

Chapter 3

Vectors
First Edition

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How to use this chapter

This chapter belongs to Part I, *Measurement and Mathematical Description*, of *Physics for Technical and Engineering Students - Volume I: Mechanics, Fluids, Waves, and Thermodynamics*. It supplies the geometric language needed to extend one-dimensional physics to space. The aim is not to memorise a collection of three-component formulas. It is to move reliably between a physical description, a drawing, components and an invariant result.

The core assumes the units, modelling habits and elementary geometry developed in *Measurement, Units, and Physical Modelling*. It also uses the distinction between signed displacement and distance from *Motion Along a Straight Line*. Sines, cosines and inverse trigonometric functions are reviewed where they enter. Sections 3.1–3.5 form the core exposition. Section 3.6 is a supporting extension on changing axes, not a prerequisite for the core problems. The core does not require matrix algebra or vector calculus. Those tools are introduced in the separate advanced section, which may be skipped on a first reading without losing access to the core exercises.

Read a figure with a pencil in hand. Locate its origin, identify what an arrow represents, and check whether its scale has physical units. Before calculating a component, predict its sign. Before normalising a vector, ask whether it can be zero. Before using a product, decide whether the answer should be a scalar or a vector, and write down its units. These small decisions prevent errors that a calculator cannot diagnose.

Pedagogical boundary

Forces appear here only as prescribed vectors whose directions can be compared. The dynamics of forces, the physical meaning of work, and the physics of torque belong to later chapters. A geometric calculation does not certify that a real bracket is strong enough, that a robot trajectory is collision-free, or that a measurement system is calibrated.

The worked examples, diagram coordinates and numerical data in this chapter are original, constructed teaching cases. Numerical inputs are stipulated model values, not reported measurements, unless a problem explicitly asks the reader to plan a measurement. Figures distinguish scaled geometric constructions, analytical plots and schematic direction indicators. Extra digits used in a calculation are guard digits, not claims about experimental precision.

Notation and reading paths

The core uses an orthonormal right-handed Cartesian basis. A component column is always understood in a stated basis and unit; it is not an origin-free point record. Angles in geometric examples are displayed in degrees unless specified. Trigonometric derivatives use radians. The symbols below distinguish a vector, a magnitude, a signed component, and a direction.

Symbol	Meaning	Unit or role
$\mathbf{a}, \mathbf{b}$	geometric vectors; context states the quantity	vector
$a = \ \mathbf{a}\ $	nonnegative magnitude	input unit
a_x, a_y, a_z	signed Cartesian components	input unit
$\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$	fixed Cartesian unit directions	dimensionless
$\hat{\mathbf{u}}, \hat{\mathbf{n}}$	a specified unit direction or normal	dimensionless
$\mathbf{r}_P$	position of point P relative to a declared origin	m
$\mathbf{d}$	a displacement or correction	length
θ, ϕ	angle between directions; axis rotation angle	radians/degrees
$R(\phi)$	passive planar component transformation	dimensionless
T	matrix transpose	matrix
t, h	time and a time increment in path examples	operation
$\mathbf{r}'$	vector derivative in the specified frame	s
$\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta$	moving polar unit directions (advanced)	m s^{-1}
∇	Cartesian spatial derivative operator (advanced)	dimensionless
		inverse length

Core route. Read geometric addition, components, projections and the two products, then the applied cases and complete solutions. Changing axes is a supporting extension. Oriented volume builds directly on the cross product. **Advanced route.** Continue with matrices, vector derivatives, sensitivity, moving bases and the optional field-calculus bridge. Partial derivatives and integrals are introduced there before use; ordinary vector-algebra practice does not require them. Problems identify extensions explicitly. All 46 end problems and their assessable subparts have reasoned solutions in this chapter.

Chapter 3

Vectors

A positioning stage can move the correct distance and still miss its target. A survey can contain individually plausible lengths and still fail to close. Two descriptions of one displacement can have completely different components and both be correct. In each case, direction and reference axes are part of the information, not optional decoration.

In the previous chapter, a sign distinguished the two directions on a line. In a plane or in space, one sign is no longer enough. A vector combines magnitude and direction with a rule for addition. We first understand that rule geometrically, then build components, projections and products. Finally we change axes without changing the underlying geometry.

Learning goals

After the core you should be able to distinguish a vector from its components; add and subtract vectors geometrically and algebraically; reconstruct a vector without losing its quadrant; separate a signed projection from a vector projection; evaluate and interpret dot and cross products. The supporting extension teaches transformations between rotated Cartesian axes. You should also be able to reject inconsistent inputs and test a numerical result with geometric invariants. The optional advanced section develops rotation matrices, moving bases, derivatives and the sensitivity of normals; a final bridge introduces spatial fields.

3.1 Scalars, vectors and geometric addition

3.1.1 What the vector represents

A *scalar* is specified by a numerical value and, when appropriate, a physical unit. Its value does not acquire a new component merely because we rotate our spatial axes. A signed temperature difference is still a scalar; a negative sign by itself does not make a quantity a vector. Conversely, the three numbers in a list of temperatures at three sensors do not form a spatial vector just because there are three of them. Their transformation rule is not that of a displacement.

A displacement from point P to point Q is represented by an oriented straight segment. Its magnitude is the straight-line separation of the endpoints; its direction is from P to Q . It says nothing by itself about the path followed between those points. We write a vector in bold, such as $\mathbf{a}$, and its magnitude as $a = \|\mathbf{a}\|$. Magnitudes are *nonnegative*; signed components can be positive, zero or negative.

Definition 3.1: Equality and the zero vector

Two free vectors are equal when they have the same magnitude and direction. Translating an arrow without rotating or rescaling it leaves that free vector unchanged. The zero vector $\mathbf{0}$ has magnitude zero and no uniquely defined direction. It is the additive identity: $\mathbf{a} + \mathbf{0} = \mathbf{a}$.

The adjective *free* matters. A displacement can be copied to another location to perform an addition.

A *position vector*, however, is drawn from a specified origin to a specified point. Changing the origin changes that position vector. Likewise, a physical force may need its point of application as well as its vector value; moving its arrow to another point need not preserve its mechanical effect. Free-vector algebra does not erase this additional physical information.

A scaled drawing also needs a unit conversion. For example, an arrow 4 cm long on paper may represent 8 m of displacement. Its paper length and its physical magnitude are not the same quantity. All arrows compared by length in one quantitative construction must use the same physical scale.

3.1.2 Addition follows endpoints, not magnitudes

To construct $\mathbf{a} + \mathbf{b}$, copy $\mathbf{b}$ so that its tail lies at the head of $\mathbf{a}$. Draw the resultant from the first tail to the last head. This is the *head-to-tail rule*. If instead both original arrows start at one point, complete a parallelogram; its diagonal from the common tail gives the same sum. The other diagonal represents a difference, with its orientation still to be specified.

Figure 3.1 uses one constructed survey in both representations. The legs are 8 m at 30° and 5 m at 150° , with bearings measured counterclockwise from the positive horizontal axis. An obstacle motivates the indirect route, but the diagonal remains a valid description of displacement even when it is not a physically accessible route.

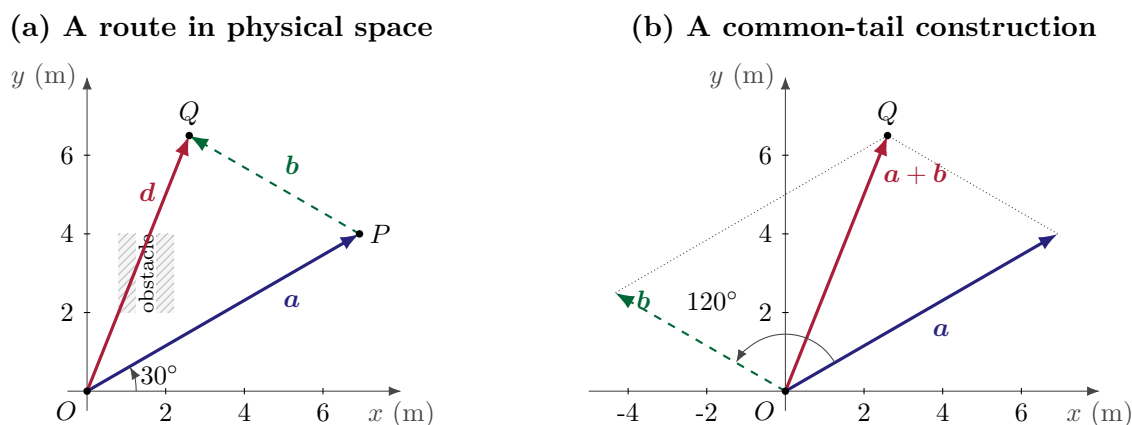


Figure 3.1: Two representations of the same constructed displacement. Both panels use the same horizontal and vertical scale: $\mathbf{a}$ has magnitude 8 m and bearing 30° ; $\mathbf{b}$ has magnitude 5 m and bearing 150° . In (a), $O \rightarrow P \rightarrow Q$ is the indirect route, and the hatched rectangle is a stipulated obstacle. In (b), translated copies complete the parallelogram. The solid resultant and the dashed second leg remain distinguishable without colour. The diagonal is a displacement, not a claim that the direct route is accessible.

The parallelogram establishes commutativity: starting with $\mathbf{b}$ instead of $\mathbf{a}$ reaches the same diagonal endpoint. A chain of three translated arrows establishes associativity: placing parentheses around either pair does not change the first or last point. Thus

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}, \quad (\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}). \quad (3.1)$$

These are statements about the resultant. They do not imply that the order of real moves is irrelevant to obstacles, travel time or equipment constraints.

The straight segment between two endpoints cannot exceed the length of a broken route joining them. This gives the triangle inequality. Applying it once more to $\mathbf{a} = (\mathbf{a} + \mathbf{b}) - \mathbf{b}$ gives the lower bound:

$$\|\mathbf{a}\| - \|\mathbf{b}\| \leq \|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|. \quad (3.2)$$

For nonzero vectors, the upper equality occurs for the same direction and the lower equality for opposite directions. Two equal opposite vectors sum to zero. There is no contradiction between zero net displacement and a nonzero distance travelled on a closed route.

3.1.3 Subtraction and multiplication by a scalar

The vector $-\mathbf{b}$ has the same magnitude as $\mathbf{b}$ but the opposite direction. Subtraction is addition of that reversed vector:

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}). \quad (3.3)$$

With common tails, $\mathbf{a} - \mathbf{b}$ points from the head of $\mathbf{b}$ to the head of $\mathbf{a}$. Reversing the order reverses the answer. For example, if $\mathbf{a}$ is 4 m east and $\mathbf{b}$ is 7 m west, then $\mathbf{a} - \mathbf{b}$ is 11 m east, not 3 m in an unspecified direction. The difference of the magnitudes is not generally the magnitude of the difference.

Multiplication by a positive dimensionless number stretches or contracts an arrow; a negative multiplier also reverses it. For any real λ ,

$$\|\lambda \mathbf{a}\| = |\lambda| \|\mathbf{a}\|. \quad (3.4)$$

The absolute value is essential. Multiplication by a dimensional scalar is also possible, but changes the physical units. Dividing a displacement by a positive time interval produces a vector with units of velocity, not another displacement. Adding vectors requires compatible physical quantities and units; a displacement cannot be added to a velocity simply because their component lists have equal length.

Guided checks

Can two vectors of magnitudes 8 m and 5 m have a zero sum? No: the smallest possible resultant is 3 m. Can three nonzero vectors sum to zero? Yes: their head-to-tail construction can form a closed triangle. In a subtraction, check the orientation by adding back the subtracted vector; the result must be the original vector.

3.1.4 Reading a difference directly from a construction

The arrow joining two vector heads is useful whenever two positions, readings, or commands are compared. It must be oriented according to the subtraction order. Figure 3.2 deliberately keeps the two original arrows at a common tail. Following $O \rightarrow B \rightarrow A$ must have the same net displacement as following $O \rightarrow A$. Consequently the arrow $B \rightarrow A$ is $\mathbf{a} - \mathbf{b}$. This endpoint argument determines its sign before any numbers are calculated. A dashed translated copy of $-\mathbf{b}$ gives a second construction of the same free vector.

Example 3.1: Comparing two displacement records

Statement and model. Two inspection records, referred to one fixed origin and the same east–north axes, give $\mathbf{a} = (4, 1) \text{ m}$ and $\mathbf{b} = (1, 3) \text{ m}$. Find the displacement from the endpoint of $\mathbf{b}$ to that of $\mathbf{a}$, its length, and a command that returns from the endpoint of $\mathbf{b}$ to the origin. Treat both records as exact planar values.

Strategy. Read the endpoint order from Figure 3.2. Subtract the coordinates of the starting point from those of the destination; do not subtract two unsigned lengths. The requested comparison displacement is

$$\mathbf{d} = \mathbf{a} - \mathbf{b} = (3\hat{x} - 2\hat{y}) \text{ m}, \quad d = \sqrt{3^2 + (-2)^2} \text{ m} = \sqrt{13} \text{ m} \simeq 3.606 \text{ m}.$$

It points east and south, as the diagram predicts. By contrast, the difference of magnitudes is $\sqrt{17} - \sqrt{10} \simeq 0.961 \text{ m}$, which is a different scalar and does not specify this displacement.

The command from B to the origin is $-\mathbf{b} = (-1, -3) \text{ m}$. The command from A to B is $-\mathbf{d} = (-3, 2) \text{ m}$; it has the same length as $\mathbf{d}$ and the opposite orientation. These two commands answer different questions despite both involving a reversal.

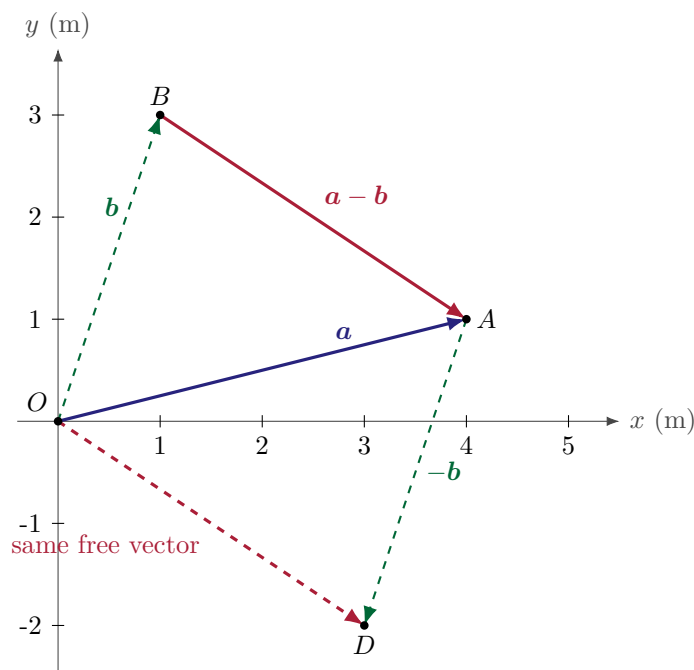


Figure 3.2: Two geometrically equivalent constructions of subtraction. Prescribed displacements are $\mathbf{a} = (4, 1)$ m and $\mathbf{b} = (1, 3)$ m. The solid red arrow is $B \rightarrow A$; the dashed red arrow is its translated copy $O \rightarrow D$, where $D = (3, -2)$ m. The additional dashed green arrow implements $\mathbf{a} + (-\mathbf{b})$. Horizontal and vertical scales are equal. The subtraction order determines the orientation.

Checks and interpretation. Adding back the subtracted record gives $\mathbf{b} + \mathbf{d} = (1, 3) + (3, -2) = (4, 1)$ in metres. The loop $O \rightarrow B \rightarrow A \rightarrow O$ closes because $\mathbf{b} + \mathbf{d} - \mathbf{a} = \mathbf{0}$. This closure is a stronger check than comparing lengths: it tests both signed components. If the two records used different origins, their direct difference would require an origin conversion before this construction could be used.

3.1.5 How the angle controls a resultant

Hold the lengths a and b fixed and vary their common-tail angle ϕ between 0° and 180° . The head-to-tail triangle contains the supplementary angle $180^\circ - \phi$. Applying its cosine rule gives

$$R^2 = a^2 + b^2 + 2ab \cos \phi, \quad R = \|\mathbf{a} + \mathbf{b}\|. \quad (3.5)$$

The plus sign belongs to a sum with a *common-tail* angle; confusing this angle with the triangle's interior angle produces an incorrect sign. At $\phi = 0^\circ$, the arrows reinforce and $R = a + b$. At 180° , they oppose and $R = |a - b|$. For an intermediate angle, the resultant falls between those limits. The familiar $\sqrt{a^2 + b^2}$ occurs at 90° and nowhere else for fixed positive a, b .

Figure 3.3 makes this dependence quantitative. It also supplies an inverse construction: a measured resultant can determine an angle, even though the two lengths alone cannot. The graph varies geometry, not time; no motion law or force model is being assumed.

Example 3.2: Inferring an angle from the resultant length

Statement. Two planar translations have lengths 5 m and 3 m, and their sum has length 4 m. Determine their unsigned common-tail angle. Explain what remains unknown and assess a proposed resultant of 9 m using the same two translation lengths.

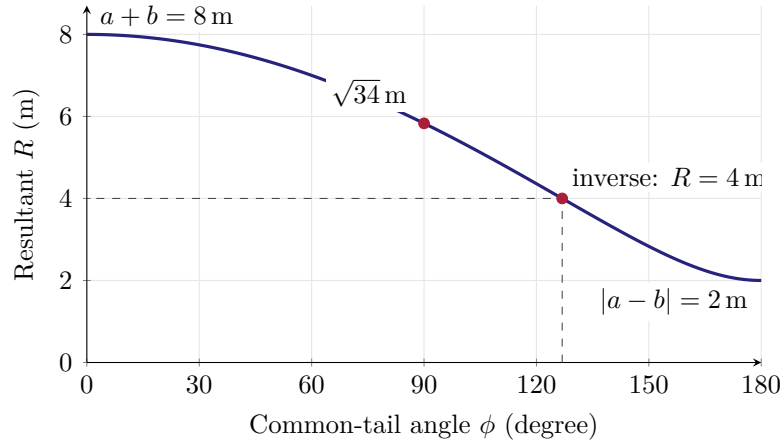


Figure 3.3: An analytical graph for two prescribed displacement lengths $a = 5$ m and $b = 3$ m. Only their included common-tail angle varies. The solid curve is $R(\phi) = \sqrt{34 + 30 \cos \phi}$ m, with the trigonometric input expressed in degrees. The marked inverse construction selects $\phi = 126.87^\circ$ from $R = 4$ m; the other point shows the perpendicular case. The axes represent unlike quantities and do not depict a spatial trajectory.

Model and choice of method. No absolute bearing is supplied. Therefore use the cosine rule for lengths rather than inventing component directions. Rearrange Eq. (3.5) before substitution:

$$\cos \phi = \frac{R^2 - a^2 - b^2}{2ab} = \frac{16 - 25 - 9}{30} = -\frac{3}{5}, \quad \phi = \arccos(-3/5) = 126.87^\circ.$$

The negative cosine requires an obtuse angle, consistent with a resultant shorter than the perpendicular value $\sqrt{34}$ m. Figure 3.3 supplies the same inference through the curve's intersection with the horizontal $R = 4$ m line.

For a component check only, choose the first vector along positive x . One possible second vector is then $\mathbf{b} = 3(-3/5, 4/5) \text{ m} = (-1.8, 2.4) \text{ m}$. The sum is $(3.2, 2.4) \text{ m}$ and its length is $\sqrt{3.2^2 + 2.4^2} = 4 \text{ m}$. Reflecting the construction gives $(-1.8, -2.4) \text{ m}$ for the second vector and the same resultant length. An arbitrary common rotation gives further possibilities. Thus the unsigned angle is fixed, but an absolute bearing and a clockwise/counterclockwise ordering are not.

Admissibility check. A 9 m resultant would imply $\cos \phi = (81 - 34)/30 = 47/30 > 1$. The equivalent triangle bound is $2 \text{ m} \leq R \leq 8 \text{ m}$. Both arguments reject the proposed exact inputs. Applying an inverse cosine to a clipped value of one would replace the inconsistent record with a different problem; it would not solve it.

3.2 Coordinates, components and unit vectors

3.2.1 Choosing a Cartesian basis

Choose an origin O and two perpendicular positive directions in a plane. In three-dimensional space choose a third perpendicular direction, with a right-handed orientation: curling the right hand from positive x toward positive y points the thumb along positive z . We use $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ for the corresponding *unit vectors*. Each has magnitude one and is dimensionless. A metre belongs to a length component, not to the unit vector multiplying it.

Drop perpendiculars from the head of an arrow to the axes. The signed lengths along the axes are its scalar components. The arrow can be reconstructed by following its axial parts head to tail:

$$\mathbf{a} = a_x \hat{\mathbf{x}} + a_y \hat{\mathbf{y}} + a_z \hat{\mathbf{z}}. \quad (3.6)$$

Here a_x is a signed scalar, whereas $a_x\hat{x}$ is a vector. A component is not the unsigned length of a projection line. For a displacement pointing partly toward negative x , $a_x < 0$ even though $|a_x| > 0$.

The component list (a_x, a_y, a_z) is shorthand for a vector *in the stated basis*. Without the basis and units, the list is incomplete. Conversely, in one fixed basis the decomposition is unique. If two different component lists represented the same vector, their difference would give a nonzero combination of perpendicular directions equal to zero, which is impossible.

Pythagoras applied first in the xy plane gives the horizontal projection length $a_{xy} = \sqrt{a_x^2 + a_y^2}$. Applying it between that projection and the z component gives

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}. \quad (3.7)$$

This Euclidean formula assumes orthonormal axes: perpendicular basis vectors of unit length. It is not a formula for arbitrary skew coordinates. Squared components have the square of the original unit; taking the square root restores the vector's unit. No direction can be reconstructed from this magnitude alone.

For points with position vectors $\mathbf{r}_P$ and $\mathbf{r}_Q$ referred to the *same* origin and basis,

$$\overrightarrow{PQ} = \mathbf{r}_Q - \mathbf{r}_P = (x_Q - x_P)\hat{x} + (y_Q - y_P)\hat{y} + (z_Q - z_P)\hat{z}. \quad (3.8)$$

The endpoint coordinates of an arrow drawn away from the origin are not its components; the endpoint *differences* are.

3.2.2 Angles and components in a plane

Let $a > 0$, and let θ be measured counterclockwise from positive x . The right-triangle definitions of cosine and sine, extended with their quadrant signs, give

$$a_x = a \cos \theta, \quad a_y = a \sin \theta. \quad (3.9)$$

An angle measured clockwise from north is a different convention. Translate that bearing into the declared convention before using this equation. For example, 20° west of north corresponds to $\theta = 110^\circ$, not 20° .

Conversely, $\tan \theta = a_y/a_x$ when $a_x \neq 0$, but a single inverse tangent does not determine the quadrant. Both $(1, 1)$ and $(-1, -1)$ have the ratio $a_y/a_x = 1$. A safe procedure is to predict the quadrant from the component signs, then use

$$\theta = \text{atan2}(a_y, a_x), \quad \mathbf{a} \neq \mathbf{0}, \quad (3.10)$$

or a reference angle with an explicit quadrant correction. The order of the two arguments is y, x . A negative returned angle can be increased by 360° to express the same bearing in $[0, 360^\circ)$. For a vertical vector use 90° or 270° according to the sign of a_y ; do not divide by zero. Software may return a number for two zero arguments, but that does not give the zero vector a physical direction.

Angles can be expressed in degrees or radians, with $180^\circ = \pi \text{ rad}$. The ratios entering trigonometric functions are dimensionless. A calculator or program must know which angle unit is supplied. The derivative formulas in the advanced section use radians.

Example 3.3: A survey around an obstacle

Use the two legs in Figure 3.1 to find the resultant displacement, its bearing and its relation to the route length. Treat the stipulated lengths and bearings as exact inputs of this geometric model; no surveying precision is being asserted.

Representation and strategy. Use east as positive x and north as positive y . Both legs

have positive y components; the second has a negative x component. Resolve each leg and add corresponding components:

$$\begin{aligned}\mathbf{a} &= (8 \cos 30^\circ, 8 \sin 30^\circ) \text{ m} = (4\sqrt{3}, 4) \text{ m}, \\ \mathbf{b} &= (5 \cos 150^\circ, 5 \sin 150^\circ) \text{ m} = (-5\sqrt{3}/2, 5/2) \text{ m}, \\ \mathbf{d} &= \mathbf{a} + \mathbf{b} = (3\sqrt{3}/2, 13/2) \text{ m}.\end{aligned}$$

Thus $d_x \simeq 2.598 \text{ m}$, $d_y = 6.500 \text{ m}$ and

$$d = \sqrt{27/4 + 169/4} \text{ m} = 7.000 \text{ m}, \quad \theta_d = \text{atan2}(6.5, 2.598 \dots) \simeq 68.21^\circ.$$

The broken route has length $(8 + 5) \text{ m} = 13 \text{ m}$; this is not the magnitude of the sum. The resultant satisfies $3 \leq d/(1 \text{ m}) \leq 13$.

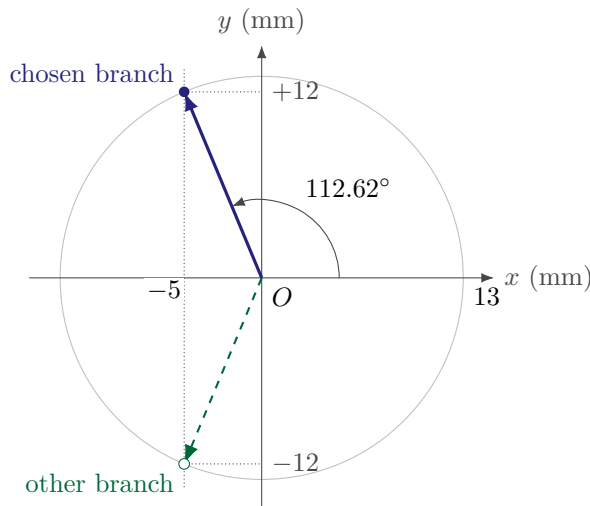
Independent geometric check. The angle between the common-tail legs is 120° . The head-to-tail triangle has included angle 60° , so the cosine rule gives $d^2 = [8^2 + 5^2 - 2(8)(5) \cos 60^\circ] \text{ m}^2 = 49 \text{ m}^2$. If the same two lengths were exactly opposite, the result would instead be 3 m along the longer leg. Equal opposite legs would close the route. The obstacle does not alter these vector identities; it restricts which route can actually be followed.

3.2.3 Reconstruction can be ambiguous or impossible

Knowing a magnitude and one component need not determine a vector. In a plane,

$$a_y = \pm \sqrt{a^2 - a_x^2}. \quad (3.11)$$

There are two branches when $|a_x| < a$, one when $|a_x| = a$, and no real solution when $|a_x| > a$. The radicand is an admissibility test before it is a calculation. A direction statement, another signed measurement or a geometric constraint must select a branch. In three dimensions the same information generally leaves a circle of possible transverse components, not just two possibilities.



What each constraint supplies

The circle specifies $\|\mathbf{d}\| = 13 \text{ mm}$. The vertical dotted line specifies $d_x = -5 \text{ mm}$. Their two intersections give $d_y = \pm 12 \text{ mm}$. The extra condition $d_y > 0$ selects the filled point.

Quadrant check

$d_x < 0$, $d_y > 0$ places the chosen vector in quadrant II. A negative answer from $\arctan(12/(-5))$ alone would select the wrong bearing.

Figure 3.4: A scaled geometric reconstruction with an explicit two-branch ambiguity. Both axes use millimetres and equal scales; the circle is calculated from $x^2 + y^2 = 169 \text{ mm}^2$. The angle is measured counterclockwise from positive x . The dashed arrow is an admissible alternative before the upward-direction observation is imposed, not an uncertainty interval.

Example 3.4: Reconstructing a measured displacement

A hypothetical inspection record specifies a displacement magnitude $L = 13$ mm and signed horizontal component $d_x = -5$ mm. An independent observation says that the endpoint lies above the horizontal through the start. Find the displacement and its bearing. Then assess a second record with $L = 10$ mm and $d_x = -11$ mm.

Solve the geometry before selecting a sign. The first record gives $d_y^2 = L^2 - d_x^2 = (169 - 25) \text{ mm}^2 = 144 \text{ mm}^2$, so $d_y = \pm 12$ mm. The observation selects +12 mm, not the square root operation alone. Therefore

$$\mathbf{d} = (-5\hat{x} + 12\hat{y}) \text{ mm}, \quad \theta = \text{atan2}(12, -5) = 112.62^\circ \quad (\text{approximately}).$$

Both the sign prediction and the angle put the vector in quadrant II. Substitution returns $\sqrt{(-5)^2 + 12^2} = 13$, and $13 \cos \theta = -5$ in millimetres.

For the second record, $L^2 - d_x^2 = -21 \text{ mm}^2$. No real planar displacement satisfies both exact values. Replacing the negative radicand by its absolute value would invent a solution. The record should be checked for units, calibration, transcription or uncertainty intervals. Bounds that overlap an admissible set could change the conclusion, but no bounds were supplied here.

Key message

A component belongs to a chosen axis; a magnitude belongs to the vector. When an inverse problem is ambiguous, retain the branches until the physical information selects one. When its inputs are inconsistent, a refusal to produce a real vector is a successful modelling check, not a failed calculation.

3.2.4 Quadrant information is part of a direction

The four vectors in Figure 3.5 have the same magnitude and the same absolute component ratio. They nevertheless have four different bearings. The signs of both components provide information that disappears if their ratio alone is formed. For $|a_x| = 3$ and $|a_y| = 4$, first compute the acute reference angle $\alpha = \arctan(4/3) = 53.13^\circ$. Then use the signs to choose α , $180^\circ - \alpha$, $180^\circ + \alpha$, or $360^\circ - \alpha$. This construction reproduces the quadrant-aware atan2 result without relying on a calculator's default output interval.

The argument order is as consequential as the signs. A routine called as $\text{atan2}(a_x, a_y)$ returns an angle associated with a different reference axis. A degree/radian mistake is a separate problem: correct quadrant handling cannot compensate for the wrong angle unit. Reconstructing both components from the computed magnitude and angle tests these choices together.

3.2.5 Three-dimensional displacement and direction cosines

A three-dimensional direction generally needs more than one angle. Define $\alpha_x, \alpha_y, \alpha_z$ as the angles between a nonzero vector and the *positive* Cartesian axes. Each is in $[0^\circ, 180^\circ]$. The signed component formulas give

$$\cos \alpha_x = \frac{a_x}{a}, \quad \cos \alpha_y = \frac{a_y}{a}, \quad \cos \alpha_z = \frac{a_z}{a}, \quad \cos^2 \alpha_x + \cos^2 \alpha_y + \cos^2 \alpha_z = 1. \quad (3.12)$$

The final relation follows by dividing $a_x^2 + a_y^2 + a_z^2 = a^2$ by a^2 ; the three axis angles are constrained, rather than independent choices. An obtuse axis angle is appropriate when that component is negative. The three angles do not generally add to 180° .

