

Chapter 4

# Motion in Two and Three Dimensions

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# Preface

A moving inspection probe must do more than reach a list of coordinates. Its direction of travel matters to an observer, its speed matters to the time spent near a surface, and its acceleration describes how rapidly that velocity changes. These are three different questions. A plot of the path answers none of them completely unless it is accompanied by a clock and a reference frame.

This chapter extends the straight-line kinematics and vector methods of the preceding chapters to motion in a plane and in space. We first differentiate a prescribed trajectory. We then study particularly useful models: constant vector acceleration, projectile motion, circular motion, and motion viewed from translating frames. A computational and measurement pathway asks how much of this information can be recovered from discrete samples. The advanced section explains why differentiating coordinates in a rotating polar basis requires more than differentiating the coordinate values.

The objective is a reliable description of motion, not yet an account of its causes. No force law is needed to differentiate a prescribed position function. Conversely, obtaining an acceleration from a path does not identify the forces or prove that a proposed machine can realize that path. Newton's laws and their applications supply that next layer in the following chapters.

## The Guideline

Name the object and observer, choose axes and a common clock, state the model and its time interval, and identify the event being sought. Use vectors for direction and components for calculation. Test the same event in every relevant coordinate before interpreting the answer.

All numerical setups and graphics in this chapter are constructed teaching models. The tabulated coordinate perturbations are synthetic, not measurements. The standard treatments in the three supplied textbooks inform the comparative reading indicated in the bibliography; the explanations, worked setups, practice tasks and illustrations here are newly developed for this chapter. Every practice problem has a complete reasoned solution in this chapter. The worked examples supply complementary methods, graphical interpretations and checks.

## Pedagogical boundary

The core assumes the units, graphs and one-dimensional derivatives of *Motion in One Dimension*, and the fixed-basis vector operations of *Vectors and Mathematical Tools*. The numerical pathway introduces its own update rules. Polar-basis differentiation is optional enrichment: no core exercise requires it. Forces, drag laws, energy methods, rotating-observer dynamics and a full theory of space-curve torsion are deferred.

# Conventions and notation

A bold symbol is a vector; an unbold symbol may be a component or a magnitude and is defined when introduced. In particular,  $v = |\mathbf{v}| \geq 0$  is speed, whereas  $v_x$  can be negative. Cartesian unit vectors  $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$  are fixed in the named frame. A dot denotes differentiation with respect to that frame's common time coordinate.

| Symbol                          | Meaning                                            | Unit                               |
|---------------------------------|----------------------------------------------------|------------------------------------|
| $\mathbf{r} = (x, y, z)$        | Position relative to the stated origin             | m                                  |
| $\mathbf{v}, \mathbf{a}$        | Velocity and acceleration                          | $\text{m s}^{-1}, \text{m s}^{-2}$ |
| $s$                             | Distance accumulated along a path                  | m                                  |
| $a_t = dv/dt$                   | Signed forward-tangential component                | $\text{m s}^{-2}$                  |
| $a_n = v^2/R_c$                 | Non-negative principal-normal component            | $\text{m s}^{-2}$                  |
| $R, R_c$                        | Circle radius; local radius of curvature           | m                                  |
| $\theta, \omega = \dot{\theta}$ | Plane angle; signed angular rate                   | rad, $\text{rad s}^{-1}$           |
| $\mathbf{v}_{A/B}$              | Velocity of A measured relative to B               | $\text{m s}^{-1}$                  |
| $h$                             | Numerical time step, not a height unless redefined | s                                  |
| $\sigma_x$                      | Standard uncertainty of a coordinate               | m                                  |

The normal unit vector  $\hat{\mathbf{n}}$  points toward the local centre of curvature, where that direction is defined. It is not generally the direction toward the coordinate origin. In the advanced polar representation,  $\hat{\mathbf{e}}_r$  points outward from the origin, so its acceleration component is negative in uniform circular motion about that origin. These are different bases, not contradictory signs.

Numerical projectile examples use the stipulated uniform value  $g = 9.81 \text{ m s}^{-2}$ , with  $y$  upward and  $a_y = -g$ . This is a teaching-model input, not an exact SI constant or a measured local value. Angles in calculus formulas and computer trigonometric functions are in radians. A degree symbol is written explicitly for quoted degree angles. Extra numerical digits in convergence tables diagnose numerical error and do not claim measurement precision.

## Note

All numerical models use SI inputs. Expressions such as  $x(t) = (0.6 \text{ m s}^{-1})t$  retain dimensional coefficients. When a table or algorithm says “SI numerical values,”  $t$  is entered in seconds and each output has the unit specified in its heading. A graph of  $y$  against  $x$  is a trajectory projection; a graph of  $y$  against  $t$  is not.

# Chapter 4

## Motion in Two and Three Dimensions

An inspection probe passes through the point (1.2, 0.8, 0.4) m. Does that information tell us which way it is moving, how fast it is moving, or whether it is turning? It does not. A probe can pass through the same point along many paths, and even along one fixed path it can follow many different schedules. The opening task of this chapter is to make those distinctions quantitative for a prescribed three-dimensional inspection path.

### Learning goals

After the core and its supporting investigations, you should be able to differentiate a vector trajectory, coordinate its component equations with one clock, solve unequal-height projectile and interception events, distinguish turning from changing speed, compose explicitly named relative velocities, and evaluate the limitations of sampled motion data. After the advanced section, you should also be able to derive and test the changing-basis terms in polar velocity and acceleration.

### 4.1 Position, velocity and acceleration vectors

#### 4.1.1 A trajectory needs a parameter and an observer

In a fixed Cartesian frame, a particle's position is

$$\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}. \quad (4.1)$$

The three component functions refer to the *same* time. A point on a parametric curve is obtained by choosing one value of  $t$  and evaluating all three functions there. Choosing  $x$  at one time and  $y$  at another usually produces a point the particle never occupied.

A reference frame specifies more than a drawing of axes: it identifies the observer relative to which the coordinates are recorded. Here the origin and axes are attached to a laboratory unless another frame is named. Shifting the fixed origin changes  $\mathbf{r}$ , but not its derivatives. Changing to a moving observer can change the derivatives; Section 4.6 treats that operation explicitly.

For observations at  $t_1$  and  $t_2 > t_1$ ,

$$\Delta\mathbf{r} = \mathbf{r}(t_2) - \mathbf{r}(t_1), \quad \mathbf{v}_{\text{avg}} = \frac{\Delta\mathbf{r}}{t_2 - t_1}. \quad (4.2)$$

The displacement joins the two endpoints directly. The distance travelled follows the intervening path. Their difference is already important on a line with reversals; a curved path makes it unavoidable even without a reversal.

**Definition 4.1: Instantaneous velocity and acceleration**

For a sufficiently differentiable trajectory in a fixed Cartesian basis,

$$\mathbf{v}(t) = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} + \dot{z}\hat{\mathbf{k}}, \quad (4.3)$$

$$\mathbf{a}(t) = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t} = \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}}. \quad (4.4)$$

Speed is  $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$ . These equations define the kinematic quantities; they do not assume a particular force or a constant acceleration.

The velocity is tangent to a smooth trajectory wherever  $v \neq 0$ . To see why, divide the displacement chord by its time interval and let that interval shrink: the chord direction approaches the local direction of travel. A component graph contains the same information in a different form. The slope of  $x(t)$  is  $v_x(t)$ ; its changing slope is  $a_x(t)$ . Neither slope is generally the slope of the projected path  $y(x)$ . When  $v_x \neq 0$ , the chain rule instead gives

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{v_y}{v_x}. \quad (4.5)$$

This ratio is dimensionless when both coordinates have the same length unit. A time-graph slope has units of velocity. Confusing these two graphs loses both the units and the physical meaning.

**Example 4.1: A three-dimensional inspection path**

A point-like probe follows, for  $0 \leq t \leq 4$  s,

$$x = (0.60 \text{ m s}^{-1})t, \quad y = (0.20 \text{ m s}^{-2})t^2, \quad z = (0.050 \text{ m s}^{-3})t^3.$$

Find its position, velocity, speed and acceleration at  $t = 2.0$  s, and interpret the arrows in Fig. 4.1.

**Model and strategy.** The formulas prescribe the laboratory coordinates of one point. Differentiate the functions before substituting the event time; differentiating the numerical coordinates of a single event would give no motion information.

$$\begin{aligned} \mathbf{v}(t) &= (0.60 \text{ m s}^{-1})\hat{\mathbf{i}} + (0.40 \text{ m s}^{-2})t\hat{\mathbf{j}} + (0.150 \text{ m s}^{-3})t^2\hat{\mathbf{k}}, \\ \mathbf{a}(t) &= (0.40 \text{ m s}^{-2})\hat{\mathbf{j}} + (0.300 \text{ m s}^{-3})t\hat{\mathbf{k}}. \end{aligned}$$

At the stated time,

$$\begin{aligned} \mathbf{r} &= (1.20\hat{\mathbf{i}} + 0.80\hat{\mathbf{j}} + 0.40\hat{\mathbf{k}}) \text{ m}, \\ \mathbf{v} &= (0.60\hat{\mathbf{i}} + 0.80\hat{\mathbf{j}} + 0.60\hat{\mathbf{k}}) \text{ m s}^{-1}, \quad v = \sqrt{1.36} \text{ m s}^{-1} = 1.17 \text{ m s}^{-1}, \\ \mathbf{a} &= (0\hat{\mathbf{i}} + 0.40\hat{\mathbf{j}} + 0.60\hat{\mathbf{k}}) \text{ m s}^{-2}, \quad |\mathbf{a}| = \sqrt{0.52} \text{ m s}^{-2} = 0.721 \text{ m s}^{-2}. \end{aligned}$$

**Interpretation.** The position arrow begins at the origin; the velocity and acceleration arrows are attached to the event point to show local directions. Their lengths cannot be compared without their different drawing scales. The increasing slopes of  $y(t)$  and  $z(t)$  explain the positive  $y$  and  $z$  accelerations. The straight  $x(t)$  graph gives zero  $x$  acceleration even though  $v_x \neq 0$ .

**Checks.** Each derivative removes one power of time from the units. At this event  $dy/dx = 0.8/0.6 = 4/3$  and  $dz/dx = 1$ , matching the tangent direction. The scalar sum of the velocity components,  $2.0 \text{ m s}^{-1}$ , is not the speed. The changing ratios of components show that this path is not one straight line.

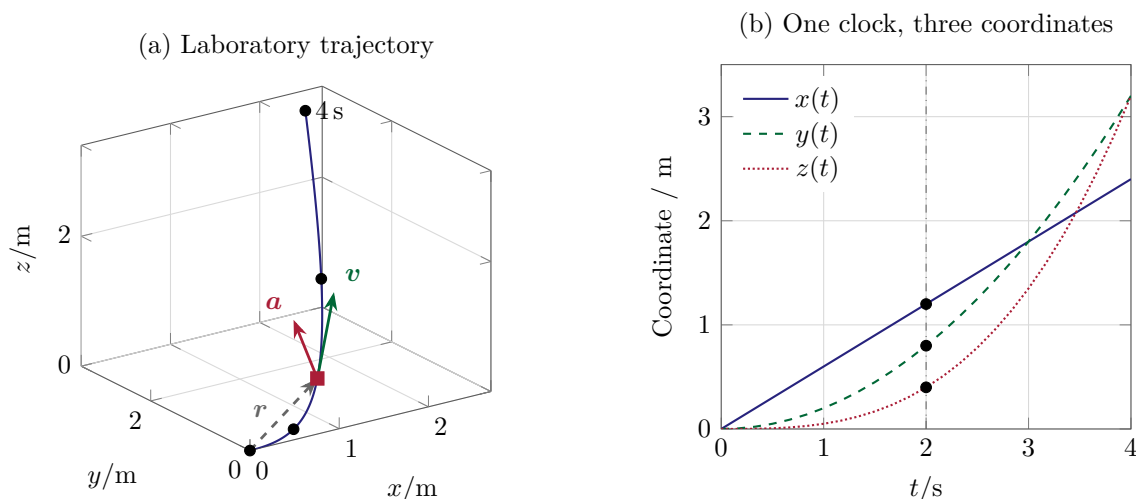


**Figure 4.1:** Calculated representations of the inspection path. The square marks the two-second event. Dots on the spatial curve are separated by one second. The local arrows at two seconds use the plotted displacement equivalents  $(1.20\text{ s})\mathbf{v}$  and  $(1.20\text{ s}^2)\mathbf{a}$ ; they are not additional position vectors. The three-dimensional panel is a projection, so apparent angles on the page are not physical three-dimensional angles. The component panel uses the same event time.

#### 4.1.2 Acceleration can change speed, direction, or both

A positive acceleration component does not always mean that an object speeds up. On a line, a negative velocity and positive acceleration initially reduce speed. In space, acceleration can also rotate the velocity without changing its magnitude. Differentiate  $v^2 = \mathbf{v} \cdot \mathbf{v}$ :

$$2v \frac{dv}{dt} = 2\mathbf{v} \cdot \mathbf{a}, \quad \frac{dv}{dt} = \frac{\mathbf{v} \cdot \mathbf{a}}{v} \quad (v > 0). \quad (4.6)$$

A positive dot product means increasing speed, a negative one decreasing speed, and a zero one instantaneous constancy of speed. A zero dot product does *not* imply zero acceleration. Section 4.5 will identify the perpendicular, turning part.

For the inspection probe at two seconds,  $\mathbf{v} \cdot \mathbf{a} = 0.68\text{ m}^2\text{ s}^{-3} > 0$ , so its speed increases. But  $\mathbf{a}$  is not parallel to  $\mathbf{v}$ : there is also a change in direction. In particular, the zero  $x$  acceleration and nonzero  $x$  velocity rule out parallel vectors here.

#### Example 4.2: Same endpoints, different averages

For the same probe, compare the average velocity from zero to four seconds with the average speed over that interval. Must either equal the instantaneous velocity at two seconds?

**Calculation.** The endpoint displacement is  $(2.4, 3.2, 3.2)\text{ m}$ , hence

$$\mathbf{v}_{\text{avg}} = (0.60, 0.80, 0.80)\text{ m s}^{-1}, \quad |\mathbf{v}_{\text{avg}}| = \sqrt{1.64}\text{ m s}^{-1} = 1.28\text{ m s}^{-1}.$$

The distance follows the curve:

$$s = \int_0^{4\text{ s}} \sqrt{(0.60\text{ m s}^{-1})^2 + [(0.40\text{ m s}^{-2})t]^2 + [(0.150\text{ m s}^{-3})t^2]^2} dt.$$

Numerical integration gives  $s = 5.47955\text{ m}$ , so  $v_{\text{avg, speed}} = s/(4\text{ s}) = 1.36989\text{ m s}^{-1} > 1.28\text{ m s}^{-1}$ : the curved path is longer than its chord. The integral, rather than a sum of coordinate changes, is the appropriate distance formula.

**Interpretation and check.** The average velocity differs from  $\mathbf{v}(2\text{ s})$  in its  $z$  component. A midpoint derivative equals a secant slope for a quadratic coordinate, but not in general for a cubic coordinate. There is no rule assigning a vector average to the midpoint of a time interval.

### 4.1.3 The geometric path and the time schedule

A curve and a motion along that curve are different pieces of information. For a fixed geometric path written as  $\mathbf{R}(q)$ , a dimensionless parameter  $q$  identifies a location. A schedule  $q = q(t)$  identifies when that location is reached. Applying the chain rule twice gives

$$\mathbf{v} = \frac{d\mathbf{R}}{dq}\dot{q}, \quad \mathbf{a} = \frac{d^2\mathbf{R}}{dq^2}\dot{q}^2 + \frac{d\mathbf{R}}{dq}\ddot{q}. \quad (4.7)$$

The first acceleration term depends on the changing tangent of the geometric curve; the second depends on the changing schedule. This distinction explains why the shape of a photograph of a path does not determine the speed or acceleration. It also prepares the separation of turning and speed change later in the chapter.

#### Example 4.3: Two schedules for the same semicircle

A marker follows the upper semicircle of radius  $R = 2.00$  m, starting at  $(R, 0)$  and ending at  $(-R, 0)$  after  $T = 4.00$  s. Compare the schedules  $\phi_A = \pi t/T$  and  $\phi_B = \pi(t/T)^2$ , using

$$\mathbf{r} = R \cos \phi \hat{\mathbf{i}} + R \sin \phi \hat{\mathbf{j}}, \quad 0 \leq t \leq T.$$

**Derivation.** Differentiation in fixed Cartesian axes gives

$$\begin{aligned} \mathbf{v} &= R\dot{\phi}(-\sin \phi \hat{\mathbf{i}} + \cos \phi \hat{\mathbf{j}}), \\ \mathbf{a} &= -R\dot{\phi}^2(\cos \phi \hat{\mathbf{i}} + \sin \phi \hat{\mathbf{j}}) + R\ddot{\phi}(-\sin \phi \hat{\mathbf{i}} + \cos \phi \hat{\mathbf{j}}). \end{aligned}$$

For A,  $\dot{\phi} = \pi/T$ ,  $\ddot{\phi} = 0$ , and the speed is  $R\pi/T = 1.571$  m s<sup>-1</sup>. For B,  $\dot{\phi} = 2\pi t/T^2$ ,  $\ddot{\phi} = 2\pi/T^2$ , so the speed rises linearly from zero to  $2R\pi/T = 3.142$  m s<sup>-1</sup>.

**Compare one event.** At the common arrival point  $(-R, 0)$ , A has  $\mathbf{v}_A = -(R\pi/T)\hat{\mathbf{j}}$  and  $\mathbf{a}_A = (R\pi^2/T^2)\hat{\mathbf{i}}$ . B has twice that velocity, four times the inward acceleration component, and an additional downward component  $-2R\pi/T^2\hat{\mathbf{j}}$ . Numerically,

$$\mathbf{a}_A = 1.234\hat{\mathbf{i}} \text{ m s}^{-2}, \quad \mathbf{a}_B = (4.935\hat{\mathbf{i}} - 0.7854\hat{\mathbf{j}}) \text{ m s}^{-2}.$$

**Interpretation and limits.** Both motions travel  $\pi R$ , so both have average speed  $\pi R/T$ , despite their different instantaneous speeds. At B's start the speed vanishes, but the Cartesian formula remains regular and gives  $\mathbf{a}_B(0) = 0.7854\hat{\mathbf{j}} \text{ m s}^{-2}$ . A tangent *direction of travel* defined by  $\mathbf{v}/v$  is unavailable at that instant. The geometrical path itself is still smooth.

