectors. The rows are separate depictions of one-dimensional rails; vertical spacing has no physical meaning. Both motions speed up after the initial rest event.

the rates are $-0.25, -0.75, -1.25, -1.75 \text{ m s}^{-1}$. Their magnitudes increase in both cases.

The four midpoints are $0.50, 1.50, 2.50, 3.50$ s. Substitution into $v_{\pm} = \pm 0.50t$ returns precisely these four averages. This equality follows because velocity is linear in time. It cannot be assumed for a general nonlinear motion.

A direct second-difference check gives $x_{i+1} - 2x_i + x_{i-1} = \pm 0.50 \text{ m}$ at each available interior dot. Dividing by $(1 \text{ s})^2$ returns $\pm 0.50 \text{ m s}^{-2}$ for the stipulated quadratic models. The five positions alone do not prove that acceleration was constant between the observations. A different smooth interpolating curve could pass through every dot and have different intermediate velocities or reversals. The exact derivative and monotonicity statements use Eq. (2.12).

2.3.6 A horizontal tangent with and without a reversal

At a differentiable interior extremum of position, velocity must vanish. The converse is not valid: a zero derivative may occur during motion that continues in the same direction. A diagnostic requires the sign of velocity on both sides of the event, with a stated neighbourhood in which the model applies.

Let $\tau = t - 1.00 \text{ s}$ be time measured from the selected event and consider, on $0 \leq t \leq 2 \text{ s}$,

$$\begin{aligned} x_R(t) &= 0.50 \text{ m} + (0.30 \text{ m s}^{-2})\tau^2, & v_R(t) &= (0.60 \text{ m s}^{-2})\tau, \\ x_P(t) &= 0.50 \text{ m} + (0.20 \text{ m s}^{-3})\tau^3, & v_P(t) &= (0.60 \text{ m s}^{-3})\tau^2. \end{aligned} \quad (2.13)$$

Both trajectories pass through $x = 0.50 \text{ m}$ with zero velocity at $t = 1 \text{ s}$. Motion R has negative velocity before that event and positive velocity after it; its position has a minimum. Motion P has positive velocity on either side, with a momentary zero value at the centre; its position increases through a horizontal inflection.

Example 2.6: What can be inferred at the zero-velocity event?

Find the acceleration of each motion at $t = 1 \text{ s}$. Determine whether the speed derivative exists there. Does zero acceleration in P mean that P has been at rest for a finite time?

Solution. Differentiation gives $a_R = 0.60 \text{ m s}^{-2}$ throughout the interval and $a_P = (1.20 \text{ m s}^{-3})\tau$. At the selected event, $a_R = 0.60 \text{ m s}^{-2}$ but $a_P = 0$. For R, speed is $(0.60 \text{ m s}^{-2})|\tau|$. Its one-sided time derivatives are -0.60 and $+0.60 \text{ m s}^{-2}$, so its speed derivative does not exist at the reversal. Its signed acceleration remains well defined.

For P, speed equals the nonnegative velocity $(0.60 \text{ m s}^{-3})\tau^2$. Its derivative is zero at $\tau = 0$

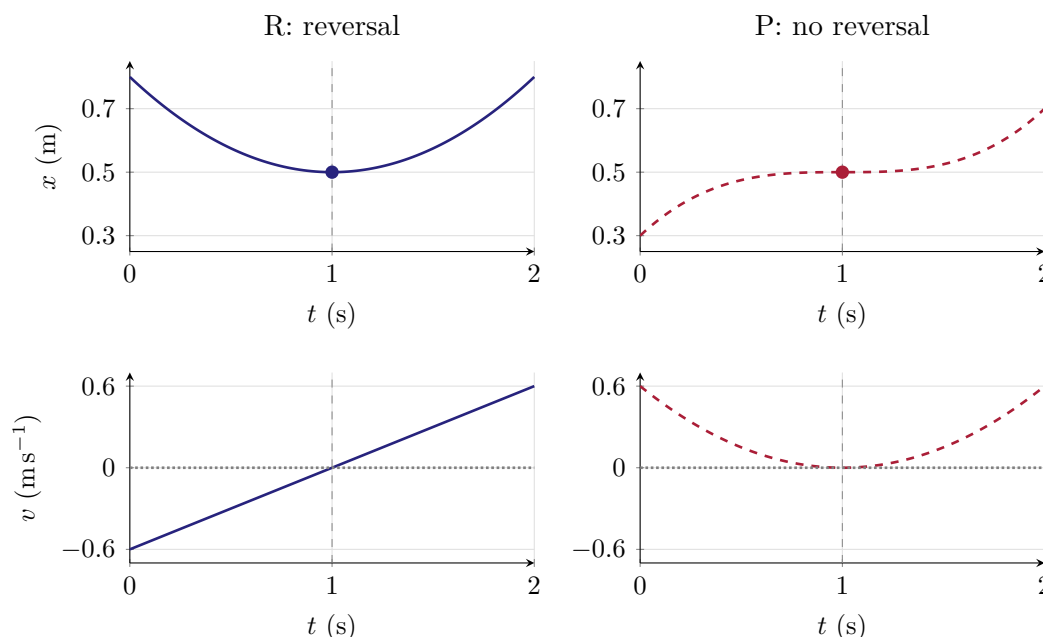


Figure 2.6: Two quantitative counterexamples to identifying a reversal from $v = 0$ alone. Each lower curve is the derivative of the position curve directly above it. At the aligned event $t = 1$ s, R changes velocity sign whereas P only touches zero. The same axis ranges in the two columns support a direct comparison.

and exists there. Nevertheless, P moves at every other nearby time. Zero velocity and zero acceleration at one instant do not imply a finite rest interval. Remaining at rest throughout a time interval requires constant position throughout that interval.

2.3.7 Average velocity uses durations as its weights

Combining interval averages requires attention to the denominator of the average. Suppose a journey is partitioned into intervals of durations $\Delta t_i > 0$ and displacements Δx_i . Define $\bar{v}_i = \Delta x_i / \Delta t_i$ for each interval. Then, without assuming that velocity is constant inside any interval,

$$\bar{v} = \frac{\sum_i \Delta x_i}{\sum_i \Delta t_i} = \sum_i w_i \bar{v}_i, \quad w_i = \frac{\Delta t_i}{\sum_j \Delta t_j}, \quad \sum_i w_i = 1. \quad (2.14)$$

This is a duration-weighted average. Short and long intervals must not be counted equally merely because each appears once in a table. The arithmetic mean is appropriate only when the durations are equal, or when another special condition happens to reproduce that value.

Example 2.7: Equal distances and equal durations are different journeys

A marker makes two forward-moving legs with leg-average velocities 2.00 and 0.50 m s^{-1} . In journey D each leg covers 2.00 m . In journey T each leg lasts 2.00 s . Calculate total duration, total displacement and whole-journey average velocity for each. Explain which journey has the arithmetic mean 1.25 m s^{-1} .

Solution. For journey D, the durations are $2.00/2.00 = 1.00 \text{ s}$ and $2.00/0.50 = 4.00 \text{ s}$. Therefore the total duration is 5.00 s and the displacement is 4.00 m , giving

$$\bar{v}_D = 0.80 \text{ m s}^{-1} = \frac{1}{5}(2.00) + \frac{4}{5}(0.50) \text{ m s}^{-1}.$$

Equal distances produce unequal duration weights. For two equal forward distances L , the general result is $\bar{v} = 2L/(L/v_1 + L/v_2) = 2v_1v_2/(v_1 + v_2)$ when both leg-average velocities are positive.

For journey T, the displacements are $(2.00)(2.00) = 4.00$ m and $(0.50)(2.00) = 1.00$ m. The total duration is 4.00 s and total displacement 5.00 m, so $\bar{v}_T = 1.25$ m s⁻¹. This is the arithmetic mean because the two duration weights are each 1/2. Both journeys move forward, so average speed equals average velocity in each journey. A return leg would require a signed velocity in the displacement calculation and a separate distance calculation.

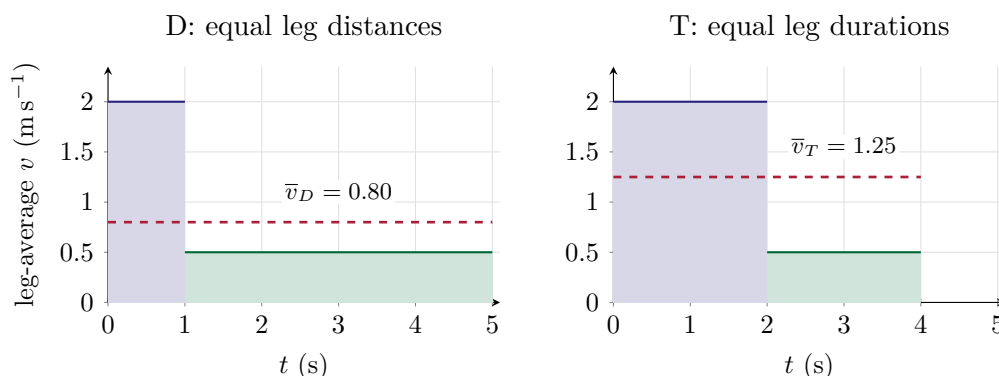


Figure 2.7: Area-preserving summaries of two journeys. Each bar width is an interval duration and its height is that interval's average velocity; hence its area equals the interval displacement. Dashed levels are the whole-journey averages in m s⁻¹. The bars do not assert that the instantaneous velocity was constant within each leg or jumped at a boundary. Acceleration cannot be read from these summary joins.

2.3.8 A quantitative three-graph consistency test

Position, velocity and acceleration graphs may use different vertical scales because their dimensions differ. What must remain consistent is the derivative relation and the time assigned to each event. A useful non-polynomial test is the prescribed motion

$$x(t) = A \sin(\omega t), \quad A = 0.20 \text{ m}, \quad \omega = \pi \text{ s}^{-1}, \quad 0 \leq t \leq 2 \text{ s}. \quad (2.15)$$

Here the sine argument is dimensionless. The angular-rate coefficient ω expresses phase change per unit time; the radian can be written as a dimensionless angle unit. The derivative follows from the angle-addition identity rather than from guessing a phase shift. Write $\theta = \omega t$ and $\delta = \omega h$. Then

$$\frac{\sin(\theta + \delta) - \sin \theta}{h} = \sin \theta \frac{\cos \delta - 1}{h} + \cos \theta \frac{\sin \delta}{h}.$$

For angle increments measured in radians, $\sin \delta / \delta \rightarrow 1$ and $(\cos \delta - 1) / \delta \rightarrow 0$ as $\delta \rightarrow 0$. Since $\delta / h = \omega$, the two terms approach zero and $\omega \cos \theta$, respectively. This is the sine derivative with its time-rate factor. Applying $\cos(\theta + \delta) = \cos \theta \cos \delta - \sin \theta \sin \delta$ in the same limit gives $d \cos(\omega t) / dt = -\omega \sin(\omega t)$. Thus differentiation twice gives

$$v(t) = A\omega \cos(\omega t), \quad a(t) = -A\omega^2 \sin(\omega t) = -\omega^2 x(t). \quad (2.16)$$

Consequently, the three amplitudes are 0.20 m, 0.20π m s⁻¹ and $0.20\pi^2$ m s⁻². Assigning all three the same numerical amplitude would not represent this physical model. The rate factors must be retained even when a schematic sketch makes the curves look equally tall. This example is a kinematic prescription; explaining what interaction could generate it belongs to later chapters.

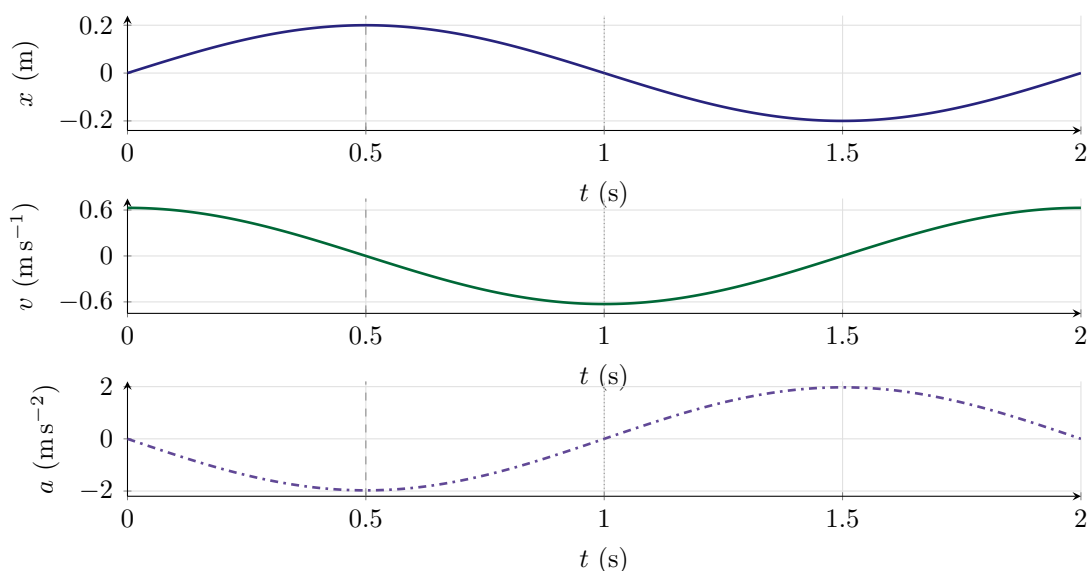


Figure 2.8: Quantitative aligned graphs derived from a single sinusoidal position model. All panels use the same time range and tick positions, while their vertical units and ranges differ. The dashed guide at 0.50 s identifies a reversal; the dotted guide at 1.00 s identifies a zero-acceleration event during motion. The trigonometric phase is expressed in radians, with the rate factors retained in both derivatives.

Example 2.8: Use the aligned graphs to check two events

Find x, v, a at 0.50 s and 1.00 s. Classify each event and calculate the displacement and distance during the complete two-second interval.

Solution. At 0.50 s the phase is $\pi/2$, so $x = +0.20$ m, $v = 0$ and $a = -0.20\pi^2 \text{ m s}^{-2} \approx -1.9739 \text{ m s}^{-2}$. Velocity changes from positive to negative across the event; it is a reversal at maximum position. At 1.00 s the phase is π , so $x = 0$, $v = -0.20\pi \text{ m s}^{-1} \approx -0.6283 \text{ m s}^{-1}$ and $a = 0$. The object is moving toward negative x . Acceleration changes from negative to positive across that event, so the signed velocity has a minimum there; zero acceleration does not mean rest.

At $t = 0, 0.50, 1.50, 2.00$ s, the positions are 0, +0.20, -0.20, 0 m. The full displacement is zero. Splitting distance at the two velocity sign changes gives $0.20 + 0.40 + 0.20 = 0.80$ m. Thus $\bar{v} = 0$ and average speed is 0.40 m s^{-1} . The absolute integral of the cosine velocity gives the same distance, while its signed integral gives the endpoint difference. These independent checks link the plots to the physical journey.

The graphical tests now identify signed rates, changes of speed and event times without introducing forces. The next section builds a restricted model in which acceleration stays constant over a specified phase, so its accumulated changes can be evaluated exactly.

2.4 Constant-acceleration models

2.4.1 Accumulating a constant rate

Knowing acceleration means knowing how velocity changes, not knowing its starting value. Suppose a particular phase begins at time t_0 , with position x_0 and velocity v_0 . Write $\tau = t - t_0$ for elapsed time in that phase. If a is constant, every interval contributes $a \Delta t$ to the velocity change. Adding intervals gives

$$v(t) = v_0 + a\tau. \quad (2.17)$$

Its graph is a straight line whose initial height is v_0 and whose slope is a . Changing v_0 shifts the graph vertically without changing the acceleration. This is why an acceleration alone cannot determine velocity.

Position requires a second accumulation. Over a very short interval, displacement is approximately velocity times the interval duration. Adding all such contributions and taking the limit defines a *definite integral*:

$$x(t) - x_0 = \lim_{\max \Delta t_j \rightarrow 0} \sum_j v(t_j) \Delta t_j = \int_{t_0}^t v(s) ds. \quad (2.18)$$

The symbol s inside the integral is a dummy time variable; it is not a distance and is not the unit symbol s . The interval endpoints and each sign are part of the calculation. Contributions from negative velocity subtract from displacement rather than becoming positive areas.

A useful way to calculate an integral is to find an *antiderivative*: a function whose derivative is the integrand. Since $d(\tau^2/2)/d\tau = \tau$, the antiderivative of $v_0 + a\tau$ is $v_0\tau + a\tau^2/2$. Evaluating it between 0 and τ gives

$$x(t) = x_0 + v_0\tau + \frac{1}{2}a\tau^2. \quad (2.19)$$

The fundamental theorem of calculus connects this endpoint evaluation with the accumulated sum when the functions have the usual continuity properties. Differentiating Eq. (2.19) returns Eq. (2.17), and substituting $\tau = 0$ returns x_0 . These are independent algebraic checks on the integration constant.

The same result follows geometrically. The signed area under a straight velocity graph is a rectangle $v_0\tau$ plus a triangle $a\tau^2/2$. Equivalently, it is the time interval multiplied by the mean of the two endpoint velocities:

$$\Delta x = \frac{v_0 + v}{2}\tau, \quad \bar{v} = \frac{v_0 + v}{2} \quad \text{for constant } a. \quad (2.20)$$

The geometry still works when the velocity changes sign, provided areas below zero are negative. It is a displacement formula, not a distance formula across a reversal.

Caution

The definition $\bar{v} = \Delta x / \Delta t$ is always valid for a nonzero elapsed interval. The shortcut $\bar{v} = (v_0 + v)/2$ is guaranteed by a linear velocity history, hence by constant acceleration. It is not generally valid for a curved velocity history, although particular non-linear histories can happen to give the same average.

2.4.2 Eliminating time, while preserving the domain

Combining Eqs. (2.17) and (2.20) without dividing by a gives

$$\begin{aligned} v^2 - v_0^2 &= (v - v_0)(v + v_0) \\ &= a\tau(v + v_0) = 2a\Delta x. \end{aligned}$$

Thus

$$v^2 = v_0^2 + 2a(x - x_0). \quad (2.21)$$

This relation is useful when an elapsed time is not known or needed. It loses velocity-sign information because of the squares. A square root determines a speed magnitude; direction must come from the motion interval or from $v = v_0 + a\tau$. Two visits to the same position can have opposite velocities.

All these equations refer to the *same constant-acceleration phase* and its initial conditions. They cannot be carried unchanged through a switch to cruising or a different acceleration. In the limit $a = 0$, Eqs. (2.17) and (2.19) become constant velocity and uniform motion. Equation (2.21) then

says $v^2 = v_0^2$ but gives no information about how long the motion lasts. No formula obtained by division by a should be used to infer that time when $a = 0$.

Method

For each phase, write its start time, position, velocity, acceleration and allowed duration. Solve within that domain. Pass the end position and velocity to the next phase, but reset its *elapsed* time. Finally check continuity, signs, units and whether a proposed stopping phase has been extended beyond the actual stop.

Example 2.9: Lift start–cruise–stop motion

An ideal lift starts from rest at $x = 0$ and $t = 0$, with upward displacement counted positive. It accelerates at $+1.20 \text{ m s}^{-2}$ for 2.00 s , cruises for 3.00 s , and then slows at -0.80 m s^{-2} until it stops. The prescribed speed cap is 2.40 m s^{-1} . Find the total duration, displacement and continuous position function, and check the cap.

Model. Acceleration is constant within each phase; changes between phases are idealized as instantaneous. The peak speed calculated below is 2.40 m s^{-1} , exactly the prescribed cap. These are prescribed kinematic parameters, not a model of the lift's drive forces or passenger comfort.

Start. For $0 \leq t \leq 2 \text{ s}$,

$$v = 1.20t, \quad x = 0.60t^2$$

using numerical SI values. At $t = 2 \text{ s}$, $v_1 = 2.40 \text{ m s}^{-1}$ and $x_1 = 2.40 \text{ m}$.

Cruise. For $2 \leq t \leq 5 \text{ s}$, the elapsed time is $t - 2$:

$$v = 2.40, \quad x = 2.40 + 2.40(t - 2).$$

The cruise displacement is $2.40 \times 3.00 = 7.20 \text{ m}$, so $x_2 = 9.60 \text{ m}$ at $t = 5 \text{ s}$.

Stop. Put $\tau = t - 5 \text{ s}$, with initial velocity 2.40 m s^{-1} . The equation $0 = 2.40 - 0.80\tau$ gives a braking duration of 3.00 s . During this phase,

$$v = 2.40 - 0.80(t - 5), \quad x = 9.60 + 2.40(t - 5) - 0.40(t - 5)^2$$

in numerical SI values. At $t = 8 \text{ s}$, $v = 0$ and $x = 13.20 \text{ m}$.

Result and check. The full duration is 8.00 s and the upward displacement is 13.20 m . Independently, the velocity graph has two triangles and a rectangle, with areas $2.40 + 7.20 + 3.60 = 13.20 \text{ m}$. All velocities are non-negative, so this is also the distance. Both x and v agree from the two sides at each join. Continuing the last quadratic beyond 8 s would predict downward motion; the intended subsequent phase instead has $x = 13.20 \text{ m}$ and $v = 0$.

An ideal step in acceleration produces a corner in the velocity graph but no jump in position or velocity. Exactly at a corner, the derivative dv/dt has different one-sided values and is not defined in this piecewise model. Assigning an arbitrary value at a single endpoint does not alter an integral or any finite displacement. A real lift changes acceleration over a finite interval; the step model is useful only when that omitted detail is not the question being asked.

2.4.3 A kinematic constraint can change the motion program

A short traverse may end before the intended cruise speed is reached. This can be decided without a force model. Consider ideal rest-to-rest motion over distance $D > 0$, constrained by a positive speed cap V and an acceleration magnitude cap A . For the simple fastest monotone program with instantaneous acceleration switches, accelerating from rest to V requires time V/A and distance

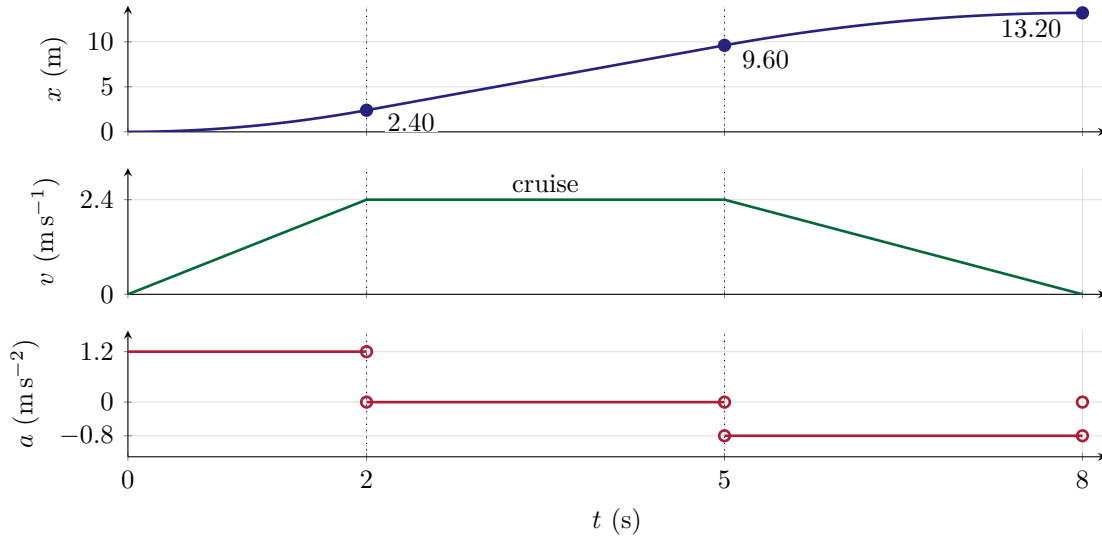


Figure 2.9: Calculated lift start–cruise–stop motion. Position and velocity join continuously at 2 and 5 s; acceleration changes discontinuously in this idealization. Open circles mark one-sided acceleration limits, not missing measured data. The lift is at rest after 8 s; the ideal acceleration is not defined by dv/dt exactly at a corner.

$V^2/(2A)$. Symmetric braking requires the same distance. Thus a cruise phase is possible only if

$$D \geq \frac{V^2}{A}. \quad (2.22)$$

For a longer traverse, $T = D/V + V/A$: the extra V/A accounts for the time penalty of the two ramps relative to cruising the entire distance. For a shorter traverse, there is no cruise. The two triangular areas give

$$v_{\text{peak}} = \sqrt{AD}, \quad T = 2\sqrt{D/A}, \quad D < V^2/A. \quad (2.23)$$

These results assume instantaneous switching of acceleration within its allowed range and no additional motion constraints. They are kinematic bounds within an ideal program, not a certified machine-control design.

Example 2.10: A short traverse cannot reach the speed cap

Take $D = 0.120$ m, $A = 0.80$ m/s² and $V = 0.40$ m/s. Decide whether the stage reaches the cap and find the ideal duration.

Threshold. The two ramps to and from the speed cap would require $V^2/A = 0.200$ m, greater than the available distance. The assumed cruise program is therefore impossible.

Triangular program. Equation (2.23) gives

$$v_{\text{peak}} = \sqrt{(0.80)(0.120)} \simeq 0.310 \text{ m/s}, \quad T = 2\sqrt{0.120/0.80} \simeq 0.775 \text{ s}.$$

Each ramp lasts 0.387 s and covers 0.0600 m.

Independent area check. Half the product of the full duration and the peak velocity is $\frac{1}{2}(0.7745967)(0.3098387) \simeq 0.120$ m. The peak is below V , as required. At the boundary $D = V^2/A$, both duration formulas give $2V/A$, so the two regimes join consistently. A naive constant-speed estimate gives $D/V = 0.300$ s for this traverse. It omits the time needed to change velocity from and back to rest, and even assumes a cap that this short motion never reaches. The phase-resolved 0.775 s therefore answers a different, correctly constrained question.

2.4.4 Two objects must be compared at the same event time

For an encounter on a line, the positions must agree at the *same* time in the same frame. Equal velocities are not an encounter condition: separated objects can have equal velocities indefinitely. Conversely, objects can meet while their velocities differ. A shared clock makes the distinction precise.

Example 2.11: Two carriages, one encounter

Two markers on parallel straight guides have longitudinal coordinates $x_A = 0.30 + 0.20t$ and $x_B = 0.10t^2$, with numerical SI values valid for $0 \leq t \leq 4$ s. When are their longitudinal positions equal, and are their velocities then equal?

Event equation. Set $x_A(t) = x_B(t)$:

$$0.10t^2 - 0.20t - 0.30 = 0 \implies (t - 3)(t + 1) = 0.$$

The root $t = -1$ s lies outside the stated domain. At the allowed root $t = 3$ s, both positions are 0.90 m.

Velocity comparison. Differentiation gives $v_A = 0.20 \text{ m s}^{-1}$ and $v_B = 0.20t = 0.60 \text{ m s}^{-1}$ at the encounter. They share a position, not a velocity. As a check, $x_B - x_A$ changes from negative to positive at 3 s: B passes the longitudinal position of A. The parallel guides avoid assuming an actual collision or modelling what contact would do.

2.4.5 Unequal acceleration and braking limits

The symmetric traverse is a useful first model, but a motion program need not have equal ramp durations. Let $A > 0$ be the allowed positive acceleration, let $B > 0$ be the magnitude of the allowed negative acceleration, and let $V > 0$ be a speed cap. The stage starts and ends at rest and travels a prescribed distance $D > 0$ in the positive direction. Assume that the ideal program uses $+A$, then possibly zero acceleration, then $-B$, with instantaneous switches. These are *kinematic constraints*: they specify allowable histories without yet explaining the drive forces that produce them.

If the cap is attained, the ramp durations and displacements are

$$t_a = \frac{V}{A}, \quad d_a = \frac{V^2}{2A}, \quad t_b = \frac{V}{B}, \quad d_b = \frac{V^2}{2B}.$$

The ramp distances follow either from the time-free equation or from the two triangular areas under $v(t)$. A cruise is possible only when

$$D \geq D_* \equiv \frac{V^2}{2} \left(\frac{1}{A} + \frac{1}{B} \right). \quad (2.24)$$

Its duration is $(D - D_*)/V$. Adding all durations gives

$$T = \frac{D}{V} + \frac{V}{2} \left(\frac{1}{A} + \frac{1}{B} \right), \quad D \geq D_*. \quad (2.25)$$

The weaker braking limit makes the braking triangle wider, not lower: both ramps join the same cruise speed. Choosing equal ramp times when $A \neq B$ would violate either continuity of velocity or one of the stipulated accelerations.

For $D < D_*$, write U for the attainable peak speed. There is no cruise, so the two ramp areas must sum to D :

$$D = \frac{U^2}{2A} + \frac{U^2}{2B}.$$

Solving gives

$$U = \sqrt{\frac{2DAB}{A+B}}, \quad T = U \left(\frac{1}{A} + \frac{1}{B} \right) = \sqrt{2D \left(\frac{1}{A} + \frac{1}{B} \right)}. \quad (2.26)$$

The switch occurs after distance $d_a = BD/(A+B)$. In particular, it is not generally halfway along the traverse. Braking must begin earlier when its acceleration magnitude is smaller. At $D = D_*$ the peak speed is V , the cruise duration is zero, and both expressions for T give $V/A + V/B$. Their first derivatives with respect to D also agree there: the short-program derivative is $(1/A + 1/B)/T = 1/V$ at the boundary. This smooth join is a useful check on the model rather than an extra law of motion.