Orthogonal views avoid a common trap in perspective drawings: an apparent screen angle need not equal a physical three-dimensional angle. Figure 3.6 uses a plan view to obtain the horizontal

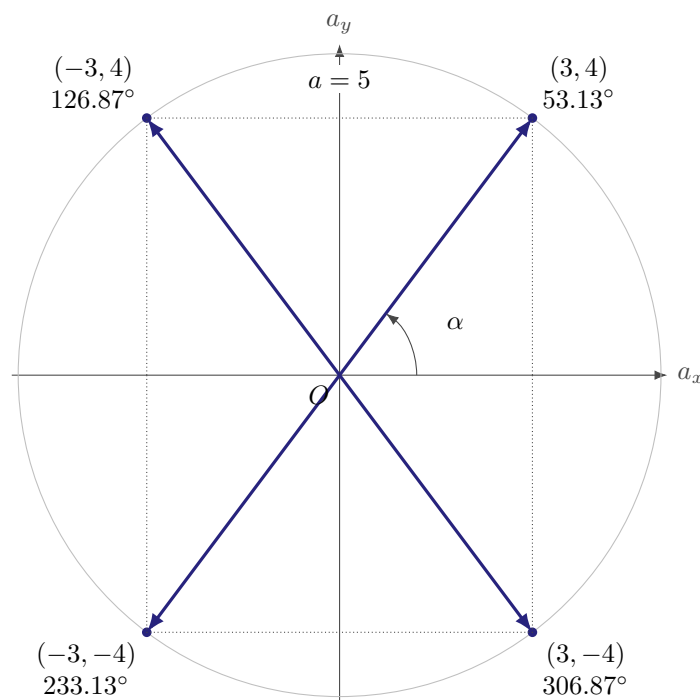


Figure 3.5: Four distinct directions with $|a_x| = 3$, $|a_y| = 4$ and $a = 5$ in a common arbitrary component unit. Equal horizontal and vertical scales make the endpoint circle quantitative. Bearings are measured counterclockwise from positive x and reported in $[0, 360^\circ)$. The acute reference angle is $\alpha = 53.13^\circ$; the component signs select the quadrant. No bearing is assigned to the origin.

displacement and an elevation view to combine it with the vertical component. Both right triangles are genuine spatial constructions.

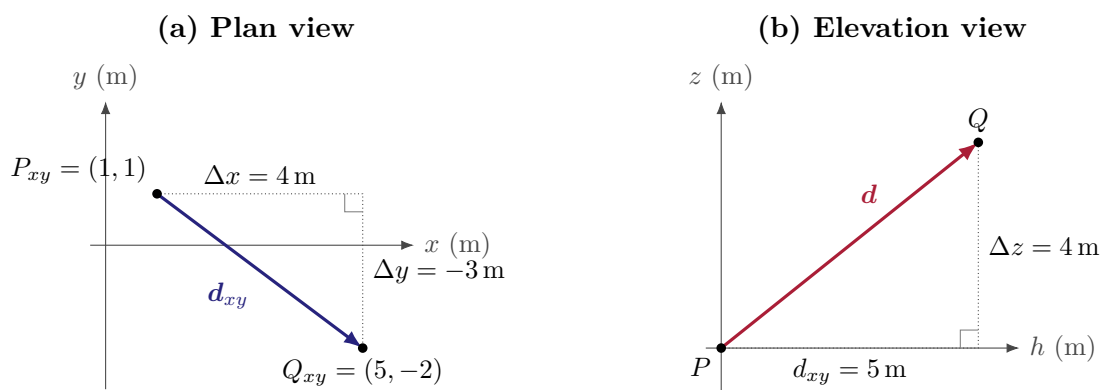


Figure 3.6: Orthogonal views of the displacement from $P = (1, 1, 1)$ m to $Q = (5, -2, 5)$ m. In the plan view, $d_{xy} = \sqrt{4^2 + (-3)^2} = 5$ m. The elevation is taken in the vertical plane containing that horizontal displacement; its horizontal coordinate h is distance along d_{xy} from P , not the original x coordinate. It combines 5 m horizontally with 4 m vertically. Each panel uses equal physical scales on its axes.

Example 3.5: A three-dimensional endpoint survey

Statement and assumptions. In one fixed orthonormal frame, two surveyed points are $P = (1, 1, 1)$ m and $Q = (5, -2, 5)$ m. Find the displacement $P \rightarrow Q$, its magnitude, its unit direction and all three positive-axis angles. Determine which results survive a change of origin. The coordinates are exact values in a Euclidean geometric model.

Strategy and calculation. Subtract endpoints before calculating a norm. Figure 3.6 provides

the two successive right triangles:

$$\mathbf{d} = \mathbf{r}_Q - \mathbf{r}_P = (4\hat{x} - 3\hat{y} + 4\hat{z}) \text{ m}, \quad d_{xy} = 5 \text{ m}, \quad d = \sqrt{5^2 + 4^2} \text{ m} = \sqrt{41} \text{ m} \simeq 6.403 \text{ m}.$$

The direction vector is dimensionless, $\hat{\mathbf{u}} = (4, -3, 4)/\sqrt{41}$, so the axis angles are

$$\alpha_x = \alpha_z = \arccos(4/\sqrt{41}) \simeq 51.34^\circ, \quad \alpha_y = \arccos(-3/\sqrt{41}) \simeq 117.94^\circ.$$

The negative y component requires the obtuse y angle. Reporting its acute supplement would describe a different direction. The elevation above the horizontal plane is $\arctan(4/5) \simeq 38.66^\circ$, which is complementary to α_z but is neither α_x nor α_y .

Invariant and dimensional checks. The squared unit-direction components sum to $(16 + 9 + 16)/41 = 1$. Multiplying each by $\sqrt{41} \text{ m}$ reconstructs $\mathbf{d}$. Translate the origin by a fixed vector $\mathbf{c}$: the positions become $\mathbf{r}'_P = \mathbf{r}_P - \mathbf{c}$ and $\mathbf{r}'_Q = \mathbf{r}_Q - \mathbf{c}$, but their difference remains $\mathbf{d}$. Its length and axis angles therefore survive this origin shift because the axes have not been rotated. A rotation of the axes would retain the length while changing the components and their individual axis angles.

3.2.6 Why perpendicular basis vectors matter

The word *component* can mean a coefficient in a basis expansion or a perpendicular scalar projection. They coincide for an orthonormal basis. They generally differ for an oblique basis. To see the distinction without introducing new mechanics, take two unit vectors separated by 60° :

$$\hat{\mathbf{e}}_1 = (1, 0), \quad \hat{\mathbf{e}}_2 = (1/2, \sqrt{3}/2), \quad \mathbf{a} = c_1\hat{\mathbf{e}}_1 + c_2\hat{\mathbf{e}}_2.$$

Because both basis vectors have length one, c_1, c_2 carry the physical unit of $\mathbf{a}$. Yet Pythagoras cannot be applied to these two coefficient arrows; their included angle is not a right angle. Expanding the Cartesian coordinates gives

$$a_x = c_1 + \frac{1}{2}c_2, \quad a_y = \frac{\sqrt{3}}{2}c_2, \quad a^2 = c_1^2 + c_2^2 + c_1c_2. \quad (3.13)$$

The extra cross term is essential. The drawn coefficient parallelogram in Figure 3.7 also makes clear why projecting the resultant onto either oblique line need not return that line's expansion coefficient.

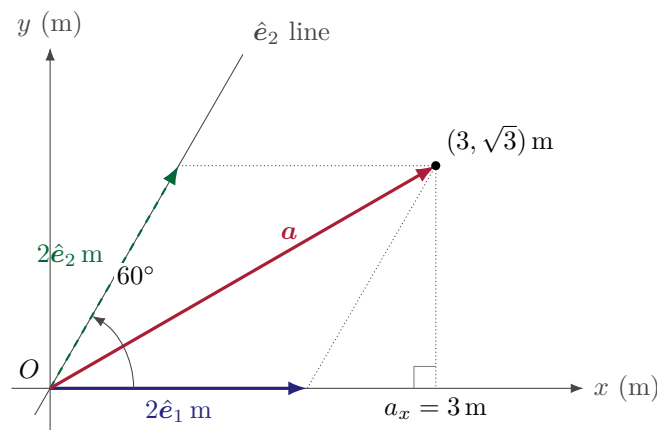


Figure 3.7: A quantitative oblique-basis construction with equal Cartesian scales. Both basis vectors are unit vectors, but their angle is 60° . Expansion coefficients $c_1 = c_2 = 2 \text{ m}$ generate $\mathbf{a} = (3, \sqrt{3}) \text{ m}$. Its projection on the horizontal basis direction is 3 m rather than the coefficient 2 m . The dotted parallelogram follows basis coefficients; the vertical dotted line is a perpendicular Cartesian projection.

Example 3.6: Diagnosing a skew-coordinate norm

Statement. A positioning command is supplied as coefficients $(c_1, c_2) = (2, 2)$ m in the two unit directions of Figure 3.7. A calculation reports its length as $\sqrt{2^2 + 2^2}$ m. Determine the correct Cartesian vector and length, and explain why the reported procedure fails.

Strategy and solution. Reconstruct the physical arrow from its declared basis. The second coefficient contributes to *both* Cartesian components:

$$\mathbf{a} = 2(1, 0) \text{ m} + 2(1/2, \sqrt{3}/2) \text{ m} = (3, \sqrt{3}) \text{ m}.$$

The magnitude is $\sqrt{9 + 3} \text{ m} = 2\sqrt{3} \text{ m} \simeq 3.464 \text{ m}$. Eq. (3.13) independently gives $a^2 = (4 + 4 + 4) \text{ m}^2 = 12 \text{ m}^2$. The reported $2\sqrt{2} \text{ m} \simeq 2.828 \text{ m}$ omits the positive cross term and underestimates the length by about 18.4%.

Once the scalar product is introduced, the same distinction can be stated as $\mathbf{a} \cdot \hat{\mathbf{e}}_1 = 3 \text{ m}$ and $\mathbf{a} \cdot \hat{\mathbf{e}}_2 = 3 \text{ m}$: neither projection is the corresponding expansion coefficient. The equality of the two projections follows from this symmetric example, not from a general oblique-basis rule.

Check and boundary. If the second basis direction were changed to $(0, 1)$, the cross term would vanish and the original Pythagorean procedure would become valid. If the two basis directions were made parallel, they would cease to provide unique coordinates for a plane. Here 60° is neither of those cases. In this chapter, ordinary unqualified component formulas refer to the stated orthonormal Cartesian basis.

3.3 Vector algebra and projections

3.3.1 Algebra in one stated basis

Because the basis decomposition is unique, vector addition and scalar multiplication become component operations:

$$\mathbf{a} + \mathbf{b} = (a_x + b_x)\hat{\mathbf{x}} + (a_y + b_y)\hat{\mathbf{y}} + (a_z + b_z)\hat{\mathbf{z}}, \quad (3.14)$$

$$\lambda\mathbf{a} - \mu\mathbf{b} = (\lambda a_x - \mu b_x)\hat{\mathbf{x}} + (\lambda a_y - \mu b_y)\hat{\mathbf{y}} + (\lambda a_z - \mu b_z)\hat{\mathbf{z}}. \quad (3.15)$$

The components being combined must refer to the same axes. A robot's local "forward" direction need not be the survey's positive x direction. Adding a local component list directly to a laboratory component list usually mixes two different descriptions. Section 3.6 supplies the required conversion.

For a closed sequence of displacements, $\sum_i \mathbf{d}_i = \mathbf{0}$, so each component sum must vanish. If a computed loop has a nonzero closing vector, either the route does not close or the inputs/model contain an inconsistency. Testing one component alone is not enough.

3.3.2 A direction without a length

Given a nonzero vector $\mathbf{b}$, its unit direction is

$$\hat{\mathbf{u}} = \frac{\mathbf{b}}{\|\mathbf{b}\|}, \quad \|\hat{\mathbf{u}}\| = 1. \quad (3.16)$$

Numerator and denominator have the same unit, so it cancels. For example, the edge vector $(3, 4) \text{ mm}$ gives $\hat{\mathbf{u}} = (3/5, 4/5)$, not $(3, 4)$ and not $(3/5, 4/5) \text{ mm}$. Scaling the edge by a positive number leaves this unit direction unchanged. Scaling by a negative number reverses it. Normalising zero is undefined. Normalising a very small measured vector may also be unreliable when its direction is dominated by measurement error.

3.3.3 Signed scalar and vector projections

Choose a unit direction $\hat{\mathbf{u}}$. The signed scalar projection of $\mathbf{a}$ along it is the signed length reached on that axis by a perpendicular from the head of $\mathbf{a}$. If θ is the angle between the two nonzero directions, this component is $a \cos \theta$. It is positive toward $\hat{\mathbf{u}}$, negative away from it, and zero for a perpendicular vector.

We use the dot notation for this projection on a unit direction. The complete scalar-product definition and its algebra are developed in the next section:

$$a_{\parallel} = \mathbf{a} \cdot \hat{\mathbf{u}}, \quad \mathbf{a}_{\parallel} = a_{\parallel} \hat{\mathbf{u}}, \quad \mathbf{a}_{\perp} = \mathbf{a} - \mathbf{a}_{\parallel}. \quad (3.17)$$

The first quantity is a scalar; the second and third are vectors. Both the scalar component and the projected vectors have the unit of $\mathbf{a}$. The perpendicular residual obeys $\mathbf{a}_{\perp} \cdot \hat{\mathbf{u}} = 0$, because subtracting the parallel part removes exactly the projection on the chosen axis.

If the reference direction is supplied as a nonunit vector $\mathbf{b} \neq \mathbf{0}$, normalise first. Equivalently,

$$\mathbf{a}_{\parallel} = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \mathbf{b}. \quad (3.18)$$

Multiplying $\mathbf{a} \cdot \mathbf{b}$ directly by $\mathbf{b}$ does not produce the projection: the magnitude and often the units would be wrong. Replacing $\hat{\mathbf{u}}$ by $-\hat{\mathbf{u}}$ reverses the signed scalar component but leaves the vector projection onto the *line* unchanged.

In a plane a useful perpendicular unit direction is $\hat{\mathbf{n}} = (-u_y, u_x)$. The pair is orthonormal, so

$$\mathbf{a} = (\mathbf{a} \cdot \hat{\mathbf{u}}) \hat{\mathbf{u}} + (\mathbf{a} \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}}, \quad \|\mathbf{a}\|^2 = a_{\parallel}^2 + \|\mathbf{a}_{\perp}\|^2. \quad (3.19)$$

The second equality is Pythagoras, not an assumption that all vector magnitudes add. In three dimensions the perpendicular residual belongs to a plane rather than to one uniquely chosen transverse axis.

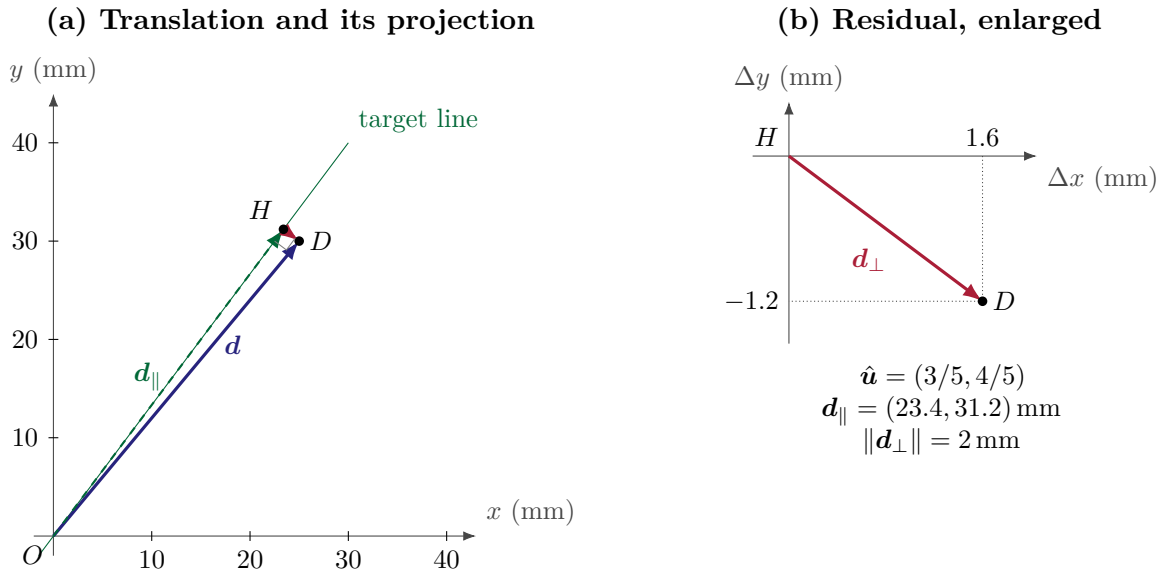


Figure 3.8: The net positioning displacement decomposed along an oblique line. The foot H is at (23.4, 31.2) mm and the endpoint D at (25, 30) mm. Panel (a) uses equal physical scales on its two axes; panel (b) translates the origin to H and enlarges the residual with its own equal millimetre scales. It is not a second displacement from the original origin. The displayed extent of the target line does not define the unit vector's magnitude.

Example 3.7: A positioning stage and an oblique target

Two commanded planar translations in one fixed laboratory basis are $\mathbf{p} = (40, 10)$ mm and $\mathbf{q} = (-15, 20)$ mm. A target line has direction $(3, 4)$. Find the net translation, its along-line component and its perpendicular error relative to a line through the starting point.

Strategy. Add physical translations, normalise the target direction, then split the net vector. Do not normalise the translation itself: its magnitude is part of the requested answer.

$$\mathbf{d} = \mathbf{p} + \mathbf{q} = (25, 30) \text{ mm}, \quad \hat{\mathbf{u}} = (3/5, 4/5), \quad \hat{\mathbf{n}} = (-4/5, 3/5).$$

The signed along-line component is $d_{\parallel} = [25(3/5) + 30(4/5)] \text{ mm} = 39 \text{ mm}$. Hence

$$\mathbf{d}_{\parallel} = (23.4, 31.2) \text{ mm}, \quad \mathbf{d}_{\perp} = (1.6, -1.2) \text{ mm} = -2 \text{ mm } \hat{\mathbf{n}}.$$

The shortest correction to the target *line* is $-\mathbf{d}_{\perp} = (-1.6, 1.2)$ mm, of magnitude 2 mm. A particular target *point* could require a further along-line correction.

Checks and interpretation. The dot product $[(1.6)(3/5) + (-1.2)(4/5)] \text{ mm} = 0 \text{ mm}$ verifies perpendicularity; the two parts reconstruct $(25, 30)$ mm. Independently, $25^2 + 30^2 = 39^2 + 2^2 = 1525$ in square millimetres. The net magnitude is about 39.051 mm, slightly larger than its 39 mm projection. The negative transverse component means the endpoint is on the $-\hat{\mathbf{n}}$ side; it does not mean the error length is negative.

A zero projection is meaningful even for the zero vector, provided the reference unit direction exists. An angle involving the zero vector is not meaningful. This distinction is useful in software: a projection may be defined when an angle routine must reject its inputs.

3.3.4 Projection as a signed response to changing direction

Keeping a nonzero vector's magnitude fixed while changing its angle to an oriented reference line gives a cosine response. This simple analytical model separates three quantities that are sometimes conflated:

$$\frac{a_{\parallel}}{a} = \cos \theta, \quad \frac{\|\mathbf{a}_{\parallel}\|}{a} = |\cos \theta|, \quad \frac{\|\mathbf{a}_{\perp}\|}{a} = \sin \theta \quad (0^\circ \leq \theta \leq 180^\circ).$$

Only the first ratio is signed. The final sine is nonnegative on this domain; it describes a perpendicular *length*, not a signed transverse component. Figure 3.9 shows why taking an absolute value too early destroys an orientation observation. The acute and obtuse marked angles have equal projection lengths but opposite signed projections.

A magnitude and signed projection determine an angle to a direction but usually do not determine a complete vector. In a plane, reflection across the reference line leaves the signed projection unchanged. In three dimensions, all transverse directions on a circle can satisfy the same two measurements. The perpendicular norm follows from Pythagoras, but its orientation still requires further data. This is a useful distinction when reconstructing a command from along-track and total-length observations.

3.3.5 The closest point on a shifted line

A target line need not pass through the coordinate origin. Let it be described by a known point $\mathbf{r}_0$ and a unit direction $\hat{\mathbf{u}}$:

$$\mathbf{r}(s) = \mathbf{r}_0 + s\hat{\mathbf{u}}, \quad -\infty < s < \infty. \quad (3.20)$$

Here s has units of length. Given a measured point with position $\mathbf{r}_P$, first form $\mathbf{a} = \mathbf{r}_P - \mathbf{r}_0$. Project *this difference*, rather than the position vector from an unrelated origin. The foot H and the residual from H to P are

$$s_* = \mathbf{a} \cdot \hat{\mathbf{u}}, \quad \mathbf{r}_H = \mathbf{r}_0 + s_*\hat{\mathbf{u}}, \quad \mathbf{e} = \mathbf{r}_P - \mathbf{r}_H. \quad (3.21)$$

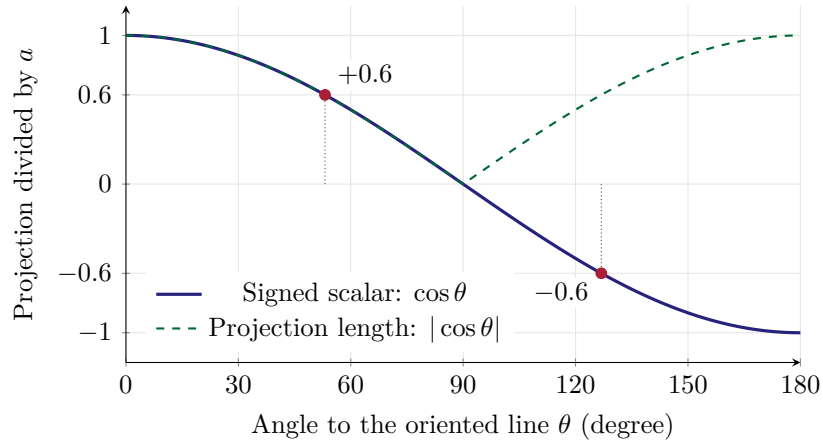


Figure 3.9: Analytical normalized projection curves for a fixed magnitude $a > 0$ and a varying angle to one oriented line. The solid signed scalar changes sign at 90° ; the dashed projection length remains nonnegative and folds the obtuse branch upward. On the acute branch the curves coincide, so the dashed line partly overlays the solid line. The points mark angles 53.13° and 126.87° , which have equal projection lengths.

The residual obeys $\mathbf{e} \cdot \hat{\mathbf{u}} = 0$. To prove that it is the shortest separation, consider an arbitrary other point on the line:

$$\mathbf{r}_P - \mathbf{r}(s) = \mathbf{e} + (s_* - s)\hat{\mathbf{u}}, \quad \|\mathbf{r}_P - \mathbf{r}(s)\|^2 = \|\mathbf{e}\|^2 + (s - s_*)^2.$$

The two terms are perpendicular. Since the final square is nonnegative, the minimum occurs at $s = s_*$. This is a geometric argument for the nearest point, and it also supplies an independent numerical check of an implementation.

Figure 3.10 shows both the physical geometry and the distance-squared function for one command. A squared distance is often convenient because it has the same minimum as the distance and avoids a square root during the minimization.

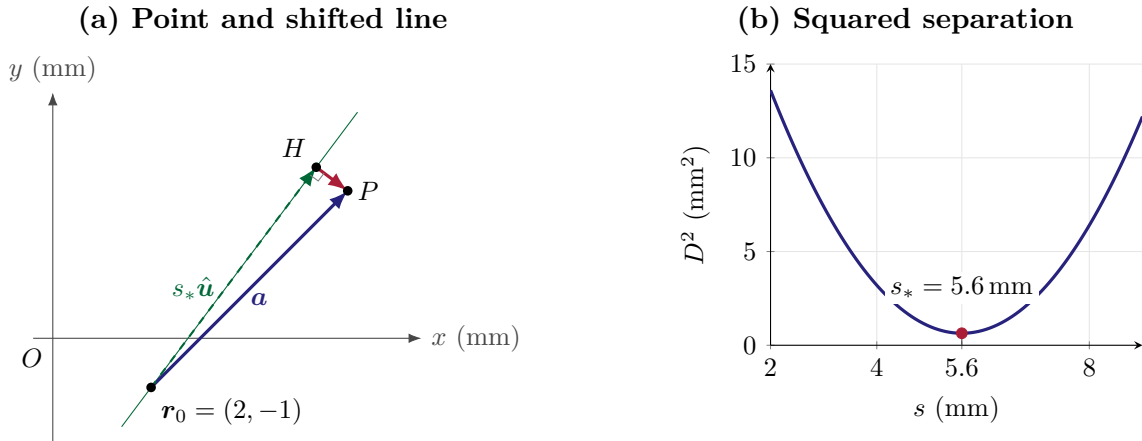


Figure 3.10: A target line through $(2, -1)$ mm with unit direction $\hat{\mathbf{u}} = (0.6, 0.8)$ and a measured point $P = (6, 3)$ mm. Panel (a) has equal spatial scales and shows the orthogonal foot $H = (5.36, 3.48)$ mm; the small red residual is $H \rightarrow P$. Panel (b) is the independently derived analytical curve $D^2(s) = 0.64 \text{ mm}^2 + (s - 5.6 \text{ mm})^2$. Its minimum verifies the foot calculation. The plot's domain is a displayed interval, not a restriction of the infinite target line.

Example 3.8: Recovering a closest command from point data

Statement and model. A planar guide is the infinite straight line through $\mathbf{r}_0 = (2, -1)$ mm in unit direction $\hat{\mathbf{u}} = (3/5, 4/5)$. A positioning record gives $\mathbf{r}_P = (6, 3)$ mm. Find the closest

point on the guide, the shortest corrective displacement from P , and the minimum distance. Then find the nearest allowed point if the guide instead permits only $0 \leq s \leq 4$ mm. All coordinates refer to one orthonormal frame; guide thickness and measurement uncertainty are neglected.

Strategy and symbolic solution. Translate to the reference point of the line, project, and translate back. The translated vector is $\mathbf{a} = \mathbf{r}_P - \mathbf{r}_0 = (4, 4)$ mm. Thus

$$s_* = (4 \cdot 3/5 + 4 \cdot 4/5) \text{ mm} = 5.6 \text{ mm},$$

$$\mathbf{r}_H = (2, -1) \text{ mm} + 5.6(3/5, 4/5) \text{ mm} = (5.36, 3.48) \text{ mm}.$$

The residual and corrective command have opposite orientations:

$$\mathbf{e} = \mathbf{r}_P - \mathbf{r}_H = (0.64, -0.48) \text{ mm}, \quad \mathbf{\delta} = \mathbf{r}_H - \mathbf{r}_P = (-0.64, 0.48) \text{ mm}.$$

Their common magnitude is $\sqrt{0.64^2 + 0.48^2} \text{ mm} = 0.80 \text{ mm}$. The distance-squared expression in Figure 3.10 is $D^2(s) = 0.64 + (s - 5.6)^2$ when all values are expressed in millimetres. It explicitly verifies the minimum and shows why another along-line displacement increases the separation.

Finite-guide extension. The perpendicular foot lies outside the allowed interval. The closest permitted value is its nearer endpoint, $s = 4$ mm, giving $\mathbf{r}_E = (4.4, 2.2)$ mm. The correction is $\mathbf{r}_E - \mathbf{r}_P = (-1.6, -0.8)$ mm and its length is $\sqrt{3.2} \text{ mm} \simeq 1.789 \text{ mm}$. This correction is not perpendicular to the infinite line because the admissible set now ends before the perpendicular foot. The squared-distance model checks it directly: $D^2(4) = [0.64 + (4 - 5.6)^2] \text{ mm}^2 = 3.2 \text{ mm}^2$.

Further checks. The infinite-line residual satisfies $\mathbf{e} \cdot \hat{\mathbf{u}} = 0.64(0.6) - 0.48(0.8) = 0$ in millimetres. Adding the correction to the original position gives the reported target point. Using $\mathbf{r}_P \cdot \hat{\mathbf{u}}$ without subtracting $\mathbf{r}_0$ would project onto a parallel line through the origin and answer a different geometric question. Constraints must therefore be stated before projecting.

3.4 The scalar product

3.4.1 The geometric definition and its sign

A projection asks how much of one vector lies along a chosen direction. To compare two vectors symmetrically, multiply that signed projection by the magnitude of the reference vector.

Definition 3.2: Scalar or dot product

For nonzero vectors with angle $0 \leq \theta \leq \pi$ between their directions,

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \theta. \quad (3.22)$$

If either vector is zero, define the product to be zero. The result is a scalar with the product of the two vectors' units.

The geometric construction is symmetric: projecting $\mathbf{a}$ along $\mathbf{b}$ and multiplying by b gives the same result as projecting $\mathbf{b}$ along $\mathbf{a}$ and multiplying by a . Thus $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.

For nonzero vectors an acute angle gives a positive product, a right angle gives zero, and an obtuse angle gives a negative product. Parallel vectors in the same direction give ab ; antiparallel vectors give $-ab$. A dot product is not itself a length or an angle, although it can be used to find both. In particular, its sign cannot be interpreted as the sign of a vector magnitude.

3.4.2 Deriving the component formula

Projection on a fixed line is additive: joining two displacements adds their signed advances along that line. This establishes distributivity of the dot product over vector addition. The products of the Cartesian unit vectors are

$$\hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1, \quad \hat{x} \cdot \hat{y} = \hat{y} \cdot \hat{z} = \hat{z} \cdot \hat{x} = 0.$$

Expand both vectors in their common basis and distribute. Of the nine pairwise terms, the six involving perpendicular unit vectors vanish; the other three give

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z. \quad (3.23)$$

This formula relies on orthonormality. It is not an instruction to multiply two arbitrary three-entry lists without checking their meanings or coordinate frames. It also explains why the signed component on an axis is $a_x = \mathbf{a} \cdot \hat{x}$, rather than a magnitude.

Setting $\mathbf{b} = \mathbf{a}$ gives $\mathbf{a} \cdot \mathbf{a} = a^2 \geq 0$. Combining the two forms of the product gives

$$\theta = \arccos\left(\frac{\mathbf{a} \cdot \mathbf{b}}{ab}\right), \quad a > 0, \quad b > 0, \quad |\mathbf{a} \cdot \mathbf{b}| \leq ab. \quad (3.24)$$

The ratio is dimensionless and lies in $[-1, 1]$. The inequality, often called the Cauchy–Schwarz inequality in this setting, follows directly from $|\cos \theta| \leq 1$. A value outside this interval can signal a wrong norm, inconsistent units, overflow or rounding. Small rounding excursions at an endpoint may be clipped in a controlled implementation; substantial excursions must not be silently hidden.

The angle from $\arccos$ is the unsigned angle between the vectors, not a planar bearing measured from positive x . It does not tell whether one planar vector is clockwise or counterclockwise from the other. A signed orientation needs the additional information developed with the cross product.

3.4.3 Useful identities and physical boundaries

Expanding a squared norm gives a general result that includes the cosine rule:

$$\|\mathbf{a} \pm \mathbf{b}\|^2 = a^2 + b^2 \pm 2\mathbf{a} \cdot \mathbf{b}. \quad (3.25)$$

For perpendicular vectors the cross term is zero; only then does the familiar Pythagorean sum apply directly to these two vectors. For opposite vectors the cross term is negative in the sum, giving the expected cancellation. This is an independent way to test a component calculation.

A dot product between two displacements has units of length squared. A dot product between a force and a unit direction has units of force. A dot product between a force and a length vector has units N m. The units alone do not prove that a physical process has transferred energy. The chapter *Kinetic Energy and Work* will explain when a force–displacement product describes work, including the role of the actual displacement and of a varying force. Here the product serves as a geometric comparison.