### 4.1.4 A genuinely three-dimensional path: the helix

A useful spatial benchmark combines uniform circular motion in a horizontal projection with uniform vertical translation. With  $R > 0$ , constant angular rate  $\omega$ , and vertical velocity  $u$ , prescribe

$$\begin{aligned} \mathbf{r}(t) &= R \cos(\omega t)\hat{\mathbf{i}} + R \sin(\omega t)\hat{\mathbf{j}} + ut\hat{\mathbf{k}}, \\ \mathbf{v}(t) &= -R\omega \sin(\omega t)\hat{\mathbf{i}} + R\omega \cos(\omega t)\hat{\mathbf{j}} + u\hat{\mathbf{k}}, \\ \mathbf{a}(t) &= -R\omega^2 \cos(\omega t)\hat{\mathbf{i}} - R\omega^2 \sin(\omega t)\hat{\mathbf{j}}. \end{aligned} \quad (4.8)$$

The velocity remains tangent to the spatial helix; it includes a vertical component. The acceleration is horizontal and points toward the vertical axis. Its dot product with velocity is zero, so the three-dimensional speed  $\sqrt{R^2\omega^2 + u^2}$  is constant. This motion is not planar circular motion when  $u \neq 0$ : after one projected revolution the marker reaches a different height. Constant speed and a constant acceleration *magnitude* do not imply constant acceleration *vector*.

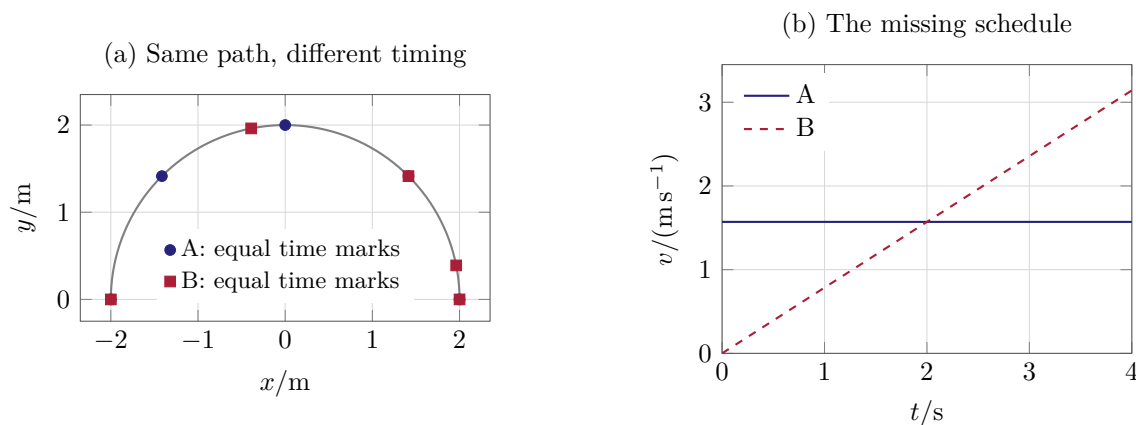


**Figure 4.2:** Two calculated motions on one semicircle. Markers in (a) occur at  $t = 0, 1, 2, 3, 4$  s, and the endpoint markers coincide. The speed graphs have equal area,  $2\pi$  m; this is the common distance travelled. Different marker spacing reveals the different schedules even though the locus is identical.

#### Example 4.4: An ascending inspection marker

Take  $R = 0.800$  m,  $\omega = \pi/2$  s $^{-1}$ , and  $u = 0.300$  m s $^{-1}$ . Find the state at one second and compare displacement with distance after a full projected turn.

**Local state.** At  $t = 1.00$  s,  $\omega t = \pi/2$ , hence

$$\begin{aligned} \mathbf{r} &= (0, 0.800, 0.300) \text{ m}, \\ \mathbf{v} &= (-1.257, 0, 0.300) \text{ m s}^{-1}, \quad v = 1.292 \text{ m s}^{-1}, \\ \mathbf{a} &= (0, -1.974, 0) \text{ m s}^{-2}. \end{aligned}$$

The vertical speed does not produce a vertical acceleration because it is constant. The nonzero horizontal acceleration changes the direction of the horizontal velocity.

**One projected turn.** Its duration is  $T = 2\pi/\omega = 4.00$  s, and its vertical rise, called the pitch, is  $uT = 1.20$  m. Horizontal coordinates return to their starting values, so  $\Delta \mathbf{r} = 1.20 \hat{\mathbf{k}}$  m. Constant speed gives distance  $s = vT = 5.168$  m.

**Checks and interpretation.** The displacement magnitude is smaller than the distance; the horizontal motion contributes to distance even though its net displacement vanishes. Setting  $u = 0$  recovers a planar circle, while setting  $\omega = 0$  produces vertical straight motion at fixed  $(x, y) = (R, 0)$ . The  $\omega = 0$  limit has no finite revolution period and must not be evaluated using  $2\pi/\omega$ .

#### 4.1.5 Why the speed integral is a path length

Divide a time interval into small pieces. The chord on piece  $i$  has length  $|\Delta \mathbf{r}_i|$ , and for smooth motion  $\Delta \mathbf{r}_i \simeq \mathbf{v}(t_i) \Delta t_i$ . Summing chord lengths and refining the partition gives

$$s = \int_{t_1}^{t_2} |\mathbf{v}(t)| dt = \int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} dt. \quad (4.9)$$

This integral includes every part of the route; endpoint subtraction does not. Vector addition and the triangle inequality give  $|\Delta \mathbf{r}| \leq s$ . Equality requires a straight path traversed without reversal, apart from intervals of rest. The average speed therefore satisfies  $s/\Delta t \geq |\mathbf{v}_{\text{avg}}|$ .

For the semicircle,  $s = \pi R$ , whereas  $|\Delta \mathbf{r}| = 2R$ . With the numerical values above, both schedules have average velocity  $-1.00 \hat{\mathbf{i}}$  m s $^{-1}$ , whose magnitude is  $1.00$  m s $^{-1}$ , and average speed  $1.571$  m s $^{-1}$ .

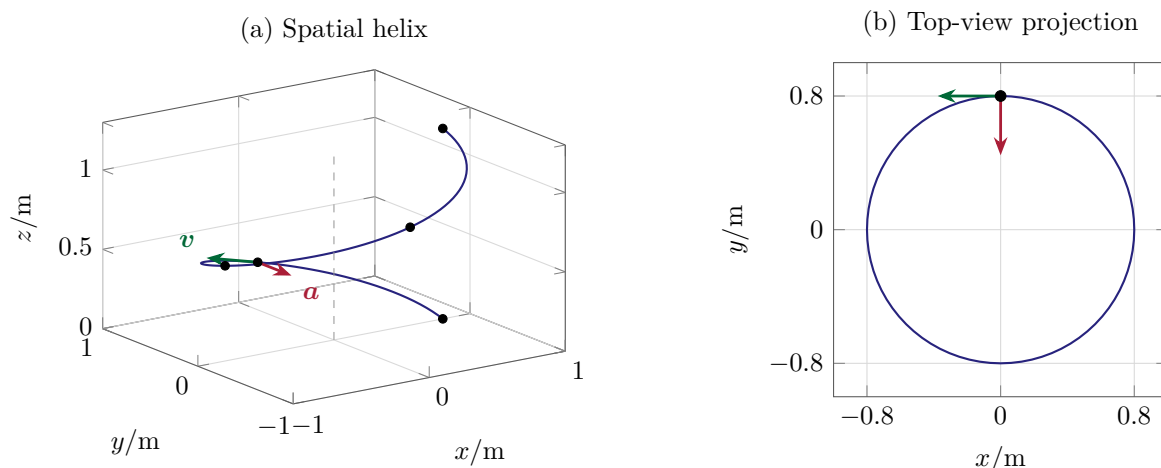


**Figure 4.3:** The verified helix and its circular top view. Dots in (a) are one second apart. At one second the velocity arrow uses  $(0.300\text{ s})\mathbf{v}$ , and the acceleration arrow uses  $(0.180\text{ s}^2)\mathbf{a}$ . Panel (b) discards the vertical component; its velocity arrow is therefore only the horizontal projection. The apparent angles in the oblique spatial view are not true three-dimensional angles.

The speed integral also resolves reversals automatically: its integrand is non-negative, so later motion cannot cancel earlier distance.

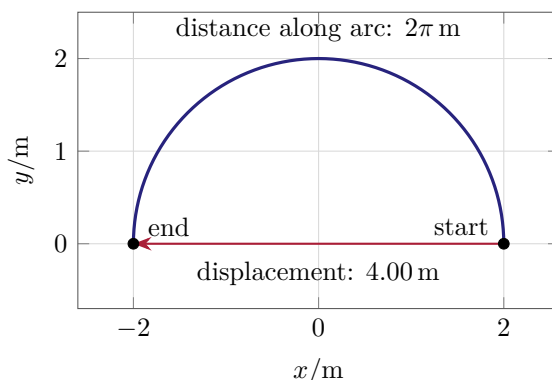


**Figure 4.4:** A calculated semicircular route with equal coordinate scales. The displacement arrow runs directly from start to end, while the distance is the length of the curved route. These geometric quantities do not depend on either of the two time schedules in Fig. 4.2.

### Guided checks

**Can a zero velocity coexist with nonzero acceleration?** Yes:  $\mathbf{r}(t) = \frac{1}{2}\mathbf{a}_0 t^2$  has  $\mathbf{v}(0) = 0$  and  $\mathbf{a}(0) = \mathbf{a}_0$ . The expression  $\mathbf{v} \cdot \mathbf{a}/v$  cannot be used at that instant.

**Does a trajectory alone fix acceleration?** No. Traversing the same circle twice as fast changes its velocity and acceleration. A parametrization supplies the missing schedule.

The next model simplifies the schedule by prescribing a constant acceleration vector. It does not change the definitions; it makes their integration especially simple.

## 4.2 Constant vector acceleration

### 4.2.1 Integrating the model

Suppose  $\mathbf{a}$  is independent of time in the chosen fixed frame over the entire interval of interest. Integrating from an initial event  $t_0$ , and writing  $\tau = t - t_0$ , gives

$$\mathbf{v}(t) - \mathbf{v}_0 = \int_{t_0}^t \mathbf{a} dt' = \mathbf{a}\tau, \quad (4.10)$$

$$\mathbf{r}(t) - \mathbf{r}_0 = \int_{t_0}^t [\mathbf{v}_0 + \mathbf{a}(t' - t_0)] dt' = \mathbf{v}_0\tau + \frac{1}{2}\mathbf{a}\tau^2. \quad (4.11)$$

These are vector equations, equivalently three copies of the one-dimensional constant-acceleration equations with a shared  $\tau$ . For example,

$$x = x_0 + v_{0x}\tau + \frac{1}{2}a_x\tau^2, \quad y = y_0 + v_{0y}\tau + \frac{1}{2}a_y\tau^2.$$

The integration constants are the components of the initial position and initial velocity. Omitting an initial height because another coordinate begins at zero changes the physical setup.

A constant *magnitude*  $|\mathbf{a}|$  is insufficient for these formulas: the direction must also be constant. Uniform circular motion, discussed below, is a counterexample with constant acceleration magnitude but continuously rotating acceleration direction. For time-varying acceleration the integral formulas still apply, but  $\mathbf{a}$  cannot be removed from the integral.

The displacement relative to  $\mathbf{r}_0$  is always a linear combination of  $\mathbf{v}_0$  and  $\mathbf{a}$ . Thus this particular model remains in the plane spanned by those two vectors, even if all three Cartesian coordinates change. When the two vectors are parallel, the path lies on a line. The inspection path of Section 4.1 is more general: its acceleration varies with time.

### 4.2.2 One clock for independent component equations

Independent integration is not permission to invent separate arrival times. A bead landing on a moving belt meets a *spatial event*. The vertical coordinate determines when the bead reaches belt height. Both its horizontal position and the belt marker's position must then be evaluated at that same time. The marker is a second moving object, not a modification of the bead's free-flight acceleration.

#### Example 4.5: Release above a moving belt

At  $t = 0$ , a small bead leaves a horizontal guide at  $(x, y) = (0, 1.80)$  m with laboratory velocity  $(2.40, 0)$  m s<sup>-1</sup>. A horizontal belt at  $y = 0$  moves toward  $+x$  at  $0.60$  m s<sup>-1</sup>. Where should a marker on the belt be initially so that the bead first meets it at belt height?

**Assumptions.** After release the bead follows  $\mathbf{a} = -g\hat{\mathbf{j}}$ , with no air drag. The belt is long enough to include the landing point. The problem ends at first contact; rebound, adhesion and subsequent transport are not modelled.

**Vertical event.** The bead has

$$x_B(t) = (2.40 \text{ m s}^{-1})t, \quad y_B(t) = 1.80 \text{ m} - \frac{1}{2}gt^2.$$

Solving  $y_B(t_f) = 0$  and choosing the positive root,

$$t_f = \sqrt{\frac{2(1.80 \text{ m})}{9.81 \text{ m s}^{-2}}} = 0.605783 \text{ s}.$$

**Horizontal matching.** The bead lands at  $x_B(t_f) = 1.45388$  m. If the marker starts at  $x_{M0}$ , then  $x_M(t) = x_{M0} + (0.60 \text{ m s}^{-1})t$ . Therefore

$$x_{M0} = (2.40 - 0.60) \text{ m s}^{-1} t_f = 1.09041 \text{ m}.$$

To the precision of the setup, place it at 1.09 m to the right of the release vertical.

**Checks and interpretation.** During flight the marker moves 0.36347 m; its initial position plus this displacement equals the bead's landing coordinate. The bead's vertical impact velocity is  $-gt_f = -5.94 \text{ m s}^{-1}$ , independent of belt speed. Reversing the belt velocity changes the required initial marker coordinate, not the ideal fall time. The relative horizontal velocity  $2.40 - 0.60 = 1.80 \text{ m s}^{-1}$  provides a second way to compute the required initial horizontal gap.

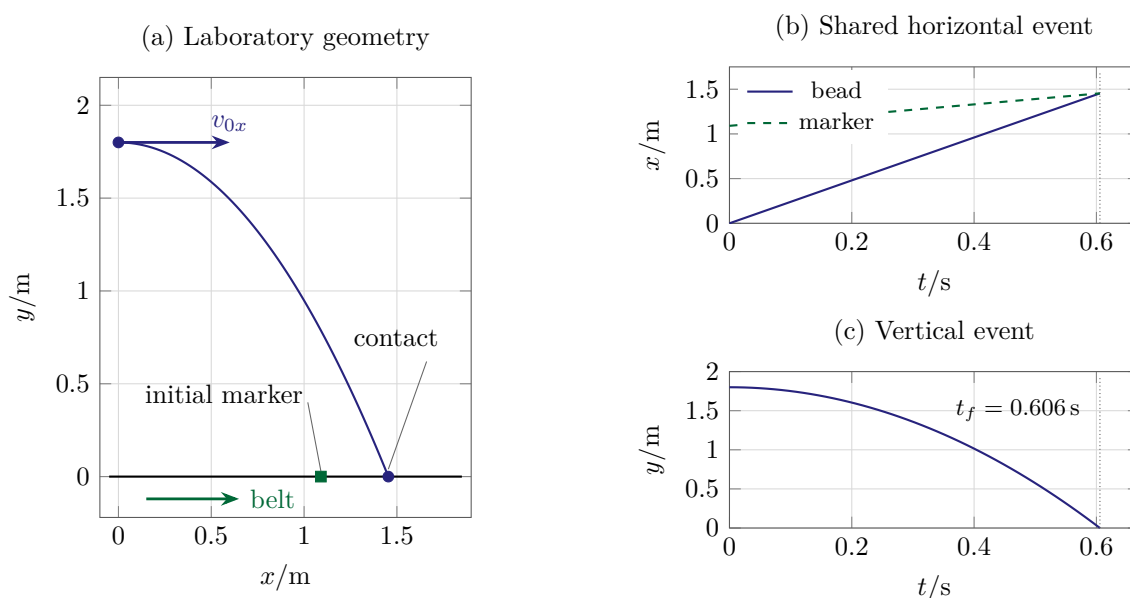


**Figure 4.5:** One landing, one time. The geometry has equal length scales; direction arrows are schematic in length. Both time panels use identical time-axis limits and the same dotted event line. The belt marker and bead coincide horizontally when the bead reaches  $y = 0$ , not when their speeds become equal.

### Caution

Component independence here follows from a *prescribed* acceleration model. In a more general physical system, an acceleration component may depend on several velocity or position components. Decomposing a vector into Cartesian components does not itself prove that the underlying dynamics are uncoupled.

### 4.2.3 Reconstructing an initial velocity from an arrival event

Sometimes the arrival time and endpoint are specified instead of the initial velocity. Equation (4.11) can then be solved component by component:

$$\mathbf{v}_0 = \frac{\mathbf{r}_f - \mathbf{r}_0}{\tau_f} - \frac{1}{2} \mathbf{a} \tau_f, \quad \tau_f > 0. \quad (4.12)$$

This is a kinematic requirement. It is not a claim that a motor or launcher can supply that velocity. A dimensional check distinguishes the displacement divided by time and acceleration multiplied by time: both are velocities. A zero arrival time cannot be inserted unless the event is reformulated.

For example, to move from  $(0, 0)$  m to  $(1.20, 0.80)$  m in 2.0 s with prescribed  $\mathbf{a} = (0.20, -0.10)$  m s<sup>-2</sup>, one needs  $\mathbf{v}_0 = (0.40, 0.50)$  m s<sup>-1</sup>. Substitution gives  $(0.80, 1.00) + (0.40, -0.20) = (1.20, 0.80)$  m. The inverse calculation checks an event, rather than selecting a formula solely because the available numbers appear in it.

## 4.3 Projectile motion

### 4.3.1 The model and the trajectory

A projectile model describes motion after release with a constant downward acceleration and negligible drag. Over a limited region, choose a flat horizontal  $x$  axis and upward  $y$  axis. With  $t = 0$  at launch,

$$\begin{aligned} x(t) &= x_0 + v_{0x}t, & v_x(t) &= v_{0x}, \\ y(t) &= y_0 + v_{0y}t - \frac{1}{2}gt^2, & v_y(t) &= v_{0y} - gt, \\ v_{0x} &= v_0 \cos \theta, & v_{0y} &= v_0 \sin \theta. \end{aligned} \quad (4.13)$$

The model assumes a point-like object, uniform  $g$ , a fixed laboratory frame and no intervening contact. It ends when the first relevant obstacle or surface is reached. Extending the mathematical parabola below a solid floor does not extend the free-flight model.

If  $v_{0x} \neq 0$ , eliminate time to obtain the path:

$$y = y_0 + \frac{v_{0y}}{v_{0x}}(x - x_0) - \frac{g}{2v_{0x}^2}(x - x_0)^2. \quad (4.14)$$

The time domain determines which part of the parabola is traversed. A vertical launch,  $v_{0x} = 0$ , cannot be described by this function  $y(x)$ ; its path is the line  $x = x_0$ , still described without difficulty by Eq. (4.13). A three-dimensional launch with a constant vertical acceleration lies in the vertical plane containing its initial horizontal velocity.