Example 2.12: An asymmetric translation-stage traverse

A stage has $A = 0.80 \text{ m s}^{-2}$, $B = 0.50 \text{ m s}^{-2}$, and $V = 0.60 \text{ m s}^{-1}$. Find its ideal rest-to-rest program for $D = 1.50 \text{ m}$. Then determine how the program changes for $D = 0.240 \text{ m}$.

Long traverse. The cap requires

$$D_* = \frac{0.60^2}{2} \left(\frac{1}{0.80} + \frac{1}{0.50} \right) = 0.585 \text{ m}.$$

Since $1.50 > D_*$, the stage accelerates for 0.750 s , cruises for $(1.50 - 0.585)/0.60 = 1.525 \text{ s}$, and brakes for 1.200 s . Hence $T = 3.475 \text{ s}$. The phase-start positions are 0 , 0.225 m and 1.140 m . Using numerical SI values, the full position function is

$$x(t) = \begin{cases} 0.40t^2, & 0 \leq t \leq 0.750, \\ 0.225 + 0.60(t - 0.750), & 0.750 \leq t \leq 2.275, \\ 1.140 + 0.60(t - 2.275) - 0.25(t - 2.275)^2, & 2.275 \leq t \leq 3.475. \end{cases}$$

Afterwards $x = 1.50 \text{ m}$ and $v = 0$. Differentiation gives the ramps and plateau in Fig. 2.10; the end of the braking phase has $v = 0$ and $x = 1.50 \text{ m}$. The one-sided first derivatives agree at each join.

Short traverse. Now $D < D_*$, so

$$U \approx 0.3843 \text{ m s}^{-1}, \quad t_a \approx 0.4804 \text{ s}, \quad t_b \approx 0.7686 \text{ s}, \quad T \approx 1.249 \text{ s}.$$

The switch position is $BD/(A+B) \approx 0.09231 \text{ m}$, leaving approximately 0.14769 m for braking. The unequal distances are consistent with the weaker braking magnitude.

Independent check. For the long traverse, the velocity areas are $0.225 + 0.915 + 0.360 = 1.500 \text{ m}$. For the short traverse, the triangular area is $\frac{1}{2}UT = 0.240 \text{ m}$, and $U < V$. These are monotone histories, so distance equals displacement. A constant-speed estimate $D/V = 2.500 \text{ s}$ for the long traverse is shorter than the phase-resolved 3.475 s by 0.975 s , or 39.0% of that estimate. It would require the stage to move at the cap for the whole traverse despite starting and ending at rest. The actual average speed is $D/T \approx 0.432 \text{ m s}^{-1}$, below the cap. Finite switching times or extra restrictions on acceleration changes would require a revised program.

2.4.6 Reaction time and braking are different phases

An observed delay before deceleration is not itself a braking acceleration. In a simple laboratory-cart model, let the cart move at $v_0 > 0$ while a trigger is detected and acted upon over a time $t_r \geq 0$. After that delay, prescribe constant acceleration $-B$ until the cart stops. The reaction phase covers $v_0 t_r$. The braking duration is v_0/B , and its displacement is $v_0^2/(2B)$. Consequently

$$D_{\text{stop}} = v_0 t_r + \frac{v_0^2}{2B}, \quad T_{\text{stop}} = t_r + \frac{v_0}{B}. \quad (2.27)$$

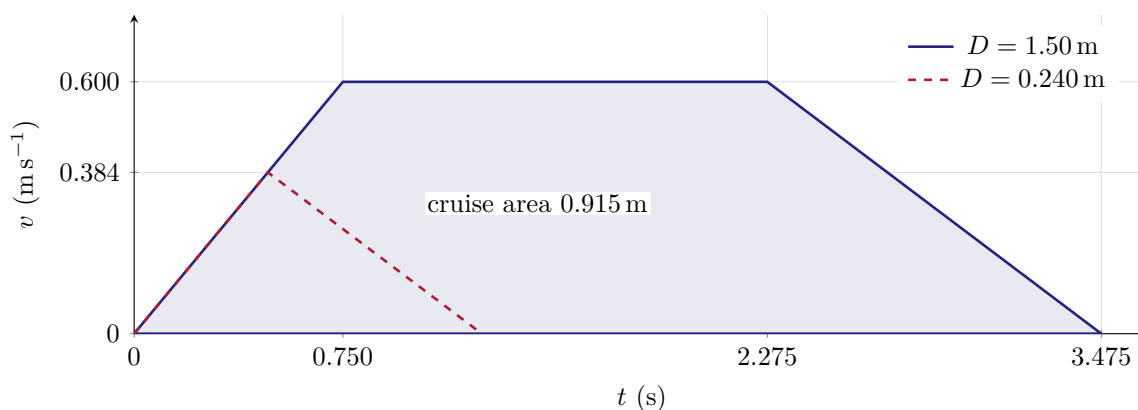


Figure 2.10: Two calculated rest-to-rest programs under the same unequal limits, $A = 0.80 \text{ m s}^{-2}$ and $B = 0.50 \text{ m s}^{-2}$. The long program reaches the 0.60 m s^{-1} cap; the short one cannot. The filled area belongs to the long program and equals 1.50 m . Both histories end at rest; extension of the descending line below zero would describe a different program.

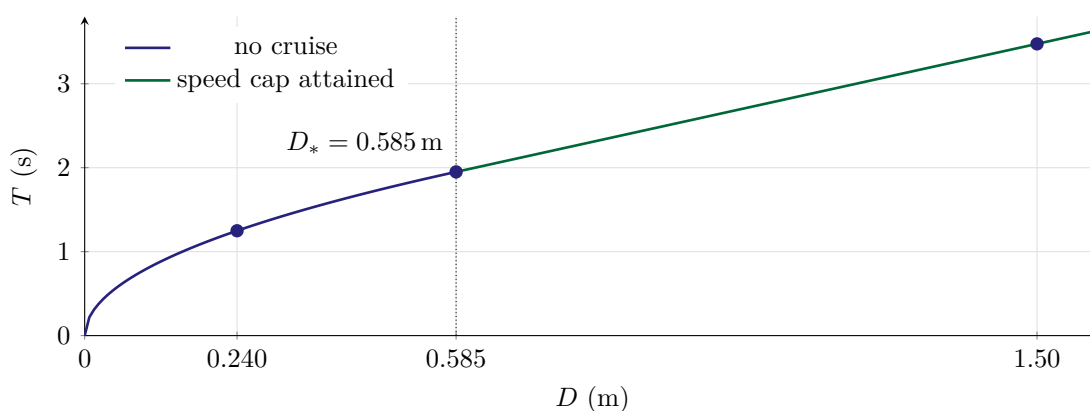


Figure 2.11: Ideal duration versus traverse distance for the asymmetric stage. The square-root branch joins the linear branch with the same value and slope at D_* . On the linear branch, an additional distance ΔD adds cruise time $\Delta D/V$. The zero-distance limit has zero duration; no nonzero motion is prescribed there.

Neither term can absorb the other: increasing the delay raises a displacement linear in speed, whereas a fixed braking magnitude gives a displacement quadratic in speed. For example, doubling v_0 doubles the reaction distance and quadruples the braking distance. The stopping time does not have the same scaling as the stopping distance.

During braking put $\tau = t - t_r$. The valid domain is $0 \leq \tau \leq v_0/B$, within which

$$x = v_0 t_r + v_0 \tau - \frac{1}{2} B \tau^2, \quad v = v_0 - B \tau.$$

The subsequent rest phase is a separate part of the model. A quadratic continued beyond its stopping time would reverse the cart, even though reversal was never prescribed. An event equation must therefore be paired with this domain restriction.

Example 2.13: Does a triggered cart stop before a checkpoint?

An ideal cart passes $x = 0$ at 12.0 m s^{-1} . A trigger delay is 0.750 s and the ensuing braking magnitude is 4.00 m s^{-2} . A checkpoint is at $x = 24.0 \text{ m}$. Determine the stopping position and time, the first checkpoint-passage time and speed, and the greatest initial speed that would stop at or before that checkpoint under these same stipulated parameters.

Stop. Reaction contributes 9.00 m , braking contributes 18.0 m , and the cart stops at 27.0 m at $t = 3.750 \text{ s}$. Thus it reaches the checkpoint before stopping. This calculation concerns

a prescribed point-cart trajectory, not contact with an obstacle or a guaranteed stopping capability of a real vehicle.

Checkpoint event. The checkpoint lies beyond the reaction phase. In the braking phase its equation is

$$24 = 9 + 12\tau - 2\tau^2, \quad \tau = \frac{12 \pm \sqrt{24}}{4}.$$

Only $\tau \approx 1.7753$ s lies in $0 \leq \tau \leq 3.00$ s. Hence the event occurs at $t \approx 2.5253$ s with positive velocity

$$v = \sqrt{24} \text{ m s}^{-1} \approx 4.899 \text{ m s}^{-1}.$$

The other root, $\tau \approx 4.2247$ s, would be a passage on an unprescribed backward continuation. The actual cart stays at $x = 27.0$ m after stopping and does not make that second passage. Independently, $v^2 = 12^2 - 2(4)(24 - 9) = 24 \text{ m}^2 \text{ s}^{-2}$, with the positive sign selected by the valid braking interval.

Inverse question. The stopping-distance inequality is $v_0(0.750) + v_0^2/8 \leq 24$. For $v_0 \geq 0$ its left side increases monotonically, so the upper bound is the positive equality root:

$$v_{0,\max} = -Bt_r + \sqrt{B^2t_r^2 + 2B(24.0 \text{ m})} \approx 11.18 \text{ m s}^{-1}.$$

Substitution returns 24.0 m; the other root is negative and outside the initial-speed domain. This bound applies only to the stated constant- B , fixed-delay model.

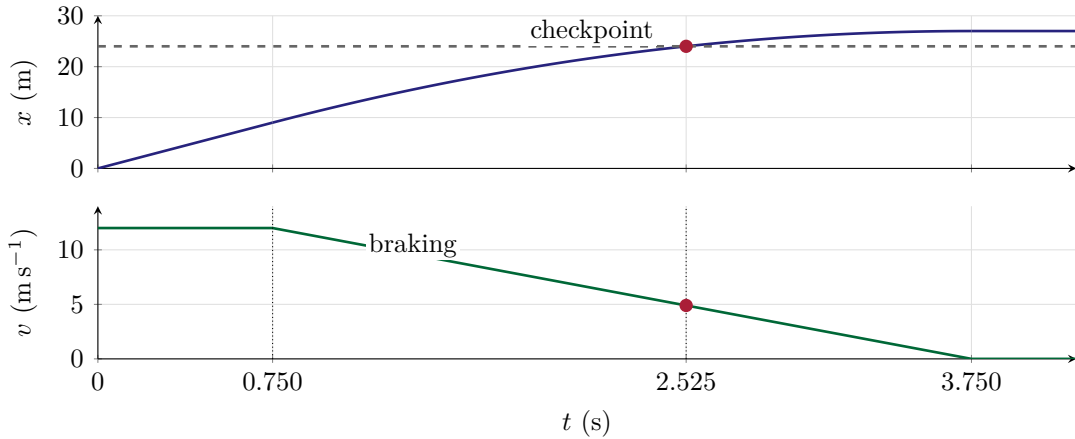


Figure 2.12: Reaction, braking and rest phases of the calculated cart trajectory. The dot denotes the first $x = 24.0$ m passage. The curves explicitly include rest after 3.750 s, preventing the quadratic braking model from being extended into a fictitious reversal. Position and velocity are continuous at both switches.

2.4.7 A delayed start requires a shared clock

Resetting elapsed time in a phase is helpful, but resetting an object's entire clock while comparing it with another object is not. Suppose object A starts at $x = 0$ at global time $t = 0$ and moves at constant speed $u > 0$. Object B remains at $x = 0$ until global time $\delta > 0$, then accelerates from rest at constant $a > 0$. Their stipulated positions are

$$x_A(t) = ut, \quad x_B(t) = \begin{cases} 0, & 0 \leq t \leq \delta, \\ \frac{1}{2}a(t - \delta)^2, & t \geq \delta. \end{cases}$$

They are coincident at release, but this is not the requested later encounter. A moves ahead while B waits. After B starts, use its local elapsed time $\tau = t - \delta \geq 0$ and write the later event as

$$\frac{1}{2}a\tau^2 = u(\tau + \delta).$$

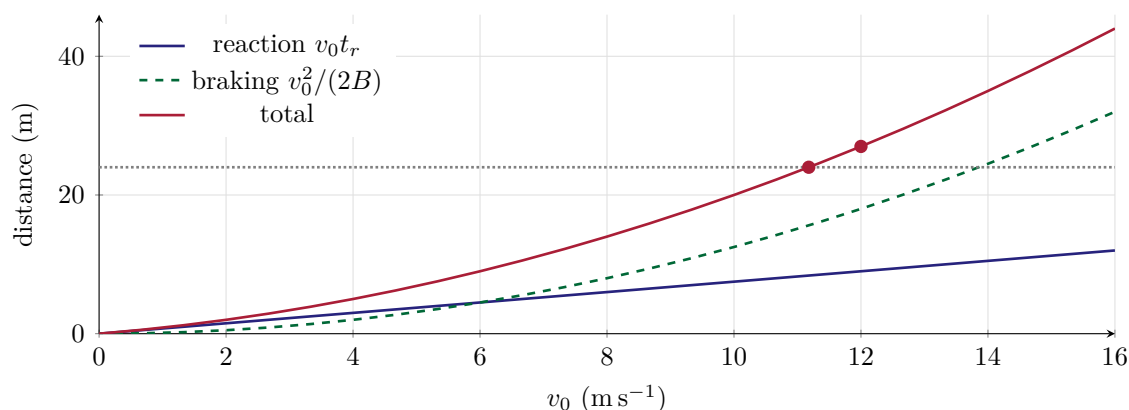


Figure 2.13: Stopping-distance components for fixed $t_r = 0.750$ s and $B = 4.00$ m s⁻². The horizontal line marks the 24.0 m checkpoint. Its intersection with the total curve gives the model's greatest compatible initial speed, 11.18 m s⁻¹. The rising curves are parameter predictions, not measurements.

Both positions are now evaluated at the same global time. The allowed solution is

$$\tau_{\text{meet}} = \frac{u + \sqrt{u^2 + 2au\delta}}{a}, \quad t_{\text{meet}} = \delta + \tau_{\text{meet}}. \quad (2.28)$$

The negative root in local time is outside the accelerating phase. There is also a useful event before the encounter: the separation $s = x_A - x_B = u(\tau + \delta) - a\tau^2/2$ is greatest when $ds/d\tau = u - a\tau = 0$. Equal velocities identify maximum separation here; equal positions identify the later encounter. These are different physical questions.

Example 2.14: A marker catches another after a delayed release

Markers on parallel straight tracks follow the model above with $u = 2.00$ m s⁻¹, $a = 1.00$ m s⁻², and $\delta = 2.00$ s. The model is stipulated until $t = 8.00$ s. Find the later longitudinal encounter, compare their velocities there, and find their greatest separation.

Encounter. Equation (2.28) gives $\tau_{\text{meet}} = 2 + \sqrt{12} \approx 5.4641$ s. Thus $t_{\text{meet}} \approx 7.4641$ s, within the stipulated interval, and the shared position is $x = 2(4 + \sqrt{12})$ m ≈ 14.928 m. A has velocity 2.00 m s⁻¹; B has velocity $a\tau \approx 5.464$ m s⁻¹. B therefore passes A's longitudinal coordinate with relative velocity 3.464 m s⁻¹.

Root domain. Expanding directly in global time gives $t^2 - 8t + 4 = 0$, with roots $4 \pm \sqrt{12}$. The smaller root, 0.5359 s, predates B's start. At that actual time B is still at $x = 0$, while A is at 1.072 m. Thus the small root solves a backward extension of B's accelerating parabola, not either object's stated position function.

Greatest separation. It occurs at $\tau = u/a = 2.00$ s, hence $t = 4.00$ s. There $x_A = 8.00$ m and $x_B = 2.00$ m, so $s_{\text{max}} = 6.00$ m. Before B's start the separation rises from zero to 4.00 m, so the later maximum is also the global maximum before the encounter. Figure 2.14 shows why equal velocities do not imply a meeting.

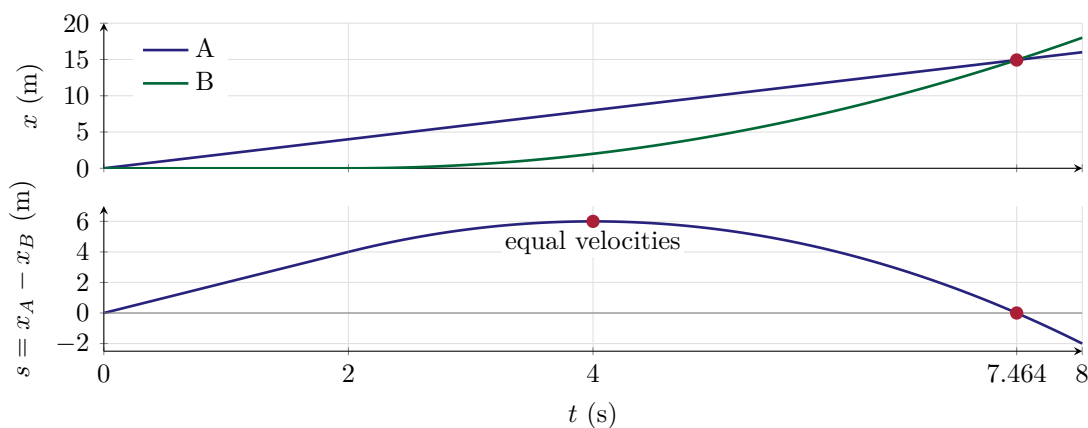


Figure 2.14: Delayed-start encounter using one global clock. B's position is constant before 2.00 s and then rises quadratically. Separation is greatest at 4.00 s, when velocities agree; the later encounter at 7.464 s occurs when separation vanishes. Negative separation after that event means B is ahead. The curves describe parallel-track coordinates and do not model a collision.

2.5 Free fall and vertical launch

2.5.1 The model and its signs

Near Earth's surface, the vertical motion of a freely moving object can often be approximated by constant downward acceleration when air resistance is negligible and the height range is small enough that variation of gravitational acceleration can be ignored. This is an empirical physical model, not a consequence of the definitions of velocity and acceleration. Mass does not enter its kinematic equations. Air resistance, buoyancy, propulsion, supports or collisions can make the model inappropriate.

Choose $+y$ upward and define $g > 0$. Then $a_y = -g$. If an object starts at y_0 with upward velocity w at $t = 0$,

$$v_y(t) = w - gt, \quad (2.29)$$

$$y(t) = y_0 + wt - \frac{1}{2}gt^2, \quad (2.30)$$

$$v_y^2 = w^2 - 2g(y - y_0). \quad (2.31)$$

An object *released from rest* has $w = 0$ relative to the chosen laboratory. An object released from a moving support need not have zero laboratory velocity. Stating the frame is part of specifying the initial condition.

For $w > 0$, the apex occurs at $t_{\text{apex}} = w/g$ and has height $y_{\text{apex}} = y_0 + w^2/(2g)$. The upward velocity falls to zero there, but the downward acceleration remains $-g$. On descent v_y is negative and a_y is negative, so the speed increases. The object is not in a finite period of rest at the top.

If instead downward displacement is positive, write $z = y_0 - y$. Then $v_z = -v_y$ and $a_z = +g$, and the initial upward launch velocity becomes $v_z(0) = -w$. Changing only the sign of g without changing the coordinates and initial velocity would mix two conventions. The physical predictions must agree after the consistent transformation.

2.5.2 A landing level is an event condition

Let the arrival level be $H > 0$ below the release position, so $y_L = y_0 - H$. Substituting into Eq. (2.30) gives

$$\frac{1}{2}gt^2 - wt - H = 0.$$

The quadratic roots are

$$t_{\pm} = \frac{w \pm \sqrt{w^2 + 2gH}}{g}. \quad (2.32)$$

For $w > 0$ and $H > 0$, the square root exceeds w , so $t_- < 0$ and $t_+ > 0$. Only t_+ lies in the post-launch model domain. The other root is a feature of the quadratic extended backward in time, not a second future landing. At arrival,

$$v_{y,L} = w - gt_+ = -\sqrt{w^2 + 2gH}. \quad (2.33)$$

The negative sign is physical: the object is descending. The time-free equation gives the same magnitude but cannot select that sign alone.

Example 2.15: Vertical launch to a lower platform

An ideal point-like object is launched vertically upward at 3.00 m s^{-1} from $y_0 = 1.80 \text{ m}$. It arrives at a lower level $y_L = 0.60 \text{ m}$. Use $g = 9.81 \text{ m s}^{-2}$ and neglect air resistance. Find the apex, arrival time and velocity, displacement and total distance before arrival.

Model boundary. These calculations end when the object first reaches the lower level. They do not describe a bounce, penetration or forces during contact. The example is a theoretical trajectory, not an instruction to launch objects from height.

Apex. The upward speed vanishes at

$$t_{\text{apex}} = 3.00/9.81 \simeq 0.3058 \text{ s}, \quad y_{\text{apex}} = 1.80 + \frac{3.00^2}{2(9.81)} \simeq 2.2587 \text{ m}.$$

Arrival. With $H = 1.20 \text{ m}$,

$$t_L = \frac{3.00 + \sqrt{3.00^2 + 2(9.81)(1.20)}}{9.81} \simeq 0.8873 \text{ s},$$

$$v_{y,L} \simeq 3.00 - 9.81(0.8873328) \simeq -5.705 \text{ m s}^{-1}.$$

The other root is -0.2757 s , outside the domain. The correct arrival is later than the return to the launch height, $2w/g \simeq 0.6116 \text{ s}$.

Displacement versus distance. Displacement is $y_L - y_0 = -1.20 \text{ m}$. Distance adds the rise and the descent:

$$\ell = (y_{\text{apex}} - y_0) + (y_{\text{apex}} - y_L) = H + \frac{w^2}{g} \simeq 2.117 \text{ m}.$$

Checks. Equation (2.31) independently gives $v_{y,L}^2 = 3.00^2 + 2(9.81)(1.20) = 32.544 \text{ m}^2 \text{ s}^{-2}$. Substitution of t_L into $y(t)$ returns 0.60 m . The ascent distance 0.4587 m is smaller than the descent distance 1.6587 m , consistent with the longer descent and larger final speed. Figure 2.15 links these events to both graphs.

2.5.3 Limits and alternative event questions

When $H \rightarrow 0$ with $w > 0$, the roots become 0 and $2w/g$: release and return to the same height. This familiar symmetry applies to equal heights under constant g , not automatically to different launch and arrival levels. When $w \rightarrow 0$ with $H > 0$, Eq. (2.32) gives $t_L = \sqrt{2H/g}$ and the arrival velocity is $-\sqrt{2gH}$, the released-from-rest limit.

For a checkpoint $d > 0$ above the launch position, the discriminant becomes $w^2 - 2gd$. There are two post-launch passages when $0 < d < w^2/(2g)$, one tangential passage at the apex, and no passage if the level is higher than the apex. A negative discriminant is a physical impossibility within the

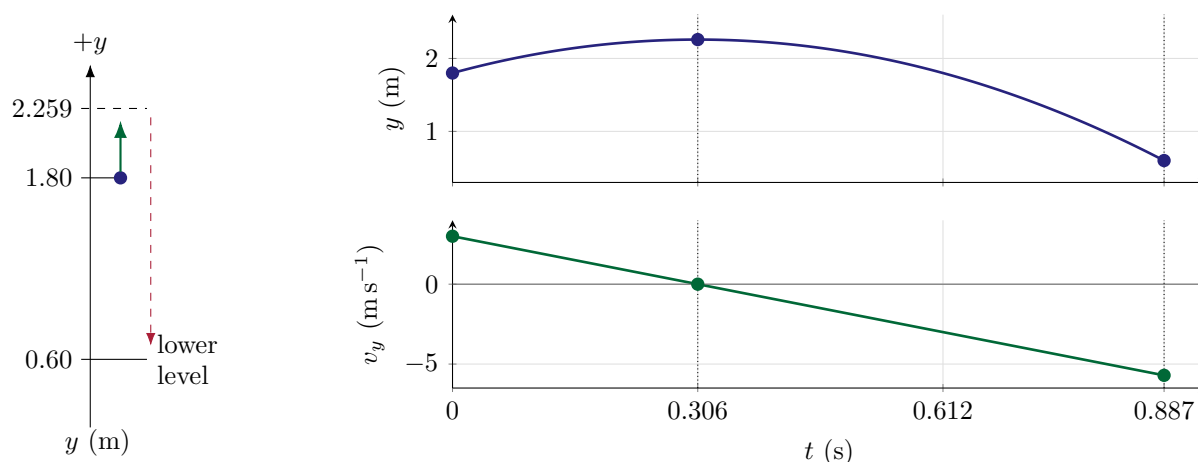


Figure 2.15: Vertical launch to a lower level, calculated with $g = 9.81 \text{ m s}^{-2}$. The left drawing uses a vertical metre scale; horizontal separation and travel arrows only distinguish ascent and descent. The two right panels share time, not the left drawing's length scale. Curves end at arrival at $y = 0.60 \text{ m}$. Motion during contact with a platform is outside the free-fall model.

stated model, not a request to report a complex arrival time. Each real root must still be tested against any restriction such as a contact that terminates the trajectory earlier.

A model with constant downward acceleration has no finite terminal speed. Its prediction of ever-increasing downward speed at long times is a warning not to extrapolate an air-resistance-free approximation arbitrarily. Explaining drag and deciding how it modifies acceleration belongs to the later dynamics treatment. Here the useful habit is to state what has been neglected and stop using the equations when that neglected effect becomes important.

Guided checks

Check. At the highest point of an upward launch, which of y , v_y , $|v_y|$ and a_y must be zero?

Answer. Only velocity and speed must be zero. The coordinate y depends on the origin, and acceleration remains $-g$. Moving the origin to the apex would also make $y = 0$ there, but would not alter the motion.

2.5.4 Two crossings of an elevated checkpoint

An arrival equation can have two future roots without either being spurious. Let $w > 0$ be the upward launch speed and $d = y_c - y_0 > 0$ the elevation of a checkpoint above the launch point. In the same constant- g , negligible-drag model,

$$d = wt - \frac{1}{2}gt^2.$$

Completing the square makes the geometry visible:

$$y(t) - y_0 = \frac{w^2}{2g} - \frac{g}{2} \left(t - \frac{w}{g} \right)^2. \quad (2.34)$$

The motion is a downward-opening parabola centred in time at its apex. A horizontal checkpoint below its maximum rise intersects it once on each side. Solving the quadratic gives

$$t_{\text{up}} = \frac{w - \sqrt{w^2 - 2gd}}{g}, \quad t_{\text{down}} = \frac{w + \sqrt{w^2 - 2gd}}{g}, \quad 0 < d < \frac{w^2}{2g}. \quad (2.35)$$

Both times are positive and straddle w/g . Substitution into $v_y = w - gt$ gives $+\sqrt{w^2 - 2gd}$ on ascent and the negative of that value on descent. Equal heights therefore have equal *speed* in this ideal model, with opposite velocities.

At $d = w^2/(2g)$ the roots coincide at the apex. This is one instant of tangential contact with a level, not two distinct passages. For a higher checkpoint there is no real root. Equivalently, the minimum upward launch speed that can reach elevation d is $\sqrt{2gd}$, with arrival at zero vertical speed in the limiting case. These classifications belong to the freely moving trajectory. If an object is caught at its first ascent passage, its free-fall model terminates there and does not predict a later descent passage through the catcher.

Example 2.16: A reachable level and an unreachable level

An ideal object is launched upward from $y_0 = 0$ at $w = 8.00 \text{ m s}^{-1}$. With $g = 9.81 \text{ m s}^{-2}$, find all post-launch passages through $y = 2.00 \text{ m}$ and their velocities. Decide whether $y = 4.00 \text{ m}$ is reached, and find the minimum launch speed needed to reach that higher level.

Reachable checkpoint. The maximum rise is $w^2/(2g) \approx 3.262 \text{ m}$ at $t = w/g \approx 0.8155 \text{ s}$. For $d = 2.00 \text{ m}$, the discriminant is $64 - 39.24 = 24.76 \text{ m}^2 \text{ s}^{-2}$, so

$$t_{\text{up}} \approx 0.3083 \text{ s}, \quad t_{\text{down}} \approx 1.3227 \text{ s}, \quad v_{y,\text{up}} \approx +4.976 \text{ m s}^{-1}, \quad v_{y,\text{down}} \approx -4.976 \text{ m s}^{-1}.$$

Their exact midpoint is $w/g \approx 0.8155 \text{ s}$, the apex time. Substituting either time into $8t - 4.905t^2$ returns 2.00 m . The signs follow from the velocity equation; using only v_y^2 would miss the distinction.