Example 3.9: A prescribed load direction

Consider two possible prescribed force vectors $\mathbf{F}_+ = (6, 8) \text{ N}$ and $\mathbf{F}_- = (-6, 8) \text{ N}$ in the same plane. A reference edge is described by $\mathbf{b} = (2, 0) \text{ m}$. Find each force’s angle to that edge, its signed component along the edge, and its dot product with the edge vector. No displacement under the force is specified.

Separate the three requested quantities. Both forces have magnitude $F = \sqrt{6^2 + 8^2} \text{ N} =$

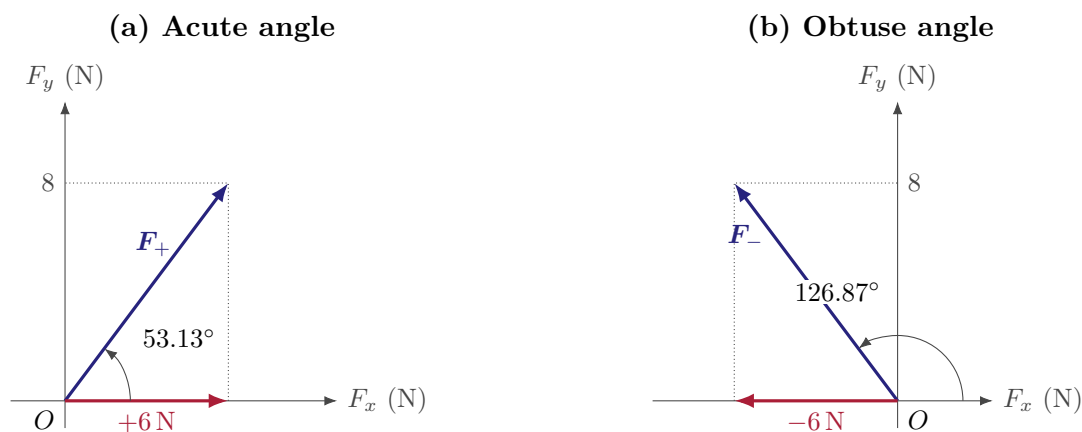


Figure 3.11: Two prescribed force vectors, each of magnitude 10 N, resolved along the same $+x$ reference direction. Both force-component plots have equal axis scales. The red horizontal arrows represent vector projections $\pm 6\hat{x}$ N; their signed scalar components are ± 6 N. The separate reference edge $\mathbf{b} = (2, 0)$ m is not drawn on these newton-valued axes. Its dot products with the forces are ± 12 N m, not ± 6 N.

10 N and the unit edge direction is $\hat{\mathbf{u}} = \hat{\mathbf{x}}$. Consequently,

$$F_{+,||} = +6 \text{ N}, \quad F_{-,||} = -6 \text{ N}, \quad \mathbf{F}_{\pm} \cdot \mathbf{b} = \pm 12 \text{ N m}.$$

The angle cosines are $+6/10$ and $-6/10$, giving $\theta_+ = 53.13^\circ$ and $\theta_- = 126.87^\circ$ approximately. The second is obtuse, in agreement with the negative scalar product. Figure 3.11 shows the corresponding signed force projections; its axes are in newtons, not metres.

Checks. Reversing the edge changes both signed scalar components and dot products to their negatives and replaces each angle by $180^\circ - \theta$. The physical force vectors are unchanged. If the edge record were $(0, 0)$ m, the dot product would still be zero, but an edge direction and the requested angles would be undefined. Finally, the values ± 12 N m are not work values for this problem: the edge has not been identified as a displacement.

Guided checks

If $\mathbf{a} \cdot \mathbf{b} = 0$ and both vectors are nonzero, they are perpendicular. If a zero vector is allowed, the equation alone does not establish two directions. If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, then $\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = 0$; this does *not* generally imply $\mathbf{b} = \mathbf{c}$. A dot product retains one projection, not all information about a vector.

3.4.4 A projection proof of the scalar-product bound

The bound $|\mathbf{a} \cdot \mathbf{b}| \leq ab$ is more than a range check for an inverse cosine. It states that a vector's projected length cannot exceed its full length. A useful algebraic proof also shows precisely when equality is possible. For $\mathbf{b} \neq \mathbf{0}$, write

$$\hat{\mathbf{u}} = \frac{\mathbf{b}}{b}, \quad p = \mathbf{a} \cdot \hat{\mathbf{u}}, \quad \mathbf{e} = \mathbf{a} - p\hat{\mathbf{u}}.$$

Distributivity gives $\mathbf{e} \cdot \hat{\mathbf{u}} = p - p = 0$. Expanding the squared residual then gives

$$\begin{aligned} \|\mathbf{e}\|^2 &= (\mathbf{a} - p\hat{\mathbf{u}}) \cdot (\mathbf{a} - p\hat{\mathbf{u}}) \\ &= a^2 - 2p(\mathbf{a} \cdot \hat{\mathbf{u}}) + p^2 = a^2 - p^2 \geq 0. \end{aligned} \quad (3.26)$$

Thus $|p| \leq a$, and multiplication by $b > 0$ yields $|\mathbf{a} \cdot \mathbf{b}| \leq ab$. If $\mathbf{b} = \mathbf{0}$, the bound holds directly since both sides are zero; normalizing that vector would be unnecessary and undefined.

Figure 3.12 represents the proof as a right triangle. Equality requires $\|\mathbf{e}\| = 0$, so $\mathbf{a}$ lies along the reference line. For two nonzero vectors this means parallel or antiparallel; both are allowed because the inequality contains an absolute value. Without that absolute value, only the same-direction case saturates the upper bound $\mathbf{a} \cdot \mathbf{b} \leq ab$. A zero vector also gives equality, but it does not establish a direction.

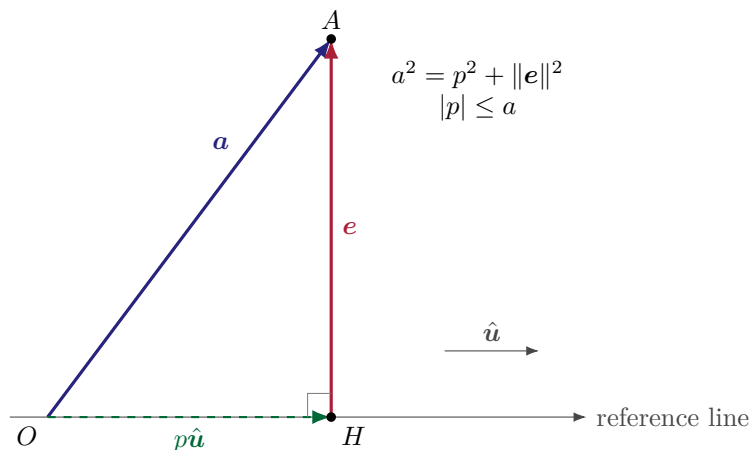


Figure 3.12: The perpendicular residual underlying the Cauchy–Schwarz bound. The original quantitative construction uses $\mathbf{a} = (3, 4)$ and $\hat{\mathbf{u}} = (1, 0)$ in a common arbitrary component unit; spatial scales are equal. The scalar projection is $p = 3$, the residual has length 4, and $a = 5$. The unit-direction arrow at the right has unit length on the same scale. For any corresponding construction the residual square is nonnegative, so the projected length cannot exceed the full vector length.

There is also an inverse interpretation. A proposed magnitude a and signed projection p are admissible only when $a^2 - p^2 \geq 0$. For example, $a = 5$ mm and $p = 3$ mm require a perpendicular norm 4 mm; they permit two planar residual signs or a circle of residual directions in three dimensions. If instead $p = 6$ mm, the residual square is -11 mm² and the exact observations are inconsistent. The inequality therefore diagnoses missing information and impossible data as well as bounding an angle formula.

Finally, the proof explains why a dot product is not cancellable like an ordinary nonzero number. Equal products $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ constrain $\mathbf{b} - \mathbf{c}$ only to be perpendicular to $\mathbf{a}$. Any transverse difference survives. One scalar comparison cannot recover every spatial component; the information lost by projection must be provided by another independent observation.

3.5 The vector product and orientation

3.5.1 An area with a chosen normal

Two nonparallel vectors with a common tail determine a plane. The parallelogram built on them has base a and perpendicular height $b \sin \theta$, so its area is $ab \sin \theta$ when both vectors are lengths. There are two opposite normal directions to the plane. An *ordered* pair selects one by the right-hand rule: curl the fingers from the first vector toward the second through the angle less than 180° ; the thumb points along the selected normal.

Definition 3.3: Vector or cross product

In an oriented, three-dimensional Euclidean space,

$$\mathbf{a} \times \mathbf{b} = ab \sin \theta \hat{\mathbf{n}}, \quad 0 \leq \theta \leq \pi, \quad (3.27)$$

where $\hat{\mathbf{n}}$ is perpendicular to both nonparallel vectors, oriented by that ordered right-hand rule. For parallel or antiparallel vectors, or if either vector is zero, the cross product is the zero vector. Its units are the product of the input units.

The zero result in a degenerate case is well defined even though a unit normal selected by the pair is not. A nonzero vector product of length vectors is an *oriented area vector*; its magnitude has units of area, not length. Drawing its arrow beside two edges therefore requires a separately declared display scale or a direction-only convention.

Reversing the order reverses the orientation but not the area:

$$\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b}). \quad (3.28)$$

For vectors in the page with $+x$ right and $+y$ up, $+z$ points toward the reader. We indicate that direction with $\odot$, suggesting an approaching arrow tip, and $\otimes$, suggesting the back of an arrow. An apparent clockwise sweep in an oblique three-dimensional picture cannot be interpreted without its axes and viewing convention.

3.5.2 From orientation to components

The right-handed basis fixes the cyclic products

$$\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}, \quad \hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}, \quad \hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}.$$

Each reversed order has a minus sign; each basis vector crossed with itself gives zero. The cross product distributes over sums and is linear in each argument. Expanding the nine terms of $\mathbf{a} \times \mathbf{b}$, the x component collects the y, z terms $a_y b_z - a_z b_y$. The other two follow cyclically:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} = & (a_y b_z - a_z b_y) \hat{\mathbf{x}} + (a_z b_x - a_x b_z) \hat{\mathbf{y}} \\ & + (a_x b_y - a_y b_x) \hat{\mathbf{z}}. \end{aligned} \quad (3.29)$$

The middle term is especially easy to reverse. Reconstruct it from $\hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}$ rather than trusting a remembered pattern.

A compact mnemonic is

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}.$$

This notation need not presuppose a course in determinants: a two-by-two minor $\begin{vmatrix} p & q \\ r & s \end{vmatrix}$ means $ps - qr$; expansion across the displayed first row uses signs $+, -, +$. The first row is a row of basis vectors, so the final object is a vector. It is a shorthand for Equation (3.29), not a new physical definition.

For readers meeting compact index notation, the same component rule is

$$(\mathbf{a} \times \mathbf{b})_i = \sum_{j=1}^3 \sum_{k=1}^3 \epsilon_{ijk} a_j b_k, \quad (1, 2, 3) \equiv (x, y, z).$$

Here ϵ_{ijk} is $+1$ for the cyclic orderings $(1, 2, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, -1 for their reversals, and zero whenever two indices coincide. For $i = 1$, only $(j, k) = (2, 3)$ and $(3, 2)$ survive, reproducing $a_y b_z - a_z b_y$. This optional shorthand adds no new physical rule; the explicit component formula is sufficient for the problems.

Two tests should accompany a computed cross product $\mathbf{c}$:

$$\mathbf{a} \cdot \mathbf{c} = 0, \quad \mathbf{b} \cdot \mathbf{c} = 0. \quad (3.30)$$

In the component expansion the paired terms cancel. These checks establish perpendicularity, but cannot distinguish $\mathbf{c}$ from $-\mathbf{c}$. An orientation check is therefore separate. A third check follows from $\sin^2 \theta + \cos^2 \theta = 1$:

$$\|\mathbf{a} \times \mathbf{b}\|^2 = a^2 b^2 - (\mathbf{a} \cdot \mathbf{b})^2. \quad (3.31)$$

This area identity checks a magnitude independently of the cyclic component calculation. It also shows that a cross product is largest, for fixed input magnitudes, when the vectors are perpendicular.

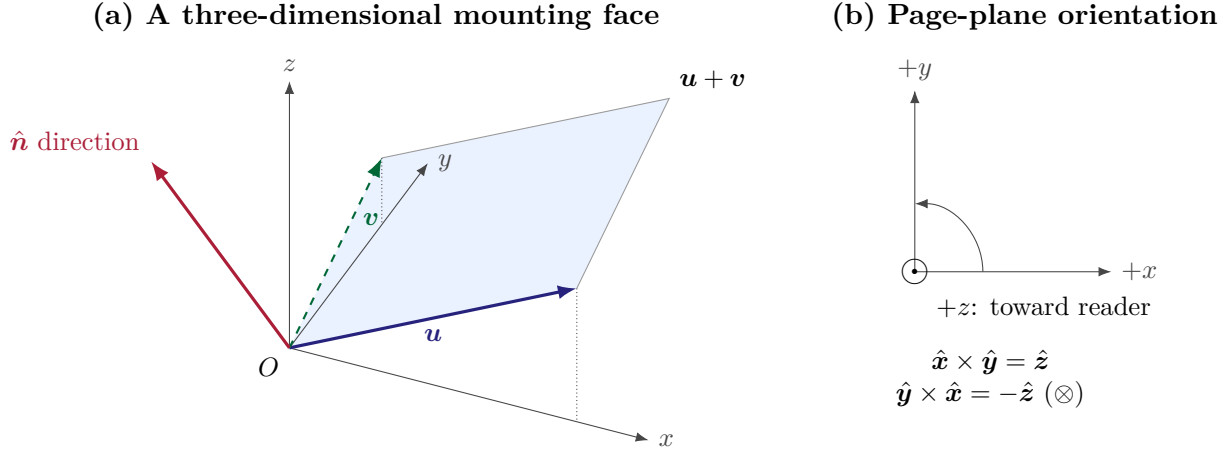


Figure 3.13: The bracket geometry and an independent orientation key. Panel (a) is an oblique linear projection of the exact Cartesian edge coordinates in Example 3.10; projected lengths and angles are not unforeshortened spatial measurements. The face area is 1400 mm^2 and the normal direction is $(-3, -2, 6)/7$. Its displayed arrow length is arbitrary, not an area scale. Panel (b) is a schematic right-handed triad for axes lying in the page; its $\odot/\otimes$ convention is not an assertion that the tilted face lies in the page.

Example 3.10: A mounting bracket's plane and normal

A hypothetical planar mounting face has adjacent edge vectors $\mathbf{u} = (40, 0, 20) \text{ mm}$ and $\mathbf{v} = (0, 30, 10) \text{ mm}$. Find the oriented area vector, area and unit normal for the ordered pair $(\mathbf{u}, \mathbf{v})$. The face is a parallelogram, not an assumed rectangle in space.

Model and calculation. With a right-handed Cartesian basis,

$$\begin{aligned} \mathbf{C} = \mathbf{u} \times \mathbf{v} &= (0(10) - 20(30), 20(0) - 40(10), 40(30) - 0(0)) \text{ mm}^2 \\ &= (-600, -400, 1200) \text{ mm}^2 = 200 \text{ mm}^2 (-3, -2, 6). \end{aligned}$$

Its magnitude is $200\sqrt{9 + 4 + 36} \text{ mm}^2 = 1400 \text{ mm}^2$, so

$$\hat{\mathbf{n}} = \frac{-3\hat{\mathbf{x}} - 2\hat{\mathbf{y}} + 6\hat{\mathbf{z}}}{7}.$$

The positive z component agrees with the positive signed area of the ordered x, y projections of the two edges. The face is tilted: neither its physical area nor its normal can be read as an uncorrected area or angle on the page.

Independent tests. The dot products with the area vector are $40(-600) + 20(1200) = 0$ and $30(-400) + 10(1200) = 0$ in cubic millimetres. Also $u^2 = 2000 \text{ mm}^2$, $v^2 = 1000 \text{ mm}^2$ and $\mathbf{u} \cdot \mathbf{v} = 200 \text{ mm}^2$, giving $u^2 v^2 - (\mathbf{u} \cdot \mathbf{v})^2 = 1\,960\,000 \text{ mm}^4 = (1400 \text{ mm}^2)^2$. Finally $\|\hat{\mathbf{n}}\| = 1$.

Reversing the order gives $-\mathbf{C}$ and $-\hat{\mathbf{n}}$, with the same area. The triangle spanned by the same two edges has area 700 mm^2 . These are geometric results only: no material strength or permissible loading has been specified.

3.5.3 What the product does and does not determine

If both nonzero vectors are planar, one often writes the signed scalar $a_x b_y - a_y b_x$. This is the z component of the three-dimensional cross product, not its nonnegative magnitude. In our orientation, a positive value corresponds to a counterclockwise turn from the first vector toward the second through the smaller signed angle. The pair $(\mathbf{a} \cdot \mathbf{b}, (\mathbf{a} \times \mathbf{b})_z)$ can distinguish signs of planar turning angles that arccos alone cannot.

A cross product also loses information. If a component parallel to $\mathbf{a}$ is added to $\mathbf{b}$, the product $\mathbf{a} \times \mathbf{b}$ does not change. One cannot in general “divide by a vector” to undo a dot or cross product. The products are not interchangeable: dot products are commutative and scalar-valued; cross products are anticommutative and vector-valued. Cross products are not generally associative, so parentheses in repeated products matter.

An oriented area normal is sometimes called an *axial vector*, in contrast to a displacement, which is a *polar vector*. Under proper rotations they are handled by the same component rules. A reflection reverses handedness, so the right-hand convention introduces an additional distinction. Our core uses only right-handed Cartesian bases; the advanced section gives a small reflection diagnostic, not a general theory of axial quantities.

Caution

Nearly parallel edges can define a mathematically nonzero normal that is very sensitive to small measurement errors. Perpendicularity checks alone cannot establish that the normal is accurately known. The advanced sensitivity example quantifies this distinction. Exact parallelism gives zero area and no unit normal selected by the pair.

3.5.4 Oriented triangles and the three projected areas

The normal to a triangular face is constructed from two *edge differences*, not from its vertices’ position vectors. For ordered vertices P, Q, R , choose $\mathbf{u} = \mathbf{r}_Q - \mathbf{r}_P$ and $\mathbf{v} = \mathbf{r}_R - \mathbf{r}_P$. The triangle’s oriented area vector is

$$\mathbf{A}_\Delta = \frac{1}{2} \mathbf{u} \times \mathbf{v}, \quad A_\Delta = \|\mathbf{A}_\Delta\|. \quad (3.32)$$

The factor one half follows because a diagonal divides the parallelogram into two congruent triangles. A common translation cancels in both edge differences; changing the origin cannot change the area or normal. Reversing the vertex order reverses $\mathbf{A}_\Delta$. A cyclic change $P, Q, R \mapsto Q, R, P$ preserves it: $(\mathbf{v} - \mathbf{u}) \times (-\mathbf{u}) = \mathbf{u} \times \mathbf{v}$.

For any point X , a unit normal $\hat{\mathbf{n}}$ supplies the signed point-to-plane distance and the perpendicular foot:

$$d_s = \hat{\mathbf{n}} \cdot (\mathbf{r}_X - \mathbf{r}_P), \quad \mathbf{r}_{X_\perp} = \mathbf{r}_X - d_s \hat{\mathbf{n}}. \quad (3.33)$$

Substitution into $\hat{\mathbf{n}} \cdot (\mathbf{r} - \mathbf{r}_P) = 0$ verifies that $X_\perp$ lies in the plane. The nonnegative distance is $|d_s|$. Any other point Y in the plane adds an in-plane segment $\mathbf{r}_Y - \mathbf{r}_{X_\perp}$ perpendicular to the normal segment. Pythagoras therefore makes its distance from X at least $|d_s|$, establishing that the normal correction is the shortest one. If a supplied normal $\mathbf{N}$ is not a unit vector, use $\hat{\mathbf{n}} = \mathbf{N}/\|\mathbf{N}\|$; the condition $\mathbf{N} \neq 0$ is essential.

Each component of the area vector has a direct geometric interpretation. For example, $u_x v_y - u_y v_x$ is the signed area of the edge parallelogram *projected onto the x, y plane*. The other components are signed projected areas on the ordered y, z and z, x planes. Taking the order z, x , rather than x, z , for the second component is essential. Positive signed area means a counterclockwise ordered turn in the displayed coordinate plane. These three signed projections determine the spatial area vector even when a perspective view makes the face orientation difficult to judge.

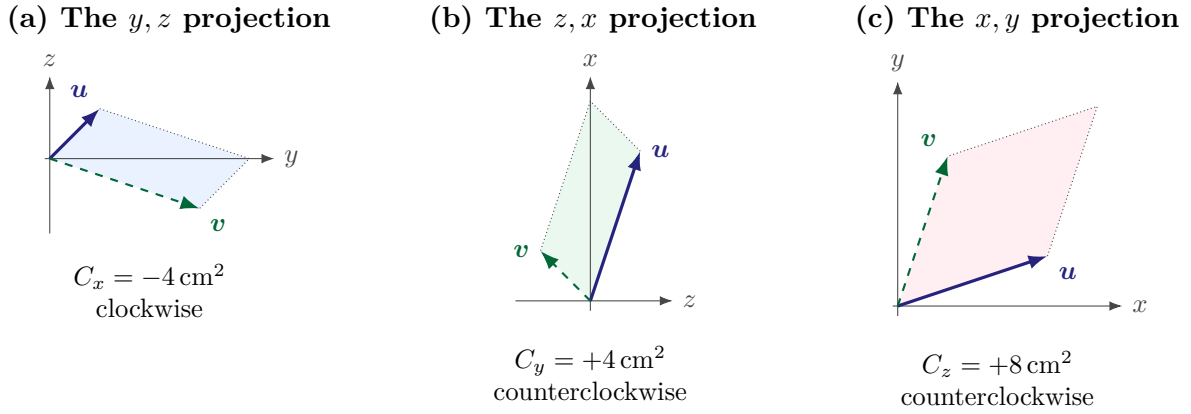


Figure 3.14: Three signed projected parallelogram areas for $\mathbf{u} = (3, 1, 1)$ cm and $\mathbf{v} = (1, 3, -1)$ cm. Each panel has equal horizontal and vertical coordinate scales. The axes are ordered y, z , z, x and x, y , so the displayed signed areas are the x, y, z components of $\mathbf{C} = \mathbf{u} \times \mathbf{v}$. A triangle would have half each signed projected area. The different panel heights reflect different component ranges, not different coordinate scales.

Example 3.11: A triangular sensor face and its chosen outward side

A hypothetical sensor face has vertices $P = (1, -1, 2)$ cm, $Q = (4, 0, 3)$ cm and $R = (2, 2, 1)$ cm. The design convention chooses the normal associated with the ordered traversal $P \rightarrow Q \rightarrow R$. Find the face area and unit normal; establish which side of the face the point $T = (1, -1, 3)$ cm occupies.

Reasoning and calculation. The two edges based at P are $\mathbf{u} = (3, 1, 1)$ cm and $\mathbf{v} = (1, 3, -1)$ cm. Their cross product is

$$\mathbf{C} = (-4, 4, 8) \text{ cm}^2, \quad \|\mathbf{C}\| = 4\sqrt{6} \text{ cm}^2.$$

Consequently $A_{\Delta} = 2\sqrt{6} \text{ cm}^2 \simeq 4.899 \text{ cm}^2$ and $\hat{\mathbf{n}} = (-1, 1, 2)/\sqrt{6}$. The displacement from the reference vertex to the test point is $\mathbf{r}_T - \mathbf{r}_P = (0, 0, 1)$ cm. Its signed projection on the chosen normal is $2/\sqrt{6} \text{ cm} > 0$, so T lies on the positive side. The face's plane is $\hat{\mathbf{n}} \cdot (\mathbf{r} - \mathbf{r}_P) = 0$; a point on that plane has zero signed projection.

Checks and interpretation. The edge-normal products are $3(-4) + 1(4) + 1(8) = 0$ and $1(-4) + 3(4) - 1(8) = 0$. Independently, $u^2 = v^2 = 11 \text{ cm}^2$ and $\mathbf{u} \cdot \mathbf{v} = 5 \text{ cm}^2$, so $\|\mathbf{C}\|^2 = (121 - 25) \text{ cm}^4 = 96 \text{ cm}^4$. The positive C_z also agrees with the counterclockwise order in the x, y projection in Figure 3.14. Reversing Q, R makes T the negative-side point while leaving its distance from the plane and the face area unchanged. Thus “outward” requires a convention or additional information about the solid; the three vertices alone determine two opposite normals.

3.5.5 A scalar triple product measures signed volume

Three edge vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ from a common vertex build a parallelepiped. Take $(\mathbf{a}, \mathbf{b})$ as the ordered base. If its area is $A = \|\mathbf{a} \times \mathbf{b}\| > 0$, its chosen unit normal is $\hat{\mathbf{n}} = (\mathbf{a} \times \mathbf{b})/A$. The signed height of the third edge is $h_s = \mathbf{c} \cdot \hat{\mathbf{n}}$. The product of base area and signed height is

$$V_s = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}), \quad V = |V_s|. \quad (3.34)$$

V is the nonnegative physical volume. V_s records whether the third edge points to the positive or negative side of the *ordered* base. A negative signed volume is not a negative amount of space. For length vectors the units are length cubed. The volume of the tetrahedron with the same three edges is $|V_s|/6$, because its parallel cross-sectional area decreases quadratically from the base to

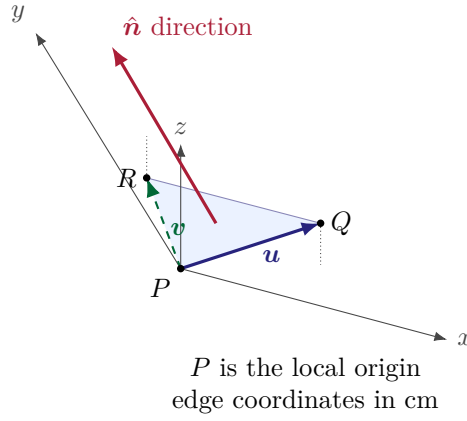


Figure 3.15: The triangular face from Example 3.11 in an oblique linear projection. The local origin is P , so the plotted vertices are $P = (0, 0, 0)$, $Q = (3, 1, 1)$ and $R = (1, 3, -1)$ in centimetres. The normal arrow is parallel to $(-1, 1, 2)$ and starts at the face centroid; its displayed length is arbitrary. Perspective angles do not supply measurements. The exact signed coordinate-plane projections in Figure 3.14 provide a separate orientation check.

the opposite vertex; integrating that area gives one third of the base-area times height, and the triangular base is half the parallelogram.

Expand the left side of Equation (3.34):

$$\begin{aligned}
 V_s &= c_x(a_y b_z - a_z b_y) + c_y(a_z b_x - a_x b_z) + c_z(a_x b_y - a_y b_x) \\
 &= a_x(b_y c_z - b_z c_y) - a_y(b_x c_z - b_z c_x) + a_z(b_x c_y - b_y c_x) \\
 &= \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}.
 \end{aligned} \tag{3.35}$$

The second line merely collects terms with the same component of $\mathbf{a}$. It both proves the cyclic triple-product identity and introduces a three-by-three determinant through three two-by-two minors. This determinant is a scalar, unlike the basis-vector mnemonic for a cross product. Cyclic permutation preserves V_s ; exchanging any two edges reverses its sign.

If $\mathbf{a}, \mathbf{b}$ are nonparallel, $V_s = 0$ says $h_s = 0$: the third vector lies in their plane. More generally, $V_s = 0$ means the three vectors are linearly dependent and can lie in a common plane through their tails; degenerate collinear or zero-edge cases are included. It does not always identify a *unique* plane. A nearly zero determinant also requires attention to measurement scale: a small signed height can be meaningful for a broad base, while a tiny base makes its estimated unit normal sensitive to errors.

Example 3.12: A skew calibration cell: volume, height and coplanarity

A calibration cell is described by three edge vectors $\mathbf{a} = (2, 0, 1)$ cm, $\mathbf{b} = (0, 3, 1)$ cm and $\mathbf{c} = (1, 1, 2)$ cm. Find its volume and signed height above the ordered $(\mathbf{a}, \mathbf{b})$ base. Then find a vector $\mathbf{c}_0$ in that base plane whose endpoint is the orthogonal projection of the endpoint of $\mathbf{c}$. **Calculation.** The oriented base area is $\mathbf{a} \times \mathbf{b} = (-3, -2, 6)$ cm², of magnitude 7 cm². Thus $\hat{\mathbf{n}} = (-3, -2, 6)/7$ and

$$V_s = (-3 - 2 + 12) \text{ cm}^3 = 7 \text{ cm}^3, \quad h_s = V_s/A = 1 \text{ cm}.$$

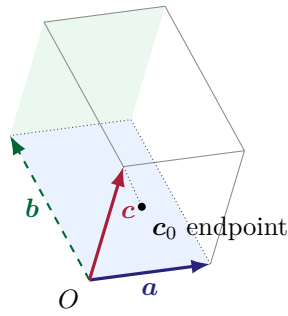
Removing the normal component gives

$$\mathbf{c}_0 = \mathbf{c} - h_s \hat{\mathbf{n}} = \left(\frac{10}{7}, \frac{9}{7}, \frac{8}{7} \right) \text{ cm}.$$

The magnitude of $\mathbf{c}$ is $\sqrt{6}$ cm, which is not the perpendicular height. The skew cell's volume is 7 cm^3 ; its associated tetrahedron has volume $7/6 \text{ cm}^3$.

Independent checks. Direct minor expansion gives $2(3 \cdot 2 - 1 \cdot 1) + (0 \cdot 1 - 3 \cdot 1) = 10 - 3 = 7$ in cubic centimetres. The projected edge satisfies $(-3, -2, 6) \cdot (10, 9, 8)/7 = (-30 - 18 + 48)/7 = 0$. It can also be written as $(5/7)\mathbf{a} + (3/7)\mathbf{b}$, confirming that it lies in the base plane. Replacing $\mathbf{c}$ with $\mathbf{c}_- = \mathbf{c} - 2h_s\hat{\mathbf{n}}$ makes its signed height -1 cm and $V_s = -7 \text{ cm}^3$, with the same positive volume. This changes the third edge's side of the base; it is different from merely relabelling the base order.

(a) The skew cell



(b) A normal section

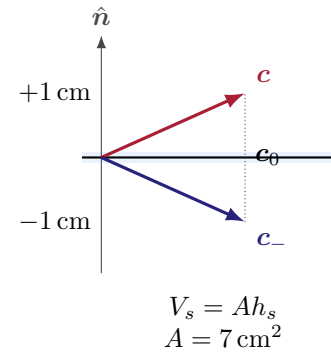


Figure 3.16: Signed volume for Example 3.12. Panel (a) uses an oblique linear projection of the exact cell vertices. The dotted segment joins the endpoint of $\mathbf{c}$ to its perpendicular projection in the base. Panel (b) is the physical section spanned by $\mathbf{c}_0$ and $\hat{\mathbf{n}}$, with equal length scales: $\|\mathbf{c}_0\| = \sqrt{5}$ cm and the normal heights are ± 1 cm. The base-area value is an annotation, not a length on this section. Both third-edge choices give the same nonnegative volume.