For  $v_{0y} > 0$ , the apex occurs at

$$t_A = \frac{v_{0y}}{g}, \quad y_A = y_0 + \frac{v_{0y}^2}{2g}. \quad (4.15)$$

This event belongs to the physical flight only if no earlier contact occurs. At the apex  $v_y = 0$ , but  $\mathbf{v} = v_{0x}\hat{\mathbf{i}}$  and  $\mathbf{a} = -g\hat{\mathbf{j}}$ . Speed is minimized there for a nonvertical launch: gravity is changing the direction of the velocity at that instant rather than its magnitude.

### 4.3.2 Unequal-height events and roots

To find when the projectile reaches a specified height  $y_f$ , solve the actual height equation rather than applying a memorized total-flight formula:

$$t_{\pm} = \frac{v_{0y} \pm \sqrt{v_{0y}^2 + 2g(y_0 - y_f)}}{g}. \quad (4.16)$$

A negative discriminant means the height is unreachable in this model. A zero discriminant places it at the apex. A positive discriminant can give two positive times, one on ascent and one on descent, or only one positive time. Negative roots describe the extrapolated trajectory before launch and are not later arrival events.

Reaching a height is not the same as reaching a target at that height. A stationary target also imposes its  $x$  coordinate. A moving target imposes its own position evaluated at the same time. A mathematically valid descending root can additionally occur after the projectile has already struck another surface; the time-domain check must come last as well as first.

**Example 4.6: Intercept a moving target at a different height**

A bead is launched from  $(0, 0.50)$  m with  $\mathbf{v}_0 = (4.00, 5.00)$  m s<sup>-1</sup>. A small target moves to the right at 1.00 m s<sup>-1</sup>, at fixed height 1.50 m. Find its required initial  $x$  coordinate for interception while the bead is descending. Then test a target initially at  $x_{T0} = 2.00$  m.

**Strategy.** First solve the bead's vertical event; then determine whether the target is at the bead's horizontal position at that time. There are no obstacles before the requested event.

$$1.50 = 0.50 + 5.00t - 4.905t^2 \quad \Rightarrow \quad t_{\pm} = \frac{5.00 \pm \sqrt{25.00 - 19.62}}{9.81} \text{ s.}$$

The roots are 0.273243 s and 0.746125 s. Since  $t_A = 5/9.81 = 0.509684$  s, the later root is the descending event. At that time,

$$x_B = 4.00t_+ = 2.98450 \text{ m}, \quad x_T = x_{T0} + 1.00t_+.$$

Thus  $x_{T0} = (4.00 - 1.00)t_+ = 2.23837$  m, approximately 2.24 m. The unrounded expression defines the exact constructed interception in Fig. 4.6.

**Velocity at contact.** The bead has

$$\mathbf{v}_f = (4.00\hat{\mathbf{i}} - 2.31948\hat{\mathbf{j}}) \text{ m s}^{-1}, \quad v_f = 4.62 \text{ m s}^{-1}, \quad \theta_v = \text{atan2}(-2.31948, 4.00) = -30.1^\circ.$$

The angle uses  $v_y/v_x$ , with its quadrant, not the reciprocal ratio. The speed is less than the initial  $\sqrt{41}$  m s<sup>-1</sup>, consistent with this event being above launch height.

**A miss despite horizontal agreement.** For a target initially at 2.00 m, the horizontal equality requires  $4t = 2 + t$ , so  $t = 2/3$  s. The bead's height then is 1.65333 m, which is 0.15333 m above the target. Neither height root satisfies the horizontal condition for this initial target position. There is no interception.

**A second impossible case.** A target at height 2.00 m is unreachable for this launch regardless of its horizontal motion: the apex is only 1.77421 m. Its height discriminant is negative. These are different failures—a timing mismatch and an unreachable height—and require different remedies.

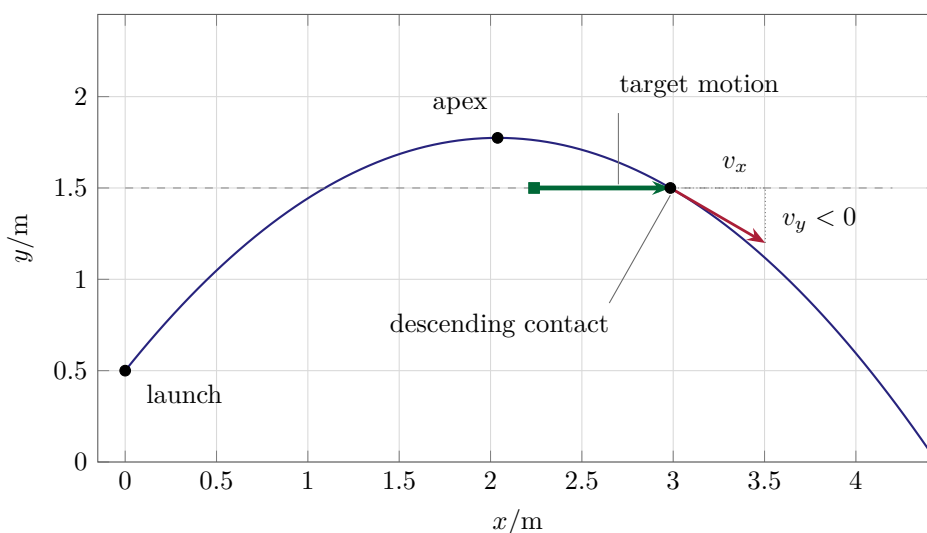


**Figure 4.6:** Analytical projectile path and target displacement during the constructed descending interception. Both axes have the same length scale. The green arrow joins the target's initial and final positions, not the bead's launch and arrival. The bead's red velocity arrow uses scale 0.13 s; the dotted component triangle belongs to that velocity, not to a force. The continued curve after interception shows only the hypothetical no-contact path.

### 4.3.3 When the familiar range formula is valid

For return to  $y_f = y_0$ , Eq. (4.16) has a launch root  $t = 0$  and, when  $v_{0y} > 0$ , a later root  $T_f = 2v_{0y}/g$ . With  $v_{0x} > 0$ , the horizontal range is

$$R = v_{0x}T_f = \frac{2v_0^2 \sin \theta \cos \theta}{g} = \frac{v_0^2 \sin(2\theta)}{g}. \quad (4.17)$$

The argument has used equal heights, fixed launch speed, uniform gravity, negligible drag, a flat landing level and an unobstructed flight. For  $0 < \theta < 90^\circ$ ,  $\sin 2\theta \leq 1$ , so this model's maximum range occurs at  $45^\circ$ . Complementary angles have the same range at the same speed, but different flight times and maximum heights.

Changing the landing height invalidates the derivation, not merely its final numerical coefficient. For a lower landing level, the positive root in Eq. (4.16) must be used and multiplied by  $v_{0x}$ . A claim that  $45^\circ$  is universally optimal ignores the conditions under which the maximum was derived. A launch-speed constraint that itself depends on angle would also create a different optimization problem.

### 4.3.4 Complementary launches: equal range does not mean equal motion

The equality of two ranges compares their final horizontal displacements. It does not say that the projectiles occupy the same place at the same time. The range formula has eliminated time, which is convenient for that one question but hides information needed for tracking, clearance and interception. Recover that information with the component time equations. Figure 4.7 compares two complete flights using one common clock; each curve ends at its own landing event.

#### Example 4.7: Two equal ranges, two different flight histories

Two point-like projectiles leave the origin simultaneously at speed  $16.0 \text{ m s}^{-1}$ , with launch angles  $25.0^\circ$  and  $65.0^\circ$ . They land on the unobstructed horizontal plane  $y = 0$ . Neglect drag and use uniform downward gravity. Find each range, flight time and maximum height. Then find both positions at the shared clock time  $t_c = 0.750 \text{ s}$ , and determine which projectile is ascending.

**Strategy and symbolic reasoning.** Resolve the same speed into two components separately for each launch. The landing condition gives  $T = 2v_0 \sin \theta / g$ , while the apex condition  $v_y = 0$  gives  $H = v_0^2 \sin^2 \theta / (2g)$ . The ranges agree because  $\sin(2 \times 25^\circ) = \sin(2 \times 65^\circ)$ , not because the component velocities agree. At the specified common time, use

$$x_c = v_0 \cos \theta t_c, \quad y_c = v_0 \sin \theta t_c - \frac{1}{2}gt_c^2, \quad v_{yc} = v_0 \sin \theta - gt_c.$$

The numerical results are

| Quantity                       | 25.0° launch | 65.0° launch |
|--------------------------------|--------------|--------------|
| $T$ / s                        | 1.379        | 2.956        |
| $R$ / m                        | 19.99        | 19.99        |
| $H$ / m                        | 2.330        | 10.72        |
| $x(t_c)$ / m                   | 10.88        | 5.071        |
| $y(t_c)$ / m                   | 2.312        | 8.117        |
| $v_y(t_c)$ / $\text{m s}^{-1}$ | −0.596       | +7.14        |

Thus the lower-angle projectile is already descending, although it is still close to its apex; the higher-angle projectile is ascending. Both values of  $t_c$  lie inside their respective flight intervals. Substitution at each individual  $T$  gives  $y(T) = 0$ , and the two  $x(T)$  values agree. The shared range therefore passes its endpoint check while the simultaneous positions are plainly different. An obstacle or a moving target would distinguish these launches even though the level-ground range calculation does not.

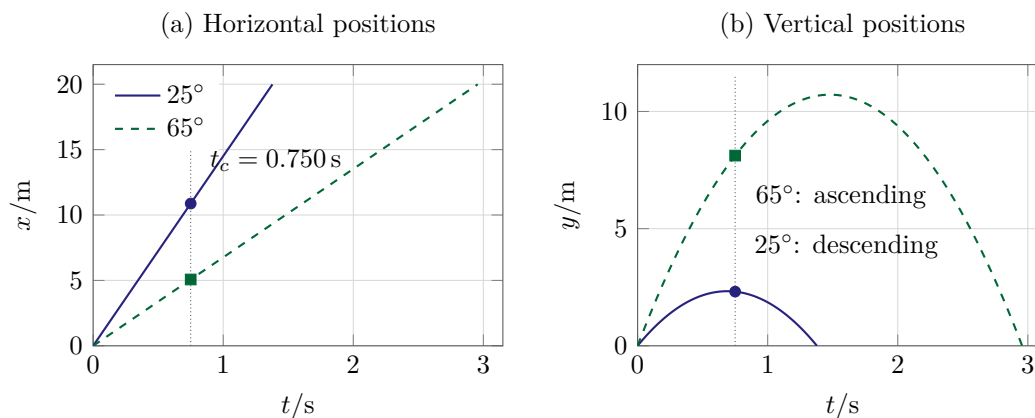


**Figure 4.7:** Equal landing ranges with different time histories. The solid curves and circular markers denote the  $25^\circ$  launch; dashed curves and square markers denote the  $65^\circ$  launch. The dotted lines identify the same clock time in both panels. Horizontal motion has a constant slope; vertical motion has slope  $v_y(t)$ , whose signs at the marked events differ. Both launches have  $v_0 = 16.0 \text{ m s}^{-1}$ , uniform  $g = 9.81 \text{ m s}^{-2}$ , and equal launch and landing heights.

### 4.3.5 Clearance is an event at a specified horizontal coordinate

A fence or an aperture poses a geometrical condition at one horizontal coordinate, rather than at one independently chosen height. If the vertical plane is  $x = L$  and  $v_{0x} > 0$ , the crossing time is fixed:

$$t_L = \frac{L - x_0}{v_{0x}}, \quad y_L = y_0 + v_{0y}t_L - \frac{1}{2}gt_L^2. \quad (4.18)$$

The projectile clears a fence extending up to  $H$  if  $y_L > H$ , provided the flight has not already ended. Solving  $y(t) = H$  alone produces the wrong event: those roots are times of crossing a horizontal height, which need not be times of passing the fence. A finite object would require a clearance margin based on its extent, but the point-particle model below tests the centre trajectory only.

#### Example 4.8: A fence test and the speed needed to pass it

A projectile is released at  $(0, 1.00) \text{ m}$  with speed  $20.0 \text{ m s}^{-1}$  and angle  $35.0^\circ$ . A narrow vertical fence at  $L = 18.0 \text{ m}$  extends from ground level to  $H = 7.00 \text{ m}$ . There are no other obstacles. In the uniform-gravity, no-drag model, find the clearance and velocity when the projectile crosses the fence plane. At this fixed launch angle, find the threshold speed for the point trajectory to touch the fence top.

**Event choice.** The given horizontal location fixes the relevant time. Resolve the launch velocity and apply Eq. (4.18):

$$v_{0x} = 16.3830 \text{ m s}^{-1}, \quad v_{0y} = 11.4715 \text{ m s}^{-1}, \quad t_L = \frac{18.0}{16.3830} \text{ s} = 1.09870 \text{ s}.$$

Consequently,

$$y_L = 1.00 + 11.4715(1.09870) - 4.905(1.09870)^2 = 7.68274 \text{ m}, \quad y_L - H = 0.68274 \text{ m}.$$

The velocity at that event is

$$\mathbf{v}_L = (16.3830\hat{\mathbf{i}} + 0.69331\hat{\mathbf{j}}) \text{ m s}^{-1}, \quad v_L = 16.3977 \text{ m s}^{-1}, \quad \theta_{vL} = 2.42^\circ.$$

The projectile is still ascending:  $t_L = 1.099 \text{ s}$  precedes  $t_A = v_{0y}/g = 1.169 \text{ s}$ . The small positive vertical component is consistent with a crossing just before the apex in Fig. 4.8.

**Inverse speed requirement.** Keep  $\theta$  fixed while varying  $v_0$ . At the fence plane, the trajectory equation gives

$$H = y_0 + L \tan \theta - \frac{gL^2}{2v_{\text{touch}}^2 \cos^2 \theta}, \quad v_{\text{touch}} = \sqrt{\frac{gL^2}{2 \cos^2 \theta (y_0 + L \tan \theta - H)}}.$$

The denominator is positive here:  $y_0 + L \tan 35^\circ - H = 6.60374 \text{ m}$ . Thus  $v_{\text{touch}} = 18.9379 \text{ m s}^{-1}$ . The given speed is larger and therefore yields positive clearance. Exactly the threshold speed produces contact, so strict clearance requires a speed greater than it. If  $y_0 + L \tan \theta \leq H$ , no finite speed at that fixed angle can pass above the fence top: even the zero-curvature limit cannot do so.

**Independent checks.** Substituting the threshold speed gives  $y_L = 7.00 \text{ m}$ . The crossing time is positive and the trajectory is above ground throughout the interval up to that crossing. Eliminating time gives the same clearance as the event calculation. Finally, expanding the velocity components verifies the purely kinematic identity  $v_L^2 = v_0^2 - 2g(y_L - y_0)$ ; no force or energy law was needed to establish it.

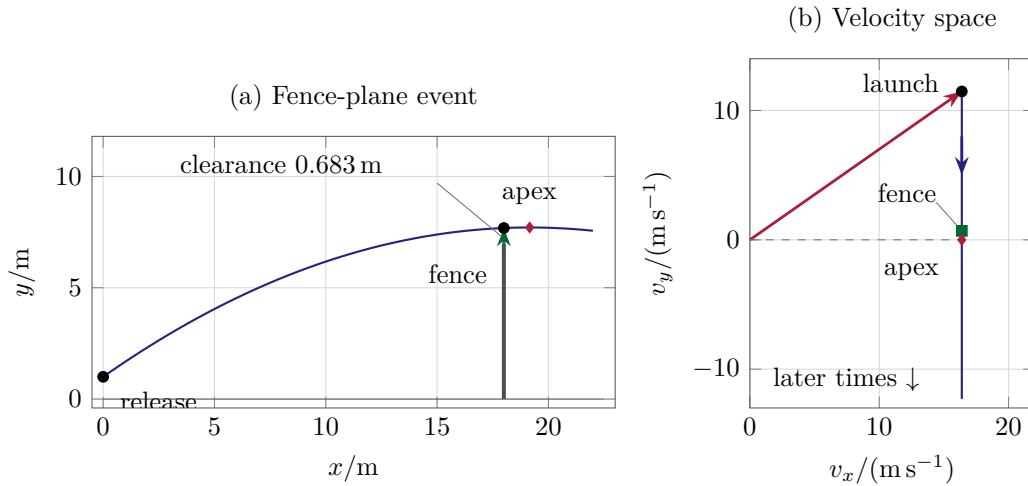


**Figure 4.8:** Two representations of the same clearance test. Panel (a) is physical geometry with equal length scales: the path passes the 7.00 m fence at  $x = 18.0 \text{ m}$  before reaching its apex. Panel (b) is an analytical plot in *velocity* space, with equal component scales. Its vertical line expresses constant  $v_x$  and decreasing  $v_y$ ; an arrow shows the direction of advancing time. Distance from its origin equals speed, so the apex velocity is the closest point on that line to the origin. The launch vector in (b) is a velocity, not a displacement or force.

#### 4.3.6 Maximum range when the landing plane is lower

Let the launch point be  $(0, h)$  above the unobstructed horizontal landing plane  $y = 0$ , with  $h \geq 0$ . The speed  $v_0$  is fixed and the angle can vary over  $0 \leq \theta < 90^\circ$ . The positive landing root gives

$$R(\theta) = \frac{v_0 \cos \theta}{g} \left[ v_0 \sin \theta + \sqrt{v_0^2 \sin^2 \theta + 2gh} \right]. \quad (4.19)$$

The extra time supplied by the initial height changes the balance between horizontal velocity and upward velocity. We can determine the optimum without a long differentiation of this square root.

Put  $u = \tan \theta$  and complete the square in the trajectory equation for a specified  $x > 0$ :

$$\begin{aligned} y(x; u) &= h + xu - \frac{gx^2}{2v_0^2}(1 + u^2) \\ &= h + \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2} - \frac{gx^2}{2v_0^2} \left( u - \frac{v_0^2}{gx} \right)^2. \end{aligned} \quad (4.20)$$

At each  $x$ , the final term is nonpositive. The first three terms give an upper bound on the height that any member of this fixed-speed launch family can reach there. Its zero determines the farthest reachable point of the landing plane. At that point the bound is attained by making the squared term zero:

$$R_{\max} = \frac{v_0}{g} \sqrt{v_0^2 + 2gh}, \quad \tan \theta_{\text{opt}} = \frac{v_0}{\sqrt{v_0^2 + 2gh}}. \quad (4.21)$$

For  $h = 0$  these reduce to  $R_{\max} = v_0^2/g$  and  $\theta_{\text{opt}} = 45^\circ$ . For  $h > 0$  the optimum angle is smaller. The bound compares different trajectories; it is not itself one projectile's path. This result still requires an unobstructed flight and an angle-independent launch speed. A hillside, drag, or a launcher whose attainable speed changes with angle creates a different optimization problem.