Higher checkpoint. For $d = 4.00 \text{ m}$ the discriminant is $64 - 78.48 = -14.48 \text{ m}^2 \text{ s}^{-2}$, so this level is not reached. The limiting speed would be

$$w_{\text{min}} = \sqrt{2(9.81)(4.00)} \approx 8.859 \text{ m s}^{-1}.$$

At that speed the object would reach 4.00 m only at its apex. A greater speed would give two future crossings provided free fall remained valid until the second one.

Symmetry check. The original 8.00 m s^{-1} trajectory returns to $y = 0$ at $2w/g \approx 1.6310 \text{ s}$. Its two 2.00 m crossings both lie within that interval, and the ascent crossing precedes the apex. Figures 2.16 and 2.17 display the root geometry and how the roots change as the checkpoint is raised.

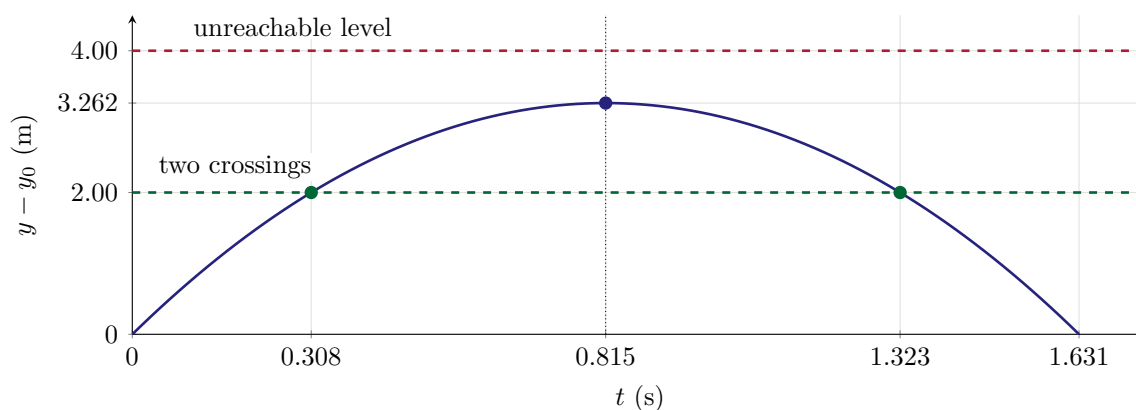


Figure 2.16: Calculated upward launch with $w = 8.00 \text{ m s}^{-1}$ and $g = 9.81 \text{ m s}^{-2}$. The 2.00 m level has an ascent and a descent crossing; the 4.00 m level lies above the apex. The trajectory is shown only until return to the launch height. The apex line identifies the time symmetry, not a period of rest.

The two roots become harder to distinguish as the checkpoint approaches the apex. Differentiating the crossing condition $d = wt - gt^2/2$ with respect to the selected checkpoint height gives

$$1 = (w - gt) \frac{dt}{dd}, \quad \frac{dt}{dd} = \frac{1}{v_y(t)}.$$

Thus raising a reachable checkpoint delays its ascent crossing and advances its descent crossing.

Near the apex the magnitude of this derivative grows because the vertical speed tends to zero. The derivative does not exist as a finite value at the merged root. A small height change there can turn two intersections into no intersection; rounding a discriminant close to zero too early can therefore change the physical classification. Figure 2.17 represents this sensitivity without extending the root formulas into their forbidden domain.

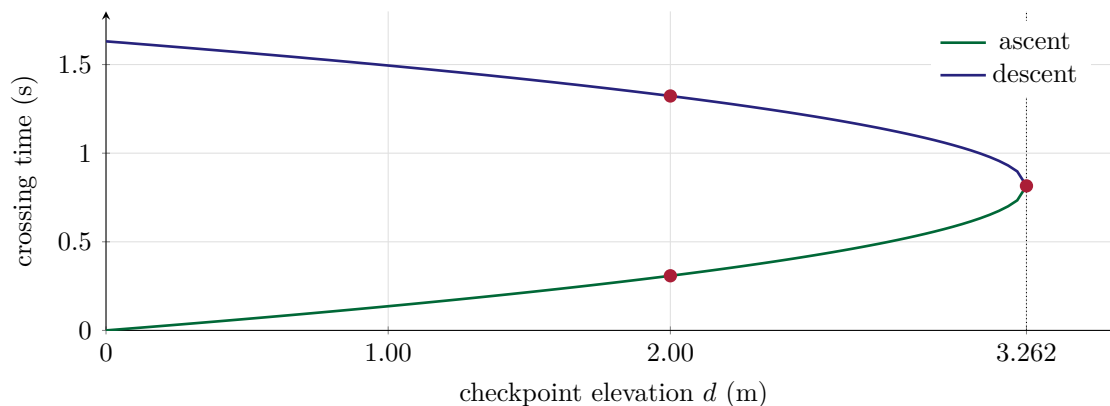


Figure 2.17: Crossing times versus checkpoint elevation for the same ideal launch. The branches merge at the maximum rise, where one apex event replaces two crossings. At $d = 0$ the ascent branch denotes release and the descent branch denotes return. No times are plotted above the maximum reachable elevation; a negative discriminant does not represent an arrival.

2.5.5 An inverse free-fall question from two passage times

Kinematics can infer an initial condition from timing data, but only when the assumptions supply enough information. Consider a point-like object released from rest and falling through two horizontal detection levels separated vertically by $L > 0$. Choose a downward-positive coordinate z with $z = 0$ at release. Let H be the drop to the first level, let $v_1 \geq 0$ be the downward velocity there, and let $\Delta t > 0$ be the measured interval between passages. The interval begins at the first detector, not at release. Its local motion equation is

$$L = v_1 \Delta t + \frac{1}{2} g (\Delta t)^2.$$

Solving for the first-passage velocity, then using the released-from-rest condition, gives

$$v_1 = \frac{L}{\Delta t} - \frac{1}{2} g \Delta t, \quad H = \frac{v_1^2}{2g}, \quad t_1 = \frac{v_1}{g}, \quad t_2 = t_1 + \Delta t. \quad (2.36)$$

The passage interval alone would still determine v_1 within the constant- g model. Determining H additionally uses zero velocity at release. Without it, an unknown downward launch speed u would instead give $v_1^2 = u^2 + 2gH$, leaving more than one possible pair (u, H) .

There is also a model-compatibility bound. For release from rest at or above the first level, $v_1 \geq 0$ requires

$$\Delta t \leq \sqrt{\frac{2L}{g}}. \quad (2.37)$$

Equality corresponds to release exactly at the first detector. A greater interval would require negative v_1 in the interval equation, which contradicts the assumed downward passage after release from rest above it. Squaring that negative value and reporting a positive H would conceal the contradiction. A failed bound instead calls for checking the assumed initial motion, detector separation, passage definitions or neglected effects.

Example 2.17: Recovering a release height from a detector interval

In an ideal calculation the detection levels are $L = 1.20$ m apart and the downward passage interval is $\Delta t = 0.250$ s. The object was released from rest above the first detector. Use $g = 9.81 \text{ m s}^{-2}$ to find v_1 , the drop H , the passage times measured from release, and the speed at the second detector.

Inverse calculation. The allowed upper interval is $\sqrt{2(1.20)/9.81} \approx 0.4946$ s, so 0.250 s is compatible. The first velocity is

$$v_1 = \frac{1.20}{0.250} - \frac{1}{2}(9.81)(0.250) = 3.57375 \text{ m s}^{-1}.$$

It follows that $H \approx 0.6510$ m, $t_1 \approx 0.3643$ s and $t_2 \approx 0.6143$ s. The second velocity is $v_2 = v_1 + g\Delta t = 6.02625 \text{ m s}^{-1}$ downward.

Forward verification. Starting again at the first detector, the calculated interval displacement is

$$(3.57375)(0.250) + \frac{1}{2}(9.81)(0.250)^2 = 0.8934375 + 0.3065625 = 1.2000 \text{ m}.$$

Starting instead at release, $z = gt^2/2$ gives H at t_1 and $H + 1.20$ m at t_2 . The independent velocity relation gives $v_2^2 - v_1^2 = 2gL$ as required. Figure 2.18 places both detector events on the same global trajectory.

What the initial condition contributes. If the same object had instead been launched downward at $u = 2.00 \text{ m s}^{-1}$, the same two-detector interval would give the same v_1 , but the inferred drop would be $(v_1^2 - u^2)/(2g) \approx 0.4471$ m. Thus the detector interval does not establish that the object was released from rest. That assumption must be independently justified.

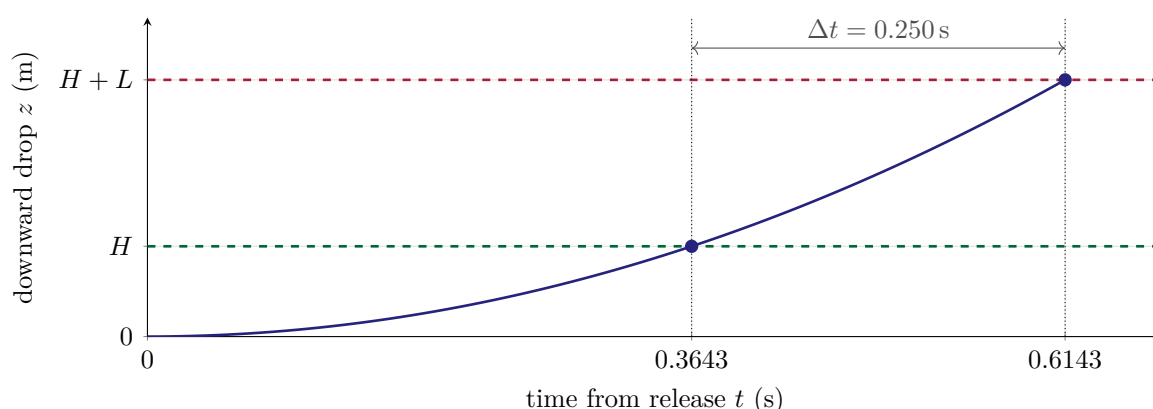


Figure 2.18: Inverse two-detector timing reconstructed as a forward trajectory. The vertical coordinate is downward drop from release, so it increases during the fall. The first detector is $H \approx 0.6510$ m below release and the second is a further $L = 1.20$ m lower. The marked passage interval differs from either time measured from release. The free-fall trajectory ends at the second detector in this displayed calculation.

2.6 General motion from integration

2.6.1 Rates and initial conditions are separate information

A prescribed velocity history need not be linear, and a prescribed acceleration need not be constant. The accumulation principles remain valid:

$$v(t) = v(t_0) + \int_{t_0}^t a(s) ds, \quad (2.38)$$

$$x(t) = x(t_0) + \int_{t_0}^t v(s) ds. \quad (2.39)$$

For continuous functions, these are the integral counterparts of the derivative definitions. Piecewise-continuous rates can be integrated phase by phase; isolated finite jumps in the rate do not spoil accumulation. Both initial values are required to reconstruct position from acceleration.

An indefinite integral displays the missing information as a constant. For example, all functions $v(t) = at + C$ have the same derivative a . The initial condition fixes $C = v(t_0) - at_0$. A second integration introduces another constant. Leaving it out is not a harmless algebraic abbreviation: it asserts a particular initial position, often without justification.

Distance is obtained from speed rather than signed velocity:

$$\ell = \int_{t_i}^{t_f} |v(s)| ds. \quad (2.40)$$

One can evaluate this by splitting at the zeros where velocity changes sign. If a zero does not change the sign, a split is optional and does not change the answer. The inequality $|\int v dt| \leq \int |v| dt$ is the continuous version of $|\Delta x| \leq \ell$.

A useful extension of the power rule is

$$\int t^n dt = \frac{t^{n+1}}{n+1} + C \quad (n = 0, 1, 2, \dots).$$

It is checked by differentiation. For a definite integral, evaluate the antiderivative at the upper limit and subtract its value at the lower limit. The order matters: reversing limits changes the sign. Polynomial coefficients must carry units that keep every accumulated term dimensionally consistent.

Example 2.18: A reversible translation program

A marker has initial position $x(0) = -0.20$ m. Its velocity varies linearly between the following programmed values:

t (s)	0	2	4	8	10
v (m s ⁻¹)	0	+0.40	+0.40	-0.40	0

Find the position at every phase boundary, the reversal event, net displacement and total distance. Assume the piecewise-linear velocity itself is the model, not just an interpolation of unknown motion.

Signed areas. The areas over 0–2, 2–4 and 4–8 s are +0.40, +0.80 and 0 m, respectively. The last interval 8–10 s contributes -0.40 m. Thus the boundary positions are

$$x(0) = -0.20, \quad x(2) = 0.20, \quad x(4) = 1.00, \quad x(8) = 1.00, \quad x(10) = 0.60 \quad \text{in m.}$$

The zero area over 4–8 s does not mean no motion. Velocity decreases through zero at 6 s. From 4 to 6 s the positive triangular area is 0.40 m, taking the marker to $x(6) = 1.40$ m; from 6 to 8 s an equal negative area brings it back.

Displacement and distance. Over the full program, the positive area is 1.60 m and the negative area is -0.80 m, so

$$\Delta x = +0.80 \text{ m}, \quad \ell = 1.60 + 0.80 = 2.40 \text{ m.}$$

The endpoint check is $0.60 - (-0.20) = +0.80$ m. Average velocity is 0.080 m s^{-1} , whereas average speed is 0.240 m s^{-1} .

Position function. Integrating each phase and carrying its initial position forward gives, with numerical SI values,

$$x(t) = \begin{cases} -0.20 + 0.10t^2, & 0 \leq t \leq 2, \\ 0.20 + 0.40(t-2), & 2 \leq t \leq 4, \\ 1.00 + 0.40(t-4) - 0.10(t-4)^2, & 4 \leq t \leq 8, \\ 1.00 - 0.40(t-8) + 0.10(t-8)^2, & 8 \leq t \leq 10. \end{cases} \quad (2.41)$$

Differentiating each branch independently recovers the stated velocity, and both positions and velocities agree at the joins. Figure 2.19 shows why the greatest position occurs at 6 s, inside a phase rather than at a listed phase boundary.

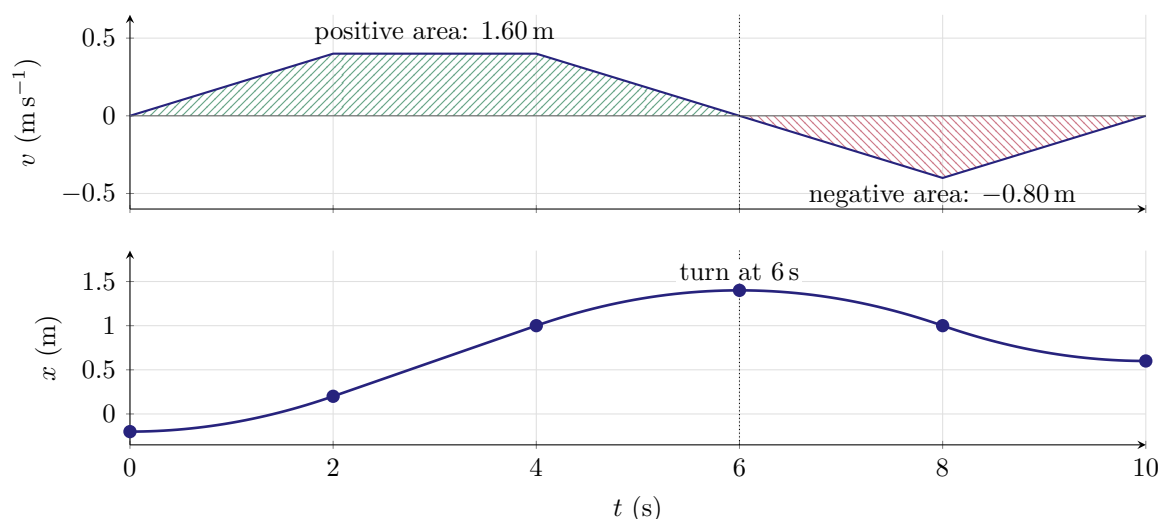


Figure 2.19: A reversible translation program and its integrated position for $x(0) = -0.20$ m. Distinct hatching identifies signed areas without relying on colour. The net area is $+0.80$ m, whereas the sum of area magnitudes is 2.40 m. The position maximum occurs when velocity changes sign, not when its magnitude is largest.

2.6.2 A nonconstant acceleration cannot be replaced indiscriminately

For any acceleration history, its interval-average value predicts the correct *endpoint velocity change*: $\Delta v = \bar{a} \Delta t$ is the definition of $\bar{a}$. It does not generally predict the correct displacement when substituted into a constant-acceleration position formula. Early changes in velocity affect position for longer than equally large changes made late in the interval. A single average acceleration discards that timing information.

Example 2.19: Integrating a linearly increasing acceleration

A stage has $a(t) = \kappa t$ with $\kappa = 0.60 \text{ m s}^{-3}$, initial velocity $v_0 = 0.40 \text{ m s}^{-1}$, and initial position $x_0 = 0.20$ m. Determine v and x during $0 \leq t \leq 2$ s. Test the endpoint-average-velocity shortcut.

Integrate with initial conditions.

$$v(t) = v_0 + \int_0^t \kappa s \, ds = v_0 + \frac{1}{2} \kappa t^2,$$

$$x(t) = x_0 + \int_0^t (v_0 + \frac{1}{2} \kappa s^2) \, ds = x_0 + v_0 t + \frac{1}{6} \kappa t^3.$$

With $\alpha = \kappa/6 = 0.10 \text{ m s}^{-3}$, this is the cubic model

$$x(t) = x_0 + v_0 t + \alpha t^3, \quad v(t) = v_0 + 3\alpha t^2. \quad (2.42)$$

At 2 s, $v = 1.60 \text{ m s}^{-1}$ and $x = 1.80$ m, so the displacement is 1.60 m and $\bar{v} = 0.80 \text{ m s}^{-1}$.

Shortcut test. The arithmetic mean of endpoint velocities is $(0.40 + 1.60)/2 = 1.00 \text{ m s}^{-1}$, not the true average. It would overestimate the displacement by 0.40 m. Likewise, $\bar{a} = \Delta v / \Delta t =$

0.60 m s^{-2} reproduces the endpoint velocity but gives the same wrong displacement if treated as constant throughout.

Checks and interpretation. Differentiating x twice gives $6\kappa t = \kappa t$ and recovers both initial conditions. The velocity curve is convex upward: its endpoint secant lies above the curve, so the trapezoidal shortcut should overestimate the area. The direction of the numerical error agrees with that independent geometrical check.

2.6.3 Inverse questions and the limits of endpoint information

A position history, when sufficiently smooth, determines its velocity and acceleration by differentiation. An acceleration history plus two initial conditions determines position by integration. By contrast, two endpoint positions do not determine a unique velocity history. The whole-interval average is known, but many functions have the same integral. Extra observations or a justified model are needed to resolve their differences.

For example, a rest-to-rest motion can have zero endpoint velocities and a positive net displacement. The endpoint average $(0 + 0)/2 = 0$ would miss all motion. The lift and reversible programs make this obvious: their nonzero intermediate areas are the information that endpoint velocities omit. Before using a shortcut, identify not just the available numbers but the relationship that makes those numbers sufficient.

Key message

Slopes convert a position history into local rates; signed areas convert rates into changes. Neither operation supplies a missing initial condition or validates a physical model. The graph, equation and measurement interpretation must refer to the same motion and the same interval.

2.6.4 Inferring a motion scale from a measured displacement

Sometimes the shape of a velocity history is prescribed, but its scale is unknown. A displacement measurement then supplies an integral constraint. This is an inverse question: one works from an observed change back to a parameter of the assumed motion. It differs from finding velocity by differentiating an observed position history. The result is as reliable as the proposed shape and the measurement used to determine its scale.

Example 2.20: A rest-to-rest traverse inferred from its area

A stage follows the smooth velocity model

$$v(t) = Kt(T - t), \quad 0 \leq t \leq T,$$

with $T = 4.00 \text{ s}$. It starts and ends at rest and moves through $D = 1.20 \text{ m}$ in the positive direction. Infer K , the acceleration history, the greatest speed and the position history for $x(0) = 0$. Compare with a triangular velocity history having the same duration and displacement.

Determine the scale from the signed area. Since $t(T - t)$ has units s^2 , K must have units m s^{-3} . Integration gives

$$D = K \int_0^T (Tt - t^2) dt = K \left(\frac{T^3}{2} - \frac{T^3}{3} \right) = \frac{KT^3}{6}.$$

Thus $K = 6D/T^3 = 0.1125 \text{ m s}^{-3}$. The independently specified duration and displacement fix one previously unknown scale; the endpoint rest conditions alone would not fix it.

Recover local motion. Differentiation gives

$$a(t) = K(T - 2t).$$

The velocity is positive inside the interval and greatest at $t = T/2 = 2.00$ s, where acceleration changes from positive to negative. Its maximum is $KT^2/4 = 0.450 \text{ m s}^{-1}$. Acceleration decreases from $+0.450$ to -0.450 m s^{-2} . Zero acceleration at the middle is an instant of greatest speed, not rest.

Recover position and check. A further integration gives

$$x(t) = K \left(\frac{Tt^2}{2} - \frac{t^3}{3} \right).$$

It returns $x(0) = 0$, $x(2) = 0.600$ m and $x(4) = 1.20$ m. Its derivative is the prescribed velocity. The distance equals the displacement here because the velocity never becomes negative. Average speed is $D/T = 0.300 \text{ m s}^{-1}$, below the greatest speed.

Test uniqueness. A symmetric triangular velocity graph of base T must have height $V = 2D/T = 0.600 \text{ m s}^{-1}$. Its rising and falling slopes are $\pm 2V/T = \pm 0.300 \text{ m s}^{-2}$. Both histories have the same duration, displacement and endpoint velocities, but different peak speeds and accelerations. Figure 2.20 displays the difference. The triangular model has an acceleration switch at the middle; the parabolic velocity model has a continuously changing acceleration within the interval. Neither history is selected by the endpoint information alone.

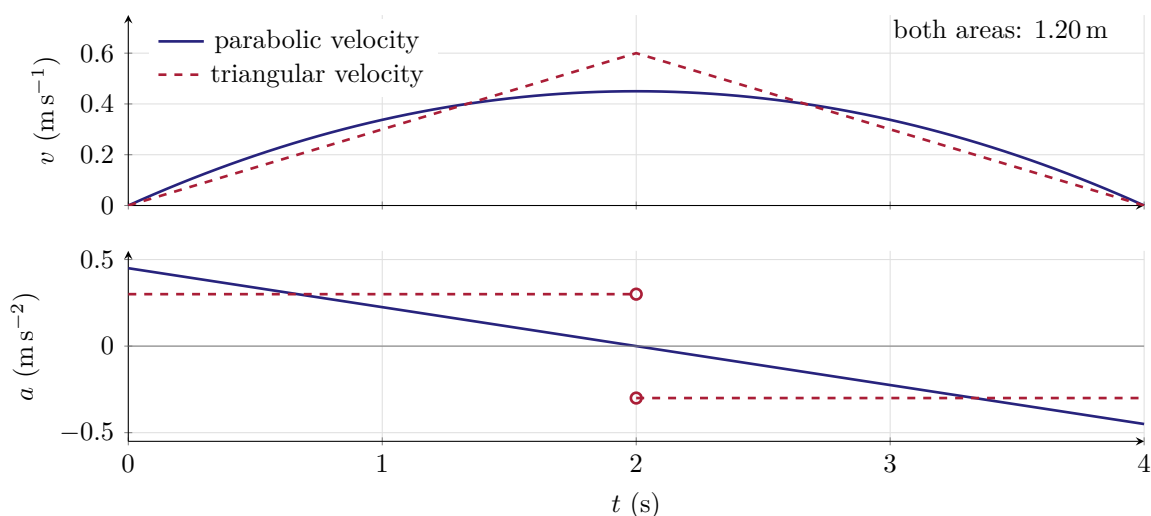


Figure 2.20: Two specified rest-to-rest models with equal displacement and duration. The common velocity area fixes the net traverse, but does not fix the peak speed or the acceleration history. Open markers show the one-sided limits at the triangular model's acceleration switch; no acceleration value at that exact switch is inferred.

2.6.5 Why the time of acceleration matters to displacement

There is a useful way to express the timing information lost by an average acceleration. Set the initial time to zero and let the final time be T . For an integrable acceleration history,

$$x(T) - x(0) = v_0 T + \int_0^T (T - s) a(s) ds. \quad (2.43)$$

The weight $T - s$ is the remaining time during which a velocity change made near time s can contribute to position. A short acceleration interval contributes approximately $a(s) \Delta s$ to velocity,

and that velocity change persists for approximately $T - s$. Summing these contributions gives the integral. An exact mathematical check is to differentiate the right-hand side with upper limit t : its first derivative is $v_0 + \int_0^t a(s) ds$, and its second is $a(t)$ wherever the rate is continuous. It also recovers the two initial conditions.

The integrand has units $(s)(m s^{-2})$ and the integration contributes another second, producing metres. For a constant a , the weighted area is $aT^2/2$, so the familiar position formula is recovered. For the preceding rest-to-rest example, the unweighted integral of acceleration is zero because final and initial velocities agree. Its weighted integral is positive because the positive acceleration occurs earlier and affects position for longer. The two facts are mutually consistent.

This identity describes a model; it does not supply missing acceleration observations. A measured final velocity fixes $\int a dt$, while a measured displacement adds a differently weighted integral. Even those two constraints usually leave many possible acceleration histories. Finding one history that satisfies them is not proof that it was the history actually followed.

2.6.6 Regular samples can conceal extra motion

Sparse positions constrain signed changes between observations, but cannot generally reveal every reversal between them. Regular sampling can also conceal an oscillatory component systematically, rather than merely missing one accidental event. Consider two stipulated numerical SI models,

$$x_A(t) = 0.20t, \quad x_B(t) = 0.20t + 0.040 \sin(4\pi t), \quad 0 \leq t \leq 2 \text{ s.}$$

In a dimensionally explicit notation the sine argument is ωt , with $\omega = 4\pi s^{-1}$. At every time $t = 0, 0.5, 1.0, 1.5, 2.0$ s, the sine term vanishes. Both models produce the identical recorded positions 0, 0.10, 0.20, 0.30, 0.40 m. Their full-interval displacement and average velocity therefore agree.

Their local motion differs strongly. Model A has the constant positive velocity $0.20 m s^{-1}$. Model B has

$$v_B(t) = 0.20 + 0.16\pi \cos(4\pi t) \quad \text{in } m s^{-1},$$

which ranges from -0.303 to $+0.703 m s^{-1}$ and changes sign twice per oscillation. Its distance exceeds 0.40 m because it repeatedly travels backwards. Figure 2.21 shows the hidden intervals of negative velocity. This example establishes ambiguity, not a claim that any particular instrument record contains such oscillations.

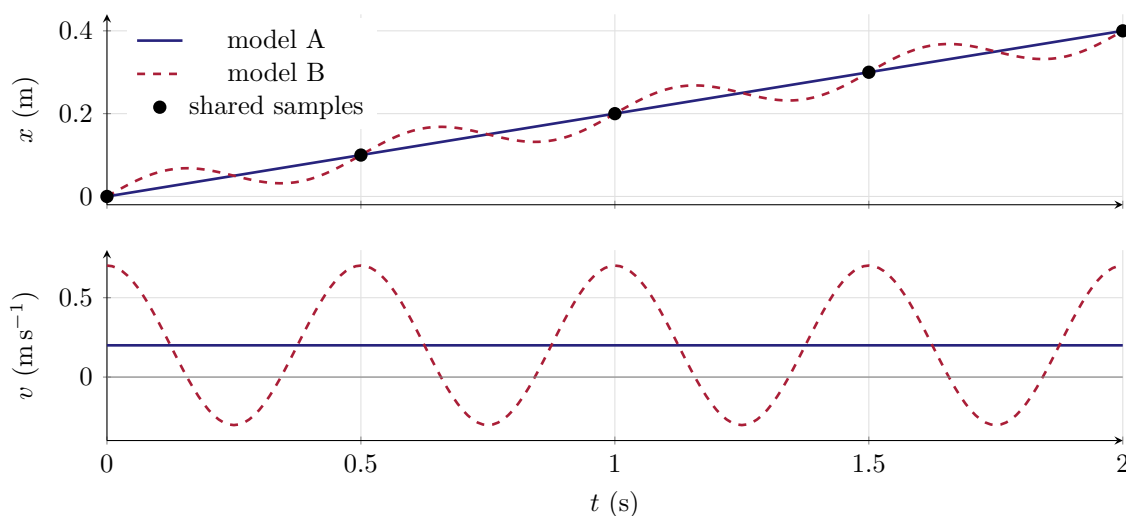


Figure 2.21: Identical position samples do not select a unique trajectory. The regularly sampled sinusoidal component vanishes at all five observation times, while its derivative creates reversals between them. Interpolating the five samples by straight lines would recover model A, but that interpolation is an added assumption.

To distinguish these histories, one needs extra observations or a justified restriction on how rapidly the motion can vary. Repeated readings only at the same five times cannot distinguish the two exact models. Denser observations can help if their times resolve the motion; an independently shifted set of sampling times also breaks this particular coincidence. A smooth line drawn through a small set of points should always be identified as an interpolation or fitted model, rather than treated as newly observed motion.