3.5.6 Repeated cross products require parentheses

An ordinary product of numbers is associative. A cross product is not: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ and $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ generally differ. The useful identity is

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}). \quad (3.36)$$

Its output is a linear combination of $\mathbf{b}, \mathbf{c}$, consistent with being perpendicular to $\mathbf{b} \times \mathbf{c}$. The identity can be verified without new notation. Its x component from the left side is

$$a_y(b_x c_y - b_y c_x) - a_z(b_z c_x - b_x c_z) = b_x(a_y c_y + a_z c_z) - c_x(a_y b_y + a_z b_z).$$

Adding and subtracting $a_x b_x c_x$ produces $b_x(\mathbf{a} \cdot \mathbf{c}) - c_x(\mathbf{a} \cdot \mathbf{b})$; the other components follow by cyclic replacement. Reversing the first outer cross product gives

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{a}(\mathbf{b} \cdot \mathbf{c}),$$

which contains a different second term.

A useful geometric special case takes a unit direction $\hat{\mathbf{u}}$:

$$\hat{\mathbf{u}} \times (\mathbf{a} \times \hat{\mathbf{u}}) = \mathbf{a} - \hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \mathbf{a}) = \mathbf{a}_\perp. \quad (3.37)$$

The double cross product removes the component parallel to $\hat{\mathbf{u}}$. A nonzero, nonunit $\mathbf{u}$ gives $u^2 \mathbf{a}_\perp$, so forgetting the unit-length condition changes both scale and, when $\mathbf{u}$ carries units, dimensions. This projection identity supplies a compact geometric tool; it does not require any force or motion model.

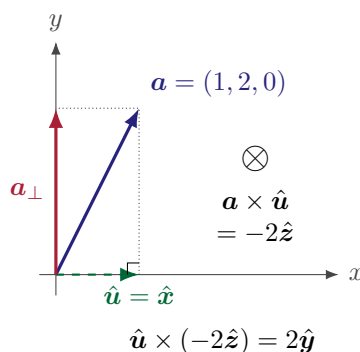


Figure 3.17: A double cross product as a perpendicular projection. Here the components are dimensionless and the x, y axes lie in the page with $+z$ toward the reader. The first product points into the page ($\otimes$); crossing it on the left with $\hat{x}$ gives the upward perpendicular component. The plane has equal horizontal and vertical scales. The $\otimes$ marker denotes direction; its printed diameter does not encode the magnitude 2.

Example 3.13: Two parenthesisations, two different vectors

For dimensionless vectors $\mathbf{a} = (1, 2, 3)$, $\mathbf{b} = (2, -1, 1)$ and $\mathbf{c} = (-1, 1, 0)$, calculate both parenthesisations of the repeated cross product and explain why their difference is allowed.

Calculation by two routes. First, $\mathbf{b} \times \mathbf{c} = (-1, -1, 1)$, giving $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (5, -4, 1)$. The scalar products are $\mathbf{a} \cdot \mathbf{c} = 1$ and $\mathbf{a} \cdot \mathbf{b} = 3$, so the identity independently gives $\mathbf{b} - 3\mathbf{c} = (5, -4, 1)$. For the other placement of parentheses, $\mathbf{a} \times \mathbf{b} = (5, 5, -5)$ and $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (5, 5, 10)$. Here $\mathbf{b} \cdot \mathbf{c} = -3$, so the second identity gives $\mathbf{b} + 3\mathbf{a} = (5, 5, 10)$.

Checks and interpretation. The first answer is perpendicular to $\mathbf{b} \times \mathbf{c}$: $-5 + 4 + 1 = 0$. The second is perpendicular to $\mathbf{a} \times \mathbf{b}$: $25 + 25 - 50 = 0$, and also to $\mathbf{c}$: $-5 + 5 = 0$. The two outer products have different input planes, so there is no reason for their normal directions to coincide. Their difference is $(0, -9, -9)$, explicitly nonzero. Associating the factors as though they were ordinary numbers would erase this geometrical information.

3.6 Changing axes and preserving invariants

Supporting extension. This section may be studied after the core vector operations. The explicitly marked frame-transformation problems use it; the remaining core problems do not.

3.6.1 Rotate the axes, not the vector

Imagine one fixed arrow on a table and two transparent coordinate grids sharing an origin. Turn the second grid counterclockwise by angle ϕ . The physical arrow has not moved, but its components along the second grid's axes usually change. This is a *passive* change of coordinates.

The new basis vectors, expressed in the old basis, are

$$\hat{x}' = \cos \phi \hat{x} + \sin \phi \hat{y}, \quad \hat{y}' = -\sin \phi \hat{x} + \cos \phi \hat{y}. \quad (3.38)$$

The sign in $\hat{y}'$ follows from its direction $90^\circ + \phi$. The new components are projections of the same $\mathbf{a}$ on these directions. Taking the two dot products gives

$$a'_x = a_x \cos \phi + a_y \sin \phi, \quad a'_y = -a_x \sin \phi + a_y \cos \phi. \quad (3.39)$$

No angle-subtraction rule needs to be guessed: the formula follows directly from the new unit directions. If the old bearing is θ , the new bearing is $\theta' = \theta - \phi$ modulo a full turn.

3.6.2 Checks that distinguish a rotation from a mistake

At $\phi = 0$, the components are unchanged. At $\phi = 90^\circ$, the new positive x' axis is the old positive y axis, and the new positive y' axis is the old negative x axis. The formula must therefore give $(a'_x, a'_y) = (a_y, -a_x)$. This simple directional test is more discriminating than checking a norm alone.

Squaring and adding Equation (3.39) cancels the mixed terms and uses $\cos^2 \phi + \sin^2 \phi = 1$:

$$(a'_x)^2 + (a'_y)^2 = a_x^2 + a_y^2. \quad (3.40)$$

The vector length is unchanged. Applying the same rotation to two vectors and expanding their dot product similarly gives

$$a'_x b'_x + a'_y b'_y = a_x b_x + a_y b_y. \quad (3.41)$$

Consequently angles between nonzero vectors are unchanged. These statements require transforming *both* component lists into the same basis. Mixing one primed and one unprimed list destroys the interpretation.

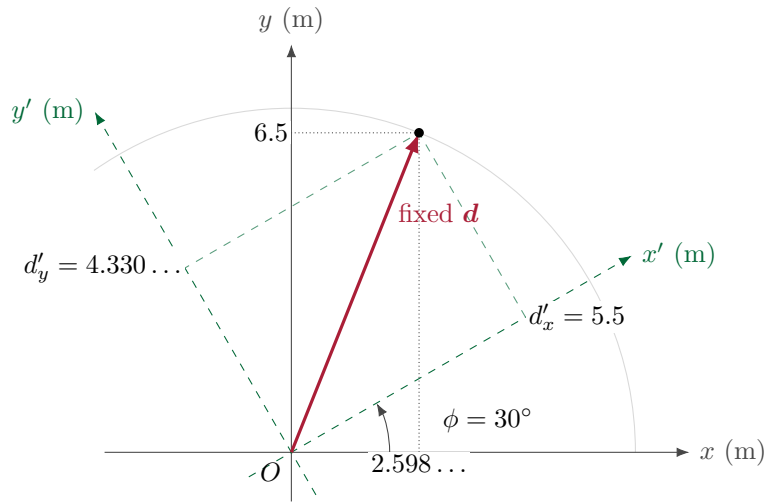


Figure 3.18: A passive axis rotation with one unchanged displacement arrow. The dashed primed axes are turned 30° counterclockwise from the solid unprimed axes. Both coordinate systems use metres and equal Cartesian scales. Dotted and dashed guides give the old and new projections respectively. The arc is part of the calculated locus of length 7 m, not an uncertainty band. Only the coordinate description changes.

Example 3.14: One survey, two coordinate grids

Return to the displacement in Figure 3.18, $\mathbf{d} = (3\sqrt{3}/2, 13/2)$ m from Example 3.3. Rotate the axes counterclockwise through $\phi = 30^\circ$, keeping the vector fixed. Then compare its dot product with the first survey leg before and after the change.

Using $\cos 30^\circ = \sqrt{3}/2$ and $\sin 30^\circ = 1/2$,

$$d'_x = \left(\frac{3\sqrt{3}}{2} \frac{\sqrt{3}}{2} + \frac{13}{2} \frac{1}{2} \right) \text{ m} = \frac{11}{2} \text{ m} = 5.500 \text{ m},$$

$$d'_y = \left(-\frac{3\sqrt{3}}{2} \frac{1}{2} + \frac{13}{2} \frac{\sqrt{3}}{2} \right) \text{ m} = \frac{5\sqrt{3}}{2} \text{ m} \simeq 4.330 \text{ m}.$$

These are different components of the *same* 7 m vector: $(11/2)^2 + (5\sqrt{3}/2)^2 = 49$. Its new bearing is $68.21^\circ - 30^\circ = 38.21^\circ$ approximately.

The first leg lies along x' , so $\mathbf{a}$ has new components (8, 0) m. In the old frame, $\mathbf{d} \cdot \mathbf{a} = [(3\sqrt{3}/2)(4\sqrt{3}) + (13/2)(4)] \text{ m}^2 = 44 \text{ m}^2$. In the new frame the same product is $[(11/2)(8) + 0] \text{ m}^2 = 44 \text{ m}^2$. Applying the inverse axis change -30° recovers the old components.

3.6.3 Active rotations and changes of origin

A different operation physically rotates an arrow counterclockwise by ϕ while keeping the axes fixed. This *active* rotation increases its bearing, so its components become

$$\tilde{a}_x = a_x \cos \phi - a_y \sin \phi, \quad \tilde{a}_y = a_x \sin \phi + a_y \cos \phi. \quad (3.42)$$

For the same positive angle, the active formula equals the passive formula with ϕ replaced by $-\phi$. Both preserve norms and mutual dot products, so those invariants alone cannot reveal a mistaken choice between them. The physical question and the 90° basis-vector test settle the sign convention.

A change of origin is separate again. If the new origin has old position vector $\mathbf{c}$ and the axes keep the same directions, a point's new position vector is $\mathbf{r}' = \mathbf{r} - \mathbf{c}$. Two positions shift by the same amount, so their difference is unchanged. A free displacement does not acquire a translation term. For a combined change, first express the position relative to the new origin and then resolve it on the new axes. This is a change in description, not motion of the object.

Take-home message

The geometric vector is not its component list. Specify what is held fixed when an angle changes: the vector or the axes. Use invariants to check lengths and angles, but use a known direction to check the active/passive sign convention.

3.7 Applied geometry, computation and measurement

The preceding tools are useful even before a dynamical law is introduced. A positioning problem asks whether a geometric target is reached. A plane reconstruction asks which normal is consistent with given edges. A survey asks whether reported displacements can fit together. Each task requires both a calculation and an account of what the available information can establish.

3.7.1 A geometric workflow for a positioning task

For a planar stage, define the starting point, the target and the laboratory axes before recording commands. A target point requires two coordinates; a target line requires a point and a direction. Confusing these targets is consequential: removing a transverse error reaches the line but not necessarily the desired point on it. If a command is expressed in local axes, transform it into the common laboratory basis before addition. If the local origin differs, transform point positions with the appropriate origin shift as well.

The correction found in Example 3.7 is shortest among straight corrections to the target line. Indeed any endpoint on that line differs from the perpendicular foot by a parallel vector. The squared correction length is then $\|\mathbf{d}_\perp\|^2$ plus a nonnegative parallel contribution, so it is minimised at the foot. This is a geometric optimum, not a claim that the shortest move is the fastest, safest or mechanically feasible move. Such claims require additional constraints and later physics.

3.7.2 An executable vector check

The listing below implements the chapter's operations for three numerical components. A planar vector is entered with zero z component. All summed inputs must represent the same kind of quantity in a common unit and basis. Unit checking is the user's responsibility: a tuple of floats carries no metre or newton tag. The rotation routine uses the supporting extension in Section 3.6; it can be omitted when testing only core sums and products. The passive rotation is about the unchanged z axis, with a positive counterclockwise turn of the x, y axes.

The unsigned angle is computed as $\text{atan2}(\|\hat{\mathbf{a}} \times \hat{\mathbf{b}}\|, \hat{\mathbf{a}} \cdot \hat{\mathbf{b}})$. Normalisation makes the inputs dimensionless, and the sine and cosine together identify an angle in $[0, \pi]$. This alternative to an inverse

cosine is useful near parallel and antiparallel directions. It does not recover a signed planar bearing. The standard-library angle functions use radians [4].

```

"""Three-vector geometry; Python >=3.10, standard library only.
Supply finite components in ONE declared unit and ONE Cartesian basis.
Angles passed to passive_z are radians. This code does not track units.
"""

from math import atan2, cos, fsum, hypot, isfinite, radians, sin
from typing import Iterable

Vec = tuple[float, float, float]

def vec(values: Iterable[float]) -> Vec:
    if isinstance(values, (str, bytes)):
        raise TypeError("Supply three numerical components, not text")
    q = tuple(float(x) for x in values)
    if len(q) != 3 or not all(isfinite(x) for x in q):
        raise ValueError("Expected three finite components")
    return q # type: ignore[return-value]

def norm(a: Vec) -> float:
    a = vec(a)
    value = hypot(*a)
    if not isfinite(value):
        raise OverflowError("Norm is outside the floating-point range")
    return value

def add(a: Vec, b: Vec) -> Vec:
    a, b = vec(a), vec(b)
    return vec(x + y for x, y in zip(a, b))

def dot(a: Vec, b: Vec) -> float:
    a, b = vec(a), vec(b)
    value = fsum(x * y for x, y in zip(a, b))
    if not isfinite(value):
        raise OverflowError("Dot product is outside the numerical range")
    return value

def cross(a: Vec, b: Vec) -> Vec:
    x, y, z = vec(a)
    p, q, r = vec(b)
    return vec((y*r-z*q, z*p-x*r, x*q-y*p))

def unit(a: Vec, floor: float = 0.0) -> Vec:
    if not isfinite(floor) or floor < 0:
        raise ValueError("The norm floor must be finite and nonnegative")
    a = vec(a)
    length = norm(a)
    if length <= floor:
        raise ValueError("No resolved direction at this norm threshold")
    return vec(x / length for x in a)

def passive_z(a: Vec, phi: float) -> Vec:
    x, y, z = vec(a)
    if not isfinite(phi):
        raise ValueError("The rotation angle must be finite (radians)")
    c, s = cos(phi), sin(phi)
    return vec((c*x+s*y, -s*x+c*y, z))

def angle(a: Vec, b: Vec, floor: float = 0.0) -> float:

```

```

    # Normalise before products to reduce avoidable magnitude problems.
    u, v = unit(a, floor), unit(b, floor)
    return atan2(norm(cross(u, v)), dot(u, v)) # unsigned, in [0, pi]

def report(a: Vec, b: Vec, phi: float) -> dict:
    a, b = vec(a), vec(b)
    return {"sum": add(a, b), "dot": dot(a, b),
            "cross": cross(a, b),
            "a_in_rotated_axes": passive_z(a, phi),
            "b_in_rotated_axes": passive_z(b, phi),
            "angle_radians": angle(a, b) if norm(a) and norm(b) else None}

if __name__ == "__main__":
    import json
    # Constructed bracket edges; components in mm, products in mm2.
    print(json.dumps(report(vec((40, 0, 20)), vec((0, 30, 10)),
                          radians(30)), indent=2))

```

For the bracket inputs, the program returns sum (40,30,30) in millimetres, dot product 200 in square millimetres and cross product (−600, −400, 1200) in square millimetres. The program's plain numbers must be interpreted with those units. It returns `None`, not an invented angle, when one input vector is zero; sums and products remain meaningful in that case. A nonzero direction below a specified `floor` is rejected by `unit`. This floor has the input component unit and must be chosen from the application, not copied from a universal constant.

Tests that expose different failures

First test exact basis cases: $\hat{x} \times \hat{y} = \hat{z}$ and a passive 90° turn maps $\hat{x}$ to $-\hat{y}$. Next test a nontrivial pair: preserve its norms and dot product under a common rotation, recover both vectors under the inverse rotation, and check cross-product perpendicularity and reversal. Include zero vectors, parallel vectors, wrong-length input lists and nonfinite entries. Finally compare one result with a geometric or symbolic calculation, rather than only running the same function twice.

Floating-point checks require tolerances. For example, compare a dot product before and after rotation using $|D' - D| \leq \epsilon_{\text{abs}} + \epsilon_{\text{rel}} ab$, where ϵ_{abs} has the product unit and ϵ_{rel} is dimensionless. A relative tolerance based only on $|D|$ is unhelpful when the true dot product is zero. These numerical tolerances describe arithmetic, not experimental uncertainty. The supplied verification uses bounded teaching inputs; arbitrary extreme inputs may overflow, and the code rejects nonfinite results rather than promising unlimited numerical range.

3.7.3 A proposed tabletop survey

A small, low-energy investigation can use marked points on paper and a ruler or other available length scale. No particular laboratory equipment is presumed. Establish a coordinate origin and an independent axis reference, identify the points clearly, and record units. Measure the component differences between successive points around a triangle or polygon. Repeat the process, including repositioning or re-reading the scale, so that repeatability is not confused with mere retyping of one reading.

Compute the closing vector $\mathbf{c}_{\text{close}} = \sum_i \mathbf{d}_i$. Independently measure at least one direct chord and compare it with the sum of the legs joining its endpoints. Check ruler alignment, angle zero, parallax and any conversion between drawing and physical scales. If bearings and lengths are measured instead of components, convert all bearings under one declared convention; a sign error in a bearing can imitate a large closing error.

For a conservative component-bound analysis, if each reported x component lies within $\pm\delta_{x,i}$ of its true value, then the closing x component lies within $\pm\sum_i \delta_{x,i}$ of the reported sum. Treat y similarly. These are bounds, not standard deviations or confidence intervals. The assumptions behind the individual bounds still require justification. Correlations do not invalidate a conservative worst-case sum, but can make it loose.

Example 3.15: What a closing vector can establish

A *constructed* record contains the following planar legs; each component is assigned a stipulated absolute error bound of 0.01 m.

Leg	d_x (m)	d_y (m)
1	1.20	0.40
2	-0.30	0.80
3	-0.89	-1.18

The closing vector is $(0.01, 0.02)$ m, with magnitude $\sqrt{0.01^2 + 0.02^2}$ m $\simeq 0.02236$ m approximately. The possible true closing components lie in $[0.01 - 0.03, 0.01 + 0.03]$ m and $[0.02 - 0.03, 0.02 + 0.03]$ m. Both intervals contain zero. Indeed the final leg $(-0.90, -1.20)$ m alone would close the nominal first two legs, but its y correction is 0.02 m, larger than that leg's bound. A valid closure instead distributes the corrections: subtract $0.00333\dots$ m from each x component and $0.00666\dots$ m from each y component. Each correction lies within its stipulated bound, and their sums remove the closing vector.

This demonstrates *compatibility* with closure, not proof of correct calibration. A common rotation of all ideal leg vectors preserves their zero sum; a common multiplicative scale also preserves closure. Independent axis and length references are needed to detect those errors. The nominal chord of the first two legs is $(0.90, 1.20)$ m, of length 1.50 m, and would provide an additional comparison if independently measured.

A report from an actual investigation should contain the raw readings, repeat procedure, calibration assumptions, closing components and the separate chord comparison, not just a percentage labelled “error”. Failure to close outside justified bounds identifies an inconsistency but generally does not identify which individual reading is wrong. Agreement within bounds supports the tested relationships; it does not validate every aspect of the apparatus or model.

3.7.4 Registering a local coordinate frame

An instrument or positioning stage often reports a point relative to its own origin and axes. A laboratory record reports the same point relative to a different origin. This combines two operations already distinguished above: an origin shift and a rotation. They must be applied in the correct order and to the correct kind of object. A point position carries an origin offset; a free displacement does not.

Let the local origin have laboratory position $\mathbf{c}$, and let the local axes be turned counterclockwise through ϕ relative to laboratory axes. For a local position column $\mathbf{q}_{\text{loc}}$, first turn its local displacement into laboratory components, then add the laboratory origin:

$$\mathbf{q}_{\text{lab}} = \mathbf{c} + \mathbf{R}(-\phi)\mathbf{q}_{\text{loc}}. \quad (3.43)$$

Here the matrix formula from the supporting extension is only an economical way to write the two component projection rules. Its negative angle is essential: the local basis directions are expressed in laboratory coordinates, whereas $\mathbf{R}(\phi)$ converts a laboratory column into the rotated frame. Subtracting two point equations removes $\mathbf{c}$. Thus the same conversion for a free displacement is $\mathbf{d}_{\text{lab}} = \mathbf{R}(-\phi)\mathbf{d}_{\text{loc}}$.

The inverse operation subtracts the origin *before* rotating: $\mathbf{q}_{\text{loc}} = \mathbf{R}(\phi)(\mathbf{q}_{\text{lab}} - \mathbf{c})$. Rotating $\mathbf{q}_{\text{lab}}$ first

and then subtracting an unrotated $\mathbf{c}$ mixes two bases. One can diagnose that error without a long calculation: the point at the local origin must have local coordinates $(0, 0)$.

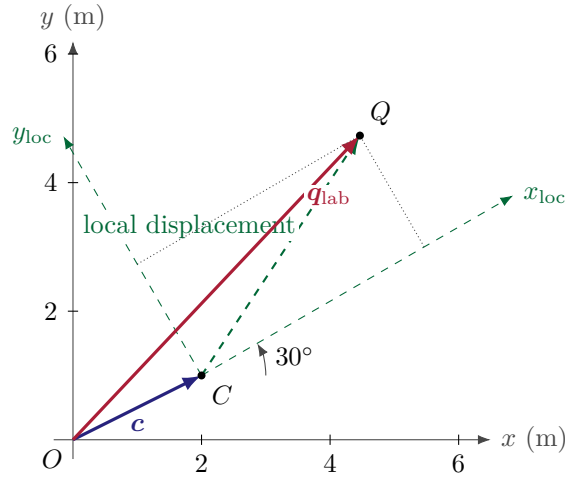


Figure 3.19: One point represented in translated and rotated frames. Equal metre scales define the laboratory geometry. The local origin is $C = (2, 1)$ m; local axes are turned 30° . The local displacement to Q has components $(4, 2)$ m. Adding its laboratory representation to $\mathbf{c}$ constructs the laboratory position. The arrow $C \rightarrow Q$ is not drawn with its two local component values treated as laboratory coordinates.

Example 3.16: A target recorded in two frames

The frame in Figure 3.19 records a stage endpoint as $(4, 2)$ m. A laboratory target has coordinates $(5, 4)$ m. Find the endpoint in the laboratory and the correction command in local axes. Assume both coordinate frames are known exactly in this stipulated geometry.

Convert the position. The local basis is turned 30° ; the laboratory components of the local displacement are $(4 \cos 30^\circ - 2 \sin 30^\circ, 4 \sin 30^\circ + 2 \cos 30^\circ)$ m. Adding the origin gives

$$\mathbf{q}_{\text{lab}} = (1 + 2\sqrt{3}, 3 + \sqrt{3}) \text{ m} \simeq (4.4641, 4.7321) \text{ m}.$$

The target minus this point is the free correction $\mathbf{d}_{\text{lab}} = (4 - 2\sqrt{3}, 1 - \sqrt{3})$ m. It is positive in laboratory x and negative in laboratory y .

Convert the displacement. No origin offset is added to this vector. Projection on the local axes yields

$$\mathbf{d}_{\text{loc}} = \left(\frac{3\sqrt{3} - 5}{2}, \frac{3\sqrt{3} - 7}{2} \right) \text{ m} \simeq (0.09808, -0.90192) \text{ m}.$$

Its length is $\sqrt{32 - 18\sqrt{3}} \text{ m} \simeq 0.90724 \text{ m}$. The correction mainly follows negative local y , despite having two substantial laboratory components. Adding it to the original local point gives $((3 + 3\sqrt{3})/2, (-3 + 3\sqrt{3})/2)$ m. Substitution in Equation (3.43) returns $(5, 4)$ m. The correction norms in the two frames agree, providing a separate invariant check. This model addresses geometry; controller limits or obstacles require additional information.

3.7.5 A closure test needs an independent reference

The survey of Example 3.15 has nominal endpoints $(0, 0)$, $(1.20, 0.40)$, $(0.90, 1.20)$ and $(0.01, 0.02)$ in metres. The final point lies near the start, but the error is much smaller than the route. A useful visual therefore combines a route view with a translated residual view; enlarging an inset does not change the underlying physical result.

Compatibility means that some corrections within the stated bounds close the loop. It does not

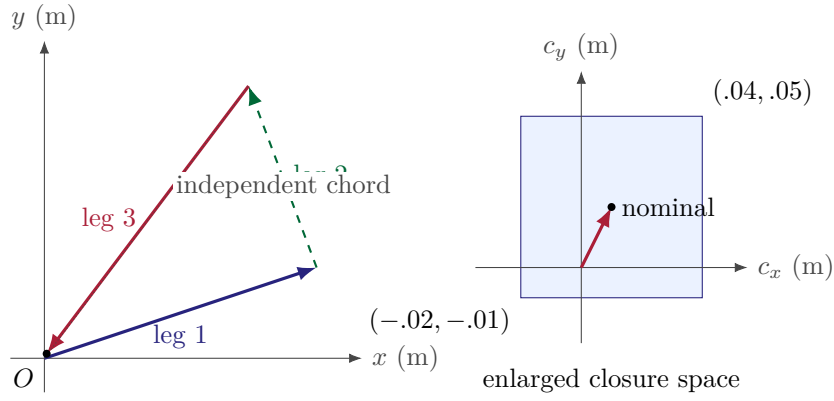


Figure 3.20: The constructed survey and its possible closing vectors. Each of three leg components has bound ± 0.01 m, so each closing component has bound ± 0.03 m. The rectangular set is centred at $(0.01, 0.02)$ m and contains zero. Axes within each panel have equal scales; the closure panel uses a separate enlargement. The chord is an additional reading to be made independently, rather than recomputed from the same legs.

prove which corrections actually occurred. A global length scale error is especially instructive. If every ideal leg is multiplied by the same $k > 0$, its sum becomes $k \sum_i \mathbf{d}_i$. A closed loop remains closed for every k , including an incorrect calibration. A common axis rotation also preserves a zero sum. These are symmetries of the closure test, and hence blind spots of that test. An independently calibrated length and orientation reference are needed to resolve them. Repeating a calculation on the same inputs supplies neither reference.

3.7.6 How bearing uncertainty depends on vector length

A small component error can produce a substantial direction error when the vector itself is small. Away from the zero vector, the local change in the bearing $\theta = \text{atan2}(y, x)$ is

$$\delta\theta \simeq \frac{x \delta y - y \delta x}{x^2 + y^2}, \quad \theta \text{ in radians.} \quad (3.44)$$

To derive it without differentiating software, write $x = r \cos \theta$, $y = r \sin \theta$ and take their small differentials. Multiply the second relation by x and the first by y , then subtract. The radial terms cancel and the angular terms give $r^2 \delta\theta$. This is a first-order approximation requiring component errors small compared with r . It fails at $r = 0$, where the direction itself is undefined.

Example 3.17: The same component bounds at two scales

A displacement record is $(30, 40)$ mm, with independent interval constraints $|\delta x|, |\delta y| \leq 0.5$ mm. Estimate the largest first-order bearing change and compare with the exact corner values. Then consider a record $(0.30, 0.40)$ mm with the same absolute bounds.

Local estimate. The nominal bearing is 53.1301° and the norm is 50 mm. The numerator in Equation (3.44) has largest absolute value $(30 + 40)(0.5) \text{ mm}^2 = 35 \text{ mm}^2$. Thus $|\delta\theta| \lesssim 0.014 \text{ rad} \simeq 0.80214^\circ$ locally. The extreme signed changes occur for component errors of opposite signs.

Exact admissible directions. The entire rectangle lies in quadrant I. Here θ decreases with x and increases with y , so its exact bounds are

$$\text{atan2}(39.5, 30.5) \leq \theta \leq \text{atan2}(40.5, 29.5), \quad 52.3264^\circ \lesssim \theta \lesssim 53.9306^\circ.$$

The exact departures are approximately -0.80370° and $+0.80049^\circ$; they need not be symmetric about the nominal angle. The linear estimate is useful because the component errors are small on the 50 mm scale.

For the small record, the allowed rectangle $[-0.20, 0.80] \times [-0.10, 0.90]$ mm contains the origin in its interior. It contains nonzero vectors in every direction near that origin. Consequently every bearing is admissible, and a zero true displacement is also admissible. There is no resolved unit direction under these bounds. Applying the linear formula far beyond its small-error domain would conceal this qualitative loss of information. These are interval bounds, not confidence probabilities.

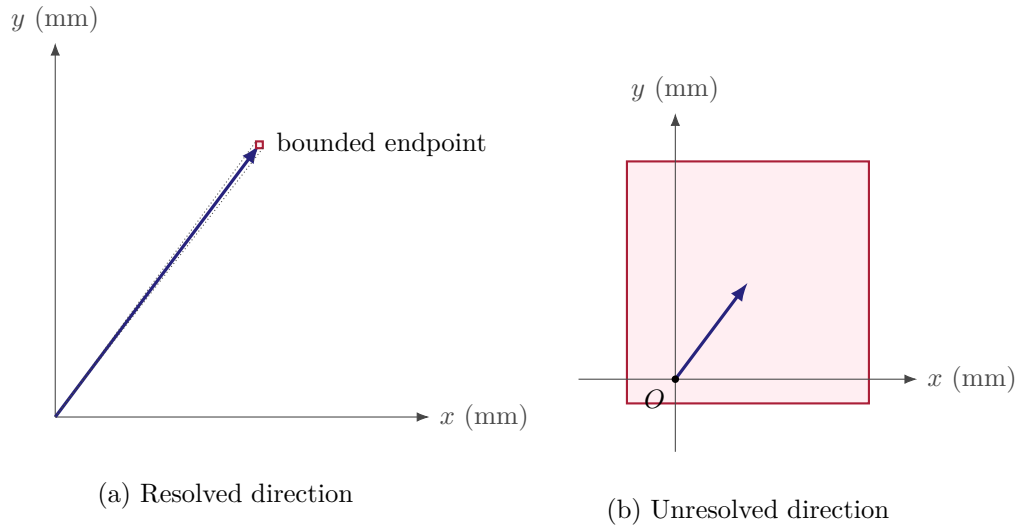


Figure 3.21: Equal absolute component bounds can have different directional consequences. Both panels have equal horizontal and vertical millimetre scales, but panel (b) is enlarged. Panel (a) has a small endpoint rectangle far from the origin; its bearings occupy a narrow interval. Panel (b)'s rectangle contains an open neighbourhood of the origin, permitting every direction and the zero vector. Shading represents an admissible set, not a statistical density.

3.8 Chapter summary and working formula sheet

The core idea is geometric. A vector has a magnitude, a direction when nonzero, and an addition rule. Components describe that vector in a chosen basis. A useful calculation preserves the connection between the two descriptions.

Formula summary

For a common right-handed orthonormal Cartesian basis,

$$\begin{aligned}
 \mathbf{a} &= a_x \hat{\mathbf{x}} + a_y \hat{\mathbf{y}} + a_z \hat{\mathbf{z}}, & a &= \sqrt{a_x^2 + a_y^2 + a_z^2}, \\
 (\mathbf{a} + \mathbf{b})_i &= a_i + b_i, & \hat{\mathbf{a}} &= \mathbf{a}/a \quad (a > 0), \\
 \mathbf{a} \cdot \mathbf{b} &= \sum_i a_i b_i = ab \cos \theta, & a_{\parallel} &= \mathbf{a} \cdot \hat{\mathbf{u}}, \\
 \mathbf{a}_{\parallel} &= a_{\parallel} \hat{\mathbf{u}}, & \mathbf{a}_{\perp} &= \mathbf{a} - \mathbf{a}_{\parallel}, \\
 \|\mathbf{a} \times \mathbf{b}\| &= ab \sin \theta, & \mathbf{b} \times \mathbf{a} &= -\mathbf{a} \times \mathbf{b}.
 \end{aligned}$$

For passive counterclockwise planar axis rotation,

$$a'_x = a_x \cos \phi + a_y \sin \phi, \quad a'_y = -a_x \sin \phi + a_y \cos \phi.$$

Magnitudes, mutual angles and dot products are invariant under this common rotation. The zero vector has no unit direction or uniquely defined angle.