#### Example 4.9: An elevated launcher: test the customary angle

A point projectile is launched at fixed speed  $16.0 \text{ m s}^{-1}$  from  $h = 8.00 \text{ m}$  above a horizontal landing plane. Uniform gravity acts and drag is negligible. Find the angle and range for maximum horizontal displacement, then compare the result with a  $45.0^\circ$  launch.

**Strategy.** Use the lower-plane event, not the equal-height range formula. Equation (4.21) gives

$$\theta_{\text{opt}} = \tan^{-1} \left( \frac{16.0}{\sqrt{16.0^2 + 2(9.81)(8.00)}} \right) = 38.215^\circ, \quad R_{\max} = 33.1440 \text{ m}.$$

At that angle, Eq. (4.19) gives a landing time  $T = 2.63652 \text{ s}$ . Evaluating the same equation at  $45.0^\circ$  gives  $R_{45} = 32.5162 \text{ m}$ , about  $0.628 \text{ m}$  less. The equal-height formula would instead give  $16.0^2/9.81 = 26.096 \text{ m}$ ; that number answers the event of returning to launch height, not the landing event  $8.00 \text{ m}$  below it.

**Checks and interpretation.** At  $x = R_{\max}$ , the upper-bound height in Eq. (4.20) is zero and its squared term vanishes. Substitution into the time equations gives  $y(T) = 0$  and  $x(T) = 33.1440 \text{ m}$ . The range curve in Fig. 4.9 confirms a maximum below  $45^\circ$ . Setting  $h = 0$  restores the familiar result continuously. The gain from choosing  $38.2^\circ$  rather than  $45^\circ$  is modest for these parameters, but the change of event is essential even when the numerical optimization gain is small.

### 4.3.7 Designing a launch to meet a moving target

For a moving target, the desired arrival point is itself a function of arrival time. This makes inverse kinematics useful. Let a launcher be at the origin and a target move on a prescribed horizontal track:

$$\mathbf{r}_T(t) = (L + Ut)\hat{\mathbf{i}} + Y\hat{\mathbf{j}}.$$

The target's track is externally specified; it is not a second freely falling projectile. If contact is to occur at a chosen positive time  $t$ , the required launch components are fixed by the two simultaneous position equations:

$$v_{0x}(t) = U + \frac{L}{t}, \quad v_{0y}(t) = \frac{Y}{t} + \frac{gt}{2}, \quad S(t) = \sqrt{\left(U + \frac{L}{t}\right)^2 + \left(\frac{Y}{t} + \frac{gt}{2}\right)^2}. \quad (4.22)$$

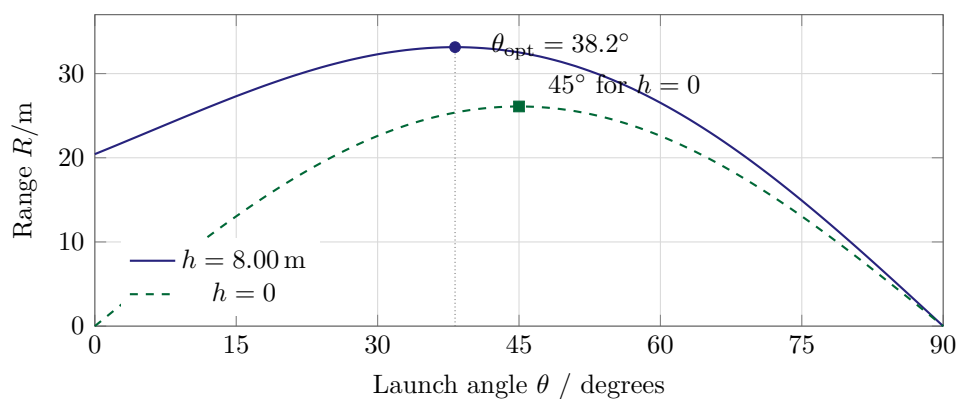


**Figure 4.9:** Range versus angle for an angle-independent launch speed  $v_0 = 16.0 \text{ m s}^{-1}$  and uniform gravity without drag. Solid: landing 8.00 m below launch. Dashed: equal launch and landing heights. The  $h > 0$  curve has nonzero range for a horizontal launch and reaches its maximum below  $45^\circ$ . At  $90^\circ$ , both launches are vertical and the horizontal range is zero. These curves describe two different landing events.

Here  $S(t)$  is the required *initial* speed for the selected time, not the projectile's speed at contact. With an unrestricted launcher, every positive time would give a kinematic candidate. With a fixed launch speed  $v_0$ , only the roots of  $S(t) = v_0$  remain. They must still be tested against any earlier ground contact and against the actual range of angles the launcher can provide.

#### Example 4.10: Two fixed-speed launches that intercept one moving target

At  $t = 0$ , a point target is at  $(15.0, 2.00) \text{ m}$  and then moves to the right at constant speed  $2.00 \text{ m s}^{-1}$ , holding its height. A point projectile is launched simultaneously from the origin with fixed speed  $20.0 \text{ m s}^{-1}$ . Its angle is selectable. Use uniform gravity, neglect drag, and assume no obstacle before contact. Find all positive interception times and launch angles; check that each event precedes return to ground.

**Strategy.** Solve for the required velocity at a candidate time, then impose the available speed. Squaring Eq. (4.22) gives the scalar equation

$$q(t) \equiv S(t)^2 = U^2 + gY + \frac{2UL}{t} + \frac{L^2 + Y^2}{t^2} + \frac{g^2 t^2}{4} = 400 \text{ m}^2 \text{ s}^{-2}.$$

Each term has the units of speed squared. With times entered in seconds and the given SI numerical parameters, the equation is

$$q(t) = 23.62 + \frac{60}{t} + \frac{229}{t^2} + 24.059025 t^2 = 400.$$

The function diverges for  $t \rightarrow 0^+$ , because covering the initial separation in vanishing time requires an unbounded velocity. It also diverges for large  $t$ , because the initial upward velocity needed to delay arrival grows with  $gt/2$ . These competing requirements create a minimum. In this case

$$q''(t) = \frac{4UL}{t^3} + \frac{6(L^2 + Y^2)}{t^4} + \frac{g^2}{2} > 0 \quad (t > 0),$$

so  $q$  is strictly convex and has only one minimum. Since  $q(2.00 \text{ s}) = 207.106 \text{ m}^2 \text{ s}^{-2} < 400 \text{ m}^2 \text{ s}^{-2}$ , there are exactly two positive roots. This establishes the number of solutions before using a numerical solver.

**Numerical evaluation.** The values  $S(0.800 \text{ s}) = 21.722 \text{ m s}^{-1}$  and  $S(0.900 \text{ s}) = 19.811 \text{ m s}^{-1}$  bracket the short-time root. Similarly,  $S(3.50 \text{ s}) = 18.820 \text{ m s}^{-1}$  and  $S(3.80 \text{ s}) = 20.067 \text{ m s}^{-1}$  bracket the long-time root. Bisection of these intervals yields

| Quantity                             | Short-time launch | Long-time launch |
|--------------------------------------|-------------------|------------------|
| Contact time / s                     | 0.888827          | 3.784223         |
| $v_{0x}$ / $\text{m s}^{-1}$         | 18.8762           | 5.96383          |
| $v_{0y}$ / $\text{m s}^{-1}$         | 6.60985           | 19.0901          |
| Launch angle                         | $19.30^\circ$     | $72.65^\circ$    |
| Contact $x$ / m                      | 16.7777           | 22.5684          |
| $v_y$ at contact / $\text{m s}^{-1}$ | -2.10954          | -18.0331         |
| Return-to-ground time / s            | 1.34757           | 3.89197          |

For each root,  $\theta = \text{atan2}(v_{0y}, v_{0x})$ , and  $v_{0x}^2 + v_{0y}^2 = 400 \text{ m}^2 \text{ s}^{-2}$ . Substitution gives  $x_P(t) = 15.0 + 2.00t = x_T(t)$  and  $y_P(t) = 2.00 \text{ m} = y_T(t)$ . Both contacts are descending events and occur before that launch's ground return. Thus both are physically available within the stipulated model.

**Interpretation and boundary.** The longer flight requires a much larger upward component, while allowing a smaller horizontal component even though the target has travelled farther. Figure 4.10 displays the two admissible times as intersections with the available-speed line. The roots would need a new calculation if the target accelerated, changed height, or became unavailable before one of these times. The derivation supplies required launch velocities; it does not establish that a real launcher can produce both angles.

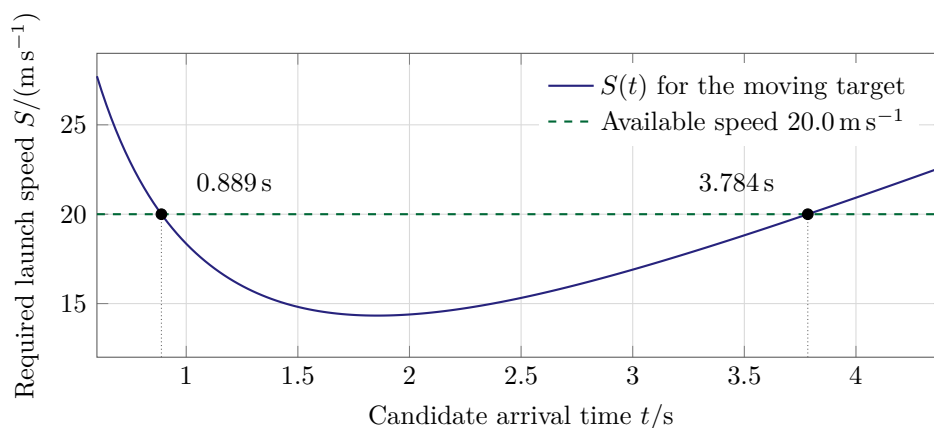


**Figure 4.10:** An inverse interception calculation. Every point on the solid curve is the *initial* speed required to meet the same target at its indicated arrival time, with  $L = 15.0 \text{ m}$ ,  $Y = 2.00 \text{ m}$ ,  $U = 2.00 \text{ m s}^{-1}$ , and uniform downward gravity. The dashed line is the fixed available launch speed, so its two intersections identify the only two positive candidate times. This is not a graph of the speed of one projectile throughout its flight. Both candidates pass the independent ground-contact and same-time coordinate checks in the example.

### Method

**Projectile event method.** Draw the datum and both coordinate directions. Record the launch components, not only the speed. Write the two time equations. Translate “lands,” “clears,” or “intercepts” into coordinate conditions. Solve for candidate times, reject negative or unavailable events, and substitute into *both* position equations. Finally compute velocity at the selected event and check the direction independently from the trajectory.

A circular path presents the complementary lesson to a projectile: velocity can change even when its magnitude stays exactly constant. The next section derives the required acceleration directly from geometry.

## 4.4 Circular motion

### 4.4.1 A rotating velocity at constant speed

Consider a point moving around a circle of fixed radius  $R > 0$  at constant speed  $v$ . Over a short interval, its radius turns through an angle  $\Delta\theta$ . The two velocity vectors, being tangent to the same circle, turn through the *same* angle. Their magnitudes remain equal, so they form an isosceles triangle when their tails are placed together.

Let  $\Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$  be the position chord and  $\Delta\mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1$  the velocity difference. The two triangles in Fig. 4.11 have equal included angles and equal-leg ratios, giving

$$\frac{|\Delta\mathbf{v}|}{v} = \frac{|\Delta\mathbf{r}|}{R} = 2 \left| \sin \frac{\Delta\theta}{2} \right|. \quad (4.23)$$

Divide by the elapsed time and take the limit. Since the chord speed approaches the instantaneous speed,

$$|\mathbf{a}| = \lim_{\Delta t \rightarrow 0} \frac{|\Delta\mathbf{v}|}{\Delta t} = \frac{v}{R} \lim_{\Delta t \rightarrow 0} \frac{|\Delta\mathbf{r}|}{\Delta t} = \frac{v^2}{R}. \quad (4.24)$$

The velocity-difference arrow points toward the inward radius at the midpoint angle. As the interval shrinks, it approaches the inward radius at the event itself. Thus the acceleration is inward and perpendicular to the instantaneous velocity. The word *centripetal* names this direction; it does not name an additional force.

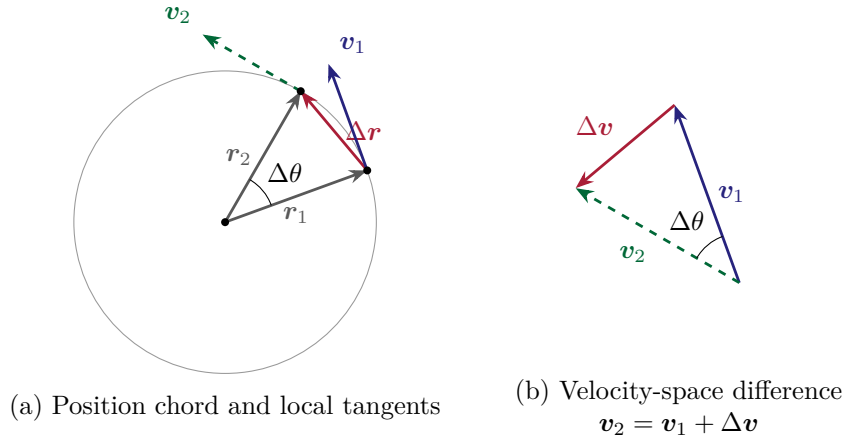


**Figure 4.11:** An original equal-angle construction with  $\Delta\theta = 40^\circ$ , enlarged to make the geometry readable. The circle radii and the two velocity magnitudes are each equal within their own scale. Position and velocity panels have different units and scales. The velocity-difference arrow joins the tip of  $\mathbf{v}_1$  to the tip of  $\mathbf{v}_2$ ; it is not their vector sum. The limiting construction, rather than the finite drawing alone, establishes acceleration at one event.

### 4.4.2 Angular rate and a Cartesian check

For a circle centred at the origin, write

$$\mathbf{r}(t) = R \cos(\theta(t)) \hat{\mathbf{i}} + R \sin(\theta(t)) \hat{\mathbf{j}}, \quad \theta(t) = \theta_0 + \omega t. \quad (4.25)$$

Positive  $\omega$  denotes counterclockwise motion viewed from  $+z$ ; negative  $\omega$  denotes clockwise motion. Differentiating with the chain rule,

$$\mathbf{v} = R\omega(-\sin \theta \hat{\mathbf{i}} + \cos \theta \hat{\mathbf{j}}), \quad (4.26)$$

$$\mathbf{a} = -R\omega^2(\cos \theta \hat{\mathbf{i}} + \sin \theta \hat{\mathbf{j}}) = -\omega^2 \mathbf{r}. \quad (4.27)$$

The squared angular rate makes the acceleration inward for either sense of rotation. The velocity changes direction when  $\omega$  changes sign at the same position; the acceleration does not. With  $\omega \neq 0$ ,

$$v = R|\omega|, \quad a_n = R\omega^2 = \frac{v^2}{R}, \quad T = \frac{2\pi}{|\omega|} = \frac{2\pi R}{v}. \quad (4.28)$$

The period  $T$  is positive. Radians arise from the ratio of arc length to radius,  $\theta = s/R$ ; inserting a numerical degree rate into the derivative formulas without conversion produces an incorrect factor.

#### Example 4.11: A marker on a rotating inspection disk

A marker is  $R = 0.300$  m from the centre of an ideal horizontal inspection disk. Its angular rate is  $\omega = 4.00$  rad s<sup>-1</sup>, counterclockwise, and it begins on the positive  $x$  axis. Find its velocity and acceleration at  $t = \pi/12$  s, and compare with the events at  $\theta = 0$  and  $\theta = 120^\circ$ .

**Strategy.** Keep the origin at the disk centre in a nonrotating laboratory frame. The event angle is  $4(\pi/12) = \pi/3 = 60^\circ$ . Equations (4.26)–(4.27) give

$$\begin{aligned} \mathbf{r} &= (0.150, 0.259808) \text{ m}, \\ \mathbf{v} &= (-1.03923, 0.600) \text{ m s}^{-1}, \\ \mathbf{a} &= (-2.400, -4.15692) \text{ m s}^{-2}. \end{aligned}$$

The speed is  $1.20$  m s<sup>-1</sup>, the acceleration magnitude  $4.80$  m s<sup>-2</sup>, and the period  $\pi/2 = 1.57$  s.

**Other events.** At  $\theta = 0$ ,  $\mathbf{v} = (0, 1.20)$  m s<sup>-1</sup> and  $\mathbf{a} = (-4.80, 0)$  m s<sup>-2</sup>. At  $120^\circ$ , they are  $(-1.03923, -0.600)$  m s<sup>-1</sup> and  $(2.400, -4.15692)$  m s<sup>-2</sup>. Neither vector is constant despite their constant magnitudes.

**Checks.** At every event,  $\mathbf{r} \cdot \mathbf{v} = 0$ ,  $\mathbf{v} \cdot \mathbf{a} = 0$ , and  $\mathbf{a} = -16 \text{ s}^{-2} \mathbf{r}$ . Reversing the rotation at a fixed angle reverses velocity but leaves acceleration unchanged. In the limit  $\omega \rightarrow 0$ , both velocity and acceleration vanish; the period diverges rather than becoming zero.

The same differentiation also shows why a marker is not launched outward along a radius when circular guidance ends. Its instantaneous velocity is tangent. Its subsequent path depends on its subsequent acceleration, not on an assumed outward continuation of its old radius. We will use Newton's laws later to decide what accelerations remain after a constraint changes.

#### Guided checks

**Is  $\mathbf{a} = -\omega^2 \mathbf{r}$  origin-independent?** Only when  $\mathbf{r}$  is measured from the fixed circle centre. If the centre is at  $\mathbf{r}_C$ , the same stationary-centre model gives  $\mathbf{a} = -\omega^2(\mathbf{r} - \mathbf{r}_C)$ .

**Does doubling speed double inward acceleration?** At fixed radius it multiplies acceleration by four. At fixed angular rate, doubling the radius doubles both speed and acceleration. State which input is held fixed before using a scaling rule.

## 4.5 Tangential and normal acceleration

### 4.5.1 Separating speed change from turning

Uniform circular motion isolates turning, but most paths include both turning and a change of speed. For a smooth path at an event where  $v > 0$ , let  $\hat{\mathbf{t}} = \mathbf{v}/v$  be the unit tangent in the direction of motion. Then  $\mathbf{v} = v\hat{\mathbf{t}}$ , so the product rule gives

$$\mathbf{a} = \frac{dv}{dt}\hat{\mathbf{t}} + v\frac{d\hat{\mathbf{t}}}{dt}. \quad (4.29)$$

The first term changes the length of the velocity vector. The second changes its direction. Because  $\hat{\mathbf{t}} \cdot \hat{\mathbf{t}} = 1$ , differentiation implies  $\hat{\mathbf{t}} \cdot d\hat{\mathbf{t}}/dt = 0$ ; hence the second term is perpendicular to the tangent.