2.7 Motion data and numerical differentiation

2.7.1 A derivative from a finite set of readings

An analytical formula can be differentiated exactly. A sensor instead supplies positions at discrete times, with finite resolution and uncertainty. One must choose both an estimator and the time to which the estimate is assigned. For uniformly spaced samples $t_i = t_0 + ih$ and positions x_i , a forward difference is

$$D_f x_i = \frac{x_{i+1} - x_i}{h}. \quad (2.44)$$

For exact positions it equals the interval-average velocity; measured positions yield an estimate. Treating it as an estimate of $v(t_i)$ is an additional approximation. Under constant acceleration, the exact interval average equals the instantaneous velocity at the *time midpoint*, $t_i + h/2$, not generally at either endpoint or at the spatial midpoint.

At an interior sample, a centred difference uses readings on both sides:

$$D_c x_i = \frac{x_{i+1} - x_{i-1}}{2h}. \quad (2.45)$$

Its time assignment is t_i . Symmetry removes some finite-step error, as the advanced section will show. There is no centred three-sample estimate at the first or last sample unless additional readings are available. One-sided endpoint estimates have different error properties and should be identified separately, not plotted as if they used the same rule.

A constant offset in all position readings cancels in either difference. An incorrect position scale does not: a one-percent scale error produces the same fractional error in the estimated velocities. Timing errors are also important because time is in the denominator. The examples below take the time stamps as exact to isolate the position-error and approximation effects.

Example 2.21: Recover velocity from constructed position readings

Use the cubic trajectory of Eq. (2.42) with $x_0 = 0.20 \text{ m}$, $v_0 = 0.40 \text{ m s}^{-1}$ and $\alpha = 0.10 \text{ m s}^{-3}$. Table 2.1 gives constructed readings at $h = 0.25 \text{ s}$. Estimate the velocity at 1 s and compare with the exact model.

Data construction. Independent zero-mean uniform errors with standard deviation $u_x = 0.002 \text{ m}$ were added to the model positions using the local Python generator `random.Random(20261002)` and its `uniform` method, then positions were rounded to 10^{-6} m . The assumed noise range before rounding was $\pm\sqrt{3}u_x$. The extra printed digits retain a reproducible teaching dataset; they do not imply micrometre measurement accuracy. All numerical differences here use the *printed rounded readings*.

Estimate. The readings at 0.75 and 1.25 s are 0.541290 and 0.898157 m, so

$$D_c x(1) = \frac{0.898157 - 0.541290}{0.50} = 0.713734 \text{ m s}^{-1}.$$

The exact derivative at 1 s is $v = 0.40 + 0.30(1)^2 = 0.700000 \text{ m s}^{-1}$. The signed error is therefore $+0.013734 \text{ m s}^{-1}$.

Separating causes. With exact unperturbed positions, the same finite difference is

$0.706250 \text{ m s}^{-1}$. Its $+0.006250 \text{ m s}^{-1}$ error comes from using a finite interval on a curved velocity history. The remaining $+0.007484 \text{ m s}^{-1}$ comes from the particular added position errors, including their tiny rounding changes. This decomposition can be made here because the generating function is known. An actual experiment would not reveal its unknown true derivative so directly.

Table 2.1: Synthetic cubic-trajectory data. Positions in the second column are the values used in every displayed numerical difference. Exact velocities are model benchmarks, not separately measured values. A dash means no centred estimate from the available data.

t (s)	x_{obs} (m)	v_{exact} (m s^{-1})	$D_c x$ (m s^{-1})
0.00	0.197651	0.400000	—
0.25	0.299079	0.418750	0.435448
0.50	0.415375	0.475000	0.484422
0.75	0.541290	0.568750	0.572026
1.00	0.701388	0.700000	0.713734
1.25	0.898157	0.868750	0.874312
1.50	1.138544	1.075000	1.075540
1.75	1.435927	1.318750	1.319754
2.00	1.798421	1.600000	—

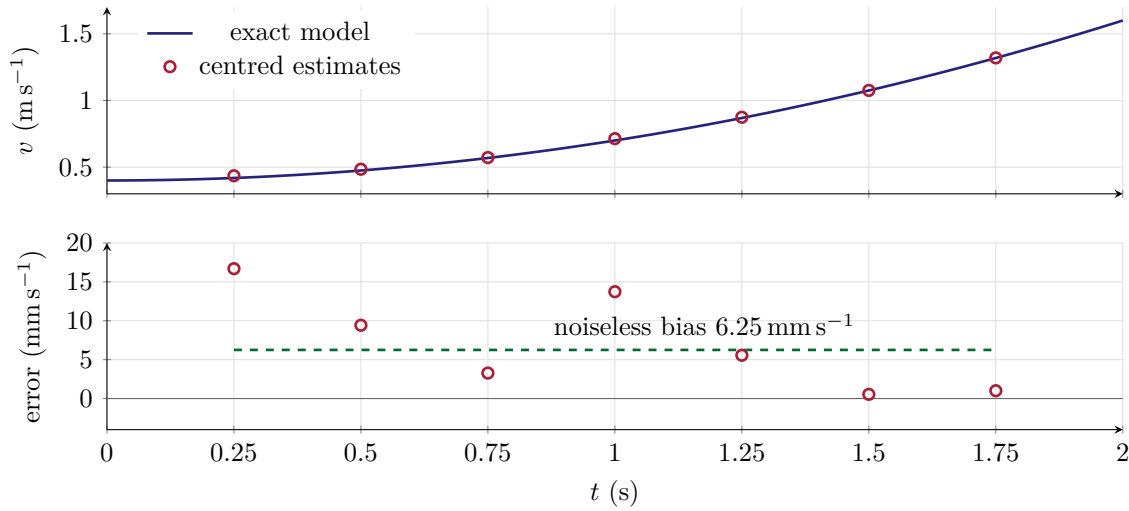


Figure 2.22: Centred derivatives of the constructed, rounded readings in Table 2.1, compared with the known cubic model. The lower panel plots estimate minus exact velocity at the same time. Scatter and deterministic finite-step bias are distinct. These are synthetic teaching values, not experimental observations; the dependence on step size is quantified in Fig. 2.25.

2.7.2 A reproducible differentiation calculation

The following standard-library Python function returns only interior centred estimates. Input times are in seconds and positions in metres; returned velocities are in metres per second. It checks lengths, finite values, increasing times and uniform spacing before using Eq. (2.45). The checks do not validate the instrument calibration or the physical model.

```

from math import isclose, isfinite

def centred_velocity(times, positions):
    t = [float(z) for z in times]
    x = [float(z) for z in positions]
    if len(t) != len(x) or len(t) < 3:
        raise ValueError("Need equal arrays with at least 3 samples")
    if not all(isfinite(z) for z in t + x):
        raise ValueError("All inputs must be finite")
    h = t[1] - t[0]
    steps = [t[i+1] - t[i] for i in range(len(t)-1)]
    if h <= 0 or any(s <= 0 for s in steps):
        raise ValueError("Times must increase")
    if not all(isclose(s, h, rel_tol=1e-10, abs_tol=0.0)
               for s in steps):
        raise ValueError("This formula requires a uniform grid")
    v = [(x[i+1] - x[i-1])/(2*h) for i in range(1, len(x)-1)]
    return t[1:-1], v

t = [0.25*i for i in range(9)]
x = [0.197651, 0.299079, 0.415375, 0.541290, 0.701388,
      0.898157, 1.138544, 1.435927, 1.798421]
ti, vi = centred_velocity(t, x)
print(vi[3])  # velocity estimate at t = 1 s: 0.713734

```

A useful verification sequence begins with constant position, for which every estimate must be zero; then a linear position, for which the exact constant velocity must be returned; and then a quadratic, for which the centred estimate at the middle time is exact in exact arithmetic. The cubic finally exposes finite-step bias. Adding the same offset to every position must leave the result unchanged, whereas reversing every position must reverse every velocity. These tests diagnose meaning, not just syntax.

The spacing test is a numerical convenience, not a definition of uniform sampling in an experiment. A timestamp record with missing frames should not be repaired by pretending that the remaining samples are equally spaced. Different formulas or a local fitted model are needed. Absolute clock values so large that adjacent times become indistinguishable in floating-point arithmetic should likewise be replaced by suitable elapsed times before computation.

2.7.3 Laboratory connection: a position record that can be audited

A possible low-energy investigation uses a marker moving along a short, safely constrained horizontal track, observed by a fixed camera or a suitable position sensor. This is a proposed method, not a claim that a particular laboratory has such equipment. The supplied synthetic data allow the mathematical investigation without any apparatus.

For a video-based record, place a calibrated length reference in the marker's motion plane and avoid an oblique view that changes scale along the track. Keep the camera fixed, choose the same marker feature in every frame, and retain actual time stamps rather than assuming a nominal frame rate. Record the positive direction, coordinate origin, conversion from image coordinates to metres, time origin and any omitted frames. A calibration marker at a different depth can introduce a systematic position-scale error even if individual pixels are selected consistently.

First plot position against time. A jump by several frame-to-frame distances may reveal a missed marker, an origin change or a timestamp error. Next estimate velocities and compare different spacings. Repeated measurements help assess scatter; they do not remove a persistent calibration bias. Where a constant-acceleration model is plausible, compare its predicted positions with

observations throughout the interval, not just with its fitted endpoints. A structured residual pattern can indicate an inadequate model, while isolated outliers can indicate a data-recording problem. Neither interpretation should be asserted without examining the record.

A useful report separates observations, assumptions and inferences. The observed quantities are positions and times. Velocity estimates depend on the differentiation rule. Acceleration inferred from a fitted quadratic depends on the constant-acceleration model. State which part of the record was used to choose a model and which independent observations, where available, were used to test it. Do not count agreement with the same points used to determine all model parameters as an independent validation.

2.7.4 Computational connection: accumulating sampled acceleration

Numerical integration provides a complementary check. On a grid, replace the acceleration over each short interval by a line between its endpoint values. The resulting trapezoidal update is

$$v_{i+1} = v_i + \frac{1}{2}(a_i + a_{i+1}) h_i, \quad (2.46)$$

$$x_{i+1} = x_i + \frac{1}{2}(v_i + v_{i+1}) h_i, \quad h_i = t_{i+1} - t_i > 0. \quad (2.47)$$

Both updates need initial conditions. The second is another approximation, even when the first happens to be exact. For the linear acceleration of Example 2.19, trapezoids integrate acceleration exactly at the sample times, but the resulting velocity is quadratic, so its trapezoids overestimate displacement.

For a uniform grid of spacing h over 0–2 s in that example, direct calculation gives the following comparison:

h (s)	numerical $x(2)$ (m)	error relative to 1.80 m
0.5000	1.825000000	+0.025000000 m
0.2500	1.806250000	+0.006250000 m
0.1250	1.801562500	+0.001562500 m
0.0625	1.800390625	+0.000390625 m

Each halving divides this error by four. For a quadratic velocity with $v'' = \kappa$, the trapezoid error over one interval is $\kappa h^3/12$; summing T/h intervals gives $\kappa T h^2/12 = 0.10 h^2$ in numerical SI units for $T = 2$ s. This analytic error is an independent check on the numerical updates, rather than merely another run of the same algorithm.

A finer grid can reduce approximation error for a prescribed smooth function. It does not remove a wrong initial velocity, an incorrect acceleration scale or missing physics. Nor does the favourable integration comparison prove that numerical *differentiation* of noisy data will improve in the same way. The advanced section addresses that distinction quantitatively.

2.7.5 Accumulation by trapezoids and by paired intervals

The same area viewpoint applies when the supplied rate is velocity rather than acceleration. With uniformly spaced readings $v_i = v(t_i)$ and $h = t_{i+1} - t_i$, the composite trapezoidal estimate is

$$\Delta x_{\text{trap}} = h \left[\frac{v_0 + v_n}{2} + \sum_{i=1}^{n-1} v_i \right]. \quad (2.48)$$

It integrates the piecewise-linear interpolation of the readings. If the velocity history itself is known to be linear on every interval, this area is exact. If the readings merely sample an unknown smooth curve, interpolation error remains.

A second method uses a quadratic over each pair of intervals. For three equally spaced velocities v_0, v_1, v_2 , the quadratic interpolant in $z = (t - t_0)/h$ is

$$p(z) = v_0 \frac{(z-1)(z-2)}{2} - v_1 z(z-2) + v_2 \frac{z(z-1)}{2}.$$

It equals the given velocity at $z = 0, 1, 2$. Integrating each coefficient over $0 \leq z \leq 2$ gives weights $1/3, 4/3, 1/3$; the factor $dt = h dz$ then gives Simpson's rule,

$$\int_{t_0}^{t_2} v(t) dt \approx \frac{h}{3} (v_0 + 4v_1 + v_2). \quad (2.49)$$

Adding disjoint pairs yields the composite rule when the number of intervals n is even. The weights alternate $4, 2, 4, 2, \dots, 4$ between the two endpoint weights of one. Uniform spacing and an even interval count are part of this particular formula. Applying its weights unchanged to uneven timestamps gives a different and generally unjustified calculation.

Example 2.22: Check accumulation against an exact velocity model

A velocity is prescribed by $v(t) = v_0 + ct^4$ on $0 \leq t \leq T$, with $v_0 = 0.300 \text{ m s}^{-1}$, $c = 0.0400 \text{ m s}^{-5}$ and $T = 2.00 \text{ s}$. Compare trapezoidal and composite Simpson accumulation as the spacing is halved.

Establish the independent reference. Polynomial integration gives

$$\Delta x = v_0 T + \frac{cT^5}{5} = 0.856000 \text{ m}.$$

The velocity remains positive, so this is also the travelled distance. At $h = 1 \text{ s}$, the three rate values are $0.300, 0.340$ and 0.940 m s^{-1} . Their trapezoidal area is 0.960000 m ; Simpson's area is $\frac{1}{3}(0.300 + 4 \cdot 0.340 + 0.940) \approx 0.866667 \text{ m}$. Neither approximation is exactly the quartic area.

Refine the actual sampling grid.

h (s)	trapezoidal (m)	Simpson (m)	Simpson error (m)
1.000	0.9600000000	0.8666666667	+0.0106666667
0.500	0.8825000000	0.8566666667	+0.0006666667
0.250	0.862656250	0.856041667	+0.000041667
0.125	0.857666016	0.856002604	+0.000002604

For this quartic, direct summation or integration of the local interpolation errors gives

$$\Delta x_{\text{trap}} - \Delta x = c \left(\frac{T^3 h^2}{3} - \frac{T h^4}{30} \right), \quad \Delta x_{\text{S}} - \Delta x = \frac{2cT h^4}{15}.$$

Each displayed expression has units of length. The leading trapezoidal term scales as h^2 ; Simpson's error here scales exactly as h^4 . Its error falls by sixteen on every halving, as Figure 2.23 confirms.

Interpret the comparison. This is convergence for newly evaluated exact model values, not the recovery of information by repeatedly interpolating a fixed noisy record. The two schemes also use different assumptions within each interval. Greater formal order is useful only where the required smoothness and data quality hold. A velocity corner, missing observation or systematic clock error requires separate treatment rather than an automatic promise of fourth-order accuracy.

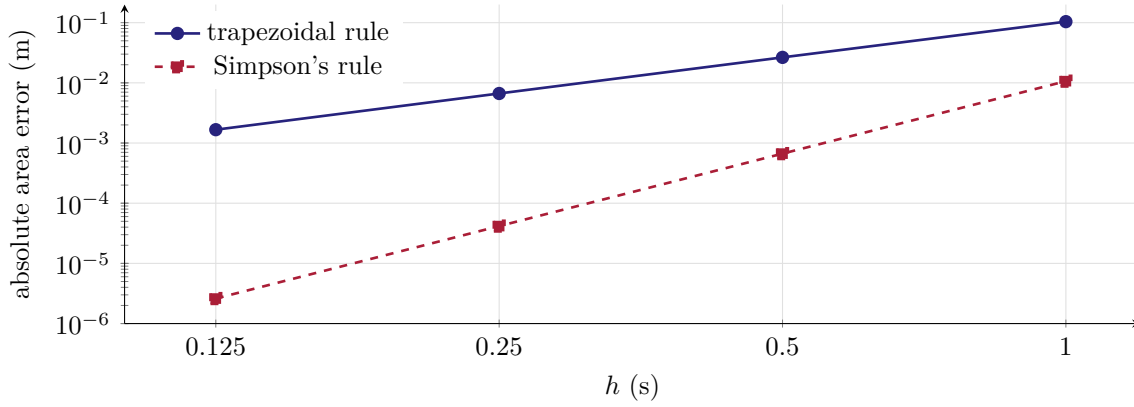


Figure 2.23: Convergence of displacement estimates for the prescribed quartic velocity in Example 2.22. Errors are measured against the independently integrated value 0.856000 m. Lines connect four calculated grids; they are not uncertainty intervals or measurements. Simpson error falls by sixteen on halving the spacing, whereas trapezoidal error approaches a factor of four.

2.7.6 Reconstructing a trajectory and inspecting its residuals

Differentiation of individual readings is one route to motion estimates. A fitted position model is another: choose a function, determine its parameters from the observations, then differentiate that function. This can reduce visibly irregular estimates, but it imports the chosen model's assumptions. A constant-acceleration quadratic and a cubic position model need not describe the same motion, even if both produce smooth curves.

Define the residual of a fitted position at sample i as

$$r_i = X_i - x_{\text{fit}}(t_i).$$

It retains the position units and sign. Positive residual means the observed coordinate lies above the fitted one. Plotting residuals on an expanded vertical scale can reveal model discrepancies hidden by the large overall traverse. A small residual is evidence of agreement at those observations; it is not a direct measurement of acceleration or proof that all intermediate motion has been resolved.

Example 2.23: Reconstruct the chapter's synthetic position record

Use all nine printed readings in Table 2.1. Fit a quadratic and a cubic by minimizing the unweighted sum $\sum_i r_i^2$. Compare their residuals, velocities and accelerations at $t = 1$ s against the known generating model.

State the models and calculation. In numerical SI values, write $x_2(t) = p + qt + rt^2$ and $x_3(t) = P + Qt + Rt^2 + St^3$. For either polynomial, setting the derivative of the sum of squares with respect to each coefficient to zero gives

$$\sum_i t_i^k \left(X_i - \sum_j c_j t_i^j \right) = 0 \quad \text{for every fitted power } k.$$

These linear equations determine the coefficients. The unweighted fit is appropriate for the stipulated equal per-reading position variance and exact times; other uncertainty models may require another objective. Coefficient units are m, m s⁻¹, m s⁻² and m s⁻³ for increasing powers.

Fitted position functions. The printed data give, retaining extra digits for reproducibility,

$$x_2(t) \simeq 0.223418461 + 0.182062297t + 0.296196485t^2,$$

$$x_3(t) \simeq 0.197400465 + 0.406312643t - 0.001152040t^2 + 0.099116175t^3.$$

The residual root-mean-square values $\sqrt{\sum r_i^2/9}$ are 0.019533 m and 0.001280 m, respectively.

These are descriptive residual sizes, not automatically standard deviations of the instrument noise. Fitting has used three or four parameters from the same nine readings.

Local rates. Differentiate each polynomial rather than reading slopes from a rounded graph. At $t = 1$ s the quadratic gives $v_2 \approx 0.774455 \text{ m s}^{-1}$ and $a_2 \approx 0.592393 \text{ m s}^{-2}$. The cubic gives $v_3 \approx 0.701357 \text{ m s}^{-1}$ and $a_3 \approx 0.592393 \text{ m s}^{-2}$. The independent generating-model values are $0.700000 \text{ m s}^{-1}$ and $0.600000 \text{ m s}^{-2}$. The equality of the two fitted accelerations at the middle time is a consequence of this symmetric sample grid, not a reason to declare both models equally accurate throughout.

Residual diagnosis and limits. Figure 2.24 shows a systematic sign-changing pattern for the quadratic, with residuals approaching 27 mm. The cubic residuals are much smaller and lack that dominant trend. This supports the cubic within this synthetic comparison; the generating law is known independently, so derivative errors can also be checked. In an experiment, that law would be unknown. A cubic has greater fitting freedom, and its lower in-sample residual is not independent validation. New observations, an independently justified shape, or a withheld portion of the record would be needed for a stronger empirical test. Neither polynomial should be extrapolated arbitrarily beyond 0–2 s.

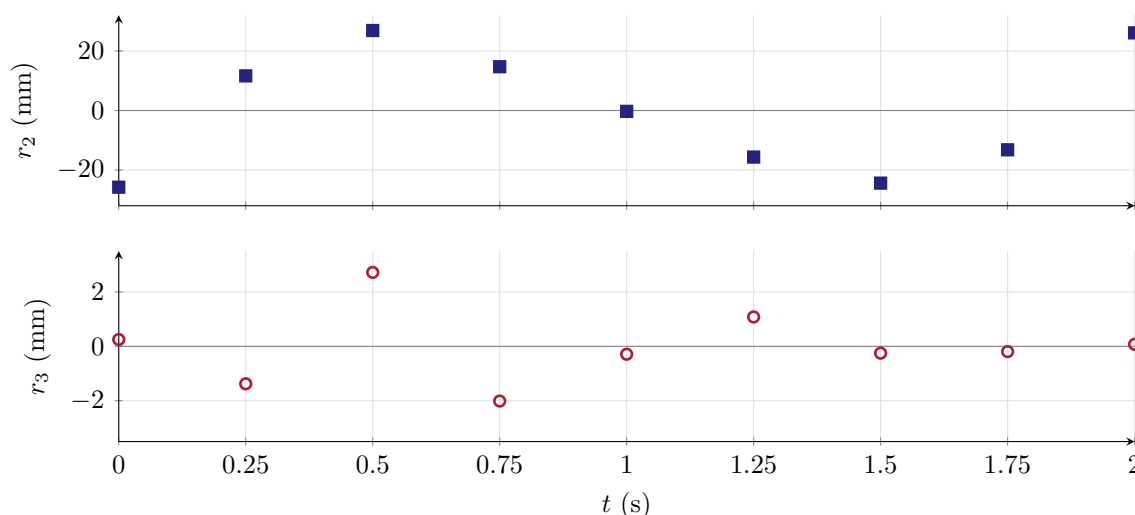


Figure 2.24: Position residuals r_2 (quadratic) and r_3 (cubic) for the same nine synthetic readings. The vertical scales differ so both residual patterns can be inspected. A quadratic leaves a much larger structured discrepancy than a cubic. Both fits were determined from these same points; this figure evaluates in-sample agreement rather than independent experimental validation.

2.7.7 Position calibration and clock calibration are distinct

Suppose an instrument reports $X = (1 + \delta)x + b$ and $\tau = (1 + \eta)t + \tau_0$, where b and τ_0 are constant origins, and δ and η are constant fractional scale errors. They need not fluctuate from reading to reading. Derivatives with respect to the reported clock then satisfy

$$\frac{dX}{d\tau} = \frac{1 + \delta}{1 + \eta} v, \quad \frac{d^2 X}{d\tau^2} = \frac{1 + \delta}{(1 + \eta)^2} a. \quad (2.50)$$

The chain rule supplies one clock factor for each derivative. The two additive origins cancel. Position scale enters velocity and acceleration once, while timing scale enters acceleration twice. These formulas compare corresponding physical events; confusing $\tau = 1$ s with $t = 1$ s would introduce a separate event-time mismatch.

For example, a position scale 1.0% high and a clock scale 0.20% high give velocity and acceleration multipliers 1.007984 and 1.005972. At an event with true $v = 0.700 \text{ m s}^{-1}$ and $a = 0.600 \text{ m s}^{-2}$, the

reported-clock derivatives are therefore $0.705589 \text{ m s}^{-1}$ and $0.603583 \text{ m s}^{-2}$. To first order in small fractional errors, the multipliers are $1 + \delta - \eta$ and $1 + \delta - 2\eta$, but the displayed numerical values use the exact ratios. Finer sampling and repeated readings do not remove these persistent biases. Calibration against an independent length and clock reference addresses a different question from estimating random point-to-point scatter.

2.8 Chapter summary and working formula sheet

Summary

Position requires a frame, axis, origin and time. Displacement is an endpoint difference; distance follows the whole route. Average velocity is displacement per elapsed time without any constant-acceleration assumption. Instantaneous velocity is the slope of position, and acceleration is the slope of signed velocity. Speeding up depends on the signs of both velocity and acceleration.

Constant-acceleration equations follow from accumulating a constant rate and applying initial conditions. A piecewise program needs a new elapsed time and transferred initial conditions at each phase. Free fall with $+y$ upward has $a_y = -g$; the model ends when an omitted interaction such as platform contact becomes relevant. General motion is reconstructed by signed integration, while distance uses the integral of speed. Discrete observations require an explicitly chosen numerical estimator and an honest uncertainty model.

Formula summary

Definitions on a fixed axis, $\Delta t > 0$

$$\Delta x = x_f - x_i, \quad \bar{v} = \frac{\Delta x}{\Delta t}, \quad \text{average speed} = \frac{\ell}{\Delta t}, \quad v = \frac{dx}{dt}, \quad a = \frac{dv}{dt}.$$

Speed is $|v|$. For $v \neq 0$, $va > 0$ means speeding up and $va < 0$ means slowing down.

Accumulation with known initial conditions

$$v(t) = v_0 + \int_{t_0}^t a(s) ds, \quad x(t) = x_0 + \int_{t_0}^t v(s) ds, \quad \ell = \int_{t_i}^{t_f} |v(s)| ds.$$

Constant acceleration on one phase, $\tau = t - t_0$

$$v = v_0 + a\tau, \quad x = x_0 + v_0\tau + \frac{1}{2}a\tau^2, \quad v^2 = v_0^2 + 2a\Delta x.$$

Then $\Delta x = (v_0 + v)\tau/2$. Across a reversal this is displacement, not distance. Squared velocities do not determine directions.

Vertical launch under constant downward acceleration

$$v_y = w - gt, \quad y = y_0 + wt - \frac{1}{2}gt^2, \quad t_{\text{apex}} = w/g \quad (w > 0).$$

Here $g > 0$, $+y$ is upward, and the apex formula presumes the free-fall phase continues that long.

Uniformly sampled positions: supporting numerical estimates

$$D_f x_i = \frac{x_{i+1} - x_i}{h}, \quad D_c x_i = \frac{x_{i+1} - x_{i-1}}{2h}.$$

The first estimates an interval average; the second is assigned to the middle time and requires neighbouring readings on both sides.

A reliable solution can answer four final questions: What is the model and its time domain? Which quantities are signed? Do the graph, equation and event conditions describe the same system? What independent check would expose an incorrect answer? The following enrichment strengthens the last question when the input is numerical data.

2.9 Advanced Topics for Technical and Engineering Students

The numerical velocity estimates in Fig. 2.22 were close to the known derivative, but not exact. A smaller interval seems an obvious improvement: after all, the derivative is defined by a limit. That intuition is correct for sufficiently smooth *exact* inputs and an appropriate scheme. It can fail for measurements because differencing subtracts imperfect positions and then divides by a small time interval.

The aim here is to build an error budget, distinguish deterministic approximation from random scatter, and decide how a usable spacing depends on both the model and the measurements. The extra tools are higher derivatives, a local Taylor expansion, expectation and variance, and a small least-squares fit. They are introduced below. This section is optional for students following the core mechanics pathway; its advanced exercises are identified separately.

2.9.1 Taylor expansion as a local model

Near a time t , a smooth function can be approximated by a polynomial in a small offset h . Its coefficients are fixed by requiring the polynomial and its derivatives to agree with the function at $h = 0$. This produces

$$x(t+h) = x(t) + v(t)h + \frac{a(t)}{2}h^2 + \frac{x^{(3)}(t)}{6}h^3 + \frac{x^{(4)}(t)}{24}h^4 + \dots, \quad (2.51)$$

where $x^{(n)}$ denotes the n th time derivative. The factorial denominators appear because differentiating h^n n times gives $n!$. For a polynomial of degree three, the displayed expansion terminates exactly after the cubic term. For a general sufficiently smooth function, omitted terms produce a remainder that must be small on the selected interval; the expansion is a local approximation, not a guarantee at arbitrarily large h .

Substitution into the forward difference yields

$$D_fx(t) = v(t) + \frac{1}{2}a(t)h + \frac{1}{6}x^{(3)}(t)h^2 + \mathcal{O}(h^3). \quad (2.52)$$

At the left endpoint, its leading error is proportional to h if $a(t) \neq 0$. The order symbol refers to the dependence on time spacing with the dimensional derivatives fixed; it is not a claim that a dimensionful time is intrinsically “small” without comparison to the motion’s variation time scale.

For the centred difference, expand $x(t-h)$ as well. Its odd powers change sign and its even powers do not. Subtracting the two expansions and dividing by $2h$ gives

$$D_cx(t) = v(t) + \frac{x^{(3)}(t)}{6}h^2 + \mathcal{O}(h^4). \quad (2.53)$$

The fourth-order remainder statement requires adequate higher smoothness, for example bounded fifth derivatives near the interval. The term proportional to ah has cancelled. A centred difference is therefore exact for a quadratic position in exact arithmetic, and its leading error for a general smooth nonquadratic trajectory is commonly second order in h .