Before accepting a result, inspect its units, quadrant, possible magnitude range, zero cases and orientation. A projection must reconstruct the vector with an orthogonal residual. A nonzero cross product must be perpendicular to both inputs and have the correct ordered orientation. A rotation must preserve the geometry and map a known basis direction correctly. These checks are complementary: passing one does not automatically pass the others.

The following section develops optional tools for more demanding geometry and kinematic descriptions. Readers following the core route may proceed directly to the exercises, leaving the explicitly marked advanced problems for later.

3.9 Advanced Topics for Technical and Engineering Students

This enrichment begins with three related questions. How can several coordinate changes be composed and checked systematically? How does a vector change when its components depend on time? How much confidence can be placed in a direction obtained by normalising a small cross product? The required additional mathematics is modest: two-by-two matrix multiplication, derivatives and a first-order small-change approximation. Each is introduced here. Further subsections explain moving bases and provide an optional, locally supported bridge to field calculus. None is a hidden requirement for core practice.

3.9.1 Rotation matrices as organised projection rules

A column of components is an efficient record of a vector relative to a basis. A two-by-two matrix transforms one column into another: each output is the sum of the products of one row with the input column. Equation (3.39) therefore becomes

$$\begin{pmatrix} a'_x \\ a'_y \end{pmatrix} = R(\phi) \begin{pmatrix} a_x \\ a_y \end{pmatrix}, \quad R(\phi) = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}. \quad (3.45)$$

The entries are dimensionless direction cosines. A matrix does not turn physical units into abstract numbers; both component columns retain the unit of the vector. In the matrix notation below, the same symbol for a vector's column is used only after its basis has been specified.

The *transpose* interchanges rows and columns. Direct multiplication gives

$$R(\phi)^T R(\phi) = I, \quad R(\phi)^{-1} = R(\phi)^T = R(-\phi), \quad (3.46)$$

where I has ones on the diagonal and zeros elsewhere. The diagonal entries of the product are $\cos^2 \phi + \sin^2 \phi = 1$; the off-diagonal entries cancel. A matrix with this property is *orthogonal*. Since the scalar product of columns can be written as $\mathbf{a}^T \mathbf{b}$, its invariance follows in one line:

$$(\mathbf{R}\mathbf{a})^T (\mathbf{R}\mathbf{b}) = \mathbf{a}^T \mathbf{R}^T \mathbf{R} \mathbf{b} = \mathbf{a}^T \mathbf{b}.$$

The result restates the geometric argument rather than replacing it.

For a second axis turn ψ relative to the first turned frame, multiply the matrices in the order in which they act:

$$R(\psi)R(\phi) = R(\phi + \psi). \quad (3.47)$$

For example, the first entry is $\cos \psi \cos \phi - \sin \psi \sin \phi = \cos(\phi + \psi)$; the other entries follow from the addition identities for sine and cosine. Planar rotations about the same origin commute. This does not imply that arbitrary rotations about different three-dimensional axes commute.

Example 3.18: Two coordinate changes and an inverse check

A displacement has laboratory components $(3, -1)$ mm. Axes are first turned counterclockwise by 30° and then by another 60° about the same origin. Find the final components and verify the inverse operation.

After the first turn the column is $((3\sqrt{3} - 1)/2, -(3 + \sqrt{3})/2)$ mm. Multiplying this column by $R(60^\circ)$ gives $(-1, -3)$ mm. Independently, the total turn is 90° , for which the geometric rule is $(a_x, a_y) \mapsto (a_y, -a_x)$, giving the same answer immediately. Both columns have squared norm 10 mm^2 . Applying $R(-90^\circ)$ returns $(3, -1)$ mm. The order of inverse operations in a general matrix chain is reversed, even though these particular planar rotations commute.

Norm preservation does not by itself imply a proper rotation. The reflection matrix $\mathbf{S} = \text{diag}(1, -1)$ also satisfies $\mathbf{S}^T \mathbf{S} = I$. But its determinant, computed as the product of diagonal entries minus the

product of off-diagonal entries, is -1 ; for $\mathbf{R}$ it is $+1$. Reflecting $\hat{\mathbf{x}}, \hat{\mathbf{y}}$ in a fixed right-handed space gives $\hat{\mathbf{x}}, -\hat{\mathbf{y}}$, whose cross product is $-\hat{\mathbf{z}}$. Thus handedness must be checked separately from lengths and dot products. This example explains why a mirrored coordinate convention cannot be treated as an unnoticed proper rotation.

3.9.2 Proper rotations, reflections and oriented normals in three dimensions

A three-dimensional orthogonal matrix $\mathbf{Q}$ obeys $\mathbf{Q}^T \mathbf{Q} = \mathbf{I}$. Applied to two component columns in the same convention, it preserves their scalar product:

$$(\mathbf{Q}\mathbf{a}) \cdot (\mathbf{Q}\mathbf{b}) = \mathbf{a}^T \mathbf{Q}^T \mathbf{Q} \mathbf{b} = \mathbf{a} \cdot \mathbf{b}.$$

Its columns are mutually perpendicular unit directions. An orthogonal matrix has determinant $+1$ or -1 , since taking the determinant of $\mathbf{Q}^T \mathbf{Q} = \mathbf{I}$ gives $(\det \mathbf{Q})^2 = 1$. A *proper rotation* has determinant $+1$ and preserves handedness. The notation $SO(3)$ denotes this set: special (determinant one), orthogonal, three-dimensional matrices. This is a classification of transformations, not an additional computational method required for the core problems.

The signed-volume identity explains how a cross product transforms. For any third vector $\mathbf{c}$, applying the same linear map to all three edges multiplies their signed volume by $\det \mathbf{Q}$. Hence

$$[(\mathbf{Q}\mathbf{a}) \times (\mathbf{Q}\mathbf{b})] \cdot (\mathbf{Q}\mathbf{c}) = \det(\mathbf{Q})(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

Orthogonality lets the right side be written $[\det(\mathbf{Q})\mathbf{Q}(\mathbf{a} \times \mathbf{b})] \cdot (\mathbf{Q}\mathbf{c})$. Because this holds for every $\mathbf{c}$, the two vectors in square brackets are identical:

$$(\mathbf{Q}\mathbf{a}) \times (\mathbf{Q}\mathbf{b}) = \det(\mathbf{Q})\mathbf{Q}(\mathbf{a} \times \mathbf{b}). \quad (3.48)$$

For a proper rotation, the area vector transforms like the input vectors. For an orientation-reversing reflection, an additional minus sign appears. An area normal produced by a cross product is consequently an *axial* vector or *pseudovector*; an ordinary displacement is a polar vector. The distinction matters when a mirrored coordinate convention is used. The cross-component rule inherited from a right-handed basis must not be silently applied as though a left-handed basis were also right-handed.

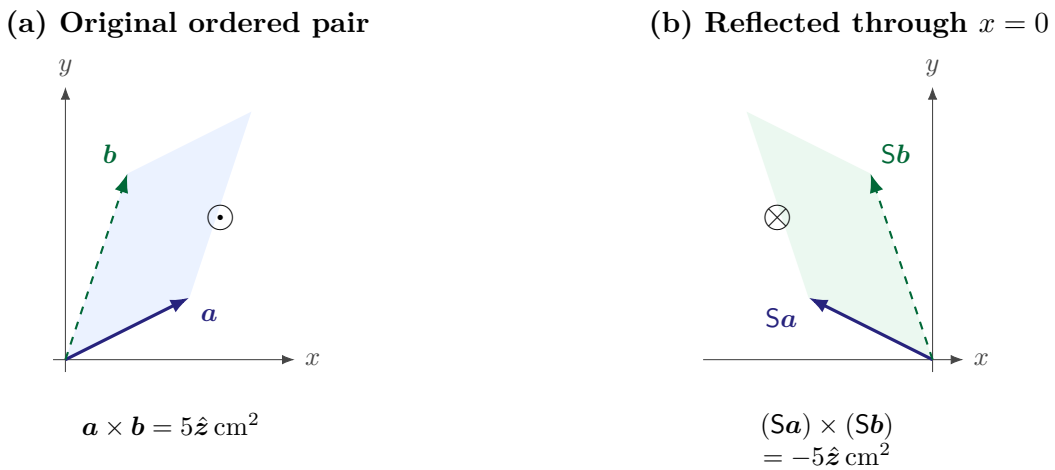


Figure 3.22: An active reflection of an ordered edge pair in a fixed right-handed space. Both panels use equal Cartesian length scales and $+z$ toward the reader. The mirror $\mathbf{S} = \text{diag}(-1, 1, 1)$ reverses x but leaves z unchanged. Nevertheless, the cross product reverses its z direction because the ordered pair's handedness changes. The unsigned area is 5 cm^2 in both panels; $\odot$ and $\otimes$ encode opposite normal directions.

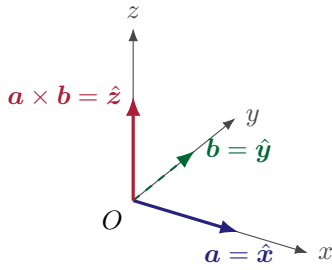
Example 3.19: A mirror preserves area but changes its orientation

Take $\mathbf{a} = (2, 1, 0)$ cm and $\mathbf{b} = (1, 3, 0)$ cm in a fixed right-handed space. Reflect both edges through the y, z plane using $\mathbf{S} = \text{diag}(-1, 1, 1)$. Compare the new cross product with $\mathbf{S}(\mathbf{a} \times \mathbf{b})$.

Calculation. Initially the area vector is $5\hat{\mathbf{z}}$ cm². The reflected edges are $(-2, 1, 0)$ cm and $(-1, 3, 0)$ cm, whose product is $[-2(3) - 1(-1)]\hat{\mathbf{z}}$ cm² = $-5\hat{\mathbf{z}}$ cm². But $\mathbf{S}(5\hat{\mathbf{z}}) = 5\hat{\mathbf{z}}$, because the mirror leaves a polar arrow along z unchanged. Multiplying by $\det \mathbf{S} = -1$ gives the actual reflected *cross-derived* area vector, as required by Equation (3.48).

Checks and interpretation. The input squared lengths remain 5 cm² and 10 cm²; their scalar product remains 5 cm². Their Gram area is therefore $\sqrt{5(10) - 25}$ cm² = 5 cm² before and after reflection. Thus preserved norms and angles do not establish preserved orientation. The reflection is specified here as an active change of the edge geometry in fixed axes. A passive description using mirrored axes needs the corresponding handedness convention; confusing the two operations is the source of the apparently contradictory normal signs.

(a) A right-handed ordered triad



(b) Active +90° about y

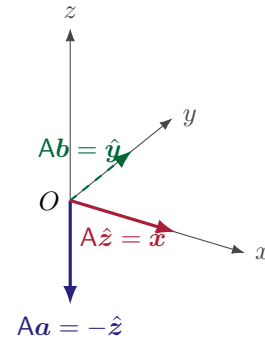


Figure 3.23: A proper active rotation carries an ordered cross product into the cross product of the rotated inputs. The original axes remain fixed in both panels. The map sends $\hat{\mathbf{x}} \mapsto -\hat{\mathbf{z}}$, $\hat{\mathbf{y}} \mapsto \hat{\mathbf{y}}$ and $\hat{\mathbf{z}} \mapsto \hat{\mathbf{x}}$, giving $(-\hat{\mathbf{z}}) \times \hat{\mathbf{y}} = \hat{\mathbf{x}}$. Every vector is dimensionless and of unit magnitude, but the oblique projection foreshortens apparent lengths and angles. Solid blue, dashed green and red arrows identify the first input, second input and output consistently.

3.9.3 A three-dimensional sequence: yaw then pitch

Planar rotations about one axis commute, but three-dimensional frame turns need not. To see what the statement means, specify every axis and convention. Define the passive component maps for a positive right-hand turn of the axes about the current z and current y directions:

$$\mathbf{P}_z(\psi) = \begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{P}_y(\theta) = \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix}. \quad (3.49)$$

Here “yaw” labels a turn about z and “pitch” a turn about y ; their precise meaning is the matrices above, rather than an unstated aircraft convention. The first yaw turns the basis about its initial z . The subsequent pitch turns it about the *intermediate* y' . A fixed vector’s final components are

$$\mathbf{a}'' = \mathbf{P}_y(\theta)\mathbf{P}_z(\psi)\mathbf{a}. \quad (3.50)$$

The rightmost map acts first. Reversing the process means a pitch about initial y , followed by a yaw about the resulting z' , giving $\mathbf{P}_z(\psi)\mathbf{P}_y(\theta)\mathbf{a}$. These are intrinsic sequences: the second axis belongs to the already changed frame. A specification using two fixed laboratory axes would describe a different sequence and must be derived with that convention.

To recover the physical vector from any final column, multiply by the inverse of the full chain. Since both matrices are orthogonal, that inverse is $P_z(-\psi)P_y(-\theta)$. The order reverses. The transpose of a passive full-frame map supplies the final basis directions as columns in the original frame. It is also the active map that carries the original basis into that final basis. An actively rotated *vector* in fixed axes must be stated as such; applying a passive component chain to a vector's old column is not a physical rotation of that vector.

Example 3.20: Two quarter-turn frame sequences do not commute

The fixed laboratory displacement vectors are $\mathbf{a} = (1, 2, 3)$ mm and $\mathbf{b} = (2, -1, 0)$ mm. Compare passive yaw $+90^\circ$ followed by pitch $+90^\circ$ with the reverse intrinsic sequence. Verify a dot product and the oriented cross product.

First sequence. A positive quarter-turn yaw sends (x, y, z) to $(y, -x, z)$; a positive quarter-turn pitch sends (x, y, z) to $(-z, y, x)$. Thus yaw then pitch gives

$$\mathbf{a}'' = (-3, -1, 2) \text{ mm}, \quad \mathbf{b}'' = (0, -2, -1) \text{ mm}.$$

In the initial basis, $\mathbf{a} \cdot \mathbf{b} = 0$ and $\mathbf{a} \times \mathbf{b} = (3, 6, -5) \text{ mm}^2$. In the final basis, $\mathbf{a}'' \cdot \mathbf{b}'' = 2 - 2 = 0$ and $\mathbf{a}'' \times \mathbf{b}'' = (5, -3, 6) \text{ mm}^2$. Applying the same passive chain to $(3, 6, -5)$ gives $(5, -3, 6)$, confirming that the ordered area vector transforms consistently.

Reverse sequence. Pitch then yaw gives instead

$$\bar{\mathbf{a}} = (2, 3, 1) \text{ mm}, \quad \bar{\mathbf{b}} = (-1, 0, 2) \text{ mm}.$$

Their scalar product remains zero and their cross product is $(6, -5, 3) \text{ mm}^2$, equal to the reverse chain applied to the original area-vector column. The squared norms remain 14 mm^2 and 5 mm^2 in either final basis; the squared area magnitude remains 70 mm^4 . These invariants do not make the two final bases identical.

Directional and inverse checks. For yaw then pitch the final basis directions, expressed in the laboratory frame, are $\hat{\mathbf{x}}'' = -\hat{\mathbf{z}}$, $\hat{\mathbf{y}}'' = -\hat{\mathbf{x}}$, $\hat{\mathbf{z}}'' = \hat{\mathbf{y}}$. Reconstruction gives $-3(-\hat{\mathbf{z}}) - 1(-\hat{\mathbf{x}}) + 2\hat{\mathbf{y}} = \mathbf{a}/\text{mm}$. For pitch then yaw they are $\bar{\hat{\mathbf{x}}} = \hat{\mathbf{y}}$, $\bar{\hat{\mathbf{y}}} = \hat{\mathbf{z}}$, $\bar{\hat{\mathbf{z}}} = \hat{\mathbf{x}}$, and $2\bar{\hat{\mathbf{y}}} + 3\bar{\hat{\mathbf{z}}} + 1\bar{\hat{\mathbf{x}}} = \mathbf{a}$ reconstructs the same fixed vector. Equivalently, applying the relevant transposed chain returns each old component column. Figure 3.24 shows the differing final bases. The vectors themselves did not move: the differing final columns express one geometry in two different coordinate grids.

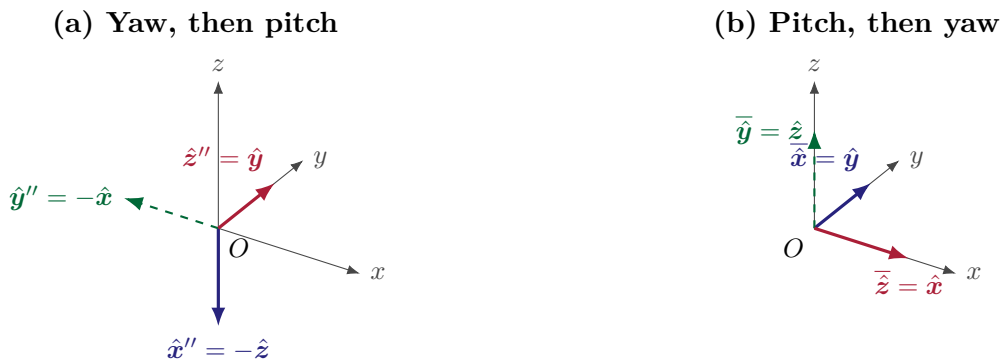


Figure 3.24: Two final right-handed bases for positive 90° intrinsic frame turns. Thin grey arrows give the same initial axes in each panel. The thick arrows give the final x, y, z unit directions, with solid blue, dashed green and red styles respectively. The second turn is about an intermediate-frame axis. Panel (a) has $\hat{\mathbf{x}}'' \times \hat{\mathbf{y}}'' = \hat{\mathbf{z}}''$; panel (b) has $\bar{\hat{\mathbf{x}}} \times \bar{\hat{\mathbf{y}}} = \bar{\hat{\mathbf{z}}}$, but the bases differ. This oblique projection is a direction diagram, not an angular measurement. No physical displacement arrow is rotated in the passive-coordinate example.

The same care with ordering will later help interpret cross-derived quantities such as torque and angular momentum, while dot products will express projections and work. Their dynamical laws

belong to the mechanics chapters. Here the established results concern geometry, orientation, units and coordinate changes.

3.9.4 A vector-valued function in a fixed basis

Let a curve be prescribed by $\mathbf{r}(t) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}} + z(t)\hat{\mathbf{z}}$, with the three basis vectors independent of time. Over a nonzero time interval h , subtract the two position vectors in the same frame. Dividing by h gives a vector rate of change. The derivative is the limit, when it exists:

$$\mathbf{r}'(t) = \lim_{h \rightarrow 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = x'(t)\hat{\mathbf{x}} + y'(t)\hat{\mathbf{y}} + z'(t)\hat{\mathbf{z}}. \quad (3.51)$$

The familiar one-dimensional limit applies to each component because the basis is fixed. Differentiability of all three components is sufficient here. If $\mathbf{r}$ is a position and t is time, the derivative has units of velocity. The chord direction approaches a tangent direction wherever the derivative is nonzero. At a zero derivative, the tangent direction cannot be obtained by normalising that derivative; more local information is needed.

The geometry does not require a force model. We prescribe the path rather than predicting it dynamically. Its coefficients and valid time interval are part of the model; extrapolation outside that interval is not justified simply because a polynomial can be evaluated there.

Example 3.21: A prescribed three-dimensional path

Consider

$$\mathbf{r}(t) = at\hat{\mathbf{x}} + bt^2\hat{\mathbf{y}} + ct^3\hat{\mathbf{z}}, \quad 0 \leq t \leq 2 \text{ s}, \quad (3.52)$$

with $a = 2 \text{ m s}^{-1}$, $b = 1 \text{ m s}^{-2}$ and $c = 0.5 \text{ m s}^{-3}$. Find the tangent rate at $t = 1 \text{ s}$ and compare it with the finite rate from 1 to 1.5 s.

Differentiate with the units intact. The fixed-basis derivative is $\mathbf{r}'(t) = a\hat{\mathbf{x}} + 2bt\hat{\mathbf{y}} + 3ct^2\hat{\mathbf{z}}$. At the stated time,

$$\mathbf{r}(1 \text{ s}) = (2, 1, 0.5) \text{ m}, \quad \mathbf{r}'(1 \text{ s}) = (2, 2, 1.5) \text{ m s}^{-1}.$$

The derivative magnitude is $\sqrt{10.25} \text{ m s}^{-1} \simeq 3.2016 \text{ m s}^{-1}$ approximately, and its unit tangent is $(4, 4, 3)/\sqrt{41}$. The velocity vector is not a line segment with metre units; when placed on the spatial curve in Figure 3.25, it is multiplied by an explicitly stated time scale.

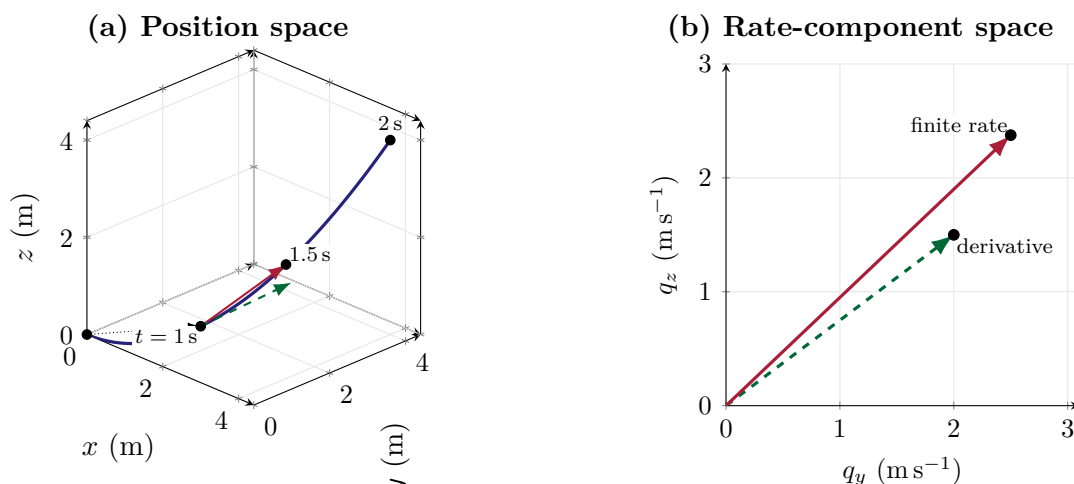
At 1.5 s the position is $(3, 2.25, 1.6875) \text{ m}$. Therefore

$$\Delta \mathbf{r} = (1, 1.25, 1.1875) \text{ m}, \quad \frac{\Delta \mathbf{r}}{0.5 \text{ s}} = (2, 2.5, 2.375) \text{ m s}^{-1}.$$

The finite rate follows the chord; it is not the instantaneous rate at the initial time. An independent expansion confirms the distinction:

$$\frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = a\hat{\mathbf{x}} + (2bt + bh)\hat{\mathbf{y}} + (3ct^2 + 3ct h + ch^2)\hat{\mathbf{z}}.$$

The extra terms vanish as $h \rightarrow 0$. At $t = 0$ the derivative is $2\hat{\mathbf{x}} \text{ m s}^{-1}$; throughout the specified interval its x component stays positive, so it is never zero. At $t = 2 \text{ s}$ it is $(2, 4, 6) \text{ m s}^{-1}$. These endpoint checks agree with a path advancing monotonically in x .



Both rates have $q_x = 2 \text{ m s}^{-1}$, which is not displayed in this two-component inset.

Figure 3.25: An analytical vector curve and two different rate vectors. Panel (a) uses $\mathbf{r}(t) = (2t, t^2, 0.5t^3)$ in the declared numerical-SI convention for $0 \leq t/(1\text{s}) \leq 2$. The solid chord is $\Delta\mathbf{r}$ from 1 to 1.5 s; the dashed tangent segment is $(0.6\text{s})\mathbf{r}'(1\text{s})$, so both displayed segments have metre units. The 3D axes use equal unit lengths before projection. Panel (b) is a separate rate-component space, not another position plot: it compares $\Delta\mathbf{r}/(0.5\text{s})$ with $\mathbf{r}'(1\text{s})$. All coordinates and 101 curve samples follow the exact polynomial, not a hand-drawn trajectory.

3.9.5 Changing a magnitude is not changing a vector

Differentiate a scalar product by expanding its component products. Ordinary one-dimensional product rules give

$$\frac{d}{dt}(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a}' \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b}'. \quad (3.53)$$

For $\mathbf{a} = \mathbf{r} \neq \mathbf{0}$, differentiating $\|\mathbf{r}\|^2 = \mathbf{r} \cdot \mathbf{r}$ yields

$$\frac{d}{dt}\|\mathbf{r}\| = \frac{\mathbf{r} \cdot \mathbf{r}'}{\|\mathbf{r}\|}. \quad (3.54)$$

This is the radial projection of the vector rate, not generally its magnitude. A vector can turn while its magnitude remains constant. As a purely kinematic example, let $\mathbf{r} = R(\cos \omega t \hat{\mathbf{x}} + \sin \omega t \hat{\mathbf{y}})$, with $R > 0$ constant and ω an inverse time. The trigonometric derivative rules for radian arguments give $\mathbf{r}' = R\omega(-\sin \omega t \hat{\mathbf{x}} + \cos \omega t \hat{\mathbf{y}})$. Its dot product with $\mathbf{r}$ vanishes, so the radius is constant even though the vector derivative is nonzero when $\omega \neq 0$. Forces and circular-motion dynamics are not needed to establish this identity.

The same component expansion gives the cross-product rule

$$\frac{d}{dt}(\mathbf{a} \times \mathbf{b}) = \mathbf{a}' \times \mathbf{b} + \mathbf{a} \times \mathbf{b}'. \quad (3.55)$$

The order of the two vectors in each term must be retained. Interchanging the factors in just one term introduces a sign error.

There is one essential basis limitation. If a basis itself depends on time, the product rule applied to $\mathbf{a} = \sum_i a_i(t)\hat{\mathbf{e}}_i(t)$ gives

$$\mathbf{a}' = \sum_i a'_i \hat{\mathbf{e}}_i + \sum_i a_i \hat{\mathbf{e}}'_i. \quad (3.56)$$

One cannot discard the second sum. A geometrically constant vector has zero vector derivative, including when described in moving coordinates; its changing components and changing basis contributions then cancel. The moving-basis derivation later in this advanced section demonstrates

this cancellation explicitly. Until that subsection, derivative calculations use fixed Cartesian bases; the dynamics of motion in polar coordinates belongs to *Motion in Two and Three Dimensions*. A constant rotation matrix may be moved through a derivative; a time-dependent one may not.

3.9.6 Finite-difference checks on a vector curve

The polynomial model supplies an exact test for a numerical derivative, not a synthetic substitute for experimental data. A forward difference has the extra terms displayed in Example 3.21. A centred difference instead gives

$$\frac{\mathbf{r}(t+h) - \mathbf{r}(t-h)}{2h} = a\hat{\mathbf{x}} + 2bt\hat{\mathbf{y}} + (3ct^2 + ch^2)\hat{\mathbf{z}}. \quad (3.57)$$

This follows by expanding $(t \pm h)^2$ and $(t \pm h)^3$ and cancelling even powers in the difference. At $t = 1$ s, both sampled points stay in the stipulated interval when $0 < h \leq 1$ s. The centred x and y components are exact for this polynomial; its z error is exactly ch^2 , not just an asymptotic claim.

h (s)	Forward y rate m s^{-1}	Forward z rate m s^{-1}	Centred z rate m s^{-1}
0.500	2.500	2.375000	1.625000
0.250	2.250	1.906250	1.531250
0.125	2.125	1.6953125	1.5078125

The exact rates are 2 and 1.5 m s^{-1} for y and z respectively. Halving h reduces the centred z error by four. The forward error is first order as $h \rightarrow 0$, but the finite ratio is not exactly two because its z component also contains a term in h^2 . In floating-point arithmetic, making h extremely small can instead magnify cancellation in the subtraction. Measurement noise would introduce another error source. A decreasing truncation term is not a promise of indefinitely improving total accuracy.

3.9.7 When a normal is poorly determined

Normalisation divides by a magnitude. If that magnitude is small because two edges are almost parallel, a small perturbation can cause a large direction change. To isolate the geometry, take

$$\mathbf{p} = (L, 0, 0), \quad \mathbf{q} = (L, \varepsilon, \delta), \quad L > 0, \varepsilon > 0, \quad (3.58)$$

where L, ε, δ are all lengths. The nominal plane has $\delta = 0$; δ represents an out-of-plane endpoint perturbation, not random noise unless an additional statistical model is supplied. Direct multiplication gives

$$\mathbf{p} \times \mathbf{q} = (0, -L\delta, L\varepsilon), \quad \hat{\mathbf{n}} = \frac{(0, -\delta, \varepsilon)}{\sqrt{\varepsilon^2 + \delta^2}}. \quad (3.59)$$

The signed tilt from $+z$ toward $-y$ is

$$\alpha = \arctan(\delta/\varepsilon), \quad \frac{\partial \alpha}{\partial \delta} = \frac{\varepsilon}{\varepsilon^2 + \delta^2}. \quad (3.60)$$

The derivative follows from $d(\arctan u)/du = 1/(1+u^2)$ and the chain rule; α is in radians. The sensitivity at the nominal plane is $1/\varepsilon$, with inverse-length units. The small-change approximation $\alpha \simeq \delta/\varepsilon$ requires $|\delta| \ll \varepsilon$. It is a local estimate, not a new exact formula.

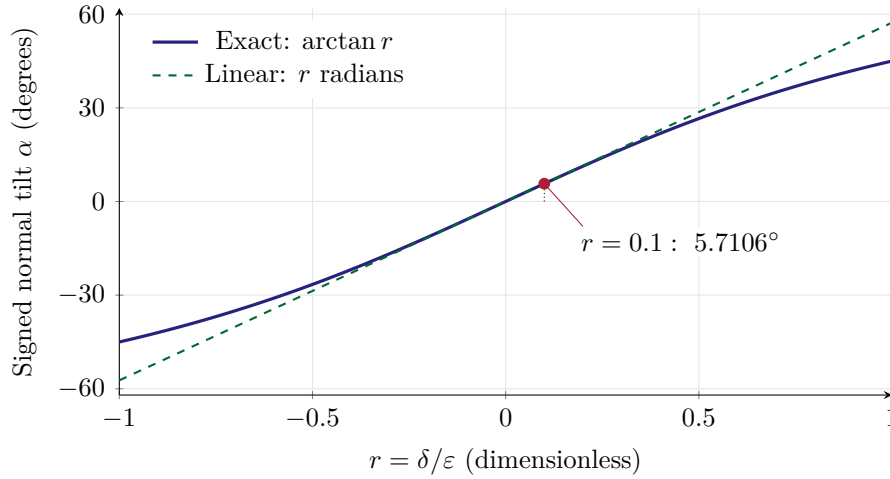


Figure 3.26: Calculated sensitivity of the normal for $\mathbf{p} = (L, 0, 0)$ and $\mathbf{q} = (L, \varepsilon, \delta)$ with $L, \varepsilon > 0$. The signed tilt is measured from $+z$ toward $-y$. The solid curve is the exact model; the dashed line is the local approximation $\alpha \simeq \delta/\varepsilon$ in radians, converted to degrees for the vertical axis. Their agreement near the origin does not justify the approximation at large $|\delta/\varepsilon|$. The marked point is the constructed perturbation of Example 3.22; no measured data or statistical band is shown.