To quantify the turning, imagine following the path through a small arc length  $ds$ . Its tangent turns by  $d\psi$ . Define the local curvature  $\kappa = |d\hat{\mathbf{t}}/ds|$ , equal to  $|d\psi/ds|$  for a plane curve. Where  $\kappa > 0$ , define  $R_c = 1/\kappa$  and let  $\hat{\mathbf{n}}$  point in the direction of  $d\hat{\mathbf{t}}/ds$ . The nearby circle sharing the path's tangent and curvature has radius  $R_c$ . It is a local construction, not necessarily a physical circular rail.

Since  $ds/dt = v$ ,

$$\frac{d\hat{\mathbf{t}}}{dt} = \frac{d\hat{\mathbf{t}}}{ds} \frac{ds}{dt} = \frac{v}{R_c} \hat{\mathbf{n}}.$$

Substitution in Eq. (4.29) yields

$$\boxed{\mathbf{a} = a_t \hat{\mathbf{t}} + a_n \hat{\mathbf{n}}, \quad a_t = \frac{dv}{dt}, \quad a_n = \frac{v^2}{R_c}.} \quad (4.30)$$

The tangential component  $a_t$  is signed. It is negative when speed decreases, so  $a_t \hat{\mathbf{t}}$  then points opposite to the motion. The principal-normal component  $a_n$  is non-negative by our choice of  $\hat{\mathbf{n}}$ . Since the components are orthogonal,

$$|\mathbf{a}| = \sqrt{a_t^2 + a_n^2}. \quad (4.31)$$

Adding their magnitudes directly would generally overestimate the total.

At a straight segment,  $d\hat{\mathbf{t}}/ds = 0$  and  $a_n = 0$ , formally the limit  $R_c \rightarrow \infty$ . There is then no unique principal-normal direction, but the normal contribution is unambiguously zero. At a stop,  $v = 0$ , the direction  $\mathbf{v}/v$  is not defined. One may sometimes extend a known geometric tangent by continuity, but the safe general calculation is Cartesian differentiation without division by speed.

#### Example 4.12: Variable-speed passage through a bend

A point-like carriage follows a horizontal circular bend of radius 5.00 m. Over  $0 \leq t \leq 5.00$  s, its speed increases according to  $v(t) = 1.00 \text{ m s}^{-1} + (0.400 \text{ m s}^{-2})t$ . Find the acceleration at five seconds. Choose the fixed axes so the carriage is at the rightmost point of the circle at that event, moving upward on the page.

**Local calculation.** At five seconds,  $v = 3.00 \text{ m s}^{-1}$ , the forward tangent is  $\hat{\mathbf{t}} = \hat{\mathbf{j}}$ , and the inward normal is  $\hat{\mathbf{n}} = -\hat{\mathbf{i}}$ . Hence

$$a_t = 0.400 \text{ m s}^{-2}, \quad a_n = \frac{(3.00 \text{ m s}^{-1})^2}{5.00 \text{ m}} = 1.80 \text{ m s}^{-2},$$

$$\mathbf{a} = (-1.80\hat{\mathbf{i}} + 0.400\hat{\mathbf{j}}) \text{ m s}^{-2}, \quad |\mathbf{a}| = 1.84 \text{ m s}^{-2}.$$

The angle from the forward tangent toward the inward normal is  $\tan^{-1}(1.80/0.400) = 77.5^\circ$ , or  $12.5^\circ$  forward of the inward direction.

**Independent Cartesian check.** The travelled arc length is  $s(t) = (1.00 \text{ m s}^{-1})t + (0.200 \text{ m s}^{-2})t^2$ . An angle consistent with the chosen event axes is

$$\theta(t) = -2.00 + \frac{s(t)}{5.00 \text{ m}}, \quad \mathbf{r} = 5.00 \text{ m}(\cos \theta, \sin \theta).$$

At five seconds  $\theta = 0$ ,  $\dot{\theta} = v/R = 0.600 \text{ s}^{-1}$ , and  $\ddot{\theta} = 0.0800 \text{ s}^{-2}$ . Differentiating  $x$  and  $y$  twice gives  $a_x = -R\dot{\theta}^2 = -1.80 \text{ m s}^{-2}$  and  $a_y = R\ddot{\theta} = 0.400 \text{ m s}^{-2}$ , agreeing with the local decomposition.

**Limits.** Keeping the same speed while setting its rate of change to zero removes only  $a_t$ . Letting the bend radius become very large removes only  $a_n$ . Neither operation is equivalent to setting the entire acceleration to zero unless the other contribution also vanishes.

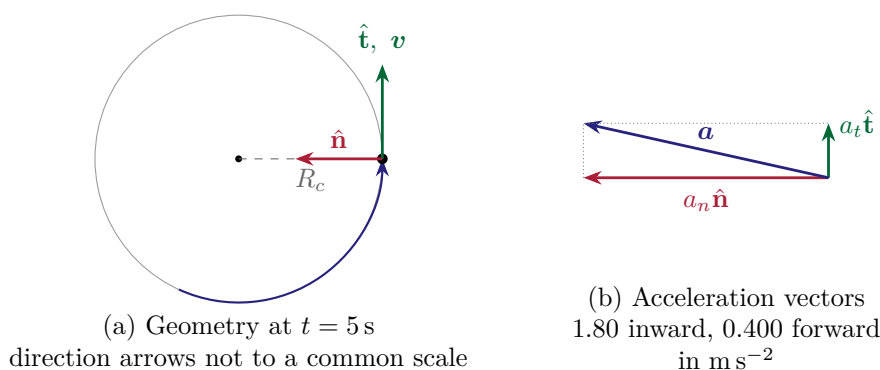


**Figure 4.12:** The bend and the acceleration triangle for the same event. The arc indicates the final 2.00 rad of the prescribed motion. In panel (b) all acceleration arrows use the same scale, so their vector sum is geometrically quantitative. No force arrows are shown.

### 4.5.2 Extracting the local components from measured vectors

A path's radius of curvature need not be supplied in advance. Given  $\mathbf{v}$  and  $\mathbf{a}$  at an event with  $v > 0$ , their tangential and normal parts follow from projection:

$$a_t = \mathbf{a} \cdot \hat{\mathbf{t}} = \frac{\mathbf{v} \cdot \mathbf{a}}{v},$$

$$\mathbf{a}_\perp = \mathbf{a} - a_t \hat{\mathbf{t}}, \quad a_n = |\mathbf{a}_\perp|, \quad R_c = \frac{v^2}{a_n} \quad (a_n > 0). \quad (4.32)$$

This construction works in space as well as in a plane. In three dimensions the direction of the principal normal may itself change along the path; a full description of that change requires further geometry, not a change to the acceleration decomposition.

#### Example 4.13: Revisiting the inspection probe's turn

At two seconds the probe of Example 4.1 has  $\mathbf{v} = (0.60, 0.80, 0.60) \text{ m s}^{-1}$  and  $\mathbf{a} = (0, 0.40, 0.60) \text{ m s}^{-2}$ . Find its speed-change rate and local radius of curvature.

**Projection.** Since  $v = \sqrt{1.36} \text{ m s}^{-1}$  and  $\mathbf{v} \cdot \mathbf{a} = 0.68 \text{ m}^2 \text{ s}^{-3}$ ,

$$a_t = \frac{0.68}{\sqrt{1.36}} \text{ m s}^{-2} = 0.5831 \text{ m s}^{-2}.$$

Here  $a_t \hat{\mathbf{t}} = \frac{1}{2} \text{ s}^{-1} \mathbf{v} = (0.30, 0.40, 0.30) \text{ m s}^{-2}$ , so

$$\mathbf{a}_\perp = (-0.30, 0, 0.30) \text{ m s}^{-2}, \quad a_n = \sqrt{0.18} \text{ m s}^{-2} = 0.4243 \text{ m s}^{-2}.$$

Thus  $R_c = 1.36 / \sqrt{0.18} \text{ m} = 3.21 \text{ m}$ .

**Interpretation and check.** The probe accelerates partly along its velocity and partly toward a local centre of curvature. That normal points along  $(-1, 0, 1)$ , not toward the laboratory origin. The dot product  $\mathbf{v} \cdot \mathbf{a}_\perp = 0$  and the identity  $0.5831^2 + 0.4243^2 \simeq 0.7211^2$  check the decomposition independently.

#### Key message

A change of speed and a change of direction are different parts of acceleration. The total acceleration, the normal component, and the signed tangential component are not interchangeable numbers. A local normal is tied to the path; a radial direction is tied to a chosen origin.

### 4.5.3 An engineering constraint expressed kinematically

Suppose a prescribed inspection motion must remain below a stipulated acceleration magnitude  $a_{\max}$ . For a known bend radius and tangential schedule, Eq. (4.31) requires

$$a_t^2 + \frac{v^4}{R_c^2} \leq a_{\max}^2. \quad (4.33)$$

If  $|a_t| < a_{\max}$ , the resulting speed bound is

$$v \leq \sqrt{R_c \sqrt{a_{\max}^2 - a_t^2}}. \quad (4.34)$$

For  $R_c = 5.00 \text{ m}$ ,  $a_t = 0.400 \text{ ms}^{-2}$  and an illustrative limit  $a_{\max} = 2.00 \text{ ms}^{-2}$ , this gives  $v \leq 3.13 \text{ ms}^{-1}$ . If  $|a_t| > a_{\max}$ , no speed can satisfy the prescribed constraint at that event. This is a consistency test for a requested trajectory, not a certified mechanical operating limit: actual force capacity, structural response and control accuracy have not been specified.

### 4.5.4 What a finite velocity difference actually measures

The equal-angle construction is also useful for assessing a finite measurement interval. Consider uniform counterclockwise circular motion, and choose two events symmetrically around a midpoint event: their angles are  $\theta_m - \delta/2$  and  $\theta_m + \delta/2$ , with  $\delta = \omega \Delta t > 0$ . Subtraction of the two Cartesian velocity vectors gives the exact finite difference

$$\frac{\Delta \mathbf{v}}{\Delta t} = -\frac{2v \sin(\delta/2)}{\Delta t} \hat{\mathbf{e}}_r(\theta_m) = -\frac{v^2 \sin(\delta/2)}{R \delta/2} \hat{\mathbf{e}}_r(\theta_m). \quad (4.35)$$

This is *average acceleration over the interval*. Its direction is precisely inward at the midpoint, and its magnitude is smaller than the instantaneous value there for  $0 < \delta < 2\pi$ . It need not point inward at the *first* event. Dividing a large velocity chord by time and assigning it to the first sample introduces both a magnitude error and a time-assignment error.

The small-angle expansion gives

$$\frac{\sin(\delta/2)}{\delta/2} = 1 - \frac{\delta^2}{24} + \mathcal{O}(\delta^4).$$

Thus halving a sufficiently small angular interval reduces the leading magnitude deficit by a factor of four. For  $\delta = 60^\circ = \pi/3$ , the ratio is  $3/\pi = 0.95493$ : even exact noiseless velocities give an average acceleration 4.51% below the instantaneous magnitude. For  $30^\circ$ , the ratio is 0.98862, a deficit of 1.14%. These are sampling effects within an exact model, not evidence that the circular acceleration law has failed.

### 4.5.5 A complete speed schedule on a circle

A speed schedule and a path shape together determine the acceleration history. For a fixed circular path of radius  $R$ , an increasing speed makes  $a_n = v^2/R$  grow quadratically even when the signed tangential acceleration is constant. A decreasing speed makes the normal contribution fall while the tangential vector continues to point opposite to motion. The two components are orthogonal at every event, but their directions rotate with the carriage.

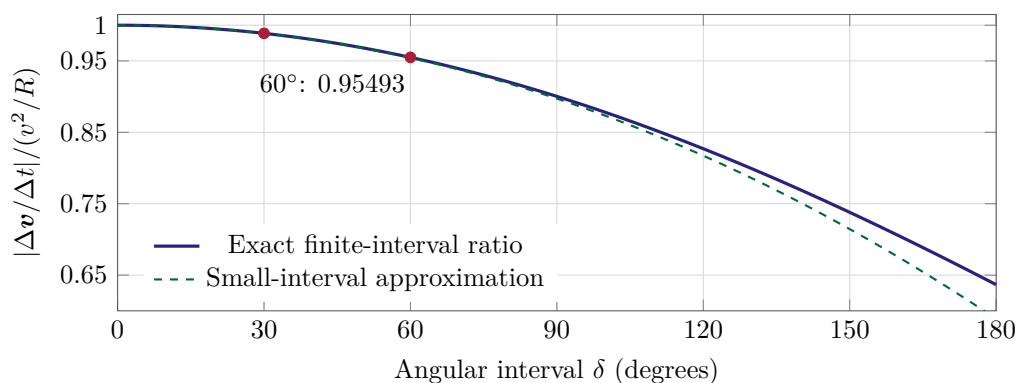


**Figure 4.13:** The exact finite-difference acceleration of uniform circular motion, assigned to the interval midpoint. The blue curve uses embedded values of the exact ratio, evaluated after converting degrees to radians. The green approximation is useful near zero and is visibly less accurate for large intervals. The analytic limit at zero is one and is included explicitly. This avoids division by zero and small-angle rounding in the drawing engine.

#### Example 4.14: A carriage slowing through a circular inspection path

A point carriage follows a fixed circle of radius  $R = 8.00$  m, counterclockwise from  $(R, 0)$ . Its speed is prescribed as

$$v(t) = 4.00 \text{ m s}^{-1} - (0.500 \text{ m s}^{-2})t, \quad 0 \leq t \leq 6.00 \text{ s}.$$

Find its acceleration history, and determine its position, velocity and acceleration at  $t = 4.00$  s. The speed remains positive on the stated interval, so the tangent always follows the counterclockwise direction.

**Integrate the angular rate.** Since  $\dot{\theta} = v/R$ , with  $\theta(0) = 0$ ,

$$\theta(t) = \frac{(4.00 \text{ m s}^{-1})t - (0.250 \text{ m s}^{-2})t^2}{8.00 \text{ m}}.$$

The local acceleration components are

$$a_t = -0.500 \text{ m s}^{-2}, \quad a_n(t) = \frac{v(t)^2}{8.00 \text{ m}}, \quad |\mathbf{a}(t)| = \sqrt{(0.500 \text{ m s}^{-2})^2 + a_n(t)^2}.$$

At  $t = 4.00$  s,  $v = 2.00 \text{ m s}^{-1}$  and  $\theta = 1.500$  rad. With  $\hat{\mathbf{t}} = (-\sin \theta, \cos \theta)$  and  $\hat{\mathbf{n}} = (-\cos \theta, -\sin \theta)$ ,

$$\begin{aligned} \mathbf{r} &= (0.56590, 7.97996) \text{ m}, \\ \mathbf{v} &= (-1.99499, 0.14147) \text{ m s}^{-1}, \\ \mathbf{a} &= -0.500\hat{\mathbf{t}} + 0.500\hat{\mathbf{n}} \quad \text{in m s}^{-2} \\ &= (0.46338, -0.53412) \text{ m s}^{-2}, \quad |\mathbf{a}| = 0.70711 \text{ m s}^{-2}. \end{aligned}$$

**Check by differentiating the prescribed path.** For  $\mathbf{r} = R\hat{\mathbf{e}}_r$ , direct differentiation gives  $\mathbf{a} = R\ddot{\theta}\hat{\mathbf{e}}_\theta - R\dot{\theta}^2\hat{\mathbf{e}}_r$ . Here  $R\dot{\theta} = -0.500 \text{ m s}^{-2}$  and  $R\dot{\theta}^2 = 0.500 \text{ m s}^{-2}$  at four seconds. Also  $\mathbf{v} \cdot \mathbf{a} = va_t = -1.00 \text{ m}^2 \text{ s}^{-3}$ , confirming that speed is decreasing.

**Interpret the history.** Initially  $a_n = 2.00 \text{ m s}^{-2}$  and  $|\mathbf{a}| = 2.06155 \text{ m s}^{-2}$ ; at six seconds they are  $0.125 \text{ m s}^{-2}$  and  $0.51539 \text{ m s}^{-2}$ . At four seconds the equal component magnitudes make acceleration point  $45^\circ$  from the inward normal toward the backward tangent. It never becomes purely tangential on this interval. Extrapolating the formula beyond its prescribed interval is unnecessary; in particular, a negative value of  $v(t)$  cannot be retained as a negative *speed*.

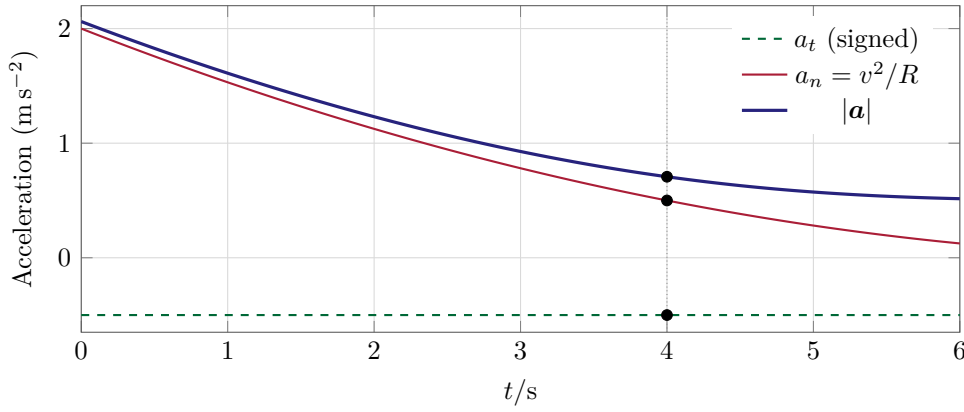


**Figure 4.14:** Acceleration history of the decelerating carriage. The signed tangential component is negative, whereas the principal-normal component and total magnitude are non-negative. The constant rate of speed decrease does not imply constant total acceleration. The marked four-second event has equal normal and tangential magnitudes.

#### 4.5.6 From local curvature to a feasible maneuver

The projection method also yields a compact invariant formula for curvature. For nonzero  $\mathbf{v}$ , the part of  $\mathbf{a}$  perpendicular to  $\mathbf{v}$  satisfies

$$|\mathbf{v} \times \mathbf{a}| = v a_n.$$

Hence, where the cross product is nonzero,

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{v^3}, \quad R_c = \frac{v^3}{|\mathbf{v} \times \mathbf{a}|}. \quad (4.36)$$

In a plane,  $|\mathbf{v} \times \mathbf{a}| = |v_x a_y - v_y a_x|$ . The sign of the determinant can identify counterclockwise or clockwise turning relative to the chosen  $+z$  direction; taking its absolute value yields the positive curvature. A vanishing cross product with  $v > 0$  implies zero normal acceleration at that event. It does not imply zero tangential acceleration.