For the cubic of Eq. (2.42), $x^{(3)} = 6\alpha$, so the bias is exactly

$$b_c(h) = D_cx - v = \alpha h^2. \quad (2.54)$$

At $h = 0.25$ s, $b_c = 0.00625$ m s⁻¹, the dashed reference in Fig. 2.22. Its sign is positive because the average of the convex quadratic velocity over a symmetric interval exceeds the velocity at the centre. This geometric explanation checks the Taylor calculation without repeating its algebra.

An abrupt switch in acceleration can invalidate a smooth Taylor approximation across the switch. A centred stencil that straddles a velocity corner samples two different phases. Reducing h away from the corner and reducing it across the corner are different tests. Numerical order statements always require a domain where the assumed smoothness applies.

2.9.2 How measurement errors enter a difference

Write a measured position as $X_i = x(t_i) + \epsilon_i$. In the first error model, the additive errors have mean zero, equal standard deviation u_x , and are independent between samples. Mean zero means that the average error over hypothetical repetitions is zero, not that every record has zero error. Variance is the mean squared deviation from the mean; its positive square root is the standard deviation.

For constants p_j and independent errors,

$$\text{Var}\left(\sum_j p_j \epsilon_j\right) = \sum_j p_j^2 u_x^2.$$

The cross terms vanish because independent zero-mean errors have zero mean product. Thus the variance of a difference is a sum of variances, not their difference. A positive and a negative coefficient do not make independent uncertainties cancel.

The random contribution to a forward estimate is $(\epsilon_{i+1} - \epsilon_i)/h$; to a centred estimate it is $(\epsilon_{i+1} - \epsilon_{i-1})/(2h)$. Consequently,

$$u_{v,f} = \frac{\sqrt{2}u_x}{h}, \quad u_{v,c} = \frac{u_x}{\sqrt{2}h}. \quad (2.55)$$

These describe noise scatter, not finite-step bias. At $u_x = 0.002 \text{ m}$ and $h = 0.25 \text{ s}$, the centred noise standard deviation is 0.00566 m s^{-1} . Halving h doubles that standard deviation if the per-reading uncertainty remains fixed. More closely spaced samples are not automatically more informative about a derivative.

Real errors need not be independent. If the two readings used by a centred difference have equal marginal variance u_x^2 and correlation coefficient ρ , then their covariance is ρu_x^2 and

$$u_{v,c}^2 = \frac{u_x^2(1 - \rho)}{2h^2}. \quad (2.56)$$

A perfectly common additive zero offset has $\rho = 1$ for that error component and cancels. Independent components still remain. A common multiplicative scale error is different: if $X_i = (1 + \delta)x_i + b$, the offset b cancels but every estimated velocity is multiplied by $1 + \delta$. Neither repeated readings nor a smaller spacing removes that scale error. Timestamp uncertainty, drift and position-dependent calibration require additional terms or a different model and are not included in Eq. (2.55).

2.9.3 A quantitative compromise between bias and noise

For the cubic trajectory, a centred estimate minus the true velocity is a deterministic bias αh^2 plus a zero-mean random contribution. Its mean-squared error is the squared bias plus the variance:

$$\text{MSE}(h) = \alpha^2 h^4 + \frac{u_x^2}{2h^2}, \quad \text{RMSE}(h) = \sqrt{\text{MSE}(h)}. \quad (2.57)$$

The cross term has zero expectation under the assumed noise model. RMSE is a performance measure averaged over hypothetical repetitions, not a guaranteed bound on the error of one estimate and not a confidence interval.

To find the best spacing under these particular assumptions, differentiate the MSE:

$$\frac{d \text{MSE}}{dh} = 4\alpha^2 h^3 - \frac{u_x^2}{h^3}.$$

For $h > 0$ and $\alpha \neq 0$, the stationary point is

$$h_{\text{opt}} = \left(\frac{u_x}{2|\alpha|} \right)^{1/3}. \quad (2.58)$$

The second derivative is positive, so this is the unique unconstrained minimum. Units check the result: u_x/α has units s^3 . Actual data availability, proximity to endpoints and the domain where the model is smooth can restrict the permitted spacings. The optimum is not a universal sample rate for all instruments.

Example 2.24: Choosing a differentiation interval

At $t = 1\text{ s}$ in the cubic model, take $\alpha = 0.10\text{ m s}^{-3}$ and independent position errors with $u_x = 0.002\text{ m}$. Compare $h = 0.50, 0.25$ and 0.125 s , and find the continuous optimum.

Bias and noise. Direct substitution gives

h (s)	bias	noise standard deviation	RMSE
	all velocity errors in m s^{-1}		
0.500	0.025000	0.002828	0.025159
0.250	0.006250	0.005657	0.008430
0.125	0.001563	0.011314	0.011421

The smallest listed spacing has the smallest bias but not the smallest expected total error.

Optimum. Equation (2.58) gives $h_{\text{opt}} \simeq 0.2154\text{ s}$, with bias 0.004642 m s^{-1} and RMSE 0.008039 m s^{-1} . The centred stencil uses positions at $t - h$ and $t + h$, a total span $2h$, not h .

Check and interpretation. At the optimum, the noise variance is twice the squared bias. Substitution in Eq. (2.57) confirms a lower error than at the three listed spacings. As $u_x \rightarrow 0$, this particular optimum tends to zero; with exact cubic data, smaller spacing reduces truncation error until numerical round-off matters. Figure 2.25 displays the two competing contributions. For the available 0.25 s grid, one cannot manufacture readings at h_{opt} simply by interpolating the existing noisy data and treating them as independent observations.

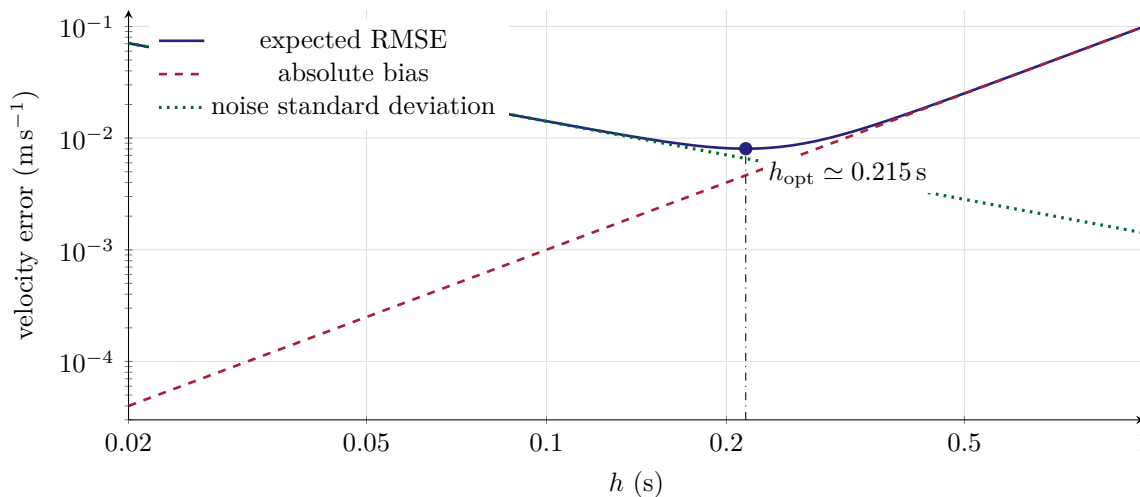


Figure 2.25: Analytically calculated error budget for a centred velocity estimate at $t = 1\text{ s}$ in the cubic model, with independent position errors of standard deviation $u_x = 0.002\text{ m}$. The same per-reading uncertainty is assumed at every spacing. Root-mean-square error (RMSE) combines squared bias and variance; it is not an error bar for this chapter's one noise realization. The optimum is model- and noise-dependent.

2.9.4 Why a wider fitted window can be worse

A local fit can use more samples to reduce noise, but it also averages over a wider portion of a changing motion. Consider five equally spaced readings X_k at offsets kh from a central time, for $k = -2, -1, 0, 1, 2$. Approximate position locally by $p + qkh$, where q is the fitted velocity. Choose

p and q to minimize the sum of squared residuals,

$$S(p, q) = \sum_{k=-2}^2 (X_k - p - qkh)^2.$$

Setting the two partial derivatives to zero gives the normal equations. Symmetry yields $\sum k = 0$, so p is the mean of the five positions and

$$q = \frac{\sum kX_k}{h \sum k^2} = \frac{-2X_{-2} - X_{-1} + X_1 + 2X_2}{10h}. \quad (2.59)$$

The zero central coefficient reflects symmetry, not an instruction to discard the centre reading from all fitting tasks. This estimator is exact for a linear or quadratic position at the central time.

For independent equal-variance position errors, its noise variance is the sum of squared weights:

$$u_q^2 = \frac{(4 + 1 + 1 + 4)u_x^2}{100h^2} = \frac{u_x^2}{10h^2}. \quad (2.60)$$

This is smaller than the three-point centred variance for the same adjacent-sample spacing h . However, for the cubic trajectory, the deterministic bias is

$$b_q = \alpha h^2 \frac{\sum k^4}{\sum k^2} = \frac{34}{10} \alpha h^2 = 3.4 \alpha h^2. \quad (2.61)$$

The wider fit sees more of the curvature of the velocity. Lower noise and lower total error are not equivalent.

Example 2.25: A quieter estimate with a larger total error

Compare the three-point centred estimate with the five-point straight-line fit at $t = 1$ s for $h = 0.25$ s in the same cubic and independent-noise model.

Three-point estimate. The preceding example gives bias 0.006250, noise standard deviation 0.005657 and RMSE 0.008430 m s⁻¹.

Five-point estimate. Its bias is $3.4(0.10)(0.25)^2 = 0.021250$ m s⁻¹. Its noise standard deviation is

$$u_q = \frac{0.002}{\sqrt{10}(0.25)} \simeq 0.002530 \text{ m s}^{-1}.$$

Combining them gives RMSE $\simeq 0.021400$ m s⁻¹, larger than the centred estimate despite visibly smaller noise scatter.

Checks and limits. Inserting exact cubic positions in Eq. (2.59) gives $q = 0.721250$ m s⁻¹, confirming the bias relative to the true 0.700000. This comparison holds adjacent-sample spacing fixed, so the five-point fit spans $4h = 1$ s rather than $2h = 0.5$ s. Holding the total window width fixed would be a different comparison. A higher-degree fit can remove this particular cubic bias but changes noise sensitivity and requires its own checks.

2.9.5 Acceleration estimates and an investigation strategy

Differencing again is even more sensitive to position errors. The familiar symmetric second difference is

$$D_2x_i = \frac{X_{i+1} - 2X_i + X_{i-1}}{h^2}. \quad (2.62)$$

Taylor expansion gives $D_2x = a + x^{(4)}h^2/12 + \mathcal{O}(h^4)$ with sufficient smoothness. For the cubic it is exact without noise. Independent position errors instead produce

$$u_{a,2} = \frac{\sqrt{6}u_x}{h^2}, \quad (2.63)$$

because the squared weights are 1, 4, 1. At $u_x = 0.002 \text{ m}$ and $h = 0.25 \text{ s}$, this is 0.0784 m s^{-2} . Halving h multiplies it by four. A jagged acceleration graph can therefore be an artefact of differentiation even when the original position record looks smooth.

A reproducible computational investigation can now separate four questions. First, use exact polynomial data and verify the predicted powers of h without noise. Second, add errors from a declared distribution with a recorded seed, calculate the same estimates, and compare their signed errors. Third, repeat with independently generated records to estimate bias and variance; do not confuse one favourable realization with expected performance. Fourth, add a common offset or a scale error and check which effects cancel. No new experimental data should be claimed in this synthetic exercise.

For an implementation test, the ensemble average of squared errors should approach Eq. (2.57) as the number of independent records grows, but finite ensembles fluctuate. It is more informative to state a numerical tolerance and the number of repetitions than to demand exact equality. The supplementary verification script performs this comparison for the stated model; readers do not need to execute any script to render this chapter.

The practical conclusion is not “differentiate as little as possible” or “always smooth first.” It is to choose an estimator and interval for the physical variation one intends to resolve, quantify the relevant uncertainty components, and test both bias and scatter against a defensible benchmark. A polished curve that erases a genuine reversal is not an improvement; a noisier curve with an honest uncertainty estimate may be more informative.

2.9.6 Independent position readings can produce correlated rate estimates

Independence belongs to a specified error model and specified variables. It is not inherited automatically by every quantity computed from them. Adjacent finite-difference estimates may share position readings; their errors are then correlated even if the original position errors are independent. This matters when deciding whether a zigzag derivative graph is evidence of a real oscillation or partly a consequence of the estimator.

For independent zero-mean position errors of variance u_x^2 , define the random parts of the centred velocity and second-difference acceleration as

$$e_i^v = \frac{\epsilon_{i+1} - \epsilon_{i-1}}{2h}, \quad e_i^a = \frac{\epsilon_{i+1} - 2\epsilon_i + \epsilon_{i-1}}{h^2}.$$

The covariance of two zero-mean errors is the expectation of their product. Expanding that product leaves only terms involving the same original ϵ_j , because distinct position errors are independent and mean zero. For centred velocities, estimates separated by one sample use disjoint original errors, but those separated by two share one with opposite signs:

$$\text{Cov}(e_i^v, e_{i+1}^v) = 0, \quad \text{Cov}(e_i^v, e_{i+2}^v) = -\frac{u_x^2}{4h^2}.$$

Division by the common variance $u_x^2/(2h^2)$ gives correlations zero and $-1/2$. A high error at one sample tends statistically to accompany a low error two samples later. This is an ensemble statement; it does not predict the sign of every particular observed pair.

The acceleration estimates overlap more strongly. The shared weight products for one-sample separation are $(-2)(1) + (1)(-2) = -4$; for two-sample separation the only shared product is 1. Hence

$$\text{Corr}(e_i^a, e_{i+1}^a) = -\frac{2}{3}, \quad \text{Corr}(e_i^a, e_{i+2}^a) = \frac{1}{6}. \quad (2.64)$$

Both correlations vanish at larger separations under this independent-input model. Figure 2.26 summarizes the finite support of these estimator-induced correlations. Deterministic truncation bias is excluded from these random-error correlations.

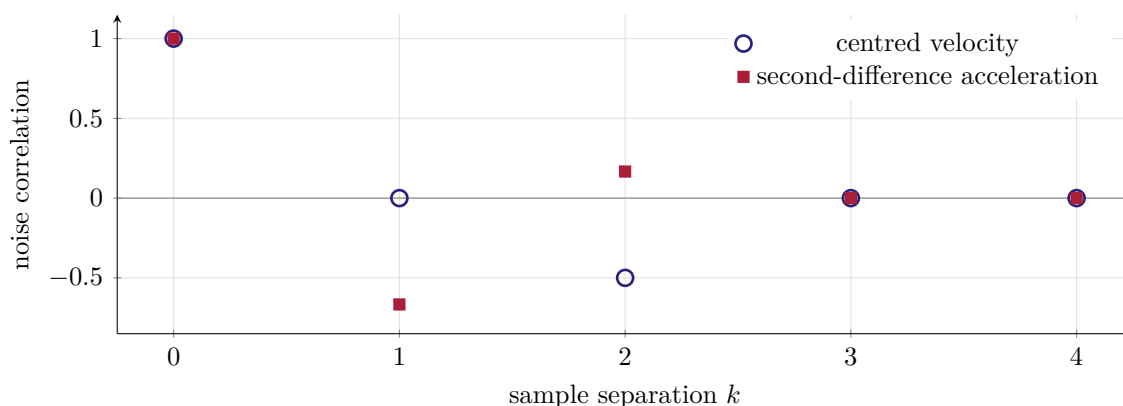


Figure 2.26: Correlations between the random errors of rate estimates separated by k grid intervals, calculated from independent equal-variance position errors. Shared input readings create correlations in the outputs. Markers coincide at $k = 0, 3, 4$; both values are identical there. A negative correlation is a property of repeated hypothetical records, not a guarantee of alternating signs in one record.

This calculation also clarifies why averaging successive derivative estimates requires care. One cannot apply an independent-error reduction proportional to the square root of the estimate count without including their covariances. Conversely, shared errors sometimes cancel in an accumulated quantity. For example, summing forward differences multiplied by h gives a telescoping displacement $X_n - X_0$; its uncertainty depends on the two endpoint errors rather than treating all interval differences as independent.

Persistent drift needs a different model. An added position error $b(t) = b_0 + b_1 t$ contributes no second difference, but contributes the velocity bias b_1 . A quadratic drift $b_2 t^2$ contributes the constant acceleration bias $2b_2$. Smooth-looking errors can therefore survive differentiation as smooth-looking false motion. These deterministic effects are not described by the independent random-noise standard deviations.

2.9.7 Acceleration reconstruction and a fourth-power noise penalty

The h^{-2} noise standard deviation of the second difference becomes an h^{-4} variance. This steep dependence makes choosing the differentiation interval especially important for acceleration. The known synthetic trajectory offers a direct check before interpreting a real record.

Example 2.26: Read acceleration without hiding the differentiation noise

Compute second-difference accelerations from the printed cubic-trajectory readings in Table 2.1, and compare with their generating acceleration. Then test the bias-noise compromise on a quartic position model, where finite-step acceleration bias is nonzero.

Use the actual three readings. At $t = 1$ s and $h = 0.25$ s,

$$D_2 X(1) = \frac{0.898157 - 2(0.701388) + 0.541290}{0.25^2} = 0.586736 \text{ m s}^{-2}.$$

The generating acceleration is $0.600000 \text{ m s}^{-2}$, so the error is $-0.013264 \text{ m s}^{-2}$. A cubic position has no noiseless second-difference bias, so this particular discrepancy comes from the stipulated position errors and their rounding. Across all seven interior times the estimates are

t (s)	D_2X (m s ⁻²)	exact a (m s ⁻²)	signed error (m s ⁻²)
0.25	0.237888	0.150000	+0.087888
0.50	0.153904	0.300000	-0.146096
0.75	0.546928	0.450000	+0.096928
1.00	0.586736	0.600000	-0.013264
1.25	0.697888	0.750000	-0.052112
1.50	0.911936	0.900000	+0.011936
1.75	1.041776	1.050000	-0.008224

The error at 0.50 s is much greater in magnitude than at 1 s, although the same rule and spacing were used. The assumed independent-error standard deviation is $\sqrt{6}(0.002)/0.25^2 \approx 0.078384 \text{ m s}^{-2}$. It describes an ensemble scatter, not a bound that every displayed error must satisfy. Figure 2.27 retains the irregularity rather than smoothing it away.

Introduce a nonzero truncation bias. For the separate stipulated quartic model $x(t) = x_0 + v_0 t + dt^4$ with $d = 0.0300 \text{ m s}^{-4}$, the acceleration is $12dt^2$. Direct expansion of $(t+h)^4 - 2t^4 + (t-h)^4$ gives $12t^2h^2 + 2h^4$. Thus

$$D_2x - a = 2dh^2 = \beta h^2, \quad \beta = 0.0600 \text{ m s}^{-4}.$$

With the same independent per-reading uncertainty $u_x = 0.002 \text{ m}$, bias and random error combine as

$$\text{MSE}_a(h) = \beta^2 h^4 + \frac{6u_x^2}{h^4}.$$

Differentiation with respect to $h > 0$ gives $4\beta^2 h^3 - 24u_x^2 h^{-5} = 0$. Its unique positive minimum is

$$h_{\text{opt}} = \left(\frac{6u_x^2}{\beta^2} \right)^{1/8} \approx 0.53455 \text{ s}.$$

The ratio inside the eighth root has units s^8 , as required. At this spacing the bias and noise standard deviation are equal, both $0.017145 \text{ m s}^{-2}$, and RMSE is $0.024246 \text{ m s}^{-2}$. At $h = 0.25 \text{ s}$ the bias is smaller, $0.003750 \text{ m s}^{-2}$, but the noise scatter is $0.078384 \text{ m s}^{-2}$, giving RMSE $0.078473 \text{ m s}^{-2}$.

State what the optimum means. The optimum pertains to the quartic model and constant per-reading uncertainty, not the chapter's cubic record. The cubic has zero second-difference truncation bias, so this particular quartic optimum must not be transferred to it. The differentiation stencil spans $2h$, and all needed observations must exist within a valid model interval. Interpolation of a fixed noisy record does not create independent readings at h_{opt} . Figure 2.28 shows the steep noise penalty for small spacings under the explicitly stated assumptions.

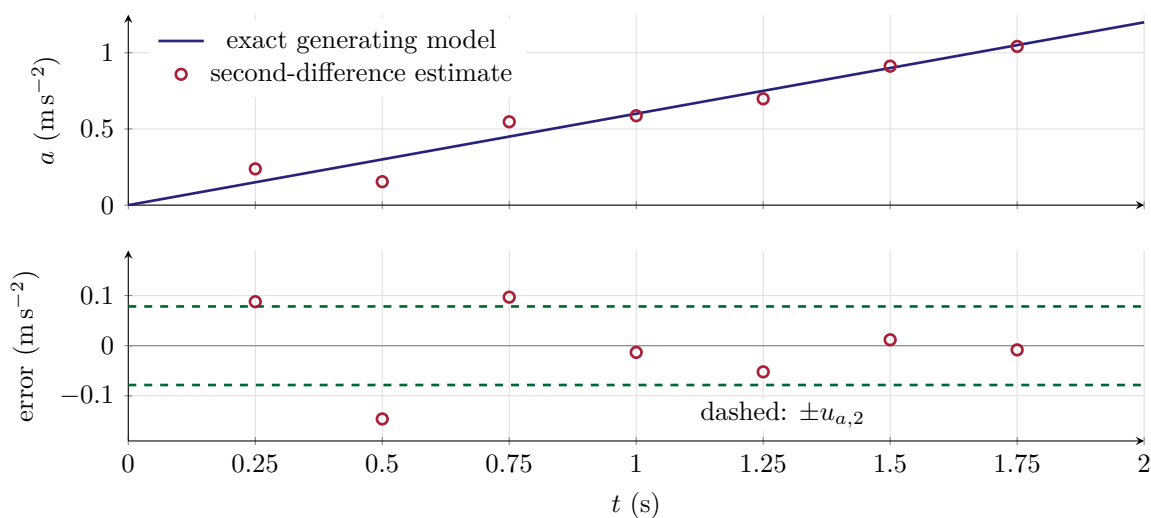


Figure 2.27: Second-difference acceleration from the printed synthetic positions, assigned to the same interior times as the exact benchmark. Lower-panel errors are estimate minus exact acceleration. Dashed lines show the independently assumed noise standard deviation $\pm 0.078384 \text{ m s}^{-2}$; they are not confidence limits or absolute bounds. No second difference is reported at either endpoint.

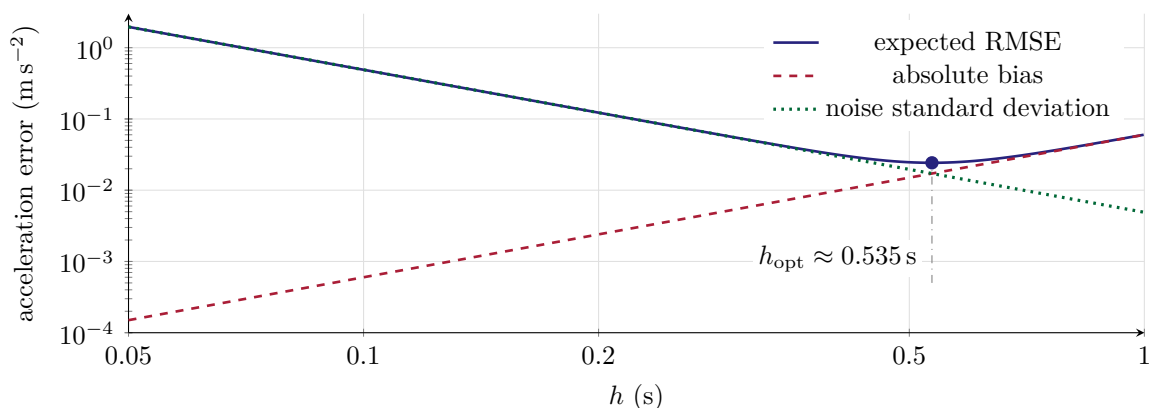


Figure 2.28: Acceleration error budget for the separate quartic position model, $d = 0.0300 \text{ m s}^{-4}$, and independent equal-variance position noise, $u_x = 0.002 \text{ m}$. The second-difference bias grows as h^2 while noise standard deviation grows as h^{-2} . The plotted optimum applies to this model and uncertainty assumption, not to every acceleration record.

2.10 Exercises and problems

Use SI units and $g = 9.81 \text{ m s}^{-2}$ for ideal vertical motion unless stated otherwise. Choose a positive direction, state the model interval, and retain signs until a magnitude is requested. Numerical positions and times in stipulated mathematical exercises may be treated as exact inputs for checking the model; they are not claims of experimental precision. Problems in group D use the optional advanced section. Complete reasoned solutions to every problem follow. Open investigations include a worked exemplar and explicit assessment criteria.

A. Coordinates, rates and representations

2.1 Relabelling a return motion

A marker is at 2.40, 1.10 and 3.00 m at times 0, 2.00 and 5.00 s, respectively, with monotone motion on each leg. Find displacement, distance, average velocity and average speed. Repeat the displacement calculation with the origin at the first position, then with the original axis reversed. Explain what changes physically. Finally, change the last position to 2.40 m at the same final time

and determine whether zero displacement implies zero distance.

2.2 What three samples leave unknown

The only known positions are $x = 0, 0.40$ and 0.10 m at $t = 0, 1$ and 2 s. No monotonicity information is supplied. What displacement and minimum possible distance are determined? Is 0.70 m necessarily the full distance? Describe an alternative motion consistent with the readings.

2.3 Equal times versus equal distances

A stage moves positively at 0.20 and 0.60 m s^{-1} . In program A it spends 2.0 s at each speed. In program B it covers 0.30 m at each speed. Find the full-interval average velocity for each. Why does the arithmetic average of the speeds work in only one of these programs?

2.4 A negative average during initially positive motion

For $0 \leq t \leq 4$ s let $x = 1 + 0.60t - 0.20t^2$ in numerical SI values. Find the reversal event, displacement, distance, average velocity, average speed and instantaneous velocity at 1 s. Draw the endpoint secant and indicate whether its slope is positive or negative. Diagnose a report of distance 0.80 m obtained by taking the absolute endpoint displacement.

2.5 Four signs, four interpretations

Classify direction and speed change for these four (v, a) pairs, in SI units:

$$(+0.80, +0.20), \quad (+0.80, -0.20), \quad (-0.80, +0.20), \quad (-0.80, -0.20).$$

Draw velocity and acceleration arrows and distinguish their meanings. An object has $v = 0$ and $a = +0.20 \text{ m s}^{-2}$ at an interior instant of a smooth constant-acceleration phase. Is it at rest for a finite interval?

2.6 Zero velocity without a reversal

Compare $x_1 = \beta t^3$ and $x_2 = -\gamma t^2$ near $t = 0$, with positive constants β and γ carrying suitable SI units. Both have zero velocity at zero. Which reverses? Explain using signs, not just the existence of a root.

2.7 An incompatible set of graphs

A report labels its position graph $x = 1 + t^2$ in numerical SI units for $0 \leq t \leq 2$ s, but labels its velocity graph $v = 2 \text{ m s}^{-1}$ and its acceleration graph $a = 0$. Diagnose and repair the report while preserving the position function. At what one instant does its incorrect velocity happen to match?

B. Constant-acceleration phases and vertical events

2.8 The phase begins at a nonzero clock time

A phase begins at $t_0 = 3$ s with $x_0 = 0.50$ m, $v_0 = -0.40 \text{ m s}^{-1}$ and constant $a = +0.20 \text{ m s}^{-2}$. It lasts 4 s. Find the turning time and position, final position and velocity, displacement and distance. Explain why using t instead of $t - t_0$ fails.

2.9 Repair a local-time error

For the lift in Example 2.9, a student writes the stopping-phase position as $x = 9.60 + 2.40t - 0.40t^2$ in numerical SI values. Test this expression at the stopping-phase start, repair it, and find the actual x and v at $t = 6$ s.

2.10 A new three-phase program

A horizontal stage starts from rest, accelerates at 0.60 m s^{-2} for 2 s, cruises for 4 s, then slows at -0.30 m s^{-2} to rest. Find phase durations, displacements, total duration and total distance. State what model applies after the stop.

2.11 An average acceleration that loses timing information

During $0 \leq t \leq 3\text{ s}$, a marker starts from rest with $a(t) = \kappa t$, where $\kappa = 0.40\text{ m s}^{-3}$. Compute its final velocity, true displacement and interval-average acceleration. Compare the displacement with the prediction obtained by treating that average acceleration as constant.

2.12 Does the kinematic stopping model fit the clearance?

A hypothetical stage enters a braking phase with $v_0 = 0.75\text{ m s}^{-1}$. The model assumes an immediate constant acceleration -1.50 m s^{-2} and permits 0.160 m further travel. Is the modelled stop inside that distance? Find the minimum constant braking magnitude that would suffice with zero delay. This is an ideal kinematic calculation, not a machine-safety specification.