Example 3.22: A small edge error and a large normal tilt

Let $L = 1.00$ m, $\varepsilon = 0.010$ m, and let an endpoint shift from $\delta = 0$ to 0.001 m. The shift is only 0.1% of L , but $\delta/\varepsilon = 0.1$. Equation (3.59) gives

$$\hat{\mathbf{n}} = \frac{(0, -1, 10)}{\sqrt{101}}, \quad \alpha = \arctan(0.1) = 5.7106^\circ \quad (\text{approximately}).$$

The linear estimate $0.1 \text{ rad} = 5.7296^\circ$ is close but not identical. The area magnitude changes from 0.0100 m^2 to $\sqrt{0.010^2 + 0.001^2} \text{ m}^2 = 0.0100499 \text{ m}^2$, only about 0.499%. An almost unchanged area can therefore accompany a significant change in its normal direction.

Checks and comparison. The perturbed normal is perpendicular to both edges and has unit magnitude. Reversing δ reverses the signed tilt. With $\varepsilon = 0.50$ m and the same perturbation, the tilt would be only $\arctan(0.002) = 0.1146^\circ$. Increasing L alone scales the area but does not improve this normal's sensitivity: L cancels on normalisation. If both ε and δ vanish, the plane selected by the two edges disappears and no unit normal can be inferred from them.

For a measurement or a geometric algorithm, check the angle between the edges as well as the absolute size of their cross product. The dimensionless ratio $\|\mathbf{p} \times \mathbf{q}\|/(pq) = |\sin \theta|$ separates near parallelism from a mere change of length unit. A physical acceptance threshold should reflect input uncertainty and the permitted orientation error. A floating-point tolerance that merely distinguishes exact zero cannot supply that engineering judgement.

What the advanced tools add

Matrices organise the same projection rules and expose inverse, composition and handedness checks. Fixed-basis differentiation turns a vector curve into a local rate without requiring a force law. Sensitivity analysis distinguishes an exactly computed direction from a well-determined direction. Geometry and validity conditions remain essential for all three tools.

3.9.8 A rotating basis and the derivative of a fixed vector

The vector derivative describes a change of the geometrical arrow. Component changes alone need not describe that change, because the directions used to measure the components may also change. This distinction matters in a rotating instrument, in polar coordinates and in the description of a curved path. It can be established with the same Cartesian projections already used in this chapter, without assuming a new law of motion.

Let $\theta(t)$ be a differentiable angle measured counterclockwise from the fixed x axis. The associated radial and transverse unit vectors are

$$\hat{e}_r = \cos \theta \hat{x} + \sin \theta \hat{y}, \quad \hat{e}_\theta = -\sin \theta \hat{x} + \cos \theta \hat{y}. \quad (3.61)$$

The transverse direction is obtained by a positive quarter turn from the radial direction: $\hat{e}_r \times \hat{e}_\theta = \hat{z}$. Both vectors have unit magnitude, and their scalar product vanishes. The names refer to directions; they do not require the geometrical vector under study to point radially outward. At the coordinate origin a polar position has no unique radial direction, so a polar position basis must be specified independently there or avoided.

Use *radians* in the differentiation. The chain rule applied to each Cartesian component gives

$$\frac{d\hat{e}_r}{dt} = \dot{\theta}(-\sin \theta \hat{x} + \cos \theta \hat{y}) = \dot{\theta}\hat{e}_\theta, \quad (3.62)$$

$$\frac{d\hat{e}_\theta}{dt} = \dot{\theta}(-\cos \theta \hat{x} - \sin \theta \hat{y}) = -\dot{\theta}\hat{e}_r. \quad (3.63)$$

A dot denotes a derivative with respect to time. Although a unit vector has constant magnitude, it has a nonzero derivative when its direction turns. Its derivative is perpendicular to itself, as differentiating $\hat{e}_r \cdot \hat{e}_r = 1$ also confirms. The unit vectors are dimensionless and their derivatives have inverse-time units.

For an arbitrary planar vector $\mathbf{a} = a_r \hat{e}_r + a_\theta \hat{e}_\theta$, apply the ordinary product rule before collecting directions:

$$\frac{d\mathbf{a}}{dt} = (\dot{a}_r - \dot{\theta}a_\theta)\hat{e}_r + (\dot{a}_\theta + \dot{\theta}a_r)\hat{e}_\theta. \quad (3.64)$$

The two terms containing $\dot{\theta}$ are basis contributions. They have the same units as the component derivatives. If the basis stops turning, $\dot{\theta} = 0$ and the fixed-basis rule is recovered. If the angle is recorded in degrees, first convert it to radians; differentiating a sine of a degree variable introduces the conversion factor $\pi/180$. A drawing may use degrees for convenience without changing the radian convention in the mathematical model.

Example 3.23: Constant vector, changing components

A fixed laboratory displacement is $\mathbf{A} = A\hat{x}$, with $A = 5$ m. Describe it in a basis turning at $\theta(t) = \Omega t$ with $\Omega = 0.40 \text{ rad s}^{-1}$, and prove that its geometrical derivative is zero.

Project first. Equations (3.61) give $A_r = A \cos \theta$ and $A_\theta = -A \sin \theta$. Their derivatives are $\dot{A}_r = -A\Omega \sin \theta$ and $\dot{A}_\theta = -A\Omega \cos \theta$. If the basis derivatives were discarded, these two nonzero numbers would incorrectly suggest that a fixed displacement is moving.

Include the basis. The radial coefficient of the full derivative is $-A\Omega \sin \theta - \Omega(-A \sin \theta) = 0$. The transverse coefficient is $-A\Omega \cos \theta + \Omega A \cos \theta = 0$. Thus $d\mathbf{A}/dt = \mathbf{0}$ for every time, including when a component crosses zero. No division by a component was used, so those crossings cause no exception.

At $\theta = \pi/6$ the displacement components are $(5\sqrt{3}/2, -5/2) \text{ m}$, approximately $(4.3301, -2.5000) \text{ m}$. The component-rate contribution is $(-1, -\sqrt{3}) \text{ m s}^{-1}$ in the moving basis. The basis contribution is $(1, \sqrt{3}) \text{ m s}^{-1}$ in that same basis. They cancel component by component. Both contributions have magnitude 2 m s^{-1} ; each is separately nonzero, whereas

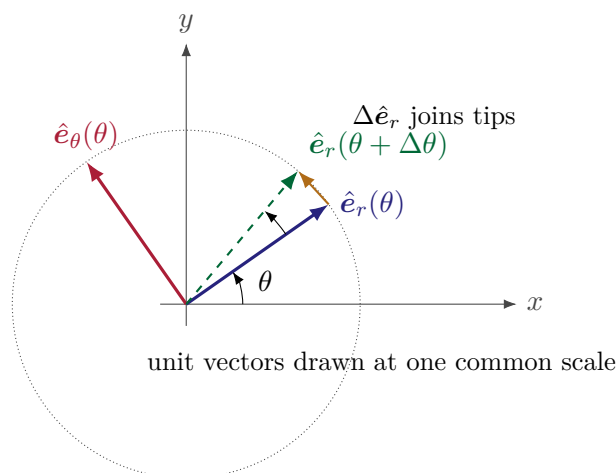


Figure 3.27: A moving orthonormal basis. The drawing uses $\theta = 35^\circ$ and $\Delta\theta = 15^\circ$ for visibility, with equal axis scales. The orange chord is the exact difference between the two radial unit vectors, not its derivative. As the angular increment tends to zero, its direction approaches $\hat{e}_\theta$ and its magnitude divided by the increment in radians approaches one. The transverse unit vector is shown at the initial angle.

their sum is zero.

Independent reconstruction. Substitute the component formulas into $A_r\hat{e}_r + A_\theta\hat{e}_\theta$. The Cartesian x coefficient is $A(\cos^2\theta + \sin^2\theta) = A$, and the y coefficient is $A\cos\theta\sin\theta - A\sin\theta\cos\theta = 0$. This proves constancy directly, before any derivative is taken.

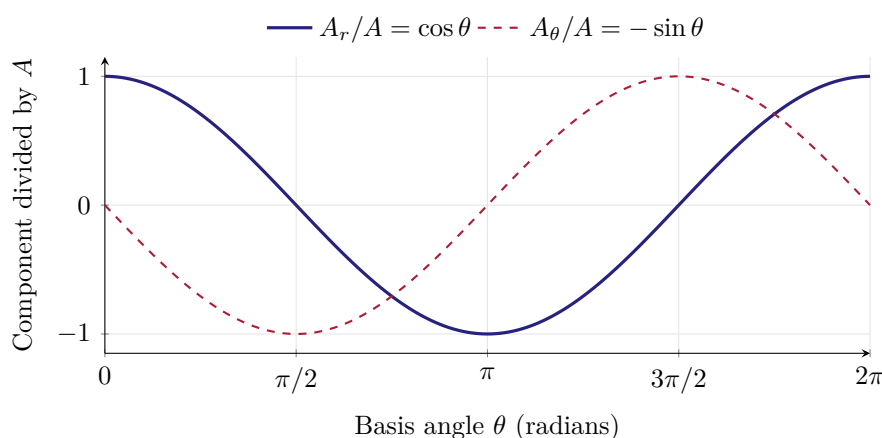


Figure 3.28: Changing projections of one geometrically fixed vector $A\hat{x}$. The horizontal variable is the moving-basis angle, not a displacement or a trajectory coordinate. The curves use the exact trigonometric model with radian arguments. At every angle the squared normalised components sum to one. Their slopes do not by themselves give the vector derivative: the basis terms in Equation (3.64) cancel those slopes.

For a polar position $\mathbf{r} = r\hat{e}_r$, the same rule yields $\mathbf{r}' = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$. It separates change of distance from change of direction. Chapter 4 develops velocity and acceleration using this geometry. Here it serves as an additional check on the product rule, rather than an assumption about a body's dynamics.

3.9.9 Optional bridge from individual vectors to spatial fields

An individual vector specifies one directed quantity. A *vector field* assigns a vector to every point in a declared spatial domain. A *scalar field* assigns a scalar there. For example, a prescribed surface height $h(x, y)$ gives one height at each horizontal location; a prescribed velocity field $\mathbf{u}(x, y, z)$ gives

a possible local velocity at each spatial location. These are mathematical descriptions. A velocity field becomes a fluid model only when its physical assumptions and evolution are supplied.

The following bridge is optional. It introduces partial derivatives, one-variable integration and the chain rule for a path before using them. It establishes geometric meanings for gradient, divergence, curl and an oriented line integral. It does not make these tools prerequisites for the core motion chapter. The examples are explicit smooth functions, so their required derivatives exist. Questions about discontinuities, singularities and nonsmooth boundaries require additional analysis.

A partial derivative changes one coordinate while holding the others fixed:

$$\frac{\partial h}{\partial x}(x, y) = \lim_{\Delta x \rightarrow 0} \frac{h(x + \Delta x, y) - h(x, y)}{\Delta x}. \quad (3.65)$$

The derivative with respect to y is defined in the same way. The symbol ∂ signals that other independent variables are held constant. If $h = x^2/L + 2y^2/L$ with a constant positive length L , then $\partial h/\partial x = 2x/L$: the term involving y is constant during that operation. Conversely, $\partial h/\partial y = 4y/L$. The height and coordinates have length units; each derivative is dimensionless, a height change per horizontal distance. A temperature field would instead have temperature per length units for its derivatives.

For smooth h , the leading change for a small displacement is

$$\Delta h = \frac{\partial h}{\partial x} \Delta x + \frac{\partial h}{\partial y} \Delta y + o(\|\Delta \mathbf{r}\|), \quad \nabla h = \frac{\partial h}{\partial x} \hat{\mathbf{x}} + \frac{\partial h}{\partial y} \hat{\mathbf{y}}. \quad (3.66)$$

The notation $o(\|\Delta \mathbf{r}\|)$ describes a remainder whose ratio to the displacement magnitude tends to zero. The *gradient* ∇h packages the first derivatives into a vector. It is not a height or a position. In three Cartesian dimensions a third term $(\partial h/\partial z)\hat{\mathbf{z}}$ is added. This simple component formula relies on an orthonormal Cartesian basis; it cannot be copied unchanged into arbitrary curved coordinates.

A direction must be normalised if the result is to represent a rate per distance. For a dimensionless unit vector $\hat{\mathbf{s}}$ and signed distance ℓ , substitute $\Delta \mathbf{r} = \ell \hat{\mathbf{s}}$ into the small-change formula:

$$D_{\hat{\mathbf{s}}} h = \lim_{\ell \rightarrow 0} \frac{h(\mathbf{r} + \ell \hat{\mathbf{s}}) - h(\mathbf{r})}{\ell} = \nabla h \cdot \hat{\mathbf{s}}. \quad (3.67)$$

This *directional derivative* is the projection of the gradient onto the travel direction. When the gradient is nonzero, the largest positive rate is $\|\nabla h\|$, obtained by travelling along it. The most negative rate occurs oppositely. A perpendicular direction has zero first-order rate. At a point with zero gradient, every directional first derivative is zero; this does not by itself establish a maximum, a minimum or a flat neighbourhood.

Example 3.24: A local slope and a contour direction

For $h = (x^2 + 2y^2)/L$, evaluate the slope at $P = (L, L)$ in direction $\mathbf{s} = (3, 4)$. Find the steepest direction and a contour tangent, and check the slope by a direct displacement calculation.

Normalise and differentiate. The unit direction is $(3/5, 4/5)$. At P , $\nabla h = 2\hat{\mathbf{x}} + 4\hat{\mathbf{y}}$, so $D_{\hat{\mathbf{s}}} h = 2(3/5) + 4(4/5) = 22/5 = 4.4$. The result is a dimensionless height change per horizontal length, not an angle. The steepest direction is $(1, 2)/\sqrt{5}$ and its rate is $2\sqrt{5} \simeq 4.4721$. A unit contour tangent is $(2, -1)/\sqrt{5}$; its scalar product with the gradient is exactly zero.

Check without the gradient formula. Set $x = L + 3\ell/5$ and $y = L + 4\ell/5$. Expanding the squares gives

$$h(x, y) - h(L, L) = \frac{22}{5}\ell + \frac{41}{25L}\ell^2.$$

Division by ℓ and the limit $\ell \rightarrow 0$ give $22/5$ again. If $\ell = 0.010L$, the exact change is $0.044164L$, whereas the first-order estimate is $0.044000L$. The residual $0.000164L$ is the explicitly calculated

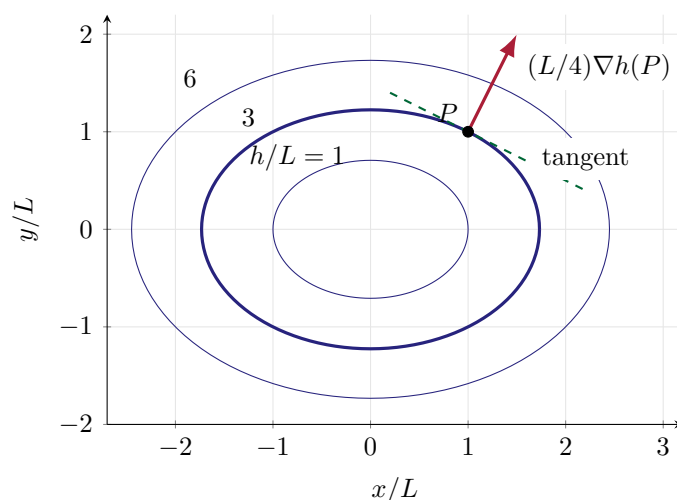


Figure 3.29: Contours and a local gradient for the original height model $h = (x^2 + 2y^2)/L$, $L > 0$. The ellipses are the exact level sets $h/L = 1, 3, 6$. At $P = (L, L)$, the gradient is $(2, 4)$ and the contour tangent has direction $(2, -1)$. The red arrow uses the explicitly stated length scale $L/4$; it is not a height displacement. Equal spatial axis scales preserve the visual right angle between gradient and tangent.

quadratic term. The local derivative is exact; extending its linear estimate to a finite step introduces the approximation.

Contour interpretation. Along a smooth constant-height curve, $dh/d\ell = \nabla h \cdot d\mathbf{r}/d\ell = 0$. Consequently the tangent is perpendicular to the gradient wherever the gradient is nonzero. At the origin of this particular model it vanishes and the lowest level set is a point, so a nonzero contour tangent cannot be assigned there.

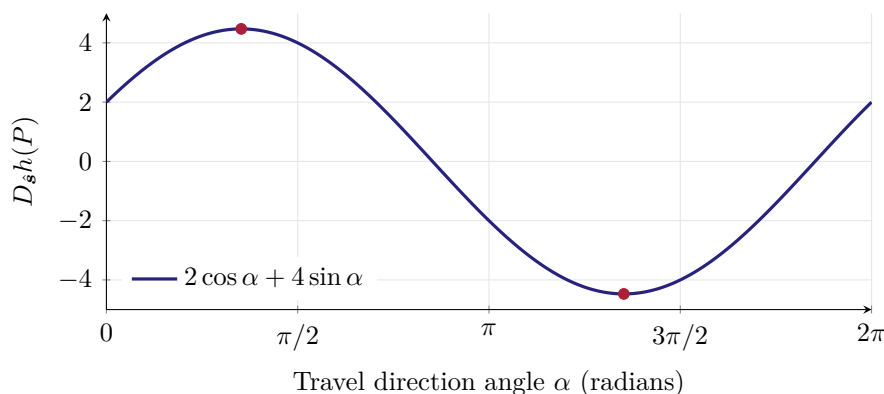


Figure 3.30: The directional derivative depends on direction at a single point. For $P = (L, L)$ in the height model, the exact curve is $2 \cos \alpha + 4 \sin \alpha$. Its positive maximum $2\sqrt{5}$ occurs at $\alpha = \text{atan2}(4, 2)$; adding π gives the negative minimum. The angle labels use radians, and both plotted quantities are dimensionless. This curve varies travel direction, not the evaluation point.

3.9.10 Divergence from the balance of opposite faces

Consider a smooth vector field $\mathbf{u} = u_x \hat{\mathbf{x}} + u_y \hat{\mathbf{y}} + u_z \hat{\mathbf{z}}$. To compare its normal components on opposite sides of a small region, form a rectangular box centred at a point, with edge lengths $\Delta x, \Delta y, \Delta z > 0$. The outward normal on the positive- x face is $\hat{\mathbf{x}}$; on the negative- x face it is $-\hat{\mathbf{x}}$. A positive value of u_x on the negative face therefore makes a *negative* contribution to outward flux. The face normal, rather than the sign of a coordinate alone, determines that contribution.

For a constant field on a planar surface of area A_s and one chosen unit normal $\hat{n}$, all surface pieces have the same normal projection. Their sum is therefore exactly $\Phi = (\mathbf{u} \cdot \hat{n})A_s = \mathbf{u} \cdot \mathbf{A}_s$, where $\mathbf{A}_s = A_s \hat{n}$ is the oriented area vector. The flux vanishes for a field tangent to the plane, is positive along the normal, and changes sign if the normal is reversed. It has units of field times area. These facts follow from projection geometry and do not assert a law about the field. For a curved surface or a varying field, the normal projection must instead be summed locally.

Flux through an oriented area is the integral of $\mathbf{u} \cdot \hat{n}$ over the area. For a velocity field its units are volume per time. For a sufficiently small box, the leading combined contribution of the x faces is

$$[u_x(x + \Delta x/2, y, z) - u_x(x - \Delta x/2, y, z)] \Delta y \Delta z \simeq \frac{\partial u_x}{\partial x} \Delta x \Delta y \Delta z.$$

The approximation samples each face near its centre and retains first-order spatial variation. Adding the y and z pairs and dividing by box volume gives

$$\lim_{\text{box shrinks}} \frac{\text{net outward flux}}{\Delta x \Delta y \Delta z} = \nabla \cdot \mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}. \quad (3.68)$$

The limit uses boxes whose aspect ratios remain bounded, so all dimensions shrink together. The *divergence* is a scalar describing this local outward flux per volume. A constant nonzero vector field has zero divergence: it passes through a box without a net excess of outward over inward flux. Zero divergence does not mean zero field.

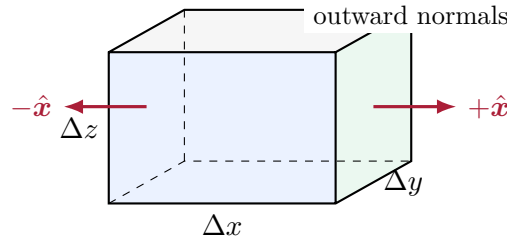


Figure 3.31: A flux box with opposite outward normals. The perspective is schematic; it establishes face orientation and edge labels rather than a quantitative 3D view. The combined x -face flux contains the difference of the two u_x values, because the normals are opposite. The omitted y and z normals obey the same outward convention. This sign structure produces the divergence.

Example 3.25: Expansion and compression in one field

Take the prescribed field $\mathbf{u} = (ax, -by, cz)$ with $a = 0.40 \text{ s}^{-1}$, $b = 0.20 \text{ s}^{-1}$, $c = 0.10 \text{ s}^{-1}$. Find its divergence and the net outward flux through a box with dimensions 0.20 m, 0.10 m, 0.30 m. The component derivatives give $\nabla \cdot \mathbf{u} = a - b + c = 0.30 \text{ s}^{-1}$. For this linear field, the opposite-face differences are exactly $a\Delta x$, $-b\Delta y$ and $c\Delta z$ at any box centre. No small-box approximation is required for the finite-box answer: each normal component is constant over its face. The box volume is $V = 0.0060 \text{ m}^3$. The three paired flux contributions are $aV = 0.0024 \text{ m}^3 \text{ s}^{-1}$, $-bV = -0.0012 \text{ m}^3 \text{ s}^{-1}$ and $cV = 0.0006 \text{ m}^3 \text{ s}^{-1}$. Their sum is $0.0018 \text{ m}^3 \text{ s}^{-1}$, agreeing with the divergence times volume. Compression along y partly offsets expansion along x and z .

All results have the correct units, and translating the box does not change the answer. If $b = a + c$, the paired fluxes cancel even though each component field is usually nonzero. The local divergence alone does not determine a density or its time evolution; such a physical conclusion additionally requires a continuity model and its assumptions.

3.9.11 Curl from circulation round an oriented rectangle

Flux compares normal projections; circulation adds tangential projections along a closed path. In an xy plane, traverse a small rectangle counterclockwise when viewed from $+z$. Its positive normal is

then $\hat{\mathbf{z}}$ by the right-hand rule. Let its lower left corner be (x, y) and its side lengths be $\Delta x, \Delta y > 0$. The two horizontal sides give, to leading order,

$$[u_x(x, y) - u_x(x, y + \Delta y)]\Delta x \simeq -\frac{\partial u_x}{\partial y}\Delta x\Delta y.$$

The bottom runs along $+x$ and the top along $-x$, explaining the minus sign. The right side runs along $+y$ and the left along $-y$; together they give $(\partial u_y/\partial x)\Delta x\Delta y$. Thus

$$\lim_{\text{rectangle shrinks}} \frac{\text{positive circulation}}{\Delta x\Delta y} = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} = (\nabla \times \mathbf{u}) \cdot \hat{\mathbf{z}}. \quad (3.69)$$

The circulation is the integral of $\mathbf{u} \cdot d\mathbf{r}$, defined explicitly in the next subsection. The derivation needs only its small-segment interpretation as tangential projection times segment length. For a velocity field, circulation has units $\text{m}^2 \text{s}^{-1}$ and circulation per area has inverse-time units.

Repeat the argument in the yz and zx planes with their normals chosen by the same right-hand rule. The three normal components define the *curl*:

$$\nabla \times \mathbf{u} = \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z} \right) \hat{\mathbf{x}} + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x} \right) \hat{\mathbf{y}} + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) \hat{\mathbf{z}}. \quad (3.70)$$

The ordering is essential. Reversing the traversal reverses the oriented normal and the signed circulation, rather than changing the field itself. The symbol ∇ here represents differential operations, not a numerical vector that can be treated as a new displacement. Component derivation fixes the signs more reliably than an unexplained determinant mnemonic.

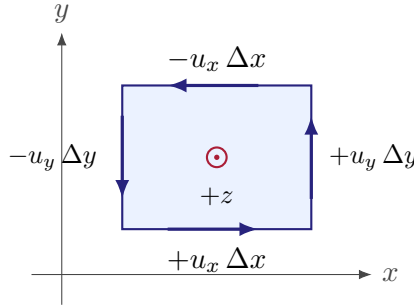


Figure 3.32: Counterclockwise rectangle orientation gives the positive z component of curl. Edge labels show leading tangential contributions; each field component is evaluated on its own side, so opposite sides need not cancel. The marked normal points toward the reader. This is an orientation schematic, not a plot of a particular field. Reversing every path arrow reverses the signed circulation and the assigned normal.

Example 3.26: A field that circulates without outward excess

Let $\mathbf{u} = \Omega(-y, x, 0)$ with constant $\Omega = 0.40 \text{ s}^{-1}$. Calculate divergence, curl and the circulation round a circle of radius $R = 0.50 \text{ m}$, oriented counterclockwise from $+z$.

The diagonal component derivatives vanish, so $\nabla \cdot \mathbf{u} = 0$. The relevant cross derivatives are $\partial u_y/\partial x = \Omega$ and $\partial u_x/\partial y = -\Omega$. Hence $\nabla \times \mathbf{u} = 2\Omega\hat{\mathbf{z}} = 0.80\hat{\mathbf{z}} \text{ s}^{-1}$. On the circle the velocity is tangential, with magnitude ΩR . Its tangential projection is positive for the declared orientation. Summing that projection over the circumference gives $\Gamma = (\Omega R)(2\pi R) = 2\pi\Omega R^2 = 0.20\pi \text{ m}^2 \text{ s}^{-1} \simeq 0.6283 \text{ m}^2 \text{ s}^{-1}$. The ratio to circle area πR^2 is 2Ω , confirming the curl component.

Check by a parameter. For $\mathbf{r}(\theta) = R(\cos \theta, \sin \theta, 0)$, $d\mathbf{r}/d\theta = R(-\sin \theta, \cos \theta, 0)$. The field along it is Ω times this derivative, so their dot product is ΩR^2 , independent of θ . Integrating from 0 to 2π reproduces the result. Clockwise traversal gives its negative. The curl magnitude is twice the angular rate in this rigid-rotation example.

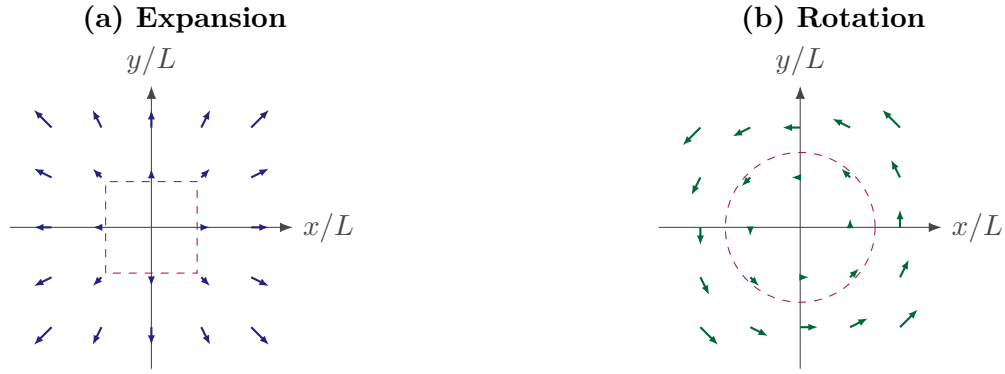


Figure 3.33: Divergence and curl describe different local geometric properties. For $k > 0$, panel (a) samples $\mathbf{u} = k(x, y, 0)$: divergence is $2k$ and curl is zero. Panel (b) samples $\mathbf{u} = k(-y, x, 0)$: divergence is zero and curl is $2k\hat{\mathbf{z}}$. Coordinates are normalised by $L > 0$. Every displayed arrow is the field multiplied by one common time factor $0.18/k$, so magnitudes are retained rather than normalised away. Equal spatial axis scales are used; the field vanishes at the origin. Dashed boundaries guide flux or circulation interpretation and do not represent material walls.

3.9.12 A path integral joins parameterisation to projection

A path is a geometrical curve with an orientation. A useful way to specify it is $\mathbf{r}(\lambda)$ over $\lambda_0 \leq \lambda \leq \lambda_1$, where λ may be a dimensionless parameter rather than time. Its small displacement is $d\mathbf{r} = \mathbf{r}'(\lambda)d\lambda$. For a smooth field $\mathbf{F}$, define

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{\lambda_0}^{\lambda_1} \mathbf{F}(\mathbf{r}(\lambda)) \cdot \mathbf{r}'(\lambda) d\lambda. \quad (3.71)$$

An integral is the limit of sums over increasingly short path pieces. Each term uses the field evaluated on that piece, dotted into its oriented displacement. The expression therefore accumulates the tangential part; a perpendicular field contributes zero. If the field is constant, it reduces to $\mathbf{F} \cdot [\mathbf{r}(\lambda_1) - \mathbf{r}(\lambda_0)]$. For a piecewise smooth path, add the integrals on the separate smooth pieces. A corner requires no artificial tangent at the corner itself.

For a linear field along a straight segment, midpoint sampling is exact. Write $\mathbf{r}(\lambda) = \mathbf{P} + \lambda(\mathbf{Q} - \mathbf{P})$ with $0 \leq \lambda \leq 1$. If $\mathbf{F}(\mathbf{r}(\lambda)) = \mathbf{b} + \lambda\mathbf{c}$, then integration gives $\int_0^1 (\mathbf{b} + \lambda\mathbf{c}) \cdot (\mathbf{Q} - \mathbf{P}) d\lambda = (\mathbf{b} + \mathbf{c}/2) \cdot (\mathbf{Q} - \mathbf{P})$. The bracket is precisely the field at the segment midpoint. A nonlinear field usually makes this a numerical approximation; exactness here depends on the stated linear variation, rather than on the segment merely being short.

A change to another monotonic parameter that preserves traversal changes $\mathbf{r}'$ and the integration element in compensating ways. The path integral therefore does not depend on how rapidly its parameter labels advance. Reversing the traversal changes its sign. These statements concern the same geometrical path; choosing a different curve between the same endpoints may change the integral. The examples below distinguish these two situations.

Suppose there exists a smooth scalar function ψ with $\mathbf{F} = \nabla\psi$ throughout the path domain. The chain rule gives $d\psi(\mathbf{r}(\lambda))/d\lambda = \nabla\psi \cdot \mathbf{r}'$. Integrating this ordinary derivative yields

$$\int_C \nabla\psi \cdot d\mathbf{r} = \psi(Q) - \psi(P). \quad (3.72)$$

Such a field is called *conservative*: its integral depends on endpoints. Here ψ is a mathematical scalar potential with units of field times length. The mechanical work interpretation and the sign convention for potential energy are developed later; no force law is assumed in this proof.

Example 3.27: Two routes and two fields

Let $P = (0, 0)$ and $Q = (L, L)$ with $L = 2$ m. Compare the diagonal route $\mathbf{r}_d(\lambda) = (L\lambda, L\lambda)$, $0 \leq \lambda \leq 1$, with a route along x from P to $(L, 0)$ followed by a route along y to Q . Take $\kappa = 0.50 \text{ N m}^{-1}$ and examine $\mathbf{F}_g = \kappa(x, y)$ and $\mathbf{F}_c = \kappa(-y, x)$. The force units provide a familiar dimensional check, but the fields are prescribed mathematical models.