#### Example 4.15: Infer a bend and revise its acceleration request

A planar inspection carriage passes a local point chosen as  $(0,0)$ , with laboratory vectors

$$\mathbf{v} = (3.00, 4.00) \text{ m s}^{-1}, \quad \mathbf{a} = (-1.00, 2.00) \text{ m s}^{-2}.$$

Infer its instantaneous curvature circle. Then assess an illustrative trajectory requirement  $|\mathbf{a}| \leq 2.00 \text{ m s}^{-2}$ , assuming the same tangential acceleration and bend radius must be retained. If it fails, calculate a permitted speed or the radius needed at the original speed.

**Resolve along the local directions.** Here  $v = 5.00 \text{ m s}^{-1}$ ,  $\hat{\mathbf{t}} = (0.600, 0.800)$ , and

$$a_t = \frac{3(-1) + 4(2)}{5} = 1.00 \text{ m s}^{-2}, \quad \mathbf{a}_\perp = (-1.60, 1.20) \text{ m s}^{-2}.$$

Thus  $a_n = 2.00 \text{ m s}^{-2}$ ,  $\hat{\mathbf{n}} = (-0.800, 0.600)$ , and  $R_c = 25/2 = 12.5 \text{ m}$ . The osculating circle has its centre at  $R_c \hat{\mathbf{n}} = (-10.0, 7.50) \text{ m}$  relative to this event. This circle describes local curvature; it does not prove that the entire trajectory is circular.

**Independent curvature check.** The determinant is  $3(2) - 4(-1) = 10.0 \text{ m}^2 \text{ s}^{-3}$ , so Eq. (4.36) gives  $R_c = 125/10 = 12.5 \text{ m}$ . Its positive sign agrees with the normal lying to the left of the direction of motion.

**Apply the magnitude constraint.** The current acceleration is  $\sqrt{1^2 + 2^2} = \sqrt{5} = 2.23607 \text{ m s}^{-2}$ , exceeding the stipulated limit. Retaining  $R_c = 12.5 \text{ m}$  and  $a_t = 1.00 \text{ m s}^{-2}$ , the

largest permitted speed is

$$v_{\max} = \sqrt{(12.5 \text{ m}) \sqrt{(2.00)^2 - (1.00)^2} \text{ m s}^{-2}} = 4.65302 \text{ m s}^{-1}.$$

Alternatively, retaining  $v = 5.00 \text{ m s}^{-1}$  requires

$$R_c \geq \frac{v^2}{\sqrt{a_{\max}^2 - a_t^2}} = 14.4338 \text{ m}.$$

**Checks and interpretation.** The vectors  $(0.600, 0.800)$  and  $(-0.800, 0.600)$  are orthonormal. At either limiting revision the normal acceleration is  $\sqrt{3} \text{ m s}^{-2}$ , so combining it with the tangential component reproduces  $2.00 \text{ m s}^{-2}$ . These are local consistency conditions for a trajectory request; a complete speed schedule must satisfy them at every event. No actuator or material model has been supplied.

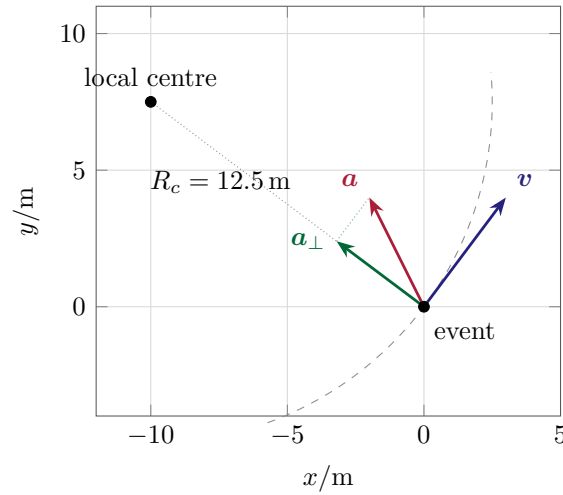


**Figure 4.15:** A prescribed event and its local curvature circle. Position axes use equal metre scales. The velocity arrow is drawn with scale  $1.00 \text{ s}$ ; the two acceleration arrows use scale  $2.00 \text{ s}^2$ . Their displayed lengths therefore represent different physical units, and only the acceleration arrows may be compared directly. The dotted closing segment is the tangential acceleration contribution. The dashed circle arc shares the event tangent and curvature, without asserting a global circular path.

Define the dimensionless variables  $q = v/\sqrt{a_{\max} R_c}$  and  $\tau = |a_t|/a_{\max}$ , for  $a_{\max} > 0$  and  $R_c > 0$ . The acceleration requirement becomes

$$\tau^2 + q^4 \leq 1, \quad q \geq 0, \quad \tau \geq 0. \quad (4.37)$$

This single graph describes all positive choices of radius and acceleration limit after rescaling. Consuming more of the allowed magnitude with speed change leaves less available for turning. At  $\tau = 1$ , a finite-radius motion with nonzero speed is excluded; at  $\tau = 0$ , the largest speed is  $\sqrt{a_{\max} R_c}$ .

## 4.6 Relative motion

### 4.6.1 Relative positions first, velocities second

An observer walking alongside a carriage and an observer fixed to the laboratory can assign different velocities to the same marker. The difference is not a disagreement about the event. It is a difference in the origins from which position changes are measured.

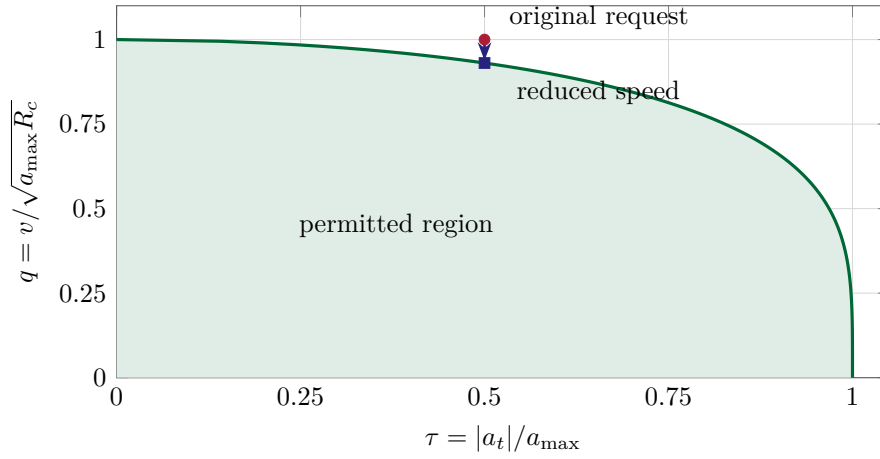


**Figure 4.16:** The kinematic acceleration budget  $\tau^2 + q^4 \leq 1$ . The shaded region includes its boundary. The example's original request  $(\tau, q) = (0.500, 1.000)$  lies outside; lowering speed to  $q = (3/4)^{1/4} = 0.93060$  reaches the boundary while holding radius and tangential acceleration fixed.

Use  $\mathbf{r}_{A/B}$  for the position of A relative to B, expressed in axes parallel to the same fixed Cartesian directions. At a common time,

$$\mathbf{r}_{A/C} = \mathbf{r}_{A/B} + \mathbf{r}_{B/C}. \quad (4.38)$$

This is a head-to-tail displacement identity: travel from C to B and then from B to A. Differentiating for translating, nonrotating frames gives

$$\mathbf{v}_{A/C} = \mathbf{v}_{A/B} + \mathbf{v}_{B/C}, \quad (4.39)$$

$$\mathbf{a}_{A/C} = \mathbf{a}_{A/B} + \mathbf{a}_{B/C}. \quad (4.40)$$

These relations use the nonrelativistic common-time description. They also imply  $\mathbf{v}_{A/B} = -\mathbf{v}_{B/A}$ . Constant relative translation makes  $\mathbf{a}_{B/C} = 0$ , so the observers then agree on acceleration although they generally disagree on velocity.

Constant translation is not required for Eq. (4.40); it is required for the *equality* of the two particle accelerations. Rotating axes need additional basis-derivative terms and cannot be treated by subtracting only an origin velocity. The advanced section introduces such basis differentiation for coordinates of a path, without developing rotating-observer dynamics.

For example, a marker moving at  $(0.40, -0.20) \text{ m s}^{-1}$  relative to a stage whose laboratory velocity is  $(1.00, 0.30) \text{ m s}^{-1}$  has laboratory velocity  $(1.40, 0.10) \text{ m s}^{-1}$ . Its speed is not the sum of the two speeds. If the stage translates at constant velocity, any laboratory acceleration of the marker is also its stage-frame acceleration.

#### 4.6.2 Heading through a medium versus track over the ground

A craft's velocity relative to water is not its velocity relative to the shore. Denote craft, water and shore by C, W and S:

$$\mathbf{v}_{C/S} = \mathbf{v}_{C/W} + \mathbf{v}_{W/S}. \quad (4.41)$$

In a uniform steady current, each of these velocities can be constant and the shore-frame path is straight. Nevertheless, the craft's heading through the water can differ from the direction of that straight ground track.

Choose  $x$  downstream and  $y$  across a straight channel. Let the current be  $w\hat{\mathbf{i}}$ , the width be  $W > 0$ , the desired downstream displacement be  $D$  (negative for an upstream destination), and the prescribed water-relative speed be  $U > 0$ . Arrival after time  $T > 0$  requires

$$\mathbf{v}_{C/S} = \left( \frac{D}{T}, \frac{W}{T} \right), \quad \mathbf{v}_{C/W} = \left( \frac{D}{T} - w, \frac{W}{T} \right). \quad (4.42)$$

Impose the water-relative speed, rather than confusing it with  $|\mathbf{v}_{C/S}|$ :

$$(D - wT)^2 + W^2 = U^2 T^2, \quad (U^2 - w^2)T^2 + 2DwT - (D^2 + W^2) = 0. \quad (4.43)$$

The equation can have no admissible positive root, one positive root, or in some geometries more than one. Every candidate must be checked for a forward crossing and for the prescribed velocity magnitude. A negative algebraic time is not an arrival.

#### Example 4.16: Crossing a channel to a prescribed point

An idealized craft crosses a 60.0 m-wide channel to a point 30.0 m upstream of its start. The current is  $1.00 \text{ m s}^{-1}$  downstream. The craft maintains a speed  $2.00 \text{ m s}^{-1}$  relative to the water. Find the constant heading and arrival time. Treat the craft as a point, with spatially uniform, steady current and no course corrections.

**Event equation.** With  $D = -30.0 \text{ m}$  and  $W = 60.0 \text{ m}$ , the numerical SI form of Eq. (4.43) is

$$3T^2 - 60T - 4500 = 0 = 3(T - 50)(T + 30).$$

Only  $T = 50.0 \text{ s}$  is a later arrival. Consequently,

$$\mathbf{v}_{C/S} = (-0.600, 1.200) \text{ m s}^{-1}, \quad \mathbf{v}_{C/W} = (-1.600, 1.200) \text{ m s}^{-1}.$$

The heading is  $\tan^{-1}(1.6/1.2) = 53.1^\circ$  upstream of directly across. The ground track is only  $\tan^{-1}(0.6/1.2) = 26.6^\circ$  upstream of directly across.

**Checks.** The water-relative speed is  $\sqrt{1.6^2 + 1.2^2} = 2.00 \text{ m s}^{-1}$ . Adding the current gives the shore velocity, and multiplying that velocity by 50.0 s gives  $(-30.0, 60.0) \text{ m}$ , the requested endpoint. Both coordinate events occur together.

**Limits and feasibility.** With zero current the same endpoint requires time  $\sqrt{30^2 + 60^2}/2 = 33.5 \text{ s}$ , and heading equals ground track. With a current  $w \geq U$ , a target upstream across the channel is impossible at constant water-relative speed: the craft cannot produce both a negative ground  $x$  component and a positive crossing component. No choice of a quadratic root can repair that physical limitation.

### 4.6.3 Different objectives give different headings

For a fixed water-relative speed  $U$ , the crossing component cannot exceed  $U$ . Thus the shortest time to reach *somewhere* on the opposite straight bank is  $W/U$ , attained by heading directly across relative to the water. The current then displaces the craft downstream by  $wW/U$ . That is not usually the shortest route between the two specified bank points.

To arrive directly opposite the start, impose  $D = 0$ . Compensation requires a water-relative upstream component  $-w$ . If  $U > w$ , the crossing component is  $\sqrt{U^2 - w^2}$ , and

$$T_{\text{opposite}} = \frac{W}{\sqrt{U^2 - w^2}}. \quad (4.44)$$

At  $U = w$ , full upstream compensation leaves zero crossing component, so arrival time is not finite. For  $U < w$ , direct-opposite arrival is impossible under this model. For the numerical speeds in Example 4.16, the fastest crossing takes 30.0 s and drifts 30.0 m downstream; a direct-opposite crossing takes 34.6 s; the specified upstream crossing takes 50.0 s. Each result answers a different question.

### 4.6.4 Closest approach as a relative-motion event

Interception is only one useful event. Two objects can miss while approaching each other very closely. Let their relative position be  $\mathbf{R}(t) = \mathbf{R}_0 + \mathbf{V}t$ , with constant relative velocity  $\mathbf{V} \neq 0$ . Their

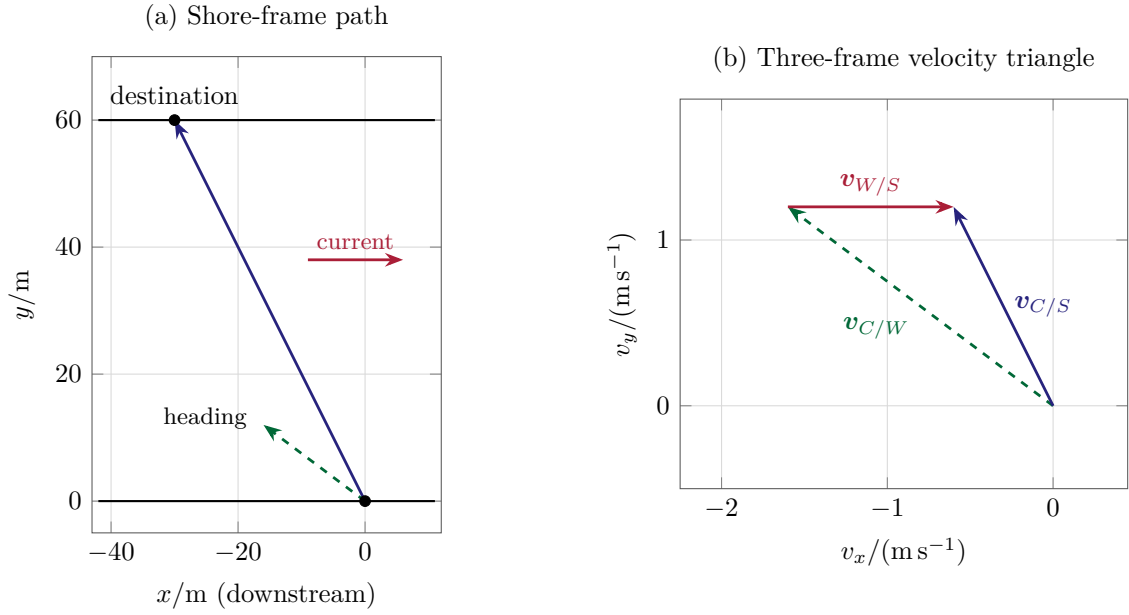


**Figure 4.17:** The upstream destination and its velocity solution. Each panel uses equal scales on its two axes, but one panel is in metres and the other in metres per second. The dashed arrow in (a) indicates the water-relative heading, not another shore-frame trajectory. Panel (b) closes exactly:  $(-1.6, 1.2) + (1, 0) = (-0.6, 1.2) m s^{-1}$ .

squared separation is

$$d^2(t) = \mathbf{R}_0 \cdot \mathbf{R}_0 + 2(\mathbf{R}_0 \cdot \mathbf{V})t + V^2 t^2.$$

Its stationary time is

$$t_* = -\frac{\mathbf{R}_0 \cdot \mathbf{V}}{V^2}. \quad (4.45)$$

The second derivative is  $2V^2 > 0$ , so this is a minimum on an unrestricted time axis. For a problem beginning at  $t = 0$ , a negative  $t_*$  instead means that the minimum future separation occurs at the start. At an interior minimum,  $\mathbf{R}(t_*) \cdot \mathbf{V} = 0$ : the remaining separation is perpendicular to the relative velocity.

#### Example 4.17: Approach without interception

Two tracked points have initial relative position  $\mathbf{R}_0 = (6, 2) m$  and constant relative velocity  $\mathbf{V} = (-1, 0.5) m s^{-1}$ . Find their closest future separation.

**Solution.** The initial dot product is  $-5 m^2 s^{-1}$ , while  $V^2 = 1.25 m^2 s^{-2}$ . Hence  $t_* = 4.0 s$ . The relative position then is  $(2, 4) m$ , whose magnitude is  $\sqrt{20} = 4.47 m$ . The dot product  $(2, 4) \cdot (-1, 0.5) = 0$  confirms the minimum.

**No collision.** The  $x$  separation would vanish at six seconds, whereas the  $y$  separation would vanish at negative four seconds. There is no common zero. For real bodies with nonzero size, point separation alone would not determine contact; their sizes and shapes would need to enter the model.

#### 4.6.5 Reachable headings when the current is stronger than the craft

The velocity triangle also answers a feasibility question before any time is calculated. In a shore-velocity diagram, every constant heading at water-relative speed  $U$  lies on a circle of radius  $U$  centred at  $(w, 0)$ . The ground velocity must point along the desired displacement  $(D, W)$ , so it lies on the forward ray through that vector. Intersections between that ray and the velocity circle correspond to admissible headings.

For  $w > U > 0$ , the circle lies entirely at positive downstream velocity. An opposite-bank endpoint must therefore be downstream. The discriminant of Eq. (4.43) is

$$\Delta = 4\{U^2 D^2 - (w^2 - U^2)W^2\}.$$

For  $D > 0$ , a finite crossing is possible precisely when

$$\frac{D}{W} \geq \frac{\sqrt{w^2 - U^2}}{U}. \quad (4.46)$$

Equality corresponds to a tangent ray and a single heading. A strictly larger ratio gives two positive arrival times and two distinct constant headings. The two-heading result is not a violation of uniqueness: the endpoint and water-relative *speed* have been prescribed, while the direction remains an unknown.

#### Example 4.18: Two headings that reach the same downstream bank point

A craft moves at constant speed  $U = 2.00 \text{ m s}^{-1}$  relative to a uniform current  $w = 3.00 \text{ m s}^{-1}$ . A channel has width  $W = 60.0 \text{ m}$ , and the destination is  $D = 90.0 \text{ m}$  downstream. Find every constant-heading crossing, under the point-craft model.