2.13 A traverse with room for cruising

An ideal rest-to-rest traverse has $D = 0.50\text{ m}$, acceleration-magnitude limit $A = 1.0\text{ m s}^{-2}$ and speed cap $V = 0.50\text{ m s}^{-1}$. For the capped fastest monotone program described in the text, find the ramp and cruise times and the total duration. Verify the distance from the velocity areas.

2.14 Meeting is not velocity matching

Two markers on parallel guides have $x_A = 0.20 + 0.10t$ and $x_B = 0.05t^2$ in numerical SI values, valid for $0 \leq t \leq 5\text{ s}$. Find every allowed longitudinal encounter and the two velocities there. Reject roots only after comparing them with the time domain.

2.15 Unequal launch and arrival heights

An ideal object is launched upward at 4.00 m s^{-1} from $y_0 = 2.00\text{ m}$ and arrives at $y = 0$. Find the apex time and height, arrival time and signed velocity. A student uses $2v_0/g$ for the arrival time. Explain the error, recover the return-time formula when the arrival height equals the launch height, and write the same trajectory using a downward-positive coordinate measured from release.

2.16 A drop and a change of acceleration scale

An ideal object is released from rest relative to the laboratory and falls 0.80 m with $+y$ upward. Find the elapsed time and arrival velocity. How would these change if the constant downward acceleration magnitude were doubled, with the distance and initial condition unchanged?

2.17 Two passages through a checkpoint

An ideal upward launch starts at $y = 0$ with $v_0 = 4.00\text{ m s}^{-1}$. Find the times and signed velocities at $y = 0.600\text{ m}$. Would a checkpoint at 1.00 m be reached? Assume the free-fall phase lasts long enough for the stated passages.

2.18 What can three heights identify?

Three stipulated height readings are $(t, y) = (0, 0.3000), (0.1000, 0.5000), (0.2000, 0.6019)$ in SI units. Treat the readings as exact for this algebraic exercise and assume one constant-acceleration phase. Infer v_0 and a_y . Does exact agreement with these three points independently establish free fall or constant acceleration between them?

C. Accumulation, measurement and computation**2.19 Zero displacement is not zero travel**

For $0 \leq t \leq 6\text{ s}$ a marker has $v = 0.30 - 0.10t$ in numerical SI values and $x(0) = 0.50\text{ m}$. A student integrates v , finds zero displacement and reports zero distance. Repair the conclusion, giving the turning position, final position, distance and average speed.

2.20 A reversal under time-dependent acceleration

Let $a(t) = (0.20 \text{ m s}^{-3})t$, $v(0) = -0.40 \text{ m s}^{-1}$ and $x(0) = 0.70 \text{ m}$ for $0 \leq t \leq 3 \text{ s}$. Derive $v(t)$ and $x(t)$, find the reversal, and compute full-interval displacement and distance.

2.21 Missing initial conditions

You are told only that $a(t) = 0$ for $0 \leq t \leq 10 \text{ s}$. What can be concluded about $v(t)$ and $x(t)$? State the additional numerical information required to determine position uniquely, and give two distinct motions consistent with the supplied acceleration.

2.22 Units belong to polynomial coefficients

A numerical fit is written $x = 0.40 + 0.30t - 0.05t^3$, with x in metres and t in seconds. Give the units of all three coefficients, derive v and a , and find $a(2 \text{ s})$. Explain why labelling the cubic coefficient 0.05 m s^{-2} is inconsistent.

2.23 Compute and diagnose two finite differences

Using Table 2.1, calculate a centred velocity at 0.50 s and a forward estimate assigned to $t = 0$. Compare each with the exact velocity at its assigned time. A student concludes that taking half the spacing must always improve these estimates. What is missing from that conclusion? Also evaluate the centred difference at 0.50 s by hand using the exact cubic positions at 0.25 and 0.75 s , and separate its finite-step bias from the noise contribution.

2.24 Position scale and clock scale

Suppose reported positions and times satisfy $X = 1.02x + 0.10 \text{ m}$ and $T = 0.99t + 3.00 \text{ s}$ for the same events. Find the multiplicative change in any corresponding interval-average velocity. Which effects cancel, and which remain in the instantaneous limit?

2.25 Check a numerical integration refinement

Apply the two trapezoidal updates to the cubic-model acceleration $a = 0.60t$ on $0 \leq t \leq 2 \text{ s}$ with the stated $v_0 = 0.40 \text{ m s}^{-1}$ and $x_0 = 0.20 \text{ m}$. Predict and, optionally, compute $x(2)$ for uniform spacings $h = 0.20$ and 0.10 s . Use the analytic trapezoid-error relation to check the results independently.

2.26 A gap in the timestamp record

A record has times $0, 0.10, 0.20$ and 0.40 s and positions $0.20, 0.24, 0.28$ and 0.36 m . What are the three interval-average velocities? A program instead assigns a duration of 0.10 s to the last interval. What false result does it produce, and why should the uniform-grid function reject this record?

2.27 Design an interpretable motion record

Propose a short, low-energy horizontal-motion investigation using either provided synthetic data or suitable supervised apparatus. Explain how you would establish length and time scales, identify a reversal, test a constant-acceleration model and report uncertainty. Distinguish what would be measured from what would be inferred; do not assume equipment availability.

2.28 A unit test that catches a factor of two

A centred-difference routine returns $(x_{i+1} - x_{i-1})/h$ on a uniform grid of adjacent spacing h . Explain the error. Propose one test that detects it and one legitimate invariance test that would not detect it by itself.

D. Advanced challenges

2.29 Forward, centred and midpoint interpretations

For the exact cubic of Eq. (2.42), evaluate the forward difference at $t = 1$ s and the centred difference there with $h = 0.20$ s. Compare the forward value also with the exact velocity at 1.10 s. Show why the midpoint interpretation is exact for constant acceleration but not for this cubic.

2.30 Noise in first and second differences

Independent position errors have standard deviation $u_x = 0.003$ m and uniform time spacing $h = 0.10$ s. Find the noise standard deviations of forward velocity, centred velocity and symmetric second-difference acceleration. Give the factors by which they change when h is halved at fixed u_x . Do not include truncation bias in these noise quantities.

2.31 Changing the uncertainty changes the optimum

For a cubic trajectory with $\alpha = 0.20$ m s⁻³ and independent position errors $u_x = 0.001$ m, find the unconstrained optimum centred-difference spacing and its RMSE. How do those two quantities scale if u_x is reduced by a factor of eight at fixed α ? State the practical domain qualification.

2.32 A correlated position-error component

The two positions in a centred difference have equal standard uncertainty 0.002 m, correlation coefficient $\rho = 0.75$, and spacing $h = 0.25$ s from the centre. Find the velocity uncertainty from this component and compare it with the independent case. Explain why a common additive offset and a common scale error behave differently.

2.33 Evaluate the five-point fit on the supplied record

Use the five readings at 0.50, 0.75, 1.00, 1.25 and 1.50 s in Table 2.1 to evaluate the five-point line-fit velocity at 1 s. Compare its actual error with the three-point centred error. Separate the known cubic bias from the noise contribution in the five-point result.

2.34 Differentiation can create correlations

Start from mutually independent equal-variance position errors ϵ_i with variance u_x^2 . The noise part of a centred velocity estimate is $\eta_i = (\epsilon_{i+1} - \epsilon_{i-1})/(2h)$. Compute $\text{Cov}(\eta_i, \eta_{i+1})$ and $\text{Cov}(\eta_i, \eta_{i+2})$, and the correlation coefficient in the second case. Why must the output velocities not all be treated as independent in a later fit?

2.35 When the smooth-error formula fails

Consider $x(t) = \beta t|t|$ near zero, with $\beta = 0.60$ m s⁻². Determine the true velocity at zero and the centred finite difference there. Does its error scale as h^2 ? Explain which smoothness assumption fails and whether the object reverses direction.

2.36 Test an error budget with synthetic repetitions

Design a repeatable numerical test of Eq. (2.57) for the cubic at $t = 1$ s, $\alpha = 0.10$ m s⁻³, $u_x = 0.002$ m and $h = 0.25$ s. Generate at least 10^4 independent pairs of zero-mean uniform errors of the stated standard deviation, record the seed, and estimate bias, variance and RMSE. Give theoretical targets and a defensible interpretation of finite-sample disagreement. No actual apparatus data are required.

E. Graph, data and modelling synthesis

The following tasks combine representations. Every graph is generated from the stated mathematical model; a connecting line represents the model only where the question explicitly says so.

2.37 Reconstruct position from a velocity record

Figure 2.29 is an exact piecewise-linear velocity program, not a sparse measurement record. Its vertices in SI units are $(t, v) = (0, -0.40), (2, +0.40), (4, +0.40), (6, -0.40)$. Set $x(0) = 0.20$ m. (a) Locate every direction reversal and write $a(t)$ on the open phases, explaining the status of acceleration at joins. (b) Find the positions at $t = 1, 2, 4, 5, 6$ s, full displacement, distance and both interval averages. (c) Derive a continuous piecewise position function and sketch its curvature and extrema. (d) Check the result using endpoint displacement and signed velocity areas.

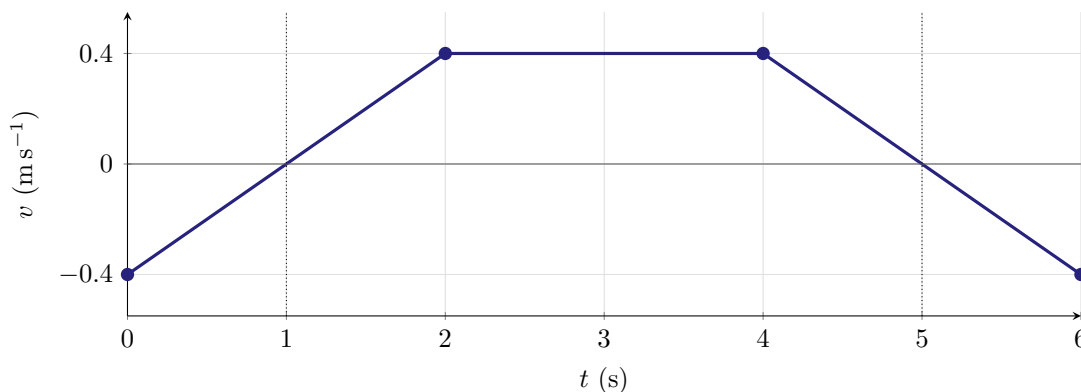


Figure 2.29: The exact velocity program for Problem 2.37. Straight segments specify constant acceleration within phases. Vertical dotted guides mark zero-velocity events; they do not assert rest intervals.

2.38 The same velocity change at different times

Two exact acceleration histories are shown in Fig. 2.30. Each marker has $x(0) = 0$ and $v(0) = -0.40$ m s⁻¹. Program A uses $a = +0.60$ m s⁻² for $0 < t < 2$ s and zero for $2 < t < 4$ s; program B applies those same phases in the opposite order. (a) Find their final velocities. (b) Derive $v(t)$ and calculate each displacement and distance, splitting at reversals. (c) Show that the common interval-average acceleration predicts the correct final velocity but gives the wrong displacement for each. (d) Derive a formula weighting acceleration by its remaining time of action, and use it to explain the difference without invoking forces.

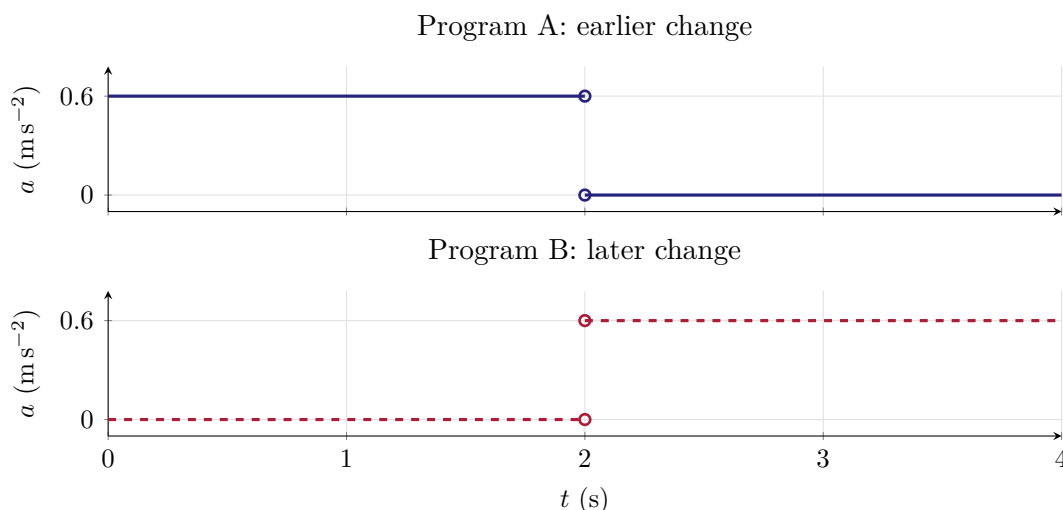


Figure 2.30: Two acceleration histories with equal signed area but different timing, for Problem 2.38. Open circles show one-sided limits at the ideal phase switch. The models define no single acceleration value at that corner.

2.39 A straight sampled record can conceal reversals

The only recorded positions are $X = 0.20, 0.60, 1.00$ m at $t = 0, 1, 2$ s. Compare the exact candidate trajectories

$$x_A(t) = 0.20 + 0.40t, \quad x_B(t) = 0.20 + 0.40t + 0.30 \sin(2\pi t), \quad 0 \leq t \leq 2,$$

where these are numerical SI formulas and the sine argument means $2\pi t/(1 \text{ s})$. Figure 2.31 shows both candidates. (a) Verify that both fit all three readings and calculate the uniquely determined displacement and average velocity. (b) Determine the minimum possible distance from the original samples. (c) Evaluate candidate B at quarter-second intervals and obtain a stronger lower bound on its distance. (d) Differentiate candidate B and use signs at $t = 0, 0.5, 1$ s to establish unobserved reversals. Explain why a visually straight interpolation alone cannot identify instantaneous velocity or total distance.

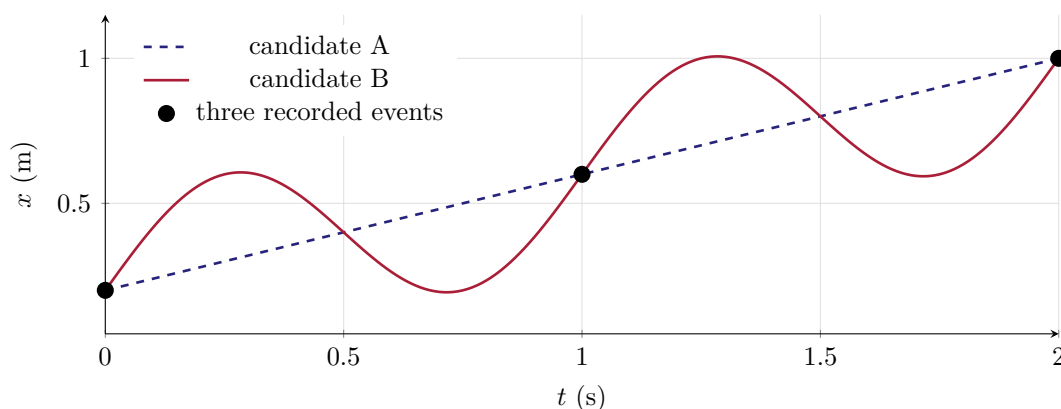


Figure 2.31: Two different motions through the same three sampled events in Problem 2.39. The oscillations are a stipulated alternative model, not reconstructed experimental motion. Neither curve is a picture of the physical track.

2.40 An encounter without overtaking

Two point markers on parallel guides have $x_A = 0.10 + 0.20t$ and $x_B = 0.10 + 0.20t + 0.05(t - 2)^2$ in numerical SI values for $0 \leq t \leq 4$ s; the last term means $(0.05 \text{ m s}^{-2})(t - 2 \text{ s})^2$. Their position curves are plotted in Fig. 2.32. (a) Find every longitudinal encounter, its position and both velocities. (b) Decide whether the markers overtake, using the sign of relative position on both sides of the event. (c) Find relative velocity and acceleration and interpret the equality of tangent slopes. (d) Show that translating or reversing the position axis leaves the physical encounter time unchanged.

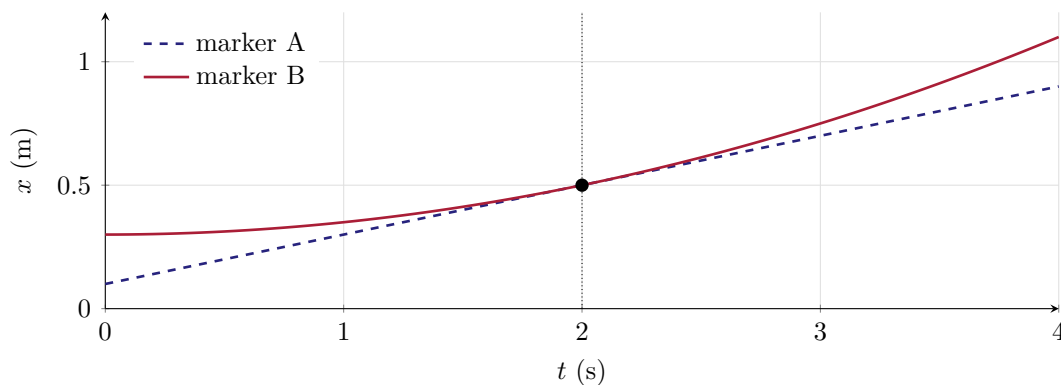


Figure 2.32: A tangential longitudinal encounter for Problem 2.40. The guides are physically parallel; the two plotted curves represent coordinates against time. Equal coordinates do not imply a collision between different guides.

2.41 The same position with opposite tangent slopes

Use the calculated position curve in Fig. 2.2, $x = -1 + 4t - t^2$ in numerical SI values for $0 \leq t \leq 5$ s. (a) Find all times at $x = 2.00$ m and the signed velocities at those events. (b) Draw or describe both tangent lines with their equations and find the secant between those two events. (c) Find the displacement and distance between the events. (d) Explain why a coordinate value alone does not specify the direction of motion.

2.42 A lift's average acceleration conceals its travel

Use the exact program in Fig. 2.9: rest initially at $x = 0$, acceleration $+1.20 \text{ m s}^{-2}$ for $0 < t < 2$ s, cruise at $+2.40 \text{ m s}^{-1}$ for $2 < t < 5$ s, and acceleration -0.80 m s^{-2} until rest at $t = 8$ s. (a) Obtain full displacement, average velocity and interval-average acceleration from areas and endpoint values. (b) Explain why substituting that average acceleration and $v_0 = 0$ into a constant-acceleration position formula fails. (c) State the continuity and differentiability of x and v at 2 and 5 s, and give the one-sided acceleration limits.

2.43 Two visits to a threshold inside a phase

Use Fig. 2.19 and its exact programmed position. On $4 \leq t \leq 8$ s, $x = 1.00 + 0.40(t - 4) - 0.10(t - 4)^2$ in numerical SI values. (a) Find both passages through $x = 1.20$ m and their signed velocities. (b) Find the elapsed time and distance travelled between the passages. (c) Explain how equal passage positions produce a horizontal secant while the position curve has a nonzero excursion. (d) Decide whether any passage through $x = 1.50$ m occurs in the full 0–10 s program, using the maximum in the figure.

2.44 Residuals are evidence, not a proof of the noise law

For the seven interior points of Table 2.1, compute the mean residual $D_c x - v_{\text{exact}}$ represented in Fig. 2.22. Use the exact cubic model $\alpha = 0.10 \text{ m s}^{-3}$ and $h = 0.25$ s to remove its common finite-step bias. (a) Tabulate the seven residuals and their bias-corrected values. (b) Find both means. (c) Explain why a nonzero bias-corrected mean in this short record does not invalidate zero-mean input errors, and why seven overlapping centred estimates are not seven independent noise observations. (d) Identify one additional test needed before using a particular noise distribution as an experimental model.

2.45 Read an error budget under changed conditions

Use the cubic model underlying Fig. 2.25. At $t = 1$ s the coefficient remains $\alpha = 0.10 \text{ m s}^{-3}$, but independent position uncertainty grows from 0.002 to 0.016 m per reading. (a) Determine the new optimum centred spacing, bias magnitude, noise standard deviation and RMSE. (b) Explain how the two component curves and the optimum move on logarithmic axes. (c) Check whether the optimum can use points entirely within $0 \leq t \leq 2$ s. (d) If only the original quarter-second grid is available, compare expected RMSE at $h = 0.25$ and 0.50 s and choose the better available estimate, explaining the assumption about position uncertainty.

2.46 Choose the correct rest-to-rest regime

An ideal one-dimensional stage must move monotonically from rest to rest with acceleration-magnitude limit $A = 0.50 \text{ m s}^{-2}$ and speed cap $V = 0.40 \text{ m s}^{-1}$. Compare traverses of $D_1 = 0.160$ m and $D_2 = 0.500$ m. (a) Derive the threshold distance for reaching the speed cap. (b) Find the fastest capped, bang–cruise–bang program for each distance, including peak velocity, ramp times, cruise time and total time. (c) Sketch both velocity graphs to a common time and velocity scale, labelling the areas that give the prescribed distances. (d) State which idealizations would require checking before implementing either program on a physical stage.

2.11 Complete reasoned solutions

Every assessable problem and subpart is addressed below. Coordinate conventions, model domains, units, event signs and independent checks are part of a complete solution. The open investigations show a worked exemplar and criteria for evaluating other valid approaches.

Problem 2.1: Relabelling a return motion

Choose the stated fixed axis. The signed leg changes are $1.10 - 2.40 = -1.30$ m and $3.00 - 1.10 = +1.90$ m. Their sum is $+0.60$ m, checked independently by the endpoint subtraction $3.00 - 2.40$. The monotone-leg assumption makes the distance $1.30 + 1.90 = 3.20$ m. Divide by the total 5.00 s to obtain $\bar{v} = +0.120$ m s⁻¹ and average speed 0.640 m s⁻¹.

Shifting the origin by 2.40 m gives positions $0, -1.30, +0.60$ m, with the same displacement and distance. Reversing the original axis gives $-2.40, -1.10, -3.00$ m, displacement -0.60 m and unchanged distance. Nothing physical changes: the sign reversal reflects the newly chosen positive direction. Taking $|\Delta x|$ as the distance would miss 2.60 m of travel.

In the modified return journey, the final position equals the initial position: displacement and average velocity are zero. Distance is $1.30 + 1.30 = 2.60$ m, so average speed is $2.60/5.00 = 0.520$ m s⁻¹.

Problem 2.2: What three samples leave unknown

Endpoint subtraction determines $\Delta x = 0.10 - 0 = +0.10$ m, regardless of what happened between samples. Each leg must cover at least the magnitude of its endpoint change. Hence $\ell \geq |0.40 - 0| + |0.10 - 0.40| = 0.70$ m. Equality is attained by any motion that is monotone on each sampled interval; neither constant speed nor constant acceleration is required.

An explicit alternative has successive positions $0, 0.80, 0.40, 0.10$ m at $0, 0.50, 1, 2$ s and monotone motion between those events. It fits every original sample but has distance $0.80 + 0.40 + 0.30 = 1.50$ m. Further unobserved excursions can increase the distance without changing the original readings. Consequently the lower bound is identifiable, whereas the actual distance has no finite upper bound from these samples alone. Average velocity is $+0.050$ m s⁻¹ in both candidates; average speed differs. A connecting line in a sampled plot is an additional interpolation assumption, not evidence excluding an excursion.

Problem 2.3: Equal times versus equal distances

In program A, the displacements are $(0.20)(2.0) = 0.40$ m and $(0.60)(2.0) = 1.20$ m. Their sum divided by 4.0 s gives $\bar{v}_A = 0.40$ m s⁻¹. The arithmetic average works because the two velocities have equal time weights.

In program B, the respective times are $0.30/0.20 = 1.50$ s and $0.30/0.60 = 0.50$ s. Therefore $\bar{v}_B = 0.60/(1.50 + 0.50) = 0.30$ m s⁻¹. Independently, the equal-distance formula is the harmonic mean $2/(1/v_1 + 1/v_2)$, which gives the same value. The slower leg occupies three times as much of the clock interval, so an equal weighting of velocities would overestimate the average. Both motions are positive throughout, making distance equal to displacement and average speed equal to average velocity in each program. That last equality depends on the absence of reversals, not on the equality of times or distances.

Problem 2.4: A negative average during initially positive motion

Differentiate before computing distance: $v = 0.60 - 0.40t$ in numerical SI units. Its root is $t_* = 1.50$ s, and the velocity changes from positive to negative there. The positions are $x(0) = 1.00$, $x(t_*) = 1.45$ and $x(4) = 0.20$ m. Thus $\Delta x = 0.20 - 1.00 = -0.80$ m while $\ell = (1.45 - 1.00) + (1.45 - 0.20) = 1.70$ m.

Divide by 4 s: $\bar{v} = -0.20 \text{ m s}^{-1}$ and average speed 0.425 m s^{-1} . The secant is $x = 1 - 0.20t$, with negative slope; Figure 2.33 supplies the requested drawing. Yet $v(1) = 0.60 - 0.40 = +0.20 \text{ m s}^{-1}$. As an independent area check, the velocity graph has positive triangular area $\frac{1}{2}(1.5)(0.6) = 0.45 \text{ m}$ and negative area $-\frac{1}{2}(2.5)(1.0) = -1.25 \text{ m}$. Their signed sum and absolute sum reproduce displacement and distance.

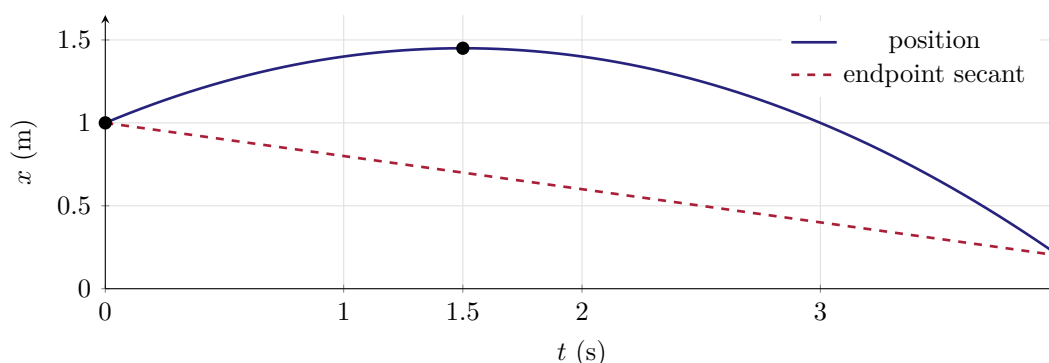


Figure 2.33: The requested secant drawing for Problem 2.4. The secant slopes downward even though the position initially rises. Its endpoints, not the turning point, set the full-interval average velocity.

Problem 2.5: Four signs, four interpretations

The velocity sign gives direction; acceleration gives the direction of velocity change. Away from zero velocity, $d|v|/dt = \text{sgn}(v)a$. The four pairs therefore imply, in order: positive motion speeding up; positive motion slowing down; negative motion slowing down; negative motion speeding up. Their signed speed derivatives are $+0.20, -0.20, -0.20, +0.20 \text{ m s}^{-2}$.

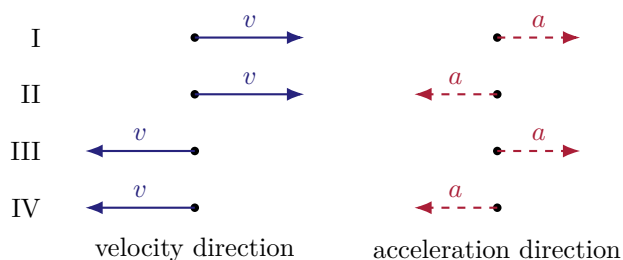


Figure 2.34: Requested velocity and acceleration directions for the four cases in Problem 2.5. Positive direction is to the right. Each arrow type uses a separate schematic scale because the quantities have different units; arrows do not represent forces.

Figure 2.34 supplies the requested arrows, which use separate schematic scales because velocity and acceleration have different dimensions. They are not force arrows. At an interior instant t_* with $v = 0$ and constant $a = +0.20 \text{ m s}^{-2}$, the local velocity is $v = 0.20(t - t_*)$ in numerical SI units. It is negative immediately before and positive immediately after: an instantaneous reversal occurs. A finite rest interval would require $v = 0$ throughout that interval, incompatible with this nonzero constant acceleration.

Problem 2.6: Zero velocity without a reversal

The coefficients have units $[\beta] = \text{m s}^{-3}$ and $[\gamma] = \text{m s}^{-2}$. Differentiating yields $v_1 = 3\beta t^2$ and $v_2 = -2\gamma t$. For every nonzero time near zero, $v_1 > 0$: the first motion slows to zero speed and then speeds up in the same positive direction. Its position is increasing through the event, and $x_1(0) = 0$ is not a local maximum or minimum.

For the second motion, $v_2 > 0$ before zero and $v_2 < 0$ afterward. It reverses from positive to negative motion, with a maximum position at $t = 0$. Its acceleration is -2γ even at the turn. In the first

motion $a_1 = 6\beta t$, changing sign at zero; that acceleration sign change does not imply a velocity sign change. Both examples obey differentiable position models, so the conclusion follows from neighbouring velocity signs rather than from uncertainty about a measured zero. A root of v is a candidate event; it becomes a reversal only if the signs on the two sides differ.