Gradient field. Since $\psi = \kappa(x^2 + y^2)/2$ has gradient $\mathbf{F}_g$, both routes must give $\psi(Q) - \psi(P) = \kappa L^2 = 2.0 \text{ N m}$. Directly on the diagonal, $\mathbf{F}_g(\mathbf{r}_d) \cdot \mathbf{r}'_d = 2\kappa L^2 \lambda$; integration from zero to one gives κL^2 . On the horizontal piece the contribution is $\int_0^L \kappa x dx = \kappa L^2/2$; on the vertical piece it is $\int_0^L \kappa y dy = \kappa L^2/2$. The sum agrees, and reversing either complete route reverses the answer.

Circulating field. On the diagonal, $\mathbf{F}_c = \kappa(-L\lambda, L\lambda)$ is perpendicular to $\mathbf{r}'_d = (L, L)$, so the integral is zero. On the horizontal piece $y = 0$ and $dy = 0$, so the integral is also zero. On the vertical piece $x = L$ and $dx = 0$, giving $\int_0^L \kappa L dy = \kappa L^2 = 2.0 \text{ N m}$. The two routes now disagree. The circulating field cannot be a gradient throughout this domain, since a gradient would satisfy Equation (3.72) for both routes.

Orientation check. Follow the two-piece route and return down the diagonal. The resulting triangle is counterclockwise from $+z$, its area is $L^2/2$, and its circulation is κL^2 . The constant curl is $2\kappa \hat{\mathbf{z}}$, so circulation divided by triangle area is 2κ , agreeing with the earlier local rectangle calculation. This is an explicit calculation for this field and boundary, rather than an assumed general integral theorem.

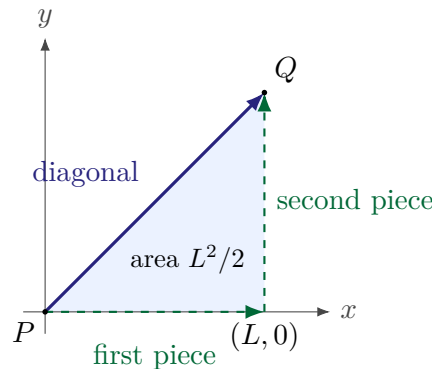


Figure 3.34: Different paths between identical endpoints. The solid blue diagonal and dashed green two-piece route both go from P to Q . Equal axis scales preserve their geometry. Following the green route and then the *reverse* blue route forms the positively oriented triangular boundary. The gradient field $\kappa(x, y)$ gives the same integral on both forward routes; the circulating field $\kappa(-y, x)$ gives zero on the diagonal and κL^2 on the two-piece route. No common speed or travel time is implied by these arrows.

Local derivative tests need a declared domain

The implication $\mathbf{F} = \nabla\psi \Rightarrow \nabla \times \mathbf{F} = \mathbf{0}$ uses equal mixed partial derivatives, valid here for smooth ψ . The reverse implication needs additional domain conditions. Zero curl at sampled points alone does not establish one global scalar potential, particularly when a domain has holes or the field is undefined somewhere. This bridge proves path independence when a potential is explicitly available, and proves path dependence by two explicit integrals. It does not substitute a memorised local test for either argument.

The four field tools extend familiar vector operations in distinct ways. A gradient represents scalar change per spatial displacement. Divergence compares normal flow through opposite faces. Curl compares tangential contributions around oriented boundaries. A line integral accumulates projection along a chosen path. In each case, the units, domain and orientation are part of the

statement. Their compact notation is useful only after these meanings are established.

3.10 Exercises and problems

Use one stated right-handed Cartesian basis unless a change of frame is explicitly requested. Include units and direction information in every physical answer. Numerical values are stipulated teaching inputs, not measurements unless bounds are explicitly given. Problems 19, 20 and 27 use the supporting frame-transformation extension; the other Problems 1–28 use the core. Problems 29–36 belong to the optional advanced pathway. Complete solutions follow for every problem and subpart, including worked exemplars and assessment criteria for investigations. Problems 37–46 extend the graphical and advanced practice; their prerequisites are stated locally.

Concepts, diagrams and admissibility

Problem 3.1. What counts as a vector?

Classify a signed temperature difference, a displacement, and a list of three sensor temperatures. Explain why the sign of a number or the length of a list is not a sufficient test for a spatial vector.

Problem 3.2. Subtraction on a line

In a fixed basis let $\mathbf{a} = -4\hat{\mathbf{x}}$ m and $\mathbf{b} = 7\hat{\mathbf{x}}$ m. Find $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$ and the magnitude of each. Draw the common-tail difference arrow and explain why subtracting the magnitudes would fail.

Problem 3.3. A possible resultant?

Two nonzero displacement vectors have magnitudes 5 m and 9 m. Which proposed resultant magnitudes are possible: 2, 4, 10, 14 or 15 m? Describe the endpoint cases.

Problem 3.4. An arrow away from the origin

An arrow starts at (2, −1) mm and ends at (−3, 5) mm. Find its components. Translate its tail to (10, 10) mm without changing the vector. Where is its new head?

Problem 3.5. The zero-vector diagnostic

For $\mathbf{a} = \mathbf{0}$ and a specified unit vector $\hat{\mathbf{u}}$, which are defined: $\|\mathbf{a}\|$, $\mathbf{a} \cdot \hat{\mathbf{u}}$, the vector projection on the line, $\hat{\mathbf{a}}$ and the angle between $\mathbf{a}$ and $\hat{\mathbf{u}}$?

Problem 3.6. Component, vector part and magnitude

A displacement has $a_x = -3$ m. Distinguish a_x , $a_x\hat{\mathbf{x}}$ and $|a_x|$. What are the units of a_x^2 ? Does this information determine the full magnitude a ?

Problem 3.7. Can a dot product be cancelled?

For a specified nonzero $\mathbf{a}$, does $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ imply $\mathbf{b} = \mathbf{c}$? Give a counterexample using Cartesian unit vectors and state the actual implication.

Problem 3.8. Can a cross product be cancelled?

For nonzero $\mathbf{a}$, does $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$ imply $\mathbf{b} = \mathbf{c}$? Give a counterexample and interpret the lost information.

Components, projections, products and frames

Problem 3.9. A branch and an inconsistent record

A planar displacement has magnitude 17 cm, component $d_x = -8$ cm and $d_y < 0$. Find d_y and the bearing from positive x . Would a record with the same magnitude and $d_x = -18$ cm be admissible as exact data?

Problem 3.10. A three-component combination

For $\mathbf{a} = (2, -1, 4)$ mm and $\mathbf{b} = (-3, 5, 1)$ mm, find $2\mathbf{a} - \mathbf{b}$ and its magnitude. State why a dimensionless multiplier is compatible with the subtraction.

Problem 3.11. The closing leg of a route

Three planar route legs are $(0.8, 0.6)$, $(-0.2, 0.9)$ and $(0.5, -0.4)$, all in kilometres and one fixed basis. Find the straight return displacement to the start and its bearing. Compare its magnitude with the length already travelled along the three legs.

Problem 3.12. A three-dimensional projection

Resolve $\mathbf{d} = (3, 4, 5)$ mm into parts parallel and perpendicular to the direction $(2, -1, 2)$. Find the signed scalar projection and check both perpendicularity and the squared-norm decomposition.

Problem 3.13. Zero advance along a tilted line

For $\mathbf{d} = (9, 12)$ mm and $\hat{\mathbf{u}} = (-4, 3)/5$, find the signed projection, the perpendicular residual and the angle between the directions.

Problem 3.14. An obtuse angle in space

Find the angle between $\mathbf{a} = (1, 2, -2)$ m and $\mathbf{b} = (-2, 1, 2)$ m. Predict acute or obtuse before evaluating the inverse trigonometric function, and check its dimensionless argument.

Problem 3.15. A calculation without component lists

Two displacement vectors have magnitudes 6 m and 10 m and dot product 30 m^2 . Find their angle, the magnitudes of their sum and difference, and the area of their parallelogram.

Problem 3.16. An ordered cross product

For $\mathbf{a} = (2, 1, 0)$ cm and $\mathbf{b} = (-1, 3, 2)$ cm, find $\mathbf{a} \times \mathbf{b}$, its magnitude and unit direction. Verify perpendicularity to both inputs and state the result on reversing the order.

Problem 3.17. A degenerate plane

For edge vectors $(3, 0, 0)$ mm and $(6, 0, 0)$ mm, determine the area vector. Does the pair determine a unique plane or unit normal? Describe the possible perpendicular unit directions.

Problem 3.18. A triangle from three points

In one right-handed frame, three vertices are $P = (1, 0, 0)$, $Q = (3, 1, 0)$ and $R = (1, 2, 2)$ in centimetres. Find the triangle area and the unit normal selected by the ordering P, Q, R . Explain the effect of exchanging Q and R .

Problem 3.19. Passive and active quarter-turns (extension)

A vector has components $(4, -3, 2)$ m. Find its components after a passive 90° counterclockwise turn of the x, y axes about z . Compare with actively rotating the vector by the same angle in fixed axes.

Problem 3.20. Axes aligned with a displacement (extension)

Choose a passive planar axis rotation that makes $\mathbf{a} = (\sqrt{3}, 1)$ mm lie along positive x' . Give its new components. Why does actively rotating it through your chosen positive angle not do the same thing?

Figures, data, computation and investigations**Problem 3.21. Read the force-projection figure**

Use Figure 3.11(b). Give the signed force component along $+x$, its vector projection and the dot product with $(2, 0)$ m. State one piece of physical information missing before this product could be interpreted as work.

Problem 3.22. Correct a transverse positioning error

Use Figure 3.8 to find the shortest correction from D to the target line. Give its vector, length and corrected laboratory endpoint. Does this necessarily reach a particular target point on the line?

Problem 3.23. Commutativity is not a route guarantee

The two survey legs in Figure 3.1 can be interchanged algebraically. Explain what is guaranteed to stay the same and what cannot be guaranteed about an actual route in an environment containing obstacles. Do not assume a reversed route is blocked without checking its geometry.

Problem 3.24. Shift the origin without moving the points

Old point coordinates are $P = (2, 3)$ mm and $Q = (8, -1)$ mm. The new origin has old coordinates $(20, -10)$ mm, with axes parallel to the old ones. Find both new point coordinates and verify that the displacement $P \rightarrow Q$ is unchanged.

Problem 3.25. A closing error outside stipulated bounds

A constructed planar survey has legs $(0.50, 0.20)$, $(-0.10, 0.70)$ and $(-0.39, -0.92)$ in metres. Every reported component has a stipulated absolute error bound of 0.004 m. Find the closing vector and decide whether exact closure is compatible with all the bounds. Can you identify the incorrect reading uniquely?

Problem 3.26. A bounded force projection

A hypothetical force record gives $F_x \in [5.8, 6.2]$ N and $F_y \in [7.8, 8.2]$ N. For the specified unit direction $(3/5, 4/5)$, find the full possible interval of the signed projection under these independent interval constraints. Is it a confidence interval?

Problem 3.27. An invariant that misses a sign error (extension)

A program advertised as a passive 90° planar rotation sends $(4, 3)$ to $(-3, 4)$. It preserves norms and mutual dot products for all pairs tested. Diagnose the error and design one test that distinguishes the intended operation from the implemented one.

Problem 3.28. Design a useful survey test

Propose a tabletop vector-addition investigation. Specify the measured quantities, an independent comparison, repeat procedure, a defensible way to handle error bounds, and one systematic error that loop closure alone cannot reveal.

Advanced problems

Problem 3.29. Rotation and its inverse (advanced)

Multiply $R(-\phi)R(\phi)$ explicitly and identify the cancellation. Test your result for $\phi = 45^\circ$ on the dimensionless vector $(1, 0)$.

Problem 3.30. A reflection is not a proper rotation (advanced)

For $S = \text{diag}(1, -1)$, show that norms and dot products are preserved. Compute its determinant and apply it to $\hat{x}, \hat{y}$ in the plane. What happens to their ordered cross product in a fixed right-handed three-dimensional space?

Problem 3.31. Vector rate and radial rate (advanced)

For the polynomial path in Equation (3.52), find $\mathbf{r}'(2\text{ s})$ and its magnitude. At $t = 1\text{ s}$ find $d\|\mathbf{r}\|/dt$ and compare it with $\|\mathbf{r}'\|$. Explain the difference.

Problem 3.32. Finite rates and convergence (advanced)

At $t = 1\text{ s}$ for the same polynomial path, compute forward and centred difference vectors using $h = 0.2\text{ s}$. Find their vector errors relative to the exact derivative. What happens to the centred error if h is halved, in exact arithmetic? State the allowed centred-step domain.

Problem 3.33. Constant length, nonzero derivative (advanced)

Let $\mathbf{r}(t) = R(\cos \omega t \hat{x} + \sin \omega t \hat{y})$, where $R = 0.40\text{ m}$ and $\omega = 3\text{ s}^{-1}$. Find $\mathbf{r}$ and $\mathbf{r}'$ at $t = (\pi/6)\text{ s}$. Compare $d\|\mathbf{r}\|/dt$ and $\|\mathbf{r}'\|$ without introducing a force law.

Problem 3.34. A perturbed normal (advanced)

Use Equation (3.58) with $L = 1\text{ m}$, $\varepsilon = 0.020\text{ m}$ and $\delta = 0.002\text{ m}$. Find the area vector, its magnitude, the unit normal and its signed tilt. What changes if only δ changes sign?

Problem 3.35. An orientation-tolerance bound (advanced)

For the same perturbation model with $\varepsilon = 0.010\text{ m}$, require $|\alpha| \leq 1^\circ$ and suppose only δ is uncertain. Find the largest admissible bound on $|\delta|$ using the exact relation. Compare with the small-angle estimate and state the model limitation.

Problem 3.36. A numerical derivative investigation (advanced)

Design and, where computing tools are available, run a step-refinement study for the polynomial path. Compare direct forward and centred differences with their exact polynomial formulas and with the analytical derivative. Explain how you would distinguish truncation from roundoff and avoid samples outside the model domain. Do not invent experimental data.

Additional graphical and advanced practice

Problems 37–40 consolidate vector geometry. Problems 41–42 use the advanced transformation and moving-basis discussion. Problems 43–46 belong to the optional field-calculus bridge and require its introduced gradient, flux and line-integral ideas; none is a prerequisite for the next chapter.

Problem 3.37. Recover the missing arrow from a component graph

Figure 3.35 shows a displacement $\mathbf{a}$ and a resultant $\mathbf{r}$ in equal metre scales. Read their components, find the missing $\mathbf{b}$ in $\mathbf{a} + \mathbf{b} = \mathbf{r}$, and give its magnitude and bearing from positive x . Draw the missing arrow both as a free vector and in the head-to-tail construction. Check the reconstructed endpoint.

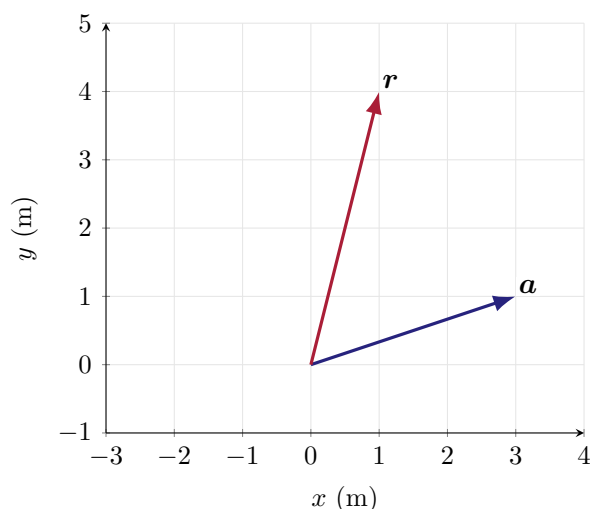


Figure 3.35: Component graph for Problem 3.37. Each grid interval is 1 m on both axes; the arrows share a tail. Recover the missing displacement by using the plotted endpoints.

Problem 3.38. Direction cosines and an inadmissible list

A displacement is $(2, -3, 6)$ mm. Find its magnitude and the cosines of its three angles with the positive Cartesian axes; give the three angles and verify the direction-cosine identity. Could $(.8, .8, 0)$ be a direction-cosine list for a vector in an orthonormal three-dimensional frame? Explain.

Problem 3.39. Oriented volume from three edges

Three edges from one vertex are $\mathbf{a} = (2, 0, 0)$ cm, $\mathbf{b} = (1, 3, 0)$ cm and $\mathbf{c} = (0, 1, 4)$ cm. Evaluate $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, determine the parallelepiped volume and the handedness of the ordered edges, and give an independent geometrical volume check. What changes when $\mathbf{b}$ and $\mathbf{c}$ exchange places?

Problem 3.40. Shortest displacement to a plane

A plane passes through $P = (1, 0, 1)$ mm with normal direction $(2, -1, 2)$. For $X = (4, 3, 5)$ mm, find the signed distance using that positive normal, the shortest correction vector into the plane and its endpoint. Verify that the endpoint lies in the plane and explain why the correction is shortest.

Problem 3.41. A cross product under reflection (advanced)

Use dimensionless vectors $\mathbf{a} = (1, 2, 0)$, $\mathbf{b} = (0, 1, 3)$ and the active reflection $\mathbf{S} = \text{diag}(1, -1, 1)$ in one fixed right-handed basis. Compute $\mathbf{a} \times \mathbf{b}$ and $(\mathbf{S}\mathbf{a}) \times (\mathbf{S}\mathbf{b})$. Compare with $\mathbf{S}(\mathbf{a} \times \mathbf{b})$ and verify the determinant factor in the axial-vector transformation rule. Explain its geometric origin.

Problem 3.42. A constant vector described in a rotating basis (advanced)

A physically constant vector is $\mathbf{A} = 5\hat{\mathbf{x}}$ m in a fixed laboratory frame. The moving basis is $\hat{\mathbf{e}}_1 = \cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}$, $\hat{\mathbf{e}}_2 = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}$, where $\phi = \omega t$ and $\omega = 2\text{ s}^{-1}$. Find its moving components and their derivatives, derive the basis derivatives, and show explicitly that both product-rule contributions cancel. Evaluate the components and component rates at $t = (\pi/8)\text{ s}$.

Problem 3.43. A directional derivative of temperature (optional field bridge)

On a planar region let $T(x, y) = T_0 + k(x^2 + 2y^2)$, with x, y lengths and $k = 1\text{ K m}^{-2}$. At $(1, 2)$ m, find the gradient, the rate of temperature change per unit distance along $(3, 4)/5$, and the maximum positive directional derivative and its direction. State the units and the first-order meaning of the directional derivative.

Problem 3.44. Flux through an oriented planar area (optional field bridge)

A uniform fluid velocity is $\mathbf{v} = (2, -1, 3) \text{ m s}^{-1}$ across a stationary planar surface with area vector $\mathbf{A} = (2, 1, 2) \text{ m}^2$. Calculate the signed volume flux, the surface area and normal velocity. Check the flux from area times normal velocity. What changes when the chosen normal is reversed, and why does the tangential velocity not contribute?

Problem 3.45. Circulation around a square (optional field bridge)

For the prescribed planar field $\mathbf{F} = k(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}})$ with $k = 3 \text{ N m}^{-1}$, evaluate the line integral counterclockwise around the square $0 \leq x, y \leq L$ with $L = 2 \text{ m}$ by treating all four edges separately. Determine the result on reversing the traversal, compare with the local circulation density times area, and decide whether this field can be a single-valued potential gradient on the whole square.

Problem 3.46. Two paths through a gradient field (optional field bridge)

For $\mathbf{F} = k(x\hat{\mathbf{x}} + y\hat{\mathbf{y}})$ with $k = 2 \text{ N m}^{-1}$, compare $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ along the straight segment and the elbow route through C shown in Figure 3.36. Here $A = (1, 0)$, $C = (3, 0)$ and $B = (3, 2)$ in metres. Evaluate both integrals, identify a scalar whose gradient is $\mathbf{F}$, and explain why the midpoint rule is exact on each straight piece.

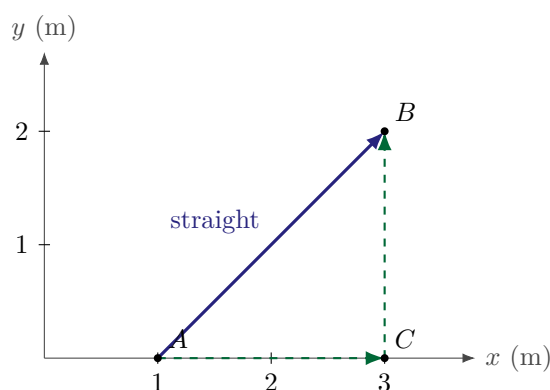


Figure 3.36: Two directed paths from A to B for Problem 3.46. Both axes use the same metre scale. The elbow path traverses the lower horizontal segment before the right vertical segment; the field is prescribed in the problem.

3.11 Complete solutions

Every end problem is solved below, including the optional advanced bridge. Use the stated basis and units throughout. Numerical investigation results are computations on the declared analytical model, not experimental measurements.

Concepts, diagrams and admissibility

Problem 3.1: What counts as a vector?

A signed temperature difference is a scalar: its sign distinguishes warming from cooling, without selecting a spatial direction. A displacement is a spatial vector because it joins two positions and obeys the component transformation law when the axes turn. Three sensor temperatures form a list of scalars; rotating the laboratory axes does not mix them as spatial components. Thus neither a minus sign nor a three-entry list proves that a quantity is a vector. The decisive test is its geometric meaning and transformation rule.

Problem 3.2: Subtraction on a line

Both signed components belong to the same axis, so

$$\mathbf{a} + \mathbf{b} = (-4 + 7)\hat{x} \text{ m} = 3\hat{x} \text{ m}, \quad \mathbf{a} - \mathbf{b} = (-4 - 7)\hat{x} \text{ m} = -11\hat{x} \text{ m}.$$

Their magnitudes are 3 m and 11 m. With common tails, $\mathbf{a} - \mathbf{b}$ points from the head of $\mathbf{b}$ at $x = 7 \text{ m}$ to the head of $\mathbf{a}$ at $x = -4 \text{ m}$, as Figure 3.37 shows. Adding $\mathbf{b}$ to this difference recovers $\mathbf{a}$. Subtracting magnitudes gives $4 - 7 = -3 \text{ m}$, which discards the directions and is neither the vector difference nor its nonnegative magnitude.

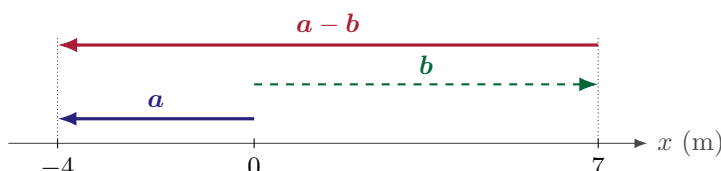


Figure 3.37: Common-tail subtraction for Problem 3.2. Vertical offsets separate collinear arrows for readability; they have no physical component. The upper arrow's head-minus-tail coordinate is $-4 - 7 = -11 \text{ m}$.

Problem 3.3: A possible resultant?

For an angle $\theta \in [0, \pi]$ between the inputs, $r^2 = 5^2 + 9^2 + 2(5)(9)\cos\theta$ in square metres. Since $-1 \leq \cos\theta \leq 1$, the allowed interval is $|9 - 5| \leq r/(1 \text{ m}) \leq 9 + 5$, namely $4 \leq r/(1 \text{ m}) \leq 14$. The proposed values 4, 10, 14 m are possible; 2 and 15 m are not. Opposite parallel inputs give the lower endpoint 4 m in the larger vector's direction; equally directed inputs give 14 m. For $r = 10 \text{ m}$, $\cos\theta = -1/15$ is admissible, confirming an interior case rather than just checking the endpoints.

Problem 3.4: An arrow away from the origin

Subtract tail coordinates from head coordinates: $\mathbf{d} = [(-3, 5) - (2, -1)] \text{ mm} = (-5, 6) \text{ mm}$. Its translated head is the new tail plus this same displacement, $(10, 10) + (-5, 6) = (5, 16) \text{ mm}$. The coordinate differences and therefore the length $\sqrt{61} \text{ mm}$ and direction are unchanged. The head coordinates alone were never the components of the arrow unless its tail was the origin.

Problem 3.5: The zero-vector diagnostic

The norm, scalar projection and vector projection are all defined: $\|\mathbf{0}\| = 0$, $\mathbf{0} \cdot \hat{\mathbf{u}} = 0$, and $(\mathbf{0} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}} = \mathbf{0}$. In contrast, $\hat{\mathbf{a}} = \mathbf{a}/\|\mathbf{a}\|$ would divide by zero, so the zero vector has no unit direction. The angle formula also divides by $\|\mathbf{a}\|$, hence the angle is undefined, not 0° or 90° . The identity $\mathbf{0} \cdot \hat{\mathbf{u}} = 0$ does not select one angle.

Problem 3.6: Component, vector part and magnitude

$a_x = -3 \text{ m}$ is a signed scalar; $a_x \hat{x} = -3\hat{x} \text{ m}$ is a vector part pointing along $-x$; $|a_x| = 3 \text{ m}$ is its nonnegative length. Squaring gives $a_x^2 = 9 \text{ m}^2$. The full magnitude satisfies $a^2 = 9 \text{ m}^2 + a_y^2 + a_z^2$, so only $a \geq 3 \text{ m}$ follows. Equality requires both remaining components to vanish. For example, $(-3, 4, 0) \text{ m}$ and $(-3, 0, 0) \text{ m}$ have the same x component but magnitudes 5 m and 3 m.

Problem 3.7: Can a dot product be cancelled?

No. In dimensionless notation take $\mathbf{a} = \hat{x}$, $\mathbf{b} = \hat{y}$ and $\mathbf{c} = 2\hat{y}$. Both dot products vanish although $\mathbf{b} \neq \mathbf{c}$. Subtracting the given equality yields $\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = 0$; thus their difference is perpendicular to $\mathbf{a}$, including the possible zero difference. Equivalently, $\mathbf{b}$ and $\mathbf{c}$ have the same component along $\mathbf{a}$ but may have different perpendicular components. A scalar projection loses transverse information, so it cannot generally be cancelled like an ordinary nonzero scalar multiplier.

Problem 3.8: Can a cross product be cancelled?

No. Let $\mathbf{a} = \hat{\mathbf{x}}$, $\mathbf{b} = \hat{\mathbf{y}}$, and $\mathbf{c} = \hat{\mathbf{x}} + \hat{\mathbf{y}}$. Then $\mathbf{a} \times \mathbf{b} = \hat{\mathbf{z}} = \mathbf{a} \times \mathbf{c}$ although the inputs differ. In general the equality implies $\mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{0}$. With $\mathbf{a} \neq \mathbf{0}$, $\mathbf{b} - \mathbf{c}$ is parallel to $\mathbf{a}$ or zero. The component parallel to the first input contributes no cross product and is precisely the information lost here. This complements Problem 3.7, which loses perpendicular information instead.

Components, projections, products and frames**Problem 3.9: A branch and an inconsistent record**

The norm equation gives $d_y^2 = [17^2 - (-8)^2] \text{ cm}^2 = 225 \text{ cm}^2$. The inequality $d_y < 0$ selects $d_y = -15 \text{ cm}$ rather than the positive root. Both components are negative, so a bearing in $[0, 360^\circ)$ is

$$\theta = 180^\circ + \arctan(15/8) = 241.93^\circ \quad \text{approximately.}$$

The check $\sqrt{64 + 225} = 17$ confirms the magnitude, and the sine and cosine of this bearing are negative as required. If $d_x = -18 \text{ cm}$, then $d_y^2 = (289 - 324) \text{ cm}^2 = -35 \text{ cm}^2$, impossible for real exact components. Taking the absolute value of the radicand would change the data. Only stated uncertainty bounds could justify a different admissibility test.

Problem 3.10: A three-component combination

Perform the same operation on each component:

$$2\mathbf{a} - \mathbf{b} = (4, -2, 8) - (-3, 5, 1) = (7, -7, 7) \text{ mm}, \quad \|2\mathbf{a} - \mathbf{b}\| = 7\sqrt{3} \text{ mm} \simeq 12.124 \text{ mm}.$$

The factor 2 is dimensionless, leaving the unit of $2\mathbf{a}$ equal to that of $\mathbf{b}$. Thus the subtraction is physically meaningful. As a reversal check, adding $\mathbf{b}$ to the result gives $(4, -2, 8) \text{ mm} = 2\mathbf{a}$.

Problem 3.11: The closing leg of a route

The sum of the three legs is $(1.1, 1.1) \text{ km}$ relative to the start. The closing displacement is its negative, $\mathbf{d}_{\text{ret}} = (-1.1, -1.1) \text{ km}$, with bearing 225° and magnitude $1.1\sqrt{2} \text{ km} \simeq 1.556 \text{ km}$. The length already travelled is the sum of the individual lengths,

$$[\sqrt{0.8^2 + 0.6^2} + \sqrt{(-0.2)^2 + 0.9^2} + \sqrt{0.5^2 + (-0.4)^2}] \text{ km} = (1 + \sqrt{0.85} + \sqrt{0.41}) \text{ km} \simeq 2.562 \text{ km}.$$

This exceeds the straight return distance, consistently with the triangle inequality. A displacement describes endpoints; the travelled length describes the specified route. Adding the closing displacement gives zero.

Problem 3.12: A three-dimensional projection

The direction list has norm 3, so $\hat{\mathbf{u}} = (2, -1, 2)/3$. The signed projection and the two vector parts are

$$d_{\parallel} = \mathbf{d} \cdot \hat{\mathbf{u}} = (6 - 4 + 10) \text{ mm}/3 = 4 \text{ mm}, \quad \mathbf{d}_{\parallel} = \frac{1}{3}(8, -4, 8) \text{ mm}, \quad \mathbf{d}_{\perp} = \frac{1}{3}(1, 16, 7) \text{ mm}.$$

The parts sum to $(3, 4, 5) \text{ mm}$. The perpendicularity check is $\mathbf{d}_{\perp} \cdot \hat{\mathbf{u}} = (2 - 16 + 14) \text{ mm}/9 = 0$. Also $\|\mathbf{d}\|^2 = 50 \text{ mm}^2$, while $\|\mathbf{d}_{\parallel}\|^2 = 16 \text{ mm}^2$ and $\|\mathbf{d}_{\perp}\|^2 = (1 + 256 + 49) \text{ mm}^2/9 = 34 \text{ mm}^2$. The squared-norm decomposition therefore holds. In space the residual is a vector in a plane; it is not one intrinsically signed transverse scalar.

Problem 3.13: Zero advance along a tilted line

First $\|\hat{\mathbf{u}}\|^2 = (16 + 9)/25 = 1$. Then $\mathbf{d} \cdot \hat{\mathbf{u}} = (-36 + 36) \text{ mm}/5 = 0$. The vector parallel part is zero and the perpendicular residual is $\mathbf{d}_{\perp} = (9, 12) \text{ mm}$, of magnitude 15 mm . Because both the input displacement and the reference direction are nonzero, $\cos \theta = 0$ implies $\theta = 90^\circ$. Zero advance along this line therefore does not mean zero displacement; the entire change is transverse.

Problem 3.14: An obtuse angle in space

The dot product is $[-2 + 2 - 4] \text{ m}^2 = -4 \text{ m}^2$. Both magnitudes are 3 m , so the negative dot product predicts an obtuse angle before evaluating any inverse function. Explicitly,

$$\cos \theta = \frac{-4 \text{ m}^2}{(3 \text{ m})(3 \text{ m})} = -\frac{4}{9}, \quad \theta \simeq 116.39^\circ.$$

The ratio is dimensionless and lies in $[-1, 1]$. Independently, $\mathbf{a} + \mathbf{b} = (-1, 3, 0) \text{ m}$ has squared norm 10 m^2 , and the norm identity gives $(9 + 9 + 2(-4)) \text{ m}^2 = 10 \text{ m}^2$. This is the unsigned angle between two directions, not a bearing of one vector.