**Check reachability first.** Here  $D/W = 1.50$ , whereas the threshold is  $\sqrt{5}/2 = 1.11803$ ; two intersections are expected. The time equation reduces, in SI numerical values, to

$$-5T^2 + 540T - 11700 = 0, \quad 5(T - 30)(T - 78) = 0.$$

Both roots are positive. For  $T_1 = 30.0 \text{ s}$ ,

$$\mathbf{v}_{C/S}^{(1)} = (3.00, 2.00) \text{ m s}^{-1}, \quad \mathbf{v}_{C/W}^{(1)} = (0, 2.00) \text{ m s}^{-1}.$$

The craft heads directly across relative to the water, while the current supplies all its downstream velocity. For  $T_2 = 78.0 \text{ s}$ ,

$$\mathbf{v}_{C/S}^{(2)} = (15/13, 10/13) \text{ m s}^{-1}, \quad \mathbf{v}_{C/W}^{(2)} = (-24/13, 10/13) \text{ m s}^{-1}.$$

The second heading is  $\tan^{-1}(24/10) = 67.38^\circ$  upstream of directly across, reducing both downstream progress and the crossing component. Both ground tracks point  $\tan^{-1}(90/60) = 56.31^\circ$  downstream of directly across.

**Check both solutions.** The second water-relative magnitude is  $\sqrt{24^2 + 10^2}/13 = 26/13 = 2.00 \text{ m s}^{-1}$ . Multiplication of either ground velocity by its own time yields  $(90.0, 60.0) \text{ m}$ . Thus the slower craft does reach the same endpoint; it is not heading to a different point. The shortest possible crossing to any point is  $W/U = 30.0 \text{ s}$ , so the first solution also attains that bound.

**Boundary case.** The smallest reachable downstream displacement is  $D_{\min} = W\sqrt{5}/2 = 67.0820 \text{ m}$ . At that endpoint the two times merge at  $T = D_{\min}w/(w^2 - U^2) = 40.2492 \text{ s}$ . A destination only  $60.0 \text{ m}$  downstream is impossible under these fixed-speed, constant-heading assumptions, despite lying downstream. Its required ground direction is too steep to intersect the velocity circle.

### 4.6.6 Closest approach in space and within a time window

The closest-approach formula is unchanged in three dimensions because it uses dot products rather than a special planar drawing. If the motion model is valid only on  $0 \leq t \leq H$ , minimize the squared

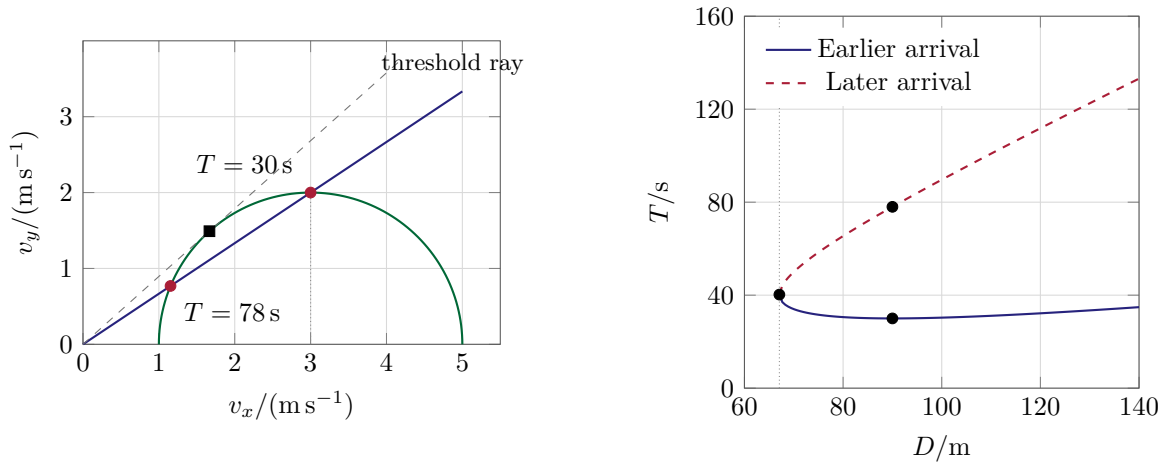


**Figure 4.18:** Strong-current crossings at  $U = 2.00 \text{ m s}^{-1}$ ,  $w = 3.00 \text{ m s}^{-1}$  and  $W = 60.0 \text{ m}$ . Left: the upper semicircle of feasible shore velocities, the requested endpoint ray (blue), and the limiting tangent ray (dashed). The plotted red intersections give two headings. Right: both positive times as the destination moves downstream. They meet at the reachability threshold; the absence of a curve to its left represents no admissible crossing.

separation on that interval. With constant nonzero  $\mathbf{V}$ , the valid minimizing time is

$$t_{\min} = \begin{cases} 0, & t_* < 0, \\ t_*, & 0 \leq t_* \leq H, \\ H, & t_* > H. \end{cases} \quad (4.47)$$

An endpoint minimum need not satisfy  $\mathbf{R} \cdot \mathbf{V} = 0$ . That perpendicularity characterizes an *interior* stationary event, not every constrained minimum. If  $\mathbf{V} = 0$ , separation is constant and every time in the interval minimizes it; the formula for  $t_*$  must not be divided by zero.

#### Example 4.19: Three-dimensional approach in a limited window

Two tracked points have relative position and relative velocity

$$\mathbf{R}_0 = (8.00, -6.00, 4.00) \text{ m}, \quad \mathbf{V} = (-2.00, 1.00, -1.00) \text{ m s}^{-1}.$$

Find their closest separation in the next  $H = 3.00 \text{ s}$ . Then find the unrestricted future minimum if this constant-velocity model remains valid.

**Locate the stationary event.** Here  $\mathbf{R}_0 \cdot \mathbf{V} = -26.0 \text{ m}^2 \text{ s}^{-1}$ ,  $V^2 = 6.00 \text{ m}^2 \text{ s}^{-2}$ , and

$$d^2(t) = 116 - 52t + 6t^2 \quad \text{in SI numerical values}, \quad t_* = \frac{13}{3} \text{ s} = 4.33333 \text{ s}.$$

Since  $t_* > H$ , the closest event in the observation window is its final event. At  $t = 3.00 \text{ s}$ ,

$$\mathbf{R} = (2.00, -3.00, 1.00) \text{ m}, \quad d = \sqrt{14} \text{ m} = 3.74166 \text{ m}.$$

The dot product  $\mathbf{R} \cdot \mathbf{V} = -8.00 \text{ m}^2 \text{ s}^{-1} < 0$  shows that separation is still decreasing as the window ends.

**Extend only under the stated conditional assumption.** At  $t_* = 13/3 \text{ s}$ ,

$$\mathbf{R}_* = \frac{1}{3}(-2, -5, -1) \text{ m}, \quad d_* = \frac{\sqrt{30}}{3} \text{ m} = 1.82574 \text{ m}.$$

The dot product  $\mathbf{R}_* \cdot \mathbf{V} = (4 - 5 + 1)/3 = 0$  confirms the interior minimum. Neither result implies a collision:  $d_* > 0$ , and zeroing different components separately gives different times.

**Independent geometric check.** Resolve  $\mathbf{R}_0$  along  $\hat{\mathbf{V}} = \mathbf{V}/V$  and perpendicular to it. Its initial longitudinal coordinate is  $q_0 = -26/\sqrt{6} = -10.6145$  m, while the perpendicular distance is constant at  $d_*$ . Consequently  $d^2 = (q_0 + Vt)^2 + d_*^2$ , reproducing the quadratic above. The separation is shortest when the longitudinal coordinate vanishes, unless the allowed interval ends first.

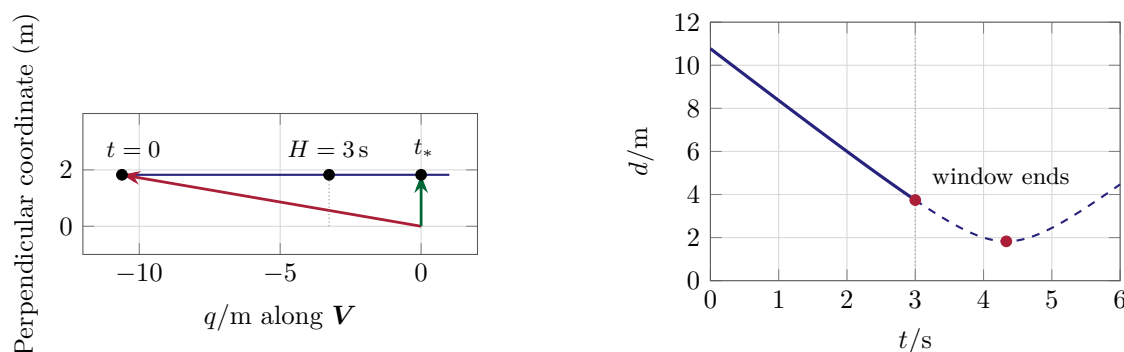


**Figure 4.19:** A closest-approach calculation in three dimensions. Left: an exact orthonormal plane containing the relative-motion line and the origin, with one axis parallel to  $\mathbf{V}$ . This is a geometrical reduction, rather than an  $x$ - $y$  projection that discards the  $z$  separation. Right: actual separation versus time. The solid curve is inside the observation window; the dashed extension requires the additional assumption that the same motion continues.

#### 4.6.7 Interception with a prescribed speed in three dimensions

In an interception problem, a fixed chaser speed does not fix its velocity until the heading has been selected. Let a chaser start at the origin, choose one constant velocity  $\mathbf{u}$  of stipulated magnitude  $U$ , and let a target have  $\mathbf{r}_T = \mathbf{R}_0 + \mathbf{V}t$ . Interception requires

$$\mathbf{u}t = \mathbf{R}_0 + \mathbf{V}t, \quad \mathbf{u} = \mathbf{V} + \frac{\mathbf{R}_0}{t}.$$

Taking the squared magnitude eliminates the unknown direction:

$$(V^2 - U^2)t^2 + 2(\mathbf{R}_0 \cdot \mathbf{V})t + R_0^2 = 0. \quad (4.48)$$

Every positive root is a candidate; reconstruct  $\mathbf{u}$  and test the full vector event. A negative root belongs to the extrapolated past. If a speed-equality case makes the quadratic coefficient zero, solve the resulting linear equation instead of applying a quadratic formula with a zero denominator.

#### Example 4.20: Aim ahead of a moving three-dimensional target

An ideal point target starts at  $\mathbf{R}_0 = (100, 50, 20)$  m and moves at constant velocity  $\mathbf{V} = (2, -1, 1)$  m s<sup>-1</sup>. A point chaser starts at the origin and may select a constant heading at speed  $U = 13.0$  m s<sup>-1</sup>. Find the future interception time and the required azimuth and elevation. Take  $+z$  upward and measure azimuth counterclockwise from  $+x$  toward  $+y$ .

**Find the common event.** With  $R_0^2 = 12900$  m<sup>2</sup>,  $\mathbf{R}_0 \cdot \mathbf{V} = 170$  m<sup>2</sup> s<sup>-1</sup>, and  $V^2 = 6$  m<sup>2</sup> s<sup>-2</sup>, the event equation becomes

$$163t^2 - 340t - 12900 = (t - 10)(163t + 1290) = 0$$

in SI numerical values. Only  $t = 10.0\text{ s}$  is a future event; the other root is  $-1290/163\text{ s}$ . The required velocity is

$$\mathbf{u} = \mathbf{V} + \frac{\mathbf{R}_0}{10.0\text{ s}} = (12.0, 4.00, 3.00)\text{ m s}^{-1}.$$

Its azimuth is  $\phi = \text{atan2}(4, 12) = 18.43^\circ$ . Its elevation above the horizontal plane is

$$\beta = \text{atan2}\left(3, \sqrt{12^2 + 4^2}\right) = 13.34^\circ.$$

The target's initial line of sight has different angles, so aiming at its initial position would not produce this common event.

**Check the speed and all coordinates.** The selected magnitude is  $\sqrt{12^2 + 4^2 + 3^2} = 13.0\text{ m s}^{-1}$ . At ten seconds the target is at  $(120, 40, 30)\text{ m}$ , and the chaser is at exactly the same vector  $10\mathbf{u}$ . The relative velocity after selecting the heading is  $\mathbf{V} - \mathbf{u} = (-10, -5, -2)\text{ m s}^{-1} = -\mathbf{R}_0/(10.0\text{ s})$ ; thus every component of relative separation vanishes together.

**Interpret the model.** Both paths are straight, and both accelerations are zero throughout the modeled motion. Three-dimensional geometry alone does not require curved motion. The result presumes the chaser already has its selected velocity at  $t = 0$ ; no maneuver to acquire that velocity has been modeled.

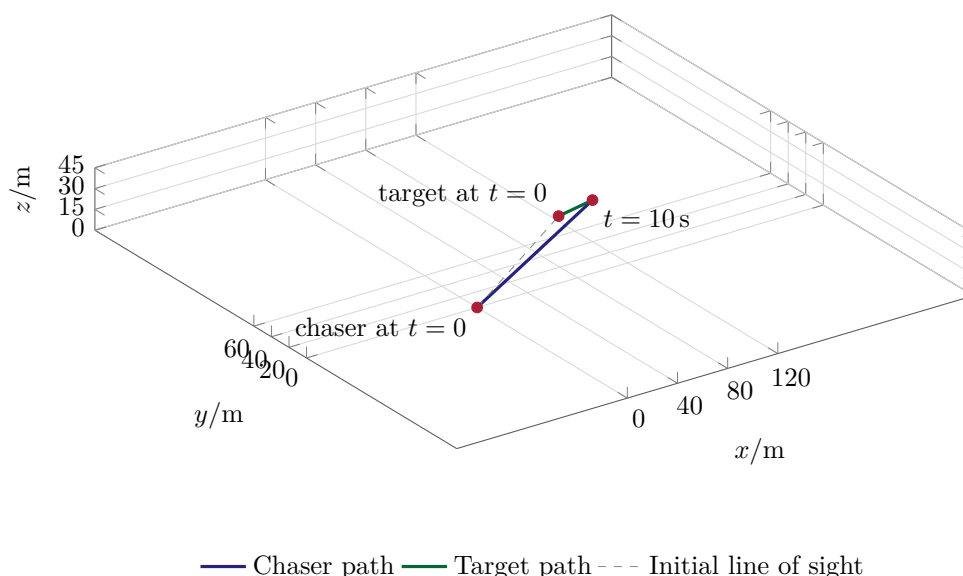


**Figure 4.20:** A three-dimensional constant-velocity interception in the common laboratory frame. The solid paths terminate at the same point  $(120, 40, 30)\text{ m}$  and the same time. The dashed initial line of sight has a different direction from the chaser path. The graph uses equal unit ratios in space; its oblique projection changes apparent angles, so the azimuth and elevation must be obtained from the stated components.

Analytical functions made these events accessible by algebra. Discrete data and prescribed accelerations rarely align with a requested event time exactly. The next pathway examines the numerical and measurement errors that arise when a continuous path is represented by samples.

## 4.7 Numerical paths beyond simple models

This supporting pathway distinguishes two tasks that are often called “trajectory reconstruction.” In a *forward calculation*, an acceleration model and an initial state are given, and a numerical method advances the state. In a *measurement analysis*, sampled coordinates are given, and differences or fits estimate their derivatives. Integration and differentiation propagate errors differently. Neither operation proves that the assumed model matches a real apparatus.

### 4.7.1 A prescribed time-dependent acceleration

If acceleration is a known function of time, the initial-value problem is

$$\dot{\mathbf{r}} = \mathbf{v}, \quad \dot{\mathbf{v}} = \mathbf{a}(t), \quad \mathbf{r}(0) = \mathbf{r}_0, \quad \mathbf{v}(0) = \mathbf{v}_0. \quad (4.49)$$

This remains kinematics: the acceleration is supplied rather than obtained from a force law. The exact solution can be written as successive integrals, but closed-form antiderivatives are not always convenient. A time grid  $t_n = nh$ , with  $h > 0$ , gives a finite set of approximations  $\mathbf{r}_n, \mathbf{v}_n$ .

The simplest forward Euler update uses initial slopes for the whole step:

$$\mathbf{r}_{n+1} = \mathbf{r}_n + h\mathbf{v}_n, \quad \mathbf{v}_{n+1} = \mathbf{v}_n + h\mathbf{a}(t_n).$$

For constant acceleration, it omits the familiar  $\frac{1}{2}\mathbf{a}h^2$  position increment. A plot of its output can look smooth while accumulating a systematic position error. This elementary limiting case already supplies a useful test.

A better update for the time-only acceleration model uses a predicted midpoint velocity:

$$\begin{aligned} \mathbf{v}_{n+1/2}^{\text{pred}} &= \mathbf{v}_n + \frac{1}{2}h\mathbf{a}(t_n), \\ \mathbf{r}_{n+1} &= \mathbf{r}_n + h\mathbf{v}_{n+1/2}^{\text{pred}}, \end{aligned} \quad (4.50)$$

$$\mathbf{v}_{n+1} = \mathbf{v}_n + h\mathbf{a}(t_n + h/2). \quad (4.51)$$

These are the explicit midpoint equations for the combined state. They are not instructions to use a midpoint acceleration in the position equation without the appropriate velocity prediction. For a smooth  $\mathbf{a}(t)$ , Taylor expansion about  $t_n$  shows that the first omitted one-step terms are proportional to  $h^3$ . Over a fixed finite interval, the accumulated state error is typically proportional to  $h^2$ , once the grid is fine enough. Constant acceleration is reproduced exactly at the grid nodes in exact arithmetic.

### 4.7.2 Events between the samples

Suppose an event is the first crossing of  $y = y_e$  while moving upward. Find a pair of adjacent samples with  $y_n < y_e \leq y_{n+1}$ . A linear interpolation gives

$$\lambda = \frac{y_e - y_n}{y_{n+1} - y_n}, \quad t_e^{(h)} = t_n + \lambda h, \quad \mathbf{r}_e^{(h)} = \mathbf{r}_n + \lambda(\mathbf{r}_{n+1} - \mathbf{r}_n). \quad (4.52)$$

The fraction must lie in  $[0, 1]$ . A zero denominator cannot be treated as a crossing by arbitrary division. If the signal is not monotone, the first valid bracket and crossing direction must be identified, and the time step must resolve any turns between samples. Endpoint sign tests alone can miss two crossings inside one step.

Even exact nodal positions need not give an exact event time under linear interpolation. The interpolation approximates the curve between nodes by a segment. For a smooth transverse crossing with nonzero  $v_y$ , this generally contributes an error of order  $h^2$ . Near a tangency,  $v_y$  is small and event-time sensitivity can be much larger; a grazing event requires special treatment. The ratio of errors on consecutive grids can also fluctuate because the event occupies different fractions of its bracket.