Problem 2.7: An incompatible set of graphs

The position function must determine velocity and acceleration by differentiation, on the stipulated interval. Thus the consistent functions are $v(t) = (2 \text{ m s}^{-2})t$ and $a(t) = 2 \text{ m s}^{-2}$. The velocity graph is a straight line from $(0, 0)$ to $(2 \text{ s}, 4 \text{ m s}^{-1})$; the acceleration graph is a horizontal line at $+2 \text{ m s}^{-2}$. Position remains the given increasing, upward-curved parabola.

The erroneous velocity value 2 m s^{-1} happens to equal the correct value only at $t = 1 \text{ s}$. Equality at one event does not make the graphs compatible. As an independent accumulation check, $x(2) - x(0) = 4 \text{ m}$, equal to the triangular velocity area $\frac{1}{2}(2)(4)$ under the corrected graph. The incorrect constant velocity has that same total area, coincidentally, but fails to reproduce intermediate positions and initial velocity. Likewise, integrating its zero acceleration would keep velocity constant. Graph consistency requires the local derivatives and initial conditions, not merely one matching value or one matching full-interval area.

Problem 2.8: The phase begins at a nonzero clock time

Set $\tau = t - 3 \text{ s}$ and restrict $0 \leq \tau \leq 4 \text{ s}$. Then $v = -0.40 + 0.20\tau$ and $x = 0.50 - 0.40\tau + 0.10\tau^2$ in numerical SI values. The root $v = 0$ gives $\tau = 2 \text{ s}$, hence clock time 5 s and position 0.10 m . Velocity changes from negative to positive, so this is the minimum position.

At $\tau = 4 \text{ s}$, $x = 0.50 \text{ m}$ and $v = +0.40 \text{ m s}^{-1}$. Endpoint subtraction gives zero displacement, whereas distance is $(0.50 - 0.10) + (0.50 - 0.10) = 0.80 \text{ m}$. The negative and positive velocity triangles each have area magnitude 0.40 m , independently checking these results. Replacing τ by the unrevised clock time would fail already at $t = 3 \text{ s}$: the formula would not return the stated initial position and velocity.

Problem 2.9: Repair a local-time error

The stopping phase begins at $t = 5 \text{ s}$ with $x = 9.60 \text{ m}$ and $v = 2.40 \text{ m s}^{-1}$. The student's expression gives $x(5) = 9.60 + 12.00 - 10.00 = 11.60 \text{ m}$, already inconsistent. Differentiating it gives $v = 2.40 - 0.80t$, which at $t = 5$ is -1.60 m s^{-1} , also inconsistent in magnitude and direction.

The constant-acceleration formula requires elapsed time $\tau = t - 5 \text{ s}$, giving $x = 9.60 + 2.40(t - 5) - 0.40(t - 5)^2$ and $v = 2.40 - 0.80(t - 5)$ in numerical SI values. At $t = 6 \text{ s}$, $\tau = 1 \text{ s}$, so $x = 11.60 \text{ m}$ and $v = 1.60 \text{ m s}^{-1}$. The one-second displacement is independently the trapezoidal area $(2.40 + 1.60)(1)/2 = 2.00 \text{ m}$, which agrees with $11.60 - 9.60$.

Problem 2.10: A new three-phase program

The first ramp ends with $v_1 = (0.60)(2) = 1.20 \text{ m s}^{-1}$ and displacement $\frac{1}{2}(0.60)(2)^2 = 1.20 \text{ m}$. The cruise lasts 4 s and contributes $(1.20)(4) = 4.80 \text{ m}$. In the stopping phase, $0 = 1.20 - 0.30\tau$ gives duration $\tau = 4 \text{ s}$ and displacement $\frac{1}{2}(1.20 + 0)(4) = 2.40 \text{ m}$. Total duration is $2 + 4 + 4 = 10 \text{ s}$; all velocities are nonnegative, so both distance and displacement are 8.40 m .

A compact continuity check, with $x(0) = 0$ chosen for convenience, is

$$x(t) = \begin{cases} 0.30t^2, & 0 \leq t \leq 2, \\ 1.20 + 1.20(t - 2), & 2 \leq t \leq 6, \\ 6.00 + 1.20(t - 6) - 0.15(t - 6)^2, & 6 \leq t \leq 10, \end{cases}$$

in numerical SI units. Both x and its derivative agree at the joins. After 10 s the intended model is a new rest phase: $x = 8.40$ m, $v = 0$, $a = 0$. Extending the negative braking acceleration beyond its stopping event would instead produce unintended reverse motion; it would describe a different program.

Problem 2.11: An average acceleration that loses timing information

Integrate with the stated initial rest condition:

$$v(t) = \int_0^t \kappa s \, ds = \frac{1}{2} \kappa t^2, \quad x(t) - x(0) = \int_0^t v(s) \, ds = \frac{1}{6} \kappa t^3.$$

At $T = 3$ s these give $v_f = 1.80$ m s⁻¹ and true displacement 1.80 m. The interval-average acceleration is $(v_f - v_0)/T = 0.60$ m s⁻², which is also the mean height of the linear acceleration graph.

Treating that average as a constant gives $\Delta x_{\text{const}} = \frac{1}{2} \bar{a} T^2 = 2.70$ m: an excess 0.90 m or 50% of the true displacement. The common endpoint velocity does not rescue the position prediction. The actual velocity rises quadratically, remaining below the straight line connecting its endpoints for interior positive times; its area is therefore smaller. Directly integrating v is an independent check, and differentiating the resulting cubic position returns the stipulated acceleration. Average acceleration preserves total velocity change but removes the timing needed for displacement.

Problem 2.12: Does the kinematic stopping model fit the clearance?

Use a positive direction along the initial motion and write $a = -A$, with $A > 0$. At the first stop, $0 = v_0^2 - 2A\Delta x$, so $\Delta x_{\text{stop}} = v_0^2/(2A) = 0.75^2/[2(1.50)] = 0.1875$ m. The stopping time is $v_0/A = 0.50$ s. Independently, the triangular area under the velocity graph is $\frac{1}{2}(0.50)(0.75) = 0.1875$ m.

Because $0.1875 > 0.160$, the assumed stop is outside the allowed distance by 0.0275 m. Requiring $v_0^2/(2A) \leq 0.160$ yields $A \geq 0.5625/0.320 = 1.7578125$ m s⁻². The result assumes immediate braking, constant achievable acceleration and no omitted dynamics. A delay, calibration error or additional constraint would require a new analysis; the equality is not a practical safety margin.

Problem 2.13: A traverse with room for cruising

Each maximum-acceleration ramp takes $V/A = 0.50$ s. The ramp displacement is $V^2/(2A) = 0.125$ m on acceleration and the same on deceleration. Together they use 0.250 m, leaving 0.250 m for cruising. At the cap 0.50 m s⁻¹, that remaining distance takes 0.50 s. Total time is therefore 1.50 s.

The velocity graph has vertices $(0, 0)$, $(0.50, 0.50)$, $(1.00, 0.50)$, $(1.50, 0)$ in SI units. Its areas are $0.125 + 0.250 + 0.125 = 0.500$ m, checking the required traverse. The condition for reaching the cap is $D \geq V^2/A = 0.250$ m, satisfied here. Within the stated monotone idealization, postponing acceleration, using less than its bound while below the cap, or braking earlier lowers feasible velocity at some times and requires extra time. The candidate thus uses the maximum feasible envelope. This argument assumes instantaneous acceleration switches and no jerk constraint; it establishes the fastest program for the given mathematical constraints, not a complete physical actuator prescription.

Problem 2.14: Meeting is not velocity matching

A longitudinal encounter requires equal positions at the same clock time:

$$0.20 + 0.10t = 0.05t^2, \quad t^2 - 2t - 4 = 0.$$

The roots are $t = 1 \pm \sqrt{5}$ s. The negative root, about -1.236 s, is outside the stipulated domain; $t_* = 1 + \sqrt{5} \simeq 3.23607$ s lies within $[0, 5]$ s. Its shared position is $0.20 + 0.10t_* \simeq 0.523607$ m.

Differentiate each trajectory: $v_A = 0.100 \text{ m s}^{-1}$ and $v_B = (0.100 \text{ m s}^{-2})t$. At the encounter $v_B \simeq 0.323607 \text{ m s}^{-1}$, different from v_A . Relative velocity $v_B - v_A \simeq +0.223607 \text{ m s}^{-1}$ shows marker B gaining position at the event. A direct substitution of t_* into both position formulas checks the equality independently. Equal coordinates on parallel guides describe a longitudinal encounter; they do not imply contact. Equal position is a condition on function values, whereas velocity matching is a separate condition on their slopes.

Problem 2.15: Unequal launch and arrival heights

Take $+y$ upward. The velocity is $4.00 - gt$, so the apex time is $4.00/9.81 \simeq 0.40775 \text{ s}$. Its height is $2.00 + 4.00^2/[2(9.81)] \simeq 2.81549 \text{ m}$. The arrival equation $0 = 2.00 + 4.00t - \frac{1}{2}(9.81)t^2$ gives

$$t_L = \frac{4.00 + \sqrt{4.00^2 + 2(9.81)(2.00)}}{9.81}, \quad v_{y,L} = -\sqrt{55.24} \text{ m s}^{-1}.$$

Numerically $t_L \simeq 1.1654 \text{ s}$ and $v_{y,L} \simeq -7.432 \text{ m s}^{-1}$. The other time root is negative. The time $2v_0/g \simeq 0.81549 \text{ s}$ is the return to $y = 2.00 \text{ m}$, not to the lower arrival level.

For an arrival height equal to the launch height, the position equation reduces to $0 = v_0t - \frac{1}{2}gt^2 = t(v_0 - gt/2)$. Its roots are release at $t = 0$ and return at $t = 2v_0/g$ for $v_0 > 0$. The formula therefore follows from a same-height event condition, which this problem does not satisfy.

Define $z = 2.00 \text{ m} - y$. Then $z = -4.00t + \frac{1}{2}gt^2$, $v_z = -4.00 + gt$ and $a_z = +g$ in numerical SI units. Arrival is $z = 2.00 \text{ m}$, which gives exactly the same quadratic and time. The velocity sign is reversed because the coordinate direction is reversed; its speed is unchanged. Substituting t_L in either position equation verifies the event independently of the velocity calculation.

Problem 2.16: A drop and a change of acceleration scale

With $+y$ upward, displacement is -0.80 m , initial velocity is zero and acceleration is $-g$. The event equation is $-0.80 = -\frac{1}{2}gt^2$, giving the physical nonnegative root

$$t = \sqrt{\frac{2(0.80 \text{ m})}{g}} \simeq 0.403855 \text{ s}, \quad v_y = -gt \simeq -3.96182 \text{ m s}^{-1}.$$

The negative time root is outside the post-release model. The independent relation $v_y^2 = 2g(0.80 \text{ m})$ checks the speed magnitude; its negative sign must still be supplied from downward motion.

If the downward acceleration magnitude is doubled, $t' = t/\sqrt{2} \simeq 0.28557 \text{ s}$ and $v'_y = \sqrt{2}v_y \simeq -5.60286 \text{ m s}^{-1}$. The distance and zero initial velocity have stayed fixed, so the scaling follows directly from $t \propto g^{-1/2}$ and speed $\propto g^{1/2}$. In either case distance equals the magnitude of displacement because there is no reversal. The changed acceleration is a hypothetical model parameter, not a claim about the gravitational field at the same laboratory location.

Problem 2.17: Two passages through a checkpoint

The event equation is $0.600 = 4.00t - \frac{1}{2}gt^2$, with roots

$$t_1 = \frac{4.00 - \sqrt{16.00 - 2(9.81)(0.600)}}{9.81}, \quad t_2 = \frac{4.00 + \sqrt{16.00 - 2(9.81)(0.600)}}{9.81}.$$

The discriminant in the square root is $4.228 \text{ m}^2 \text{ s}^{-2} > 0$, and both times are positive: about 0.1981 and 0.6174 s . Substituting into $v_y = 4.00 - gt$ gives $+\sqrt{4.228}$ on ascent and $-\sqrt{4.228}$ on descent, approximately $\pm 2.056 \text{ m s}^{-1}$.

The time-free relation independently gives $v_y^2 = 16.00 - 2g(0.600) = 4.228$. The largest height is $v_0^2/(2g) \simeq 0.81549 \text{ m}$, below 1.00 m . At that higher checkpoint the discriminant would be negative, confirming no event within the model. Squaring velocity cannot distinguish the two lower-checkpoint passages; the time and sign analysis is essential.

Problem 2.18: What can three heights identify?

A constant-acceleration position model is $y(t) = y_0 + v_0 t + \frac{1}{2} a_y t^2$. Here $y_0 = 0.3000$ m is fixed by the first event. Subtract the two equal-interval displacement changes, or take the symmetric second difference:

$$a_y = \frac{0.6019 - 2(0.5000) + 0.3000}{(0.1000)^2} = -9.81 \text{ m s}^{-2}.$$

The first interval gives $0.2000 = 0.1000 v_0 + \frac{1}{2}(-9.81)(0.1000)^2$ in numerical SI values, hence $v_0 = 2.4905 \text{ m s}^{-1}$. Substitution at 0.2000 s returns 0.6019 m and checks the fit.

The three readings determine the three coefficients of the assumed quadratic exactly. They do not independently test its behaviour between readings: adding a nonzero smooth term proportional to $t(t - 0.1000 \text{ s})(t - 0.2000 \text{ s})$ leaves all readings unchanged while changing acceleration. Nor does a fitted value numerically equal to $-g$ identify the cause of acceleration. Additional observations, uncertainty analysis and physical conditions are required to justify free fall. The algebraic fit is valid under its declared assumption; the assumption is not established by that fit alone.

Problem 2.19: Zero displacement is not zero travel

Integrating the given velocity and imposing $x(0) = 0.50$ m gives $x(t) = 0.50 + 0.30t - 0.05t^2$ in numerical SI values. Velocity changes sign at $t_* = 3$ s, where $x = 0.95$ m. At 6 s, position returns to 0.50 m; consequently $\Delta x = 0$.

Distance is the sum of the two monotone changes, $(0.95 - 0.50) + (0.95 - 0.50) = 0.90$ m. Average speed is $0.90/6 = 0.150 \text{ m s}^{-1}$, while average velocity is zero. The velocity graph provides an independent check: its positive triangle on $[0, 3]$ s has area 0.45 m and its negative triangle on $[3, 6]$ s has signed area -0.45 m. Integrating signed velocity cancels those areas; integrating its magnitude adds them. The student's integral was correct for displacement, but the interpretation as total travel was incorrect. Neither returning to the initial position nor having zero average velocity implies having remained at rest.

Problem 2.20: A reversal under time-dependent acceleration

Integrating acceleration from zero gives $v = -0.40 + 0.10t^2$. Integrating this velocity and applying $x(0) = 0.70$ m gives $x = 0.70 - 0.40t + t^3/30$ in numerical SI values. The roots of $v = 0$ are $t = \pm 2$ s; only $+2$ s lies in the given domain. Velocity is negative before it and positive afterward, so position has a minimum there.

At the turn, $x(2) = 0.70 - 0.80 + 8/30 = 1/6$ m. At the endpoint, $x(3) = 0.70 - 1.20 + 27/30 = 0.40$ m. Therefore $\Delta x = 0.40 - 0.70 = -0.30$ m and

$$\ell = (0.70 - 1/6) + (0.40 - 1/6) = \frac{23}{30} \text{ m} \simeq 0.7667 \text{ m}.$$

As a separate check, the signed integrals of v on the two intervals are $-8/15$ m and $+7/30$ m. Their signed sum is -0.30 m and their magnitude sum is $23/30$ m. Differentiating the position twice returns the stipulated acceleration.

Problem 2.21: Missing initial conditions

Integrating zero acceleration gives $v(t) = C_1$, a constant that need not vanish. Integrating once more gives $x(t) = C_2 + C_1 t$, on the stated interval. To determine a numerical position history uniquely in a chosen frame, supply one initial velocity and one initial position, for example $v(0)$ and $x(0)$; specifying a different reference time with both values works equally well.

Two distinct examples are rest at $x = 2.0$ m, so $v = 0$, and uniform negative motion $x = 2.0 \text{ m} - (0.30 \text{ m s}^{-1})t$, so $v = -0.30 \text{ m s}^{-1}$. Both have exactly zero acceleration for $0 \leq t \leq 10$ s, yet their

positions at 10 s differ. In each example differentiating x reproduces its constant velocity and differentiating again yields zero. Giving only an initial position would leave C_1 unknown; giving only an initial velocity would leave C_2 unknown. Acceleration governs changes in velocity, not velocity's absolute value or the origin-dependent position.

Problem 2.22: Units belong to polynomial coefficients

Each term must have the unit of position. The constant coefficient has unit m, the coefficient of t has unit m s^{-1} , and the cubic coefficient has unit m s^{-3} . A dimensionally explicit form is

$$x(t) = 0.40 \text{ m} + (0.30 \text{ m s}^{-1})t - (0.05 \text{ m s}^{-3})t^3.$$

Differentiation gives $v = 0.30 - 0.15t^2$ and $a = -0.30t$ in numerical SI values, so $a(2 \text{ s}) = -0.60 \text{ m s}^{-2}$. The coefficient 0.15 in the velocity formula has unit m s^{-3} and the coefficient 0.30 in the acceleration formula also has unit m s^{-3} ; the powers of time determine the units of the complete terms.

Labelling the original cubic coefficient m s^{-2} would make its product with t^3 have unit m s , which cannot be added to a length. A negative coefficient does not by itself state a negative position or velocity at every time: the complete polynomial and its valid domain must be used. Integrating the acceleration with the correct initial values returns the original position, providing another consistency check.

Problem 2.23: Compute and diagnose two finite differences

The centred estimate uses the printed readings at 0.25 and 0.75 s:

$$D_c x(0.50) = \frac{0.541290 - 0.299079}{0.50} = 0.484422 \text{ m s}^{-1}.$$

The exact velocity there is $0.40 + 0.30(0.50)^2 = 0.475000 \text{ m s}^{-1}$, so the signed error is $+0.009422 \text{ m s}^{-1}$.

At the left endpoint, the forward estimate is $(0.299079 - 0.197651)/0.25 = 0.405712 \text{ m s}^{-1}$, compared with exact $v(0) = 0.400000 \text{ m s}^{-1}$. Its error is $+0.005712 \text{ m s}^{-1}$. It estimates the average over the first quarter-second, not the initial instantaneous velocity exactly.

The two errors belong to different stencils, times and particular noise combinations. Their sizes do not prove a universal ranking. Reducing spacing reduces certain smooth-function approximation errors but divides position errors by a smaller time. Moreover, the existing table does not contain half-spacing observations; interpolating it does not create new independent measurements. The advanced error budget quantifies the competition, but this diagnosis already follows from the difference quotient.

For exact cubic positions, $x(0.25) = 0.3015625 \text{ m}$ and $x(0.75) = 0.5421875 \text{ m}$. Their difference divided by 0.50 s gives $0.481250 \text{ m s}^{-1}$. Subtracting $v(0.50) = 0.475000 \text{ m s}^{-1}$ gives finite-step bias $0.006250 \text{ m s}^{-1}$. The rounded noisy estimate is 0.484422, so this record's noise contribution is $0.003172 \text{ m s}^{-1}$. A smooth-looking curve does not demonstrate convergence or remove either error contribution.

Problem 2.24: Position scale and clock scale

For corresponding endpoints, subtraction cancels the constant offsets:

$$\Delta X = 1.02\Delta x, \quad \Delta T = 0.99\Delta t, \quad \frac{\Delta X}{\Delta T} = \frac{1.02}{0.99} \frac{\Delta x}{\Delta t}.$$

Thus every corresponding interval-average velocity is multiplied by $1.030303\dots$, an increase of about 3.03% in magnitude. The sign is unchanged because both scale factors are positive. The 0.10 m position offset and 3.00 s clock offset affect coordinate labels but not these increments.

Taking a derivative gives the same factor: $dX/dT = (dX/dt)/(dT/dt) = (1.02/0.99)v$. Scale errors therefore remain in the instantaneous limit; shrinking the interval cannot correct calibration. Corresponding events must be compared: a numerical clock reading $T = 1$ s is not the original event $t = 1$ s. As a simple check, true motion at 0.40 m s^{-1} would be reported as approximately $0.412121 \text{ m s}^{-1}$. If an acceleration were needed, its scale factor would be $1.02/0.99^2$, illustrating that additional time derivatives introduce additional time-scale factors.

Problem 2.25: Check a numerical integration refinement

The first trapezoidal update integrates $a(t) = 0.60t$ exactly on each interval because acceleration is linear. With the exact initial velocity it therefore gives $v_i = 0.40 + 0.30t_i^2$ at every grid point. The second update integrates this quadratic velocity by trapezoids, introducing a position error.

For one interval $[t, t + h]$, expanding the quadratic shows

$$\frac{h}{2}\{v(t) + v(t + h)\} - \int_t^{t+h} v(s) ds = \frac{0.30h^3}{6} = 0.05h^3$$

in numerical SI values. Summing $N = 2/h$ intervals gives error $0.10h^2$ m when h is inserted in seconds. The exact endpoint is $x(2) = 0.20 + 0.40(2) + 0.10(2)^3 = 1.800000$ m. Hence $x_h(2) = 1.804000$ m for $h = 0.20$ s and 1.801000 m for $h = 0.10$ s.

The signed errors are positive because trapezoid chords lie above the convex velocity curve. Halving the spacing reduces the error by four, matching the second-order prediction. A direct loop over the two updates reproduces both values, independently checking the analytic error sum. That convergence concerns exact model data; noisy acceleration would add a different uncertainty contribution.

Problem 2.26: A gap in the timestamp record

Use the actual duration of each interval:

$$\frac{0.24 - 0.20}{0.10 - 0} = 0.40, \quad \frac{0.28 - 0.24}{0.20 - 0.10} = 0.40, \quad \frac{0.36 - 0.28}{0.40 - 0.20} = 0.40$$

in m s^{-1} . All samples are consistent with $x(t) = 0.20 \text{ m} + (0.40 \text{ m s}^{-1})t$. The last displacement is twice the first two, but its elapsed time is also twice as long.

Replacing the last duration by 0.10 s yields the false result $0.08/0.10 = 0.80 \text{ m s}^{-1}$, a spurious doubling of velocity. A uniform-grid routine assumes one common adjacent spacing in its derivation; the durations $0.10, 0.10, 0.20$ s violate that assumption. It should raise an informative error rather than silently process the data as equally spaced. A general interval-average routine can still handle the record by using each actual time difference. The total interval provides another check: $(0.36 - 0.20)/(0.40 - 0) = 0.40 \text{ m s}^{-1}$. Gaps in a record are changes in sampling, not evidence of a physical acceleration.

Problem 2.27: Design an interpretable motion record

Worked synthetic exemplar. Use the declared trajectory $x(t) = 0.50 + 0.30t - 0.05t^2$ in numerical SI values for $0 \leq t \leq 6$ s, sampled every 0.25 s. It is a synthetic benchmark, not an apparatus measurement. Add reproducible independent position errors with standard deviation 0.002 m and a recorded seed. Keep the original timestamps. The exact model predicts constant acceleration -0.10 m s^{-2} and reversal at 3 s.

A possible supervised apparatus version uses one clearly identified marker on a low-energy horizontal stage, a fixed ruler in the plane of motion and a stationary camera. Establish length scale from several ruler intervals and time scale from reliable frame timestamps; avoid assuming that a nominal

frame rate excludes dropped frames. State the origin, positive direction, marker definition and observation interval. Record positions and timestamps as observations; fitted velocity, acceleration and reversal time are inferences.

Fit a quadratic position model and plot residuals against time. Estimate repeatability from repeated readings or records, and separately report scale and timing uncertainty. A quadratic fit's good appearance alone does not establish constant acceleration; inspect systematic residual structure and, if possible, check withheld observations. Identify a reversal from a supported sign change of inferred velocity, with its timing uncertainty, rather than a single noisy zero. For the synthetic benchmark, verify recovery against the known model before discussing apparatus data.

Assessment criteria. Award credit for (i) a reproducible record and explicit frame, (ii) credible scale and timestamp checks, (iii) a declared model and interval, (iv) a sign-based reversal test, (v) residual and uncertainty analysis, and (vi) a clear distinction between measurement, inference and simulation. A different suitable plan can earn full credit if it meets these six criteria and does not invent available equipment or measured results.

Problem 2.28: A unit test that catches a factor of two

On an adjacent grid of spacing h , the two positions used in a centred difference occur at $t_i - h$ and $t_i + h$. Their separation is $2h$, so the correct estimate is $(x_{i+1} - x_{i-1})/(2h)$. The faulty routine divides by only half the actual elapsed time, doubling every returned value. Its unit is still velocity, so dimensional checking alone will not detect the factor.

For a detection test, input $x(t) = 0.20 \text{ m} + (0.40 \text{ m s}^{-1})t$ at $t = 0, 0.25, 0.50 \text{ s}$. The positions are $0.20, 0.30, 0.40 \text{ m}$; at the centre, the correct answer is $(0.40 - 0.20)/0.50 = 0.40 \text{ m s}^{-1}$, while the faulty routine returns 0.80 m s^{-1} .

For a legitimate invariance test, add a common 7 m offset to every position and require unchanged output. Both correct and faulty formulas pass because the offset cancels in their numerator. Reversing all position signs likewise reverses both outputs without revealing the factor. Invariance tests check useful structure but need known-answer tests to establish quantitative correctness.

Problem 2.29: Forward, centred and midpoint interpretations

At $t = 1 \text{ s}$ the exact cubic positions are $x(1) = 0.7000$, $x(1.20) = 0.8528$ and $x(0.80) = 0.5712 \text{ m}$. Therefore

$$D_f x(1) = \frac{0.8528 - 0.7000}{0.20} = 0.764, \quad D_c x(1) = \frac{0.8528 - 0.5712}{0.40} = 0.704$$

in m s^{-1} . The true velocity is 0.700 there. Independently, the Taylor formulas give forward error $a(1)h/2 + \alpha h^2 = (0.60)(0.20)/2 + 0.10(0.20)^2 = 0.064$ and centred error $\alpha h^2 = 0.004 \text{ m s}^{-1}$.

The forward value is exactly the average on $[1, 1.20] \text{ s}$, whose time midpoint is 1.10 s . At that midpoint the exact velocity is $0.40 + 0.30(1.10)^2 = 0.763 \text{ m s}^{-1}$, smaller by 0.001 . For constant acceleration, write $v(t_m + s) = v(t_m) + as$; integrating symmetrically around $s = 0$ cancels the odd as term, so a linear velocity has an interval average exactly equal to its midpoint value. Here velocity is quadratic; its symmetric average exceeds the central value by $3\alpha h^2/12 = \alpha h^2/4 = 0.001 \text{ m s}^{-1}$.

Problem 2.30: Noise in first and second differences

Let each position error have zero mean and independent variance u_x^2 . A forward velocity error is $(\epsilon_{i+1} - \epsilon_i)/h$, giving variance $2u_x^2/h^2$ and standard deviation

$$u_f = \frac{\sqrt{2}u_x}{h} \simeq 0.042426 \text{ m s}^{-1}.$$

For a centred velocity error, the denominator is $2h$, so $u_c = u_x/(\sqrt{2}h) \simeq 0.021213 \text{ m s}^{-1}$. These stencils use different total time spans even at the same adjacent spacing.

A symmetric second-difference error is $(\epsilon_{i+1} - 2\epsilon_i + \epsilon_{i-1})/h^2$. Independence makes the squared weights add: $1 + 4 + 1 = 6$. Thus $u_a = \sqrt{6}u_x/h^2 \simeq 0.734847 \text{ m s}^{-2}$. Halving h at unchanged per-reading uncertainty multiplies u_f, u_c, u_a by 2, 2, 4, respectively. The two velocity uncertainties have different numerical factors but the same inverse-spacing power. These are noise standard deviations, not RMSE values: deterministic truncation bias must be computed separately for the chosen trajectory. A smoother exact function cannot remove random amplification by a small denominator.

Problem 2.31: Changing the uncertainty changes the optimum

The MSE is $\alpha^2 h^4 + u_x^2/(2h^2)$. Setting its derivative to zero gives

$$h_{\text{opt}} = \left(\frac{u_x}{2|\alpha|} \right)^{1/3} = \left(\frac{0.001}{0.40} \right)^{1/3} \text{ s} \simeq 0.135721 \text{ s}.$$

The bias magnitude is $|\alpha|h_{\text{opt}}^2 \simeq 0.00368403 \text{ m s}^{-1}$. At the minimum the noise variance is twice the squared bias, so the RMSE is $\sqrt{3}|\alpha|h_{\text{opt}}^2 \simeq 0.006381 \text{ m s}^{-1}$.

Thus $h_{\text{opt}} \propto u_x^{1/3}$ and the minimum RMSE scales as $u_x^{2/3}$ for fixed α . An eightfold decrease in u_x gives factors $1/2$ and $1/4$, respectively. Both neighbouring samples must remain inside a smooth valid-model interval, and actual independent readings must exist at the desired times. Discrete available spacings or endpoint constraints can prevent attaining the unconstrained optimum.