Problem 3.15: A calculation without component lists

The angle follows from $\cos \theta = 30/(6 \times 10) = 1/2$, hence $\theta = 60^\circ$. Expanding the norm squares using the dot product gives

$$\|\mathbf{a} + \mathbf{b}\| = \sqrt{36 + 100 + 60} \text{ m} = 14 \text{ m}, \quad \|\mathbf{a} - \mathbf{b}\| = \sqrt{36 + 100 - 60} \text{ m} = 2\sqrt{19} \text{ m}.$$

The parallelogram area is $ab \sin \theta = 60(\sqrt{3}/2) \text{ m}^2 = 30\sqrt{3} \text{ m}^2$. Its square also equals $a^2b^2 - (\mathbf{a} \cdot \mathbf{b})^2 = (3600 - 900) \text{ m}^4$. The sum lies between $|10 - 6| = 4 \text{ m}$ and 16 m , providing a separate admissibility check.

Problem 3.16: An ordered cross product

The ordered cyclic component rule gives

$$\mathbf{a} \times \mathbf{b} = (1(2) - 0(3), 0(-1) - 2(2), 2(3) - 1(-1)) \text{ cm}^2 = (2, -4, 7) \text{ cm}^2.$$

Its magnitude is $\sqrt{69} \text{ cm}^2$ and its unit direction is $(2, -4, 7)/\sqrt{69}$. Dot products with the inputs give $2(2) + 1(-4) + 0(7) = 0$ and $(-1)(2) + 3(-4) + 2(7) = 0$, with cubic-centimetre units. The independent magnitude identity uses $a^2 = 5 \text{ cm}^2$, $b^2 = 14 \text{ cm}^2$ and $\mathbf{a} \cdot \mathbf{b} = 1 \text{ cm}^2$: $5(14) - 1^2 = 69$ in fourth-power centimetres. Reversing the order gives $\mathbf{b} \times \mathbf{a} = (-2, 4, -7) \text{ cm}^2$, the same magnitude and opposite direction.

Problem 3.17: A degenerate plane

The second edge is twice the first, so their cross product and area vector vanish. They specify one line along x , not a unique plane. Every plane containing that line is compatible with the data. A unit normal must be perpendicular to x , so its complete family is $(0, \cos \beta, \sin \beta)$ for arbitrary β . Each has unit norm and zero dot product with both edges. Dividing the zero area vector by its norm would be invalid; an extra independent edge or plane condition is needed to select a particular normal.

Problem 3.18: A triangle from three points

Form edge differences from the common vertex P : $\mathbf{u} = \mathbf{Q} - \mathbf{P} = (2, 1, 0) \text{ cm}$ and $\mathbf{v} = \mathbf{R} - \mathbf{P} = (0, 2, 2) \text{ cm}$. Then $\mathbf{u} \times \mathbf{v} = (2, -4, 4) \text{ cm}^2$, of norm 6 cm^2 . The triangle area is half this value, 3 cm^2 , and the ordered unit normal is $(1, -2, 2)/3$. Its squared norm is $(1 + 4 + 4)/9 = 1$ and both edge dot products vanish. Exchanging $\mathbf{Q}$ and $\mathbf{R}$ reverses the cross product and normal but preserves the area. Translating all vertices together cancels out of both edge differences, confirming that position relative to the origin is irrelevant to the triangle's area.

Problem 3.19: Passive and active quarter-turns

For a passive positive quarter-turn, $\hat{\mathbf{x}}' = \hat{\mathbf{y}}$ and $\hat{\mathbf{y}}' = -\hat{\mathbf{x}}$. Projection onto these new directions therefore gives $(a'_x, a'_y, a'_z) = (a_y, -a_x, a_z) = (-3, -4, 2) \text{ m}$. An active positive quarter-turn in fixed axes instead gives $(-a_y, a_x, a_z) = (3, 4, 2) \text{ m}$. Both preserve the squared magnitude $(16 + 9 + 4) \text{ m}^2 = 29 \text{ m}^2$; length is not a test of the sign convention. An old $+x$ arrow has passive components $(0, -1, 0)$ but becomes an active $+y$ arrow with components $(0, 1, 0)$, which distinguishes the operations.

Problem 3.20: Axes aligned with a displacement

The input has bearing $\arctan(1/\sqrt{3}) = 30^\circ$ and norm 2 mm. Turn the axes passively by $\phi = 30^\circ$. The projection rules give

$$a'_x = \sqrt{3} \cos 30^\circ + \sin 30^\circ = 2, \quad a'_y = -\sqrt{3} \sin 30^\circ + \cos 30^\circ = 0$$

in millimetres. Thus the new components are (2, 0) mm, directed along $+x'$. An active positive turn adds to the old bearing, producing 60° and fixed-axis components $(1, \sqrt{3})$ mm. It moves the arrow away from the old $+x$ axis; an active -30° turn would instead align it with that old axis.

Figures, bounded data and investigations**Problem 3.21: Read the force-projection figure**

Panel (b) shows $\mathbf{F} = (-6, 8)$ N. Therefore the signed scalar component along $+x$ is $F_x = -6$ N, while its vector projection is $-6\hat{x}$ N. The separate length vector $\mathbf{b} = (2, 0)$ m gives $\mathbf{F} \cdot \mathbf{b} = (-6)(2)$ N m = -12 N m. The negative sign records opposing x projections. To interpret this as work, one must know that $\mathbf{b}$ describes an actual displacement of the point where the force acts and that this force describes the process along that motion (for example, it is constant there). A force and an arbitrary reference edge do not establish an energy transfer from units alone.

Problem 3.22: Correct a transverse positioning error

The plot gives $D = (25, 30)$ mm and the target unit direction $\hat{\mathbf{u}} = (3, 4)/5$. The nearest point on the line through the origin is

$$H = (\mathbf{d} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}} = 39(3, 4) \text{ mm}/5 = (23.4, 31.2) \text{ mm}.$$

Consequently the correction from D is $\mathbf{c} = H - D = (-1.6, 1.2)$ mm, of length 2 mm. It is perpendicular to the line since $(-1.6)(.6) + (1.2)(.8) = 0$. Any other line point is $H + s\hat{\mathbf{u}}$ and requires a squared correction length $\|\mathbf{c}\|^2 + s^2 \geq \|\mathbf{c}\|^2$; this proves minimality. A particular target point can differ from H , in which case an additional along-line displacement is required.

Problem 3.23: Commutativity is not a route guarantee

For legs $\mathbf{a}$ and $\mathbf{b}$, both orders give $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$. They therefore have the same resultant, its magnitude and the endpoint relative to a common start. The waypoint changes: first $\mathbf{a}$ puts it at $O + \mathbf{a}$; first $\mathbf{b}$ puts it at $O + \mathbf{b}$. Thus the two piecewise straight paths can meet different obstacles, boundaries or surface conditions. For ideal free translations their summed leg lengths are also the same, but travel time and feasibility need additional physical information. One must test the actual segments against the environment; vector commutativity itself says neither that both routes are feasible nor that one is blocked.

Problem 3.24: Shift the origin without moving the points

With parallel axes, subtract the new origin's old coordinates $\mathbf{o} = (20, -10)$ mm from every old point coordinate: $P' = (-18, 13)$ mm and $Q' = (-12, 9)$ mm. Then $Q' - P' = (6, -4)$ mm = $Q - P$. Algebraically $(Q - \mathbf{o}) - (P - \mathbf{o}) = Q - P$; the origin change cancels. Positions change because they are referred to an origin. The displacement joins the same two physical points and is unchanged, with magnitude $\sqrt{52}$ mm in either description.

Problem 3.25: A closing error outside stipulated bounds

Summing the reported legs gives $\mathbf{c} = (0.01, -0.02)$ m, with norm $\sqrt{0.0005}$ m $\simeq 0.02236$ m. Each closing component can differ from its reported value by at most $3(0.004)$ m = 0.012 m. Therefore the possible true components lie in

$$c_x \in [-0.002, 0.022] \text{ m}, \quad c_y \in [-0.032, -0.008] \text{ m}.$$

Zero belongs to the x interval but not the y interval. Exact closure is therefore incompatible with all of the stipulated bounds. This is a deterministic bound test, not a statistical significance calculation. A wrong component, underestimated error bound, inconsistent axes, or genuinely open route can all produce this discrepancy; closure alone cannot uniquely identify one reading.

Problem 3.26: A bounded force projection

The specified direction is a unit vector because $(3/5)^2 + (4/5)^2 = 1$. Its signed projection is $F_{\parallel} = .6F_x + .8F_y$. Both coefficients are positive, so the minimum uses both lower interval endpoints and the maximum uses both upper endpoints:

$$F_{\parallel} \in [.6(5.8) + .8(7.8), .6(6.2) + .8(8.2)] \text{ N} = [9.72, 10.28] \text{ N}.$$

Independent interval constraints permit these corner combinations, making the bounds attainable. No probability distribution or coverage probability was given, so this is a possible-value interval, not a confidence interval. The nominal value 10 N lies inside, with worst-case deviation 0.28 N.

Problem 3.27: An invariant that misses a sign error

The intended passive rule is $(x, y) \mapsto (y, -x)$, giving $(3, -4)$. The output $(-3, 4)$ instead follows the active positive quarter-turn $(x, y) \mapsto (-y, x)$, equivalent to a passive negative turn. Both matrices are proper rotations, so both preserve norms and dot products; those invariants do not determine which operation was intended. A decisive test fixes an old $+x$ arrow and turns the axes by $+90^\circ$. Its passive components must be $(0, -1)$ because the new $+y'$ direction is old $-x$. An output $(0, 1)$ exposes the opposite convention without relying on a length check.

Problem 3.28: Design a useful survey test

Worked design. Mark three points A, B, C on paper and establish one fixed orthogonal basis. Measure the two components of $A \rightarrow B$, $B \rightarrow C$ and $C \rightarrow A$ using a ruler and perpendicular grid lines. Separately measure $A \rightarrow C$ directly. Compare this independent chord with the sum $(A \rightarrow B) + (B \rightarrow C)$; also test whether the three-leg loop sum is zero. Repeat the entire measurement at least five times, re-reading positions rather than copying previous components. Reversing a ruler or using a second ruler can help expose offsets.

Bounds and interpretation. If every measured component has a justified absolute reading bound e , each loop component has bound $3e$ about its reported sum. The sum-minus-independent-chord comparison also has bound $3e$ because it contains three independently bounded readings. For example, with $e = 1$ mm, a reported loop $(1, -2)$ mm is compatible with closure, whereas $(1, -4)$ mm is not. Repetition reveals spread but does not automatically reduce a stipulated absolute bound by $1/\sqrt{n}$. A common scale factor error multiplies every leg and still leaves a truly closed loop closed; an external known length is needed to reveal it. A common rotated axis convention similarly requires an external direction reference.

Assessment criteria. A defensible report specifies the basis, measures all vector components, includes an independent chord, repeats actual readings, justifies bounds and tests both components, and discusses a systematic error that closure misses. Both agreement within bounds and a documented discrepancy are plausible outcomes; unexplained disagreement is not repaired by deleting a reading.

Advanced problems

Problem 3.29: Rotation and its inverse

Write $c = \cos \phi$ and $s = \sin \phi$. Direct multiplication gives

$$R(-\phi)R(\phi) = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix} = \begin{pmatrix} c^2 + s^2 & cs - sc \\ sc - cs & s^2 + c^2 \end{pmatrix} = \mathbf{I}.$$

The off-diagonal terms cancel and the diagonal terms use the Pythagorean trigonometric identity. At 45° , the first passive rotation sends $(1, 0)$ to $(1/\sqrt{2}, -1/\sqrt{2})$; the inverse sends it back to $(1, 0)$. The intermediate vector also has unit norm, confirming that the inverse restores direction rather than correcting a changed length.

Problem 3.30: A reflection is not a proper rotation

The reflection maps (a_x, a_y) to $(a_x, -a_y)$. Thus its norm square remains $a_x^2 + a_y^2$ and the dot product of two reflected lists is $a_x b_x + (-a_y)(-b_y) = a_x b_x + a_y b_y$. Equivalently $\mathbf{S}^T \mathbf{S} = \mathbf{I}$. Its determinant is $1(-1) - 0 = -1$, unlike the $+1$ determinant of a proper rotation. In fixed right-handed three-dimensional space it sends $\hat{\mathbf{x}} \mapsto \hat{\mathbf{x}}$ and $\hat{\mathbf{y}} \mapsto -\hat{\mathbf{y}}$, so their ordered cross product changes from $\hat{\mathbf{z}}$ to $-\hat{\mathbf{z}}$. Preserving lengths and dot products therefore does not establish preservation of handedness.

Problem 3.31: Vector rate and radial rate

For the prescribed path, $\mathbf{r}' = a\hat{\mathbf{x}} + 2bt\hat{\mathbf{y}} + 3ct^2\hat{\mathbf{z}}$. At $t = 2$ s this is $(2, 4, 6) \text{ m s}^{-1}$, of magnitude $\sqrt{56} \text{ m s}^{-1} \simeq 7.4833 \text{ m s}^{-1}$. At $t = 1$ s, $\mathbf{r} = (2, 1, .5) \text{ m}$ and $\mathbf{r}' = (2, 2, 1.5) \text{ m s}^{-1}$, so

$$\frac{d\|\mathbf{r}\|}{dt} = \frac{\mathbf{r} \cdot \mathbf{r}'}{\|\mathbf{r}\|} = \frac{6.75}{\sqrt{5.25}} \text{ m s}^{-1} \simeq 2.94594 \text{ m s}^{-1}, \quad \|\mathbf{r}'\| = \sqrt{10.25} \text{ m s}^{-1} \simeq 3.20156 \text{ m s}^{-1}.$$

The radial rate is the component of the vector rate along the instantaneous position direction. The remaining perpendicular part changes direction, which explains the unequal rates and the inequality $|d\|\mathbf{r}\|/dt| \leq \|\mathbf{r}'\|$.

Problem 3.32: Finite rates and convergence

The exact derivative at 1 s is $(2, 2, 1.5) \text{ m s}^{-1}$. Polynomial expansion gives the forward error $(0, bh, 3cth + ch^2)$ and the centred error $(0, 0, ch^2)$, retaining the stated coefficient units. At $h = .2$ s the results are

	Rate (m s^{-1})	Vector error (m s^{-1})
Forward	(2, 2.2, 1.82)	(0, .2, .32)
Centred	(2, 2, 1.52)	(0, 0, .02)

Their error norms are $\sqrt{.2^2 + .32^2} \simeq .37736 \text{ m s}^{-1}$ and $.02 \text{ m s}^{-1}$. Halving h gives a centred error $(0, 0, .005) \text{ m s}^{-1}$, exactly one quarter as large. Both samples $1 \text{ s} \pm h$ must lie in $[0, 2] \text{ s}$, hence $0 < h \leq 1 \text{ s}$. The quadratic error reduction is an exact-arithmetic property of this polynomial; roundoff can spoil it at very small steps.

Problem 3.33: Constant length, nonzero derivative

At $t = (\pi/6) \text{ s}$ the radian angle is $\omega t = \pi/2$. Thus $\mathbf{r} = .40\hat{\mathbf{y}} \text{ m}$ and differentiation in fixed axes gives

$$\mathbf{r}' = R\omega(-\sin \omega t \hat{\mathbf{x}} + \cos \omega t \hat{\mathbf{y}}) = -1.20\hat{\mathbf{x}} \text{ m s}^{-1}.$$

The magnitude $\|\mathbf{r}\| = R$ is constant, so its derivative is zero. The vector derivative nevertheless has magnitude $R|\omega| = 1.20 \text{ m s}^{-1}$. Its dot product with $\mathbf{r}$ is zero, consistent with a purely directional change. This conclusion uses a prescribed curve and differentiation, not a force law.

Problem 3.34: A perturbed normal

For the ordered edges, the area vector is $(0, -L\delta, L\varepsilon) = (0, -.002, .020) \text{ m}^2$. Therefore

$$A = .002\sqrt{101} \text{ m}^2 \simeq .0200998 \text{ m}^2, \quad \hat{\mathbf{n}} = \frac{(0, -1, 10)}{\sqrt{101}}, \quad \alpha = \arctan(.002/.020) \simeq +5.7106^\circ.$$

The positive sign means tilt from $+z$ toward $-y$, as defined by the model. The normal has unit norm and zero dot products with $(1, 0, 0)$ and $(1, .020, .002)$ in metre units. Changing only δ to $-.002 \text{ m}$ reverses the y component of the area vector and normal, and makes $\alpha = -5.7106^\circ$. The area magnitude and positive z component remain unchanged.

Problem 3.35: An orientation-tolerance bound

With fixed $\varepsilon > 0$, $|\alpha| = \arctan(|\delta|/\varepsilon)$ is increasing. Consequently the exact condition is

$$|\delta| \leq \varepsilon \tan 1^\circ = .010 \text{ m} \tan 1^\circ = 1.7455065 \times 10^{-4} \text{ m} \simeq .174551 \text{ mm}.$$

The small-angle estimate uses radians, $|\delta| \lesssim .010(\pi/180) \text{ m} = .174533 \text{ mm}$. It is about .0102% smaller than the exact bound, a negligible difference for many purposes at 1° , but the exact relation is available. This model constrains one out-of-plane perturbation with fixed L and ε . Errors in both edges, the origin, or coordinate calibration would require a broader uncertainty model.

Problem 3.36: A numerical derivative investigation

Study performed. Evaluate the path at $t = 1 \text{ s}$ using binary 64-bit floating-point arithmetic. For each positive $h \leq 1 \text{ s}$, compute $\mathbf{q}_F = [\mathbf{r}(1+h) - \mathbf{r}(1)]/h$ and $\mathbf{q}_C = [\mathbf{r}(1+h) - \mathbf{r}(1-h)]/(2h)$ in the numerical-SI convention. Compare with $\mathbf{q} = (2, 2, 1.5) \text{ m s}^{-1}$ and with the expanded exact formulas

$$\mathbf{q}_F = \mathbf{q} + (0, h, 1.5h + .5h^2) \text{ m s}^{-1}, \quad \mathbf{q}_C = \mathbf{q} + (0, 0, .5h^2) \text{ m s}^{-1},$$

where each h in the last display is its numerical value in seconds. Selected computed norm errors are:

h (s)	Direct forward	Direct centred Error norm in m s^{-1}	Exact centred truncation
.5	1.00778	.125	.125
.05	.0911815	.00125000	.00125
.005	.00902428	1.25000×10^{-5}	1.25×10^{-5}
5×10^{-4}	9.01492×10^{-4}	1.25000×10^{-7}	1.25×10^{-7}
5×10^{-5}	9.01398×10^{-5}	1.25161×10^{-9}	1.25×10^{-9}
10^{-7}	1.81514×10^{-7}	9.20630×10^{-11}	5×10^{-15}
10^{-9}	2.64898×10^{-7}	8.71760×10^{-8}	5×10^{-19}
10^{-13}	.00255894	.000781714	5×10^{-27}

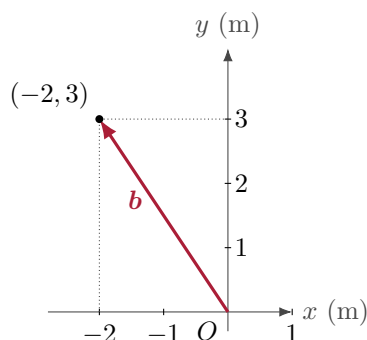
Interpretation and check. In the initial range, reducing h by ten reduces the forward error by roughly ten and the centred error by one hundred, as their first- and second-order formulas predict. At $h = 10^{-7} \text{ s}$ the direct centred error greatly exceeds its exact truncation term; still smaller steps worsen it. Subtraction of nearly equal floating-point positions, followed by division by h , amplifies roundoff. Comparing the direct result with the expanded formula isolates this computational error from truncation. Small-step details can vary with implementation and precision. No experimental noise was introduced, and all sample times remained inside the model interval.

Assessment criteria. A successful investigation records units, arithmetic precision, both sampled times, component and norm errors, and predictions of first- versus second-order behaviour. It shows actual outputs, checks the expanded formulas, and reports cancellation when present. A correctly justified range of h with truncation dominating is an acceptable outcome; indefinitely monotonic improvement is not required.

Additional graphical and advanced practice**Problem 3.37: Recover the missing arrow from a component graph**

Read the equal-scale graph: $\mathbf{a} = (3, 1) \text{ m}$ and $\mathbf{r} = (1, 4) \text{ m}$. Since $\mathbf{a} + \mathbf{b} = \mathbf{r}$, $\mathbf{b} = \mathbf{r} - \mathbf{a} = (-2, 3) \text{ m}$. Its magnitude is $\sqrt{13} \text{ m} \simeq 3.606 \text{ m}$ and its bearing is $180^\circ - \arctan(3/2) \simeq 123.690^\circ$. Draw it from the head $(3, 1)$ of $\mathbf{a}$ to $(1, 4)$; this constructs the head-to-tail route without relocating the resultant. A free copy from the origin ends at $(-2, 3)$ and has the same components. Adding the components recovers $(1, 4)$, checking both orientation and reconstruction.

(a) Free copy of the missing vector



(b) Head-to-tail reconstruction

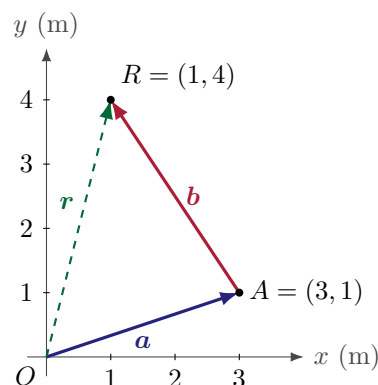


Figure 3.38: Solution construction for Problem 3.37. All coordinates are numerical values in metres, and both panels use equal metre scales. The free arrow has components $(-2, 3)$ m; the translated arrow's head-minus-tail coordinates are $(1, 4) - (3, 1) = (-2, 3)$ m. Panel (b) therefore shows $\mathbf{a} + \mathbf{b} = \mathbf{r}$.

Problem 3.38: Direction cosines and an inadmissible list

The vector length is $\sqrt{2^2 + (-3)^2 + 6^2}$ mm = 7 mm. The cosines of the angles with $+x, +y, +z$ are therefore $(2/7, -3/7, 6/7)$. Their squares sum to $(4 + 9 + 36)/49 = 1$. The associated unsigned angles are approximately $73.40^\circ, 115.38^\circ$, and 31.00° ; the y angle is obtuse because the y component is negative. The proposed list $(.8, .8, 0)$ cannot be direction cosines in an orthonormal three-dimensional frame because its squared norm is 1.28, not one. Each entry lying within $[-1, 1]$ is a necessary but insufficient test.

Problem 3.39: Oriented volume from three edges

First $\mathbf{b} \times \mathbf{c} = (12, -4, 1)$ cm². Then $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 24$ cm³. Its absolute value is the parallelepiped volume, 24 cm³, and its positive sign identifies the given ordering as right-handed. Geometrically $\mathbf{a}$ is a 2 cm edge along x , while the projection of the other two edges onto the yz plane spans area $3(4)$ cm² = 12 cm², giving the same volume. Exchanging $\mathbf{b}$ and $\mathbf{c}$ reverses the triple-product sign to -24 cm³ but leaves the nonnegative volume unchanged.

Problem 3.40: Shortest displacement to a plane

The plane passes through P and has normal direction $\mathbf{n} = (2, -1, 2)$, whose norm is 3. With $\mathbf{X} - \mathbf{P} = (3, 3, 4)$ mm, the signed distance in the $+\mathbf{n}$ sense is $[(\mathbf{X} - \mathbf{P}) \cdot \mathbf{n}] / \|\mathbf{n}\| = 11/3$ mm > 0 . The shortest correction is its negative normal projection,

$$\mathbf{c} = -\frac{(\mathbf{X} - \mathbf{P}) \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} \mathbf{n} = -\frac{11}{9}(2, -1, 2) \text{ mm}, \quad \mathbf{H} = \mathbf{X} + \mathbf{c} = (14, 38, 23) \text{ mm}/9.$$

The distance is $\|\mathbf{c}\| = 11/3$ mm. Checking $\mathbf{H} - \mathbf{P} = (5, 38, 14)$ mm/9 gives normal dot product $(10 - 38 + 28)$ mm/9 = 0, so $\mathbf{H}$ lies on the plane. A correction to any other plane point adds a tangent vector perpendicular to $\mathbf{c}$, increasing its squared norm. This proves shortest distance rather than merely finding one admissible point.

Problem 3.41: A cross product under reflection

For the dimensionless inputs, $\mathbf{a} \times \mathbf{b} = (6, -3, 1)$. The reflection sends $\mathbf{a}$ to $(1, -2, 0)$ and $\mathbf{b}$ to $(0, -1, 3)$. Their cross product is $(-6, -3, -1)$. However $\mathbf{S}(\mathbf{a} \times \mathbf{b}) = (6, 3, 1)$, so treating the cross product as a polar vector under this reflection gives the wrong sign. Since $\det \mathbf{S} = -1$, the correct identity is

$$(\mathbf{S}\mathbf{a}) \times (\mathbf{S}\mathbf{b}) = \det(\mathbf{S}) \mathbf{S}(\mathbf{a} \times \mathbf{b}) = -\mathbf{S}(\mathbf{a} \times \mathbf{b}).$$

The extra determinant encodes the reversal of handedness. This check concerns actively reflected arrows expressed in the same fixed right-handed basis.

Problem 3.42: A constant vector described in a rotating basis

With $\phi = \omega t$, passive components of $\mathbf{A} = 5\hat{x}$ m are $A_1 = 5 \cos \phi$ m and $A_2 = -5 \sin \phi$ m. Differentiate both components and the stated basis vectors:

$$A'_1 = -5\omega \sin \phi \text{ m}, \quad A'_2 = -5\omega \cos \phi \text{ m}, \quad \hat{e}'_1 = \omega \hat{e}_2, \quad \hat{e}'_2 = -\omega \hat{e}_1.$$

The component contribution is $-5\omega(\sin \phi \hat{e}_1 + \cos \phi \hat{e}_2)$ m. The basis contribution is its exact opposite, so $\mathbf{A}' = \mathbf{0}$ at all times. At $t = (\pi/8)$ s with $\omega = 2 \text{ s}^{-1}$, $\phi = \pi/4$ and the components are $(5/\sqrt{2}, -5/\sqrt{2})$ m. Their derivatives are $(-5\sqrt{2}, -5\sqrt{2}) \text{ m s}^{-1}$; these are nonzero coordinate rates, while the complete physical vector derivative is zero. Omitting basis derivatives would incorrectly predict a changing vector.

Problem 3.43: A directional derivative of temperature

The offset T_0 has no spatial derivative. From $T = T_0 + k(x^2 + 2y^2)$, $\nabla T = (2kx, 4ky)$. At $(1, 2)$ m with $k = 1 \text{ K m}^{-2}$, this is $(2, 8) \text{ K m}^{-1}$. Along $\hat{\mathbf{u}} = (3, 4)/5$, the temperature change per unit length is $\nabla T \cdot \hat{\mathbf{u}} = (6 + 32)/5 \text{ K m}^{-1} = 7.6 \text{ K m}^{-1}$. The largest positive directional derivative is the gradient norm, $\sqrt{68} \text{ K m}^{-1} \simeq 8.246 \text{ K m}^{-1}$, attained along $(1, 4)/\sqrt{17}$. Its negative occurs in the opposite direction. The directional derivative has temperature-per-length units; it is not itself a temperature. For a sufficiently small displacement $s\hat{\mathbf{u}}$, $\Delta T = 7.6 \text{ K m}^{-1} s + O(s^2)$.

Problem 3.44: Flux through an oriented planar area

For a uniform velocity field, the volume flux is $Q = \mathbf{v} \cdot \mathbf{A}$, where the area vector selects the positive crossing sense. Thus

$$Q = (2, -1, 3) \cdot (2, 1, 2) \text{ m}^3 \text{ s}^{-1} = (4 - 1 + 6) \text{ m}^3 \text{ s}^{-1} = 9 \text{ m}^3 \text{ s}^{-1}.$$

The area magnitude is 3 m^2 , with normal $(2, 1, 2)/3$, so the normal velocity is 3 m s^{-1} . Multiplying these independently recovers the flux. Reversing the normal replaces $\mathbf{A}$ by $-\mathbf{A}$ and changes Q to $-9 \text{ m}^3 \text{ s}^{-1}$ without changing the physical flow. The tangential velocity contributes no crossing volume. This use of velocity assumes a stationary planar surface and uniform flow over it.

Problem 3.45: Circulation around a square

Let the square traversal be counterclockwise: $(0, 0) \rightarrow (L, 0) \rightarrow (L, L) \rightarrow (0, L) \rightarrow (0, 0)$. For $\mathbf{F} = k(-y, x)$, the lower edge contributes zero because $y = 0$; the right edge contributes $\int_0^L kL dy = kL^2$. The upper edge contributes $\int_L^0 (-kL) dx = kL^2$; the left edge contributes zero because $x = 0$. Therefore

$$\oint \mathbf{F} \cdot d\mathbf{r} = 2kL^2 = 2(3 \text{ N m}^{-1})(2 \text{ m})^2 = 24 \text{ N m}.$$

Clockwise traversal changes the sign to -24 N m . The local circulation density is $\partial F_y/\partial x - \partial F_x/\partial y = 2k$; multiplying it by the square area gives the same value. The nonzero closed-loop integral rules out a single-valued potential gradient for this smooth field on the entire square. The sign depends on traversal, not on an unsigned perimeter length.

Problem 3.46: Two paths through a gradient field

On the straight segment $\mathbf{r}(s) = \mathbf{A} + s(\mathbf{B} - \mathbf{A})$ for $0 \leq s \leq 1$, $\mathbf{B} - \mathbf{A} = (2, 2)$ m. Since the field is linear in position, its average along the segment equals its midpoint value: $\mathbf{F}((\mathbf{A} + \mathbf{B})/2) = 2(2, 1) \text{ N} = (4, 2) \text{ N}$. Consequently the integral is $(4, 2) \cdot (2, 2) \text{ N m} = 12 \text{ N m}$. For the elbow path through $\mathbf{C} = (3, 0)$ m, the horizontal contribution is $\int_1^3 kx dx = 8 \text{ N m}$ and the vertical contribution is $\int_0^2 ky dy = 4 \text{ N m}$, again totalling 12 N m .

Here $\mathbf{F} = \nabla \Phi$ with $\Phi = k(x^2 + y^2)/2$, so the endpoint check gives $\Phi(\mathbf{B}) - \Phi(\mathbf{A}) = k(13 - 1) \text{ m}^2/2 = 12 \text{ N m}$. The symbol Φ denotes a scalar whose gradient is $\mathbf{F}$; a potential energy for a force would be $-\Phi$. The midpoint rule is exact on each straight piece because the integrand is linear in its segment parameter. Agreement here is therefore explained analytically, rather than inferred from two numerical path values alone.

3.12 Bridge to motion in two and three dimensions

A vector description now has a clear meaning: position refers to an origin, displacement joins two positions, and components belong to specified axes. The next chapter, *Motion in Two and Three Dimensions*, applies these tools to velocity, acceleration and complete kinematic models. The optional derivative preview here prepares that language without assuming its later physical applications. Keep the geometric checks: a change of direction is a vector change, a component sign is not a speed, and a transformed description must refer to the same physical situation before its numbers can be compared.

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