Problem 2.32: A correlated position-error component

For endpoint errors ϵ_+ and ϵ_- with equal variance u_x^2 and covariance ρu_x^2 ,

$$\text{Var} \left(\frac{\epsilon_+ - \epsilon_-}{2h} \right) = \frac{2u_x^2 - 2\rho u_x^2}{4h^2} = \frac{u_x^2(1 - \rho)}{2h^2}.$$

Substituting $u_x = 0.002 \text{ m}$, $h = 0.25 \text{ s}$ and $\rho = 0.75$ gives $u_v \simeq 0.00282843 \text{ m s}^{-1}$. For independence it is $0.00565685 \text{ m s}^{-1}$, so this positive correlation halves the difference uncertainty. This statement concerns the specified correlated component, not all possible uncertainties.

A common additive position offset has exactly the same value at both events and cancels: $(x_+ + b) - (x_- + b) = x_+ - x_-$. A common scale error instead gives $X = (1 + s)x$ and a centred estimate $(1 + s)D_c x$. It multiplies a generally nonzero displacement and survives subtraction. If s is uncertain, its velocity uncertainty depends on the displacement or velocity being estimated; it cannot generally be represented as equal absolute endpoint errors with the above fixed covariance model. Correlation reduces some difference errors without making calibration irrelevant.

Problem 2.33: Evaluate the five-point fit on the supplied record

Assign the readings at 0.50, 0.75, 1.00, 1.25, 1.50 s to indices $-2, -1, 0, 1, 2$ around the centre, with $h = 0.25 \text{ s}$. Equation (2.59) gives

$$q = \frac{-2(0.415375) - 0.541290 + 0.898157 + 2(1.138544)}{2.50} = 0.721282 \text{ m s}^{-1}.$$

The true velocity is $0.700000 \text{ m s}^{-1}$, so the observed signed error is $+0.021282 \text{ m s}^{-1}$. The exact cubic's fit bias is $3.4\alpha h^2 = 0.021250 \text{ m s}^{-1}$; subtraction leaves this record's noise contribution $+0.000032 \text{ m s}^{-1}$.

The three-point estimate from the two inner noncentral readings is $(0.898157 - 0.541290)/0.50 = 0.713734 \text{ m s}^{-1}$, with error $+0.013734 \text{ m s}^{-1}$. It happens to be closer in this realization. Independently

inserting exact cubic positions into both estimators gives 0.721250 and $0.706250 \text{ m s}^{-1}$, confirming their different deterministic biases. The five-point estimate has lower expected noise variance at fixed adjacent spacing but covers a wider window and has larger cubic bias. A favourable noise realization or a visibly quieter curve is not a universal guarantee of smaller total error.

Problem 2.34: Differentiation can create correlations

Expand products of the two noise combinations and use $E(\epsilon_j \epsilon_k) = 0$ for $j \neq k$, with $E(\epsilon_j^2) = u_x^2$. The estimates η_i and η_{i+1} use disjoint readings $\{i-1, i+1\}$ and $\{i, i+2\}$, so their covariance is zero.

The estimate two steps later is $\eta_{i+2} = (\epsilon_{i+3} - \epsilon_{i+1})/(2h)$. Its shared reading ϵ_{i+1} appears with the opposite sign, leaving

$$\text{Cov}(\eta_i, \eta_{i+2}) = -\frac{u_x^2}{4h^2}.$$

Each individual variance is $u_x^2/(2h^2)$, so dividing by the product of standard deviations gives correlation $-1/2$. Independent input readings therefore do not imply an independent output sequence when stencils overlap. A later weighted fit should account for the covariance structure or work directly with the original positions using a suitable model. The result concerns the random contributions, not the deterministic approximation biases.

Problem 2.35: When the smooth-error formula fails

The velocity at zero must be computed from the derivative definition:

$$v(0) = \lim_{\delta \rightarrow 0} \frac{\beta\delta|\delta| - 0}{\delta} = \lim_{\delta \rightarrow 0} \beta|\delta| = 0.$$

For $t \neq 0$, direct differentiation gives $v(t) = 2\beta|t|$. It is positive on both sides of zero, so the marker does not reverse; position remains increasing through the instant of zero speed.

A centred quotient at zero uses $x(h) = \beta h^2$ and $x(-h) = -\beta h^2$, hence $D_c x(0) = \beta h$. At $h = 0.10 \text{ s}$ its value and signed error are 0.060 m s^{-1} . Halving h halves this error: it is first order, not second order. Although x has a continuous first derivative, acceleration is -2β on the negative side and $+2\beta$ on the positive side, so velocity is not differentiable at zero and acceleration itself is undefined there. The smooth Taylor expansion used for a centred h^2 error requires derivatives that do not exist here. A symmetric stencil does not by itself guarantee second-order error for every trajectory; the regularity of the function at the stencil centre matters.

Problem 2.36: Test an error budget with synthetic repetitions

Theory. For the exact cubic the centred bias is $b = \alpha h^2 = 0.006250 \text{ m s}^{-1}$. Independent position errors contribute variance $\sigma^2 = u_x^2/(2h^2) = 0.000032 \text{ m}^2 \text{ s}^{-2}$. Therefore the target MSE is $b^2 + \sigma^2 = 0.0000710625 \text{ m}^2 \text{ s}^{-2}$ and target RMSE is $0.00842986 \text{ m s}^{-1}$. Uniform errors with standard deviation u_x must have half-width $\sqrt{3}u_x$, not u_x .

Reproducible implementation. The following standard-Python calculation was executed with $N = 100000$ independent pairs and seed 20261006. Algebraically, its error expression is the same as first generating the two exact cubic positions, adding their errors and taking the centred quotient.

```
import math, random
N, seed = 100000, 20261006
ux, h, alpha = 0.002, 0.25, 0.10
rng = random.Random(seed)
L = math.sqrt(3)*ux
b = alpha*h*h
err = [b + (rng.uniform(-L, L)-rng.uniform(-L, L))/(2*h)
```

```

    for _ in range(N)]
mean = math.fsum(err)/N
variance = math.fsum((e-mean)**2 for e in err)/(N-1)
rmse = math.sqrt(math.fsum(e*e for e in err)/N)
print(mean, variance, rmse)

```

The executed results, rounded here, were:

quantity	observed ensemble	theoretical target
bias (m s^{-1})	0.00623996	0.00625000
variance ($\text{m}^2 \text{s}^{-2}$)	3.19967×10^{-5}	3.20000×10^{-5}
RMSE (m s^{-1})	0.00842220	0.00842986

These are synthetic ensemble results, not experimental observations. Keep all generated pairs independent; reusing one endpoint error across different repetitions would change this ensemble design.

Sampling check. The mean's theoretical standard error is $\sigma/\sqrt{N} \simeq 0.0000178885 \text{ m s}^{-1}$. For this symmetric uniform-pair model, the noise fourth moment is $\mu_4 = 0.6u_x^4/h^4$. Large- N standard-error estimates for the variance and MSE are

$$\text{SE}(s^2) \simeq \sqrt{\frac{\mu_4 - \sigma^4}{N}}, \quad \text{SE}(\widehat{\text{MSE}}) = \sqrt{\frac{4b^2\sigma^2 + \mu_4 - \sigma^4}{N}}.$$

Their values are 1.1973×10^{-7} and $2.5365 \times 10^{-7} \text{ m}^2 \text{s}^{-2}$. The executed mean, variance and MSE each differed from its target by less than five of these stated standard errors. This declared tolerance allows finite-ensemble fluctuations; it is a diagnostic, not a proof of an experimental error law.

Assessment criteria. A complete submission declares the distribution and half-width, seed, repetition count, independence, theoretical targets, actual statistics, and a sampling tolerance with an interpretation. Exact equality is not expected, and running more repetitions does not reduce the finite-step bias of each individual velocity estimator.

Problem 2.37: Reconstruct position from a velocity record

(a) Events and acceleration. On the three open phases, $v = -0.40 + 0.40t$, $v = 0.40$, and $v = 0.40 - 0.40(t - 4)$, in numerical SI units. Thus $a = +0.40, 0, -0.40 \text{ m s}^{-2}$, respectively. The zero events are $t = 1$ and 5 s ; each changes the velocity sign, so each is a reversal. At 2 and 4 s velocity is continuous but has different one-sided slopes, making a single derivative-based acceleration undefined there.

(b) Accumulation. Integrate signed velocity from the initial position. At the requested times the positions are

$$x(1) = 0, \quad x(2) = 0.20, \quad x(4) = 1.00, \quad x(5) = 1.20, \quad x(6) = 1.00 \quad \text{in m.}$$

The full displacement is $1.00 - 0.20 = +0.80 \text{ m}$. Distance is $0.20 + 1.20 + 0.20 = 1.60 \text{ m}$, separating the negative–positive–negative legs. Over 6 s , average velocity is approximately $+0.1333 \text{ m s}^{-1}$ and average speed approximately 0.2667 m s^{-1} .

(c) Position and shape. Carry each ending position into the next phase:

$$x(t) = \begin{cases} 0.20 - 0.40t + 0.20t^2, & 0 \leq t \leq 2, \\ 0.20 + 0.40(t - 2), & 2 \leq t \leq 4, \\ 1.00 + 0.40(t - 4) - 0.20(t - 4)^2, & 4 \leq t \leq 6. \end{cases}$$

Figure 2.35 supplies the requested sketch. Positive curvature in the first phase gives a minimum at 1 s; the middle phase is straight; negative curvature in the last gives a maximum at 5 s. The extrema are extrema of position, not extrema of speed.

(d) **Independent checks.** The two negative velocity triangles have total signed area -0.40 m; the positive parts have area $0.20 + 0.80 + 0.20 = 1.20$ m. Their signed sum 0.80 m agrees with endpoint subtraction; their magnitude sum 1.60 m agrees with the monotone-leg calculation. Differentiation of every position branch recovers the velocity record, and both position and velocity agree at the joins.

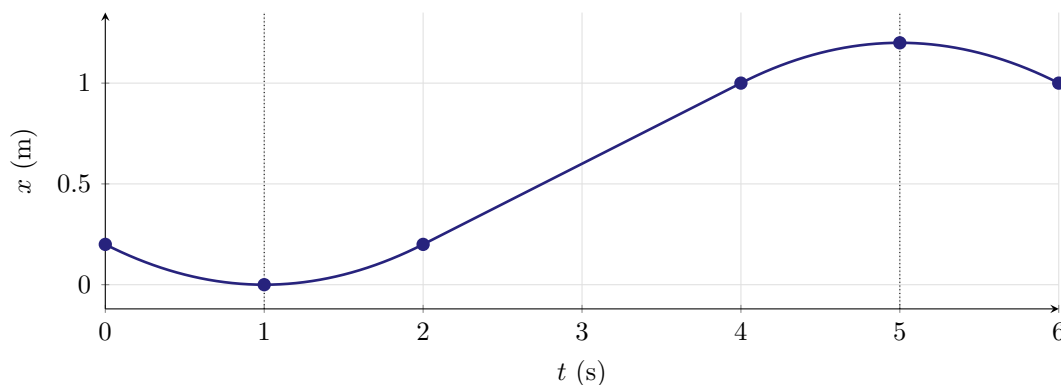


Figure 2.35: Position reconstructed by exact integration of the velocity in Problem 2.37. Dots identify requested events; they are not assumed equally spaced in time. The minima and maxima coincide with velocity sign changes.

Problem 2.38: The same velocity change at different times

(a) **Final velocities.** Both acceleration graphs have area $(0.60)(2) = 1.20 \text{ m s}^{-1}$. Adding $v_0 = -0.40 \text{ m s}^{-1}$ gives the same final velocity $+0.80 \text{ m s}^{-1}$.

(b) **Trajectories and travel.** In A, $v_A = -0.40 + 0.60t$ for $0 \leq t \leq 2$ and $v_A = 0.80$ for $2 \leq t \leq 4$. The reversal is at $2/3$ s, with position $-2/15$ m. Position at 2 s is 0.40 m; the cruise adds 1.60 m. Hence $\Delta x_A = 2.00$ m and $\ell_A = 2/15 + (2 + 2/15) = 34/15 \text{ m} \simeq 2.2667 \text{ m}$.

In B, $v_B = -0.40$ for $0 \leq t \leq 2$ and $v_B = -0.40 + 0.60(t - 2)$ afterward. At 2 s, $x_B = -0.80$ m; reversal occurs at $t = 8/3$ s, at $x_B = -14/15$ m. The last two seconds add 0.40 m, so $\Delta x_B = -0.40$ m. Distance is $14/15 + (14/15 - 0.40) = 22/15 \text{ m} \simeq 1.4667 \text{ m}$. Splitting at those turns is essential; signed phase areas alone would give displacement.

(c) **The shared average.** Both have $\bar{a} = 1.20/4 = 0.30 \text{ m s}^{-2}$. A constant acceleration of that value gives the correct v_f but predicts $v_0 T + \frac{1}{2} \bar{a} T^2 = -1.60 + 2.40 = 0.80$ m, equal to neither actual displacement.

(d) **Timing weight.** Starting from $v(t) = v_0 + \int_0^t a(s) ds$ and reversing the order of accumulation gives

$$\Delta x = v_0 T + \int_0^T (T - s) a(s) ds.$$

An acceleration increment at time s changes velocity for the remaining time $T - s$. For A the added displacement is $0.60 \int_0^2 (4 - s) ds = 3.60$ m; for B it is $0.60 \int_2^4 (4 - s) ds = 1.20$ m. Adding -1.60 m checks the earlier results. Earlier velocity change has longer to affect position, even though the unweighted acceleration areas are identical.

Problem 2.39: A straight sampled record can conceal reversals

(a) **Shared observations.** At every integer second, $\sin(2\pi t) = 0$, so both candidates return 0.20, 0.60, 1.00 m. Endpoint subtraction fixes $\Delta x = +0.80$ m and $\bar{v} = +0.40$ m s⁻¹ over the two-second interval, independent of interpolation.

(b) **Original lower bound.** The sum of absolute sampled changes is $0.40 + 0.40 = 0.80$ m. Candidate A attains this distance by monotone motion, proving it is the minimum possible. It is not necessarily the actual distance.

(c) **Candidate B.** Quarter-second evaluations give the following positions and interval travel lower bounds:

t (s)	x_B (m)	$ \Delta x_B $ from preceding row (m)
0.00	0.20	—
0.25	0.60	0.40
0.50	0.40	0.20
0.75	0.20	0.20
1.00	0.60	0.40
1.25	1.00	0.40
1.50	0.80	0.20
1.75	0.60	0.20
2.00	1.00	0.40

Summing the eight absolute changes gives 2.40 m. This is a stronger lower bound for B, not its exact distance: its true extrema occur between some quarter-second events. Candidate B must therefore travel at least three times as far as A while matching all original observations.

(d) **Unseen reversals.** Differentiate with the sine argument's time unit retained: $v_B = 0.40 + 0.60\pi \cos(2\pi t)$ in numerical SI values. At 0 and 1 s its velocity is approximately $+2.285$ m s⁻¹, whereas at 0.5 s it is approximately -1.485 m s⁻¹. Continuity therefore guarantees a positive-to-negative reversal in the first half-second and a negative-to-positive reversal in the second. The pattern repeats in the second second. The samples determine endpoint changes but cannot exclude this smooth higher-frequency motion. Straight interpolation chooses candidate A by assumption; additional sampling or justified bandwidth information is needed to reject candidate B.

Problem 2.40: An encounter without overtaking

(a) **Event and velocities.** Set $x_B - x_A = 0.05(t - 2)^2 = 0$. The only event in the domain is $t_* = 2$ s, at $x_* = 0.50$ m. Differentiation gives $v_A = 0.20$ m s⁻¹ and $v_B = 0.20 + 0.10(t - 2)$ in numerical SI values, so both velocities at the event are $+0.20$ m s⁻¹.

(b) **Ordering.** Relative position is strictly positive before and after t_* and zero only at it. Marker B has the greater longitudinal coordinate on both sides. Thus the markers meet longitudinally without overtaking; there is no sign change of their relative position. This differs from a crossing of position curves.

(c) **Relative rates.** Relative velocity is $v_B - v_A = 0.10(t - 2)$ and relative acceleration is $a_B - a_A = 0.10$ m s⁻². Their longitudinal separation first shrinks, reaches zero and then grows, while neither marker reverses its own motion in the interior. The tangent slopes coincide at the meeting, explaining the tangential contact of the plotted curves. Equal slope does not imply equal acceleration or identical future trajectories.

(d) **Coordinates.** Under a common shift $x' = x - b$, the difference remains $x'_B - x'_A = x_B - x_A$. Under common reversal $x' = -x$, it becomes the negative of that difference. Its zeros, and hence encounter times, remain unchanged. The signs of ordering and velocities change with the reversed axis, but the absence of an ordering exchange and the shared physical event persist.

Problem 2.41: The same position with opposite tangent slopes

(a) **Events.** Solving $-1 + 4t - t^2 = 2$ gives $(t - 1)(t - 3) = 0$, so both 1 and 3 s lie in the domain. Since $v = 4 - 2t$, the corresponding velocities are +2 and -2 m s^{-1} .

(b) **Tangents and secant.** The tangent through the first event is $x = 2 + 2(t - 1) = 2t$ in numerical SI units, ascending through (1, 2). The second is $x = 2 - 2(t - 3) = 8 - 2t$, descending through (3, 2). Their slopes have opposite signs and equal magnitudes. The secant joining the two equal-position events is the horizontal line $x = 2 \text{ m}$, with slope zero; it is not tangent to the trajectory at either endpoint.

(c) **Travel.** The turn occurs at $v = 0$, hence $t = 2 \text{ s}$ and $x = 3 \text{ m}$. The two-event displacement is zero, but the distance is $(3 - 2) + (3 - 2) = 2 \text{ m}$. The positive and negative triangular velocity areas between 1 and 3 s each have magnitude 1 m, independently checking both conclusions. Average speed is 1 m s^{-1} over that interval.

(d) **Meaning.** A position value identifies a coordinate, not the time derivative there. The marker visits the same location while moving in opposite directions. To infer direction one needs event time and a neighbouring position history or signed velocity measurement, not merely the coordinate or the full-interval secant.

Problem 2.42: A lift's average acceleration conceals its travel

(a) **Areas and endpoints.** The velocity graph's first triangle, cruise rectangle and last triangle have areas $\frac{1}{2}(2)(2.40) = 2.40$, $(3)(2.40) = 7.20$ and $\frac{1}{2}(3)(2.40) = 3.60 \text{ m}$. Thus $\Delta x = 13.20 \text{ m}$ and $\bar{v} = 13.20/8 = 1.65 \text{ m s}^{-1}$. The initial and final velocities are both zero, giving $\bar{a} = 0$. The acceleration areas $+1.20(2) = +2.40$ and $-0.80(3) = -2.40 \text{ m s}^{-1}$ provide the same endpoint check.

(b) **Inapplicable substitution.** Constant $a = 0$ with $v_0 = 0$ would predict no motion and zero displacement. The actual interval-average acceleration says only that final and initial velocities are equal. It loses the intervening positive velocity and its nonzero area. Average acceleration is not generally a substitute in a constant-acceleration position equation.

(c) **Joins.** At 2 and 5 s, the position is continuous and differentiable because velocity has matching one-sided values, 2.40 m s^{-1} . Velocity is continuous but not differentiable at either join. Acceleration's left and right limits are $+1.20$ and 0 m s^{-2} at 2 s, and 0 and -0.80 m s^{-2} at 5 s. Thus position is continuously differentiable but not twice differentiable at those corners. A realistic smooth switching model would round the corners and require its own quantitative parameters.

Problem 2.43: Two visits to a threshold inside a phase

(a) **Passages.** Set $\tau = t - 4 \text{ s}$. The threshold equation is $1.20 = 1.00 + 0.40\tau - 0.10\tau^2$, giving $\tau^2 - 4\tau + 2 = 0$ in numerical SI values. Both roots $\tau = 2 \pm \sqrt{2} \text{ s}$ lie in $[0, 4] \text{ s}$. Clock times are $t_- = 6 - \sqrt{2} \text{ s} \simeq 4.58579 \text{ s}$ and $t_+ = 6 + \sqrt{2} \text{ s} \simeq 7.41421 \text{ s}$. Since $v = 0.40 - 0.20\tau$, velocities are $+0.20\sqrt{2}$ and $-0.20\sqrt{2} \text{ m s}^{-1}$, approximately $\pm 0.282843 \text{ m s}^{-1}$.

(b) **Time and distance.** The elapsed time is $2\sqrt{2} \text{ s} \simeq 2.82843 \text{ s}$. Position reaches its maximum 1.40 m at $t = 6 \text{ s}$, so distance between passages is $(1.40 - 1.20) + (1.40 - 1.20) = 0.40 \text{ m}$. Their displacement is zero.

(c) **Secant.** Equal endpoint positions give the horizontal secant $x = 1.20 \text{ m}$, whose slope is zero average velocity. The curve rises 0.20 m above that line and returns, so it has nonzero excursion and nonzero average speed. The opposite signed velocity areas cancel while their magnitudes add.

(d) **Unreachable threshold.** The full program's maximum is 1.40 m ; its other phases never exceed it. Therefore no passage through 1.50 m occurs in $0\text{--}10 \text{ s}$. Extending one polynomial branch beyond its stated phase to create a root would be an invalid event calculation.

Problem 2.44: Residuals are evidence, not a proof of the noise law

(a) **Bias separation.** The exact cubic centred bias is $b = \alpha h^2 = 0.006250 \text{ m s}^{-1}$ at each interior point. Using the printed estimates and benchmarks gives:

$t \text{ (s)}$	residual	residual minus b m s^{-1}
0.25	+0.016698	+0.010448
0.50	+0.009422	+0.003172
0.75	+0.003276	-0.002974
1.00	+0.013734	+0.007484
1.25	+0.005562	-0.000688
1.50	+0.000540	-0.005710
1.75	+0.001004	-0.005246

(b) **Means.** The mean raw residual is approximately $+0.00717657 \text{ m s}^{-1}$; the mean bias-corrected residual is approximately $+0.000926571 \text{ m s}^{-1}$. Removing the model bias does not force the remaining sample mean to vanish.

(c) **Interpretation.** Zero-mean input errors predict an ensemble mean, not exact cancellation within one short record. Here centred estimates two grid steps apart share one endpoint reading with opposite signs; their noise covariance is $-u_x^2/(4h^2)$ under independent input errors. Thus the seven output errors are not all independent, even when the original position errors are. Adjacent centred stencils happen to have disjoint input readings, but that does not make the whole sequence independent. A formula for the standard error of seven independent residuals would therefore need modification.

(d) **Additional evidence.** Before assigning an experimental distribution, repeat calibrated measurements or independent records to examine scatter, bias, correlations and drift. Check ruler and clock scale separately. Neither this synthetic record nor a near-zero corrected mean establishes a uniform, Gaussian or any other physical noise law; the distribution must be supported by observations and the measurement process.

Problem 2.45: Read an error budget under changed conditions

(a) **Optimum.** With the new u_x , minimization of $\alpha^2 h^4 + u_x^2/(2h^2)$ gives

$$h_{\text{opt}} = \left(\frac{0.016}{2(0.10)} \right)^{1/3} \text{ s} \simeq 0.430887 \text{ s}.$$

Bias magnitude $|\alpha| h_{\text{opt}}^2 \simeq 0.0185664 \text{ m s}^{-1}$; noise standard deviation $u_x/(\sqrt{2} h_{\text{opt}}) \simeq 0.0262568 \text{ m s}^{-1}$; RMSE $\simeq 0.0321579 \text{ m s}^{-1}$. The noise standard deviation is $\sqrt{2}$ times the bias magnitude at the optimum, checking the derivative condition.

(b) **Plot changes.** The bias curve is unchanged because α is fixed. The noise curve rises by factor eight at every h , a vertical shift of $\log_{10} 8$ on base-ten logarithmic axes. Optimum spacing doubles and minimum RMSE quadruples, from their $u_x^{1/3}$ and $u_x^{2/3}$ scaling. The total curve consequently moves toward a wider preferred stencil.

(c) **Domain.** The required positions are at $1 \pm 0.430887 \text{ s}$, approximately 0.569113 and 1.430887 s . Both lie in the stated interval. Existence in the model domain does not ensure actual sampled observations at those times.

(d) **Available spacings.** For $h = 0.25 \text{ s}$, expected RMSE is approximately $0.0456844 \text{ m s}^{-1}$; for $h = 0.50 \text{ s}$ it is approximately $0.0337194 \text{ m s}^{-1}$. The latter is better among those two choices. This comparison assumes independent readings with the same 0.016 m standard deviation at both

spacings. It does not reuse the original low-noise realization as if it had new uncertainty, and interpolation would not manufacture independent observations at the unconstrained optimum.

Problem 2.46: Choose the correct rest-to-rest regime

(a) **Regime threshold.** Reaching V with acceleration $+A$ takes V/A and covers $V^2/(2A)$. Braking from V with $-A$ uses the same distance. Hence the two ramps require $D_* = V^2/A = 0.320$ m. A shorter traverse cannot reach the cap and still stop within D .

(b) **Programs.** For $D_1 = 0.160$ m $< D_*$, the triangular program has peak velocity $v_p = \sqrt{AD_1} \simeq 0.282843$ m s⁻¹. Each ramp lasts $v_p/A \simeq 0.565685$ s, cruise time is zero, and total time $2\sqrt{D_1/A} \simeq 1.13137$ s. The peak is below the speed cap.

For $D_2 = 0.500$ m $> D_*$, each ramp lasts $V/A = 0.800$ s and covers 0.160 m. Cruise distance is $0.500 - 0.320 = 0.180$ m, giving cruise time $0.180/0.400 = 0.450$ s and total time 2.050 s. The velocity envelope uses the largest feasible positive acceleration until the cap or switch and the latest feasible maximum braking, so within these ideal constraints it minimizes traversal time.

(c) **Common-scale graph and areas.** Figure 2.36 shows the two programs. For the triangle, each ramp area is $D_1/2 = 0.080$ m. For the trapezoid, the areas are $0.160 + 0.180 + 0.160 = 0.500$ m. This independently verifies both distances without using the position formula. After each finish, the intended rest phase has $v = 0$.

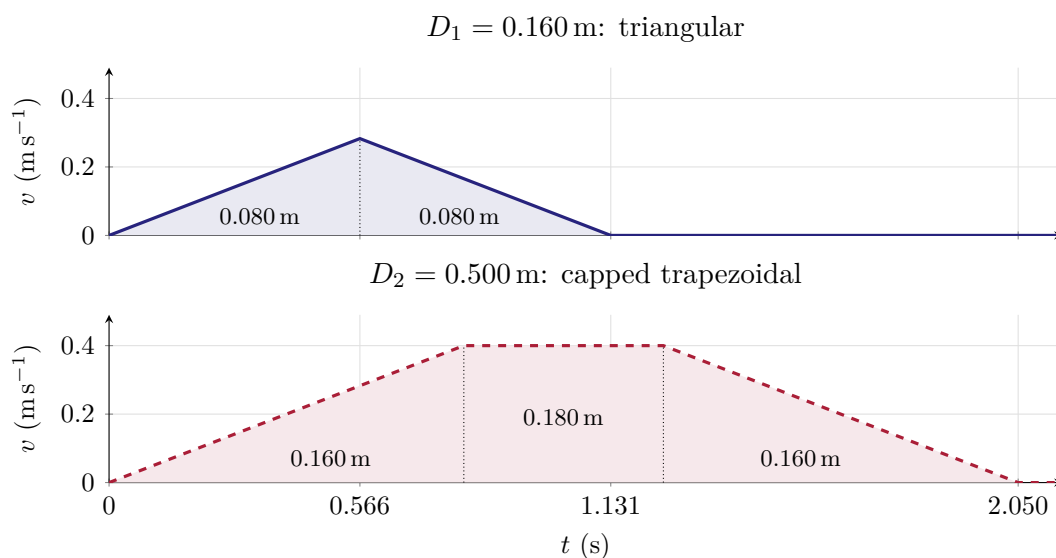


Figure 2.36: The triangular and capped trapezoidal fastest ideal velocity programs for Problem 2.46, on common scales. Both initial ramp segments have slope A ; panels share the same time and velocity scales. The labelled geometric areas carry distance units. Stopping completes each prescribed traverse; the subsequent zero-velocity segment depicts rest.

(d) **Implementation qualifications.** Check that the chosen velocity and acceleration bounds are physically achievable for the load and entire travel, that length and timing calibration support the clearance, and that acceleration switching is adequately modelled. A finite jerk limit, delayed control, velocity-dependent bounds or nonzero initial/final velocity changes the optimization problem. The ideal model assumes exact one-dimensional monotone motion and instantaneous switching; it supplies a benchmark that a real stage model must justify.

2.12 Bridge to the next chapter

A signed component describes direction only along a chosen line. The next chapter, *Vectors*, supplies the language of magnitude, direction, components and projections needed for multidimensional motion.

The modelling principles remain: identify the frame and interval, interpret derivatives as local rates, and integrate with initial conditions. A motion graph is not a picture of the track; a numerical result still needs units and validation. The optional error analysis can be revisited for measured vector trajectories without becoming a prerequisite for the vector core.

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