#### Example 4.21: Reconstruct a planar path from samples

Use the midpoint update to reconstruct a planar path from sampled values of the prescribed

acceleration

$$\mathbf{a}(t) = [(0.600 \text{ m s}^{-3})t]\hat{\mathbf{i}} + (1.00 \text{ m s}^{-2})\hat{\mathbf{j}}, \quad \mathbf{r}_0 = 0, \quad \mathbf{v}_0 = (1.00, 0) \text{ m s}^{-1}.$$

Estimate the first upward crossing of  $y = 0.500 \text{ m}$  and test grid refinement.

**Analytical benchmark.** Direct integration, with dimensional coefficients understood as stated above, gives the SI numerical formulas

$$v_x = 1 + 0.3t^2, \quad v_y = t, \quad x = t + 0.1t^3, \quad y = 0.5t^2.$$

Thus the exact event is  $t_e = 1.000 \text{ s}$ ,  $x_e = 1.100 \text{ m}$ . We deliberately choose grids on which this time is *not* a node.

**One coarse step sequence.** With  $h = 0.300 \text{ s}$ , nodes at  $t = 0.9 \text{ s}$  and  $1.2 \text{ s}$  have  $(x, y) = (0.9648, 0.4050) \text{ m}$  and  $(1.3620, 0.7200) \text{ m}$ . They bracket the event. Therefore

$$\lambda = \frac{0.500 - 0.405}{0.720 - 0.405} = 0.301587,$$

$$t_e^{(h)} = 0.990476 \text{ s}, \quad x_e^{(h)} = 0.9648 + 0.301587(1.3620 - 0.9648) = 1.084590 \text{ m}.$$

The event is predicted too early and too far left. Reporting the first node above the crossing instead would give  $1.2 \text{ s}$ , a much coarser event estimate.

**Independent discrete check.** For this particular acceleration, the numerical velocities and  $y_n$  are exact at every node. Summing the missing cubic increment in  $x$  gives the exact discrete identity

$$x_n = t_n + 0.1t_n^3 - 0.1t_nh^2 \quad (\text{SI numerical values}).$$

At the fixed endpoint  $1.2 \text{ s}$ , the absolute  $x$  error is therefore  $0.12h^2$  in metres. This identity checks the update implementation without rerunning the same numerical calculation as its own benchmark.

**Interpretation.** The event-time error here comes entirely from interpolation of the exact quadratic  $y$  values, whereas the event-position error contains both state and interpolation error. Table 4.1 separates these quantities. None of the displayed errors is measurement uncertainty.

**Table 4.1:** Computed refinement for the prescribed-acceleration model.  $E_x^{\text{end}}$  is the absolute error at fixed time  $1.2 \text{ s}$ ;  $E_t$  and  $E_x^{\text{event}}$  refer to the interpolated  $y = 0.500 \text{ m}$  event. Extra digits expose numerical behaviour.

| $h/\text{s}$ | $t_e^{(h)}/\text{s}$ | $E_x^{\text{end}}/\text{m}$ | $E_t/\text{s}$ | $E_x^{\text{event}}/\text{m}$ |
|--------------|----------------------|-----------------------------|----------------|-------------------------------|
| 0.30000      | 0.99047619           | 0.01080000                  | 0.00952381     | 0.01540952                    |
| 0.15000      | 0.99743590           | 0.00270000                  | 0.00256410     | 0.00406603                    |
| 0.07500      | 0.99938272           | 0.00067500                  | 0.00061728     | 0.00099124                    |
| 0.03750      | 0.99984277           | 0.00016875                  | 0.00015723     | 0.00025106                    |
| 0.01875      | 0.99996106           | 0.00004219                  | 0.00003894     | 0.00006237                    |

### 4.7.3 A reproducible implementation and its tests

The following code implements exactly the model and event rule above. It uses only the Python standard library. Its numeric inputs and outputs are SI values. The model is monotone in  $y$  after release; the code does not pretend to find arbitrary hidden crossings of other models.

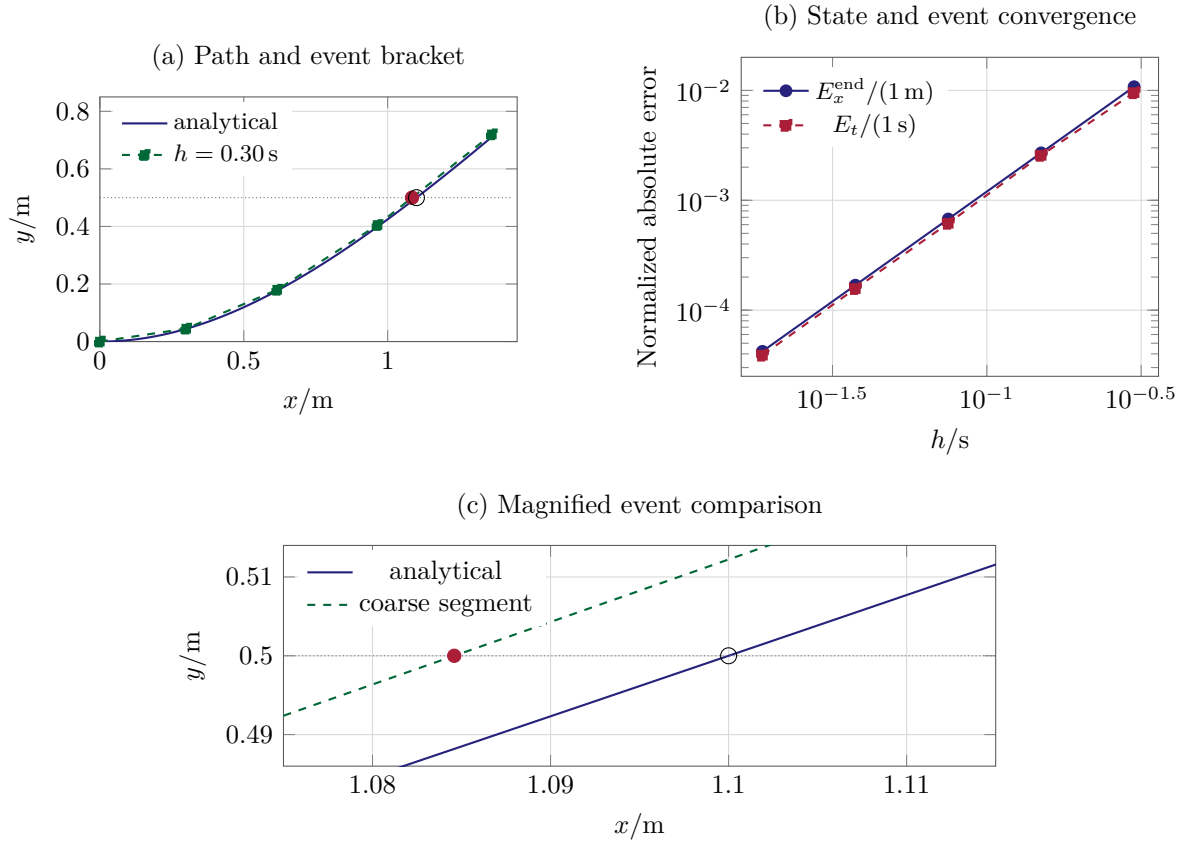


**Figure 4.21:** The analytical path, a coarse numerical reconstruction and verified error curves. In (a), the two final square nodes bracket the dotted event line; the filled dot is the interpolated event and the open circle the exact event. Their small separation should not be interpreted from line thickness alone. Panel (b) uses dimensionless errors normalized by the explicitly stated reference units. The magnified panel (c) resolves the distinct event positions using the same analytical curve and coarse bracket segment; its horizontal and vertical scales differ. The fixed-time position error decreases exactly by four under halving; the event error approaches the second-order pattern with a varying bracket fraction.

```

from math import isfinite

def path_and_event(h: float, end: float = 1.2):
    if not (isfinite(h) and isfinite(end) and h > 0 and end > 0):
        raise ValueError("Positive finite step and end time required")
    n = round(end / h)
    if n < 1 or abs(n*h - end) > 1e-10:
        raise ValueError("End time must be an integer number of steps")
    x, y, vx, vy = 0.0, 0.0, 1.0, 0.0
    nodes = [(0.0, x, y)]
    event = None
    for k in range(n):
        t = k*h
        xn = x + h*(vx + 0.5*h*(0.6*t))
        yn = y + h*(vy + 0.5*h)
        vxn = vx + h*(0.6*(t + 0.5*h))
        vyn = vy + h
        if event is None and y < 0.5 <= yn:
            fraction = (0.5 - y)/(yn - y)
            event = (t + fraction*h, x + fraction*(xn - x))
        x, y, vx, vy = xn, yn, vxn, vyn
        nodes.append(((k + 1)*h, x, y))
    if event is None:
        raise ValueError("The requested upward event was not bracketed")
    return nodes, event

for h in (0.3, 0.15, 0.075, 0.0375, 0.01875):
    nodes, (te, xe) = path_and_event(h)
    ex_end = abs(nodes[-1][1] - (1.2 + 0.1*1.2**3))
    print(h, te, ex_end, abs(te - 1.0), abs(xe - 1.1))

```

A numerical investigation should record the differential equations, initial state, step rule, event rule and comparison quantity. For this model, useful tests are zero acceleration, a separately specified constant acceleration, the exact velocity nodes, the fixed-time cubic-position error, and the interpolated event. A smooth plotted curve or a larger number of digits is not a substitute for any of these tests.

Reducing  $h$  reduces the discretization error in the present deterministic model until other numerical effects become relevant. It cannot remove a wrong coefficient, a mistaken initial condition or an omitted physical influence. A converged no-drag calculation remains a no-drag calculation. Conversely, a disagreement with measured coordinates can come from measurement error rather than a failure of the numerical integrator.

#### 4.7.4 Estimating motion from sampled coordinates

Now reverse the information flow. Suppose a fixed camera or position sensor records coordinates  $x_i = x(t_i)$  at uniform intervals  $h$ . A Taylor expansion gives

$$\begin{aligned}
 x(t+h) &= x(t) + h\dot{x} + \frac{1}{2}h^2\ddot{x} + \frac{1}{6}h^3x^{(3)} + \dots, \\
 x(t-h) &= x(t) - h\dot{x} + \frac{1}{2}h^2\ddot{x} - \frac{1}{6}h^3x^{(3)} + \dots.
 \end{aligned}$$

Subtracting and adding these equations produce central estimates

$$v_{x,i}^{\text{est}} = \frac{x_{i+1} - x_{i-1}}{2h}, \quad (4.53)$$

$$a_{x,i}^{\text{est}} = \frac{x_{i+1} - 2x_i + x_{i-1}}{h^2}. \quad (4.54)$$

For a sufficiently smooth coordinate function, both have truncation error of order  $h^2$ . They estimate quantities at the *central* sample time, not at the final sample. At endpoints one needs a different formula or must report that the central estimate is unavailable.

The denominators explain why differentiation is sensitive to coordinate noise. Under a simple model of independent equal standard uncertainties  $\sigma_x$ , with negligible timing uncertainty, variance addition gives

$$\sigma_{v_x} = \frac{\sigma_x}{\sqrt{2}h}, \quad \sigma_{a_x} = \frac{\sqrt{6}\sigma_x}{h^2}. \quad (4.55)$$

The second coefficient comes from the squared weights  $1^2 + (-2)^2 + 1^2 = 6$ . These are uncertainty formulas under the stated statistical assumptions, not bounds on every possible error. A common coordinate offset cancels; a calibration scale error generally does not. Correlated position errors or uncertain time intervals require their own propagation calculation.

#### Example 4.22: Synthetic coordinate samples and an apparent acceleration

Consider the following synthetic coordinates, sampled at  $h = 0.200$  s. They are generated from the analytical path of Example 4.21, with a deterministic alternating  $x$  perturbation  $0.001(-1)^i$  m, where  $i = 0$  at the first row. The  $y$  coordinates are unperturbed.

| $t/\text{s}$ | $x_i/\text{m}$ | $y_i/\text{m}$ |
|--------------|----------------|----------------|
| 0.0          | 0.0010         | 0.0000         |
| 0.2          | 0.1998         | 0.0200         |
| 0.4          | 0.4074         | 0.0800         |
| 0.6          | 0.6206         | 0.1800         |
| 0.8          | 0.8522         | 0.3200         |
| 1.0          | 1.0990         | 0.5000         |
| 1.2          | 1.3738         | 0.7200         |

Estimate the velocity and acceleration at  $t = 0.600$  s, then compare with the known model.

**Central differences.**

$$v_x^{\text{est}} = \frac{0.8522 - 0.4074}{0.400} = 1.112 \text{ m s}^{-1}, \quad a_x^{\text{est}} = \frac{0.8522 - 2(0.6206) + 0.4074}{0.200^2} = 0.460 \text{ m s}^{-2}.$$

The  $y$  values give  $v_y^{\text{est}} = 0.600 \text{ m s}^{-1}$  and  $a_y^{\text{est}} = 1.00 \text{ m s}^{-2}$ . The exact model has  $v_x = 1.108 \text{ m s}^{-1}$  and  $a_x = 0.360 \text{ m s}^{-2}$ .

**What caused the discrepancy?** Equal perturbations at the two outer  $x$  samples cancel in the velocity difference; its  $0.004 \text{ m s}^{-1}$  error is the cubic truncation term  $0.1h^2$  in SI units. The acceleration's cubic truncation error is zero, but its alternating errors contribute  $0.004 \text{ m}/h^2 = 0.100 \text{ m s}^{-2}$ . A small coordinate disturbance can therefore create a much larger relative acceleration error.

**Separate uncertainty model.** If instead three position errors were independent random quantities with  $\sigma_x = 0.001$  m, Eq. (4.55) would give  $\sigma_{v_x} = 0.00354 \text{ m s}^{-1}$  and  $\sigma_{a_x} = 0.0612 \text{ m s}^{-2}$ . The actual alternating perturbations above are specified numbers, not a random realization certified by this statistical model.

Halving the sample interval improves smooth-function truncation error but, at fixed coordinate uncertainty, doubles the velocity uncertainty and quadruples the acceleration uncertainty in Eq. (4.55). Thus “sample more often” is not by itself a complete differentiation strategy. Fitting a locally smooth function to several samples can reduce random fluctuations, but it introduces a window length and a model choice. A window spanning an abrupt change can erase the very event one wants to detect.

### 4.7.5 A laboratory or video investigation

A feasible teaching investigation can track a visible marker moving in one fixed plane. Place a length reference in that plane, keep the camera fixed, identify a time origin, and verify the actual time spacing of the frames used in the analysis. A scale reference in a different plane can create a perspective error. A camera view of three-dimensional motion supplies only a projection unless additional geometry or cameras are available. These observations are part of the model, not optional image-processing details.

Begin by plotting the recorded  $x(t)$  and  $y(t)$  values and marking uncertain points. Estimate interior velocities and accelerations using the formulas above, or fit local polynomials and differentiate the fits. Compare several spacings or window widths, retain units, and examine residuals against the original coordinates. A repeatable pattern in residuals may reveal a model mismatch; isolated large derivatives may instead reflect a tracking failure. The supplied synthetic table gives a complete analysis exercise even when no apparatus is available.

A useful report states what was directly observed, how pixels or sensor readings became coordinates, what was assumed about perspective and timing, and how derivative estimates changed under a reasonable analysis variation. It should not report synthetic values as observations or infer a force law from a kinematic fit alone.

### 4.7.6 A fitted trajectory and the evidence in its residuals

A fit uses several coordinate samples to estimate a small number of motion parameters. For a constant vertical acceleration, choose a central reference time  $t_c$ , write  $\tau = t - t_c$ , and fit

$$y_{\text{fit}}(t) = A + B\tau + C\tau^2, \quad v_{y,\text{fit}} = B + 2C\tau, \quad a_{y,\text{fit}} = 2C. \quad (4.56)$$

Here  $A$  is a position,  $B$  a velocity, and  $C$  half an acceleration. Centering the time variable makes these roles easy to interpret and reduces unnecessary correlations. It does not turn an approximate model into a physical law.

For equal position uncertainties, least squares minimizes the sum of squared residuals  $e_i = y_i - y_{\text{fit}}(t_i)$ . Differentiating this sum with respect to  $A, B, C$  gives three equations: the residuals must have zero sums when weighted by  $1, \tau_i, \tau_i^2$ . For a symmetric time grid, the odd sums vanish. Defining  $S_2 = \sum \tau_i^2$  and  $S_4 = \sum \tau_i^4$ , the equations simplify to

$$NA + S_2C = \sum y_i, \quad S_2B = \sum \tau_i y_i, \quad S_2A + S_4C = \sum \tau_i^2 y_i. \quad (4.57)$$

These sums are directly reproducible from a table. Unequal uncertainties require appropriate weights; missing or nonuniform times require the general equations rather than these symmetric-grid simplifications.

#### Example 4.23: A synthetic video track: fit, differentiate, inspect

Seven synthetic vertical coordinates are generated from  $y = 0.700 + 3.200t - 4.905t^2$ , with numerical inputs in SI units, and the deliberately specified perturbations in the final column. The marker continues below the arbitrary height origin; no floor or impact is part of this model. Fit a quadratic and compare its acceleration with a three-point estimate at  $t_c = 0.600$  s.

| $t/\text{s}$ | $y_i/\text{m}$ | Specified perturbation / mm |
|--------------|----------------|-----------------------------|
| 0.0          | 0.7020         | +2                          |
| 0.2          | 1.1378         | −6                          |
| 0.4          | 1.2012         | +6                          |
| 0.6          | 0.8502         | −4                          |
| 0.8          | 0.1268         | +6                          |
| 1.0          | −1.0110        | −6                          |
| 1.2          | −2.5212        | +2                          |

**Fit calculation.** For  $\tau = t - 0.6$ ,  $N = 7$ ,  $S_2 = 1.12 \text{ s}^2$ , and  $S_4 = 0.3136 \text{ s}^4$ . In SI numerical values, the three right-hand sums in Eq. (4.57) are 0.4858,  $-3.00832$ , and  $-0.581504$ . Solving the equations gives

$$A = 0.8542 \text{ m}, \quad B = -2.686 \text{ m s}^{-1}, \quad C = -4.905 \text{ m s}^{-2}, \quad a_{y,\text{fit}} = -9.810 \text{ m s}^{-2}.$$

The specified perturbations are orthogonal to all three fitting columns, so they do not shift these fitted coefficients. That property was designed into this example; it is not a general benefit guaranteed for measured data.

**Three-point comparison.** The central acceleration estimate is

$$a_y^{\text{est}} = \frac{0.1268 - 2(0.8502) + 1.2012}{(0.200)^2} = -9.310 \text{ m s}^{-2}.$$

Its  $+0.500 \text{ m s}^{-2}$  error comes from the combination of the three specified perturbations. The central velocity estimate is  $-2.686 \text{ m s}^{-1}$ : the two outer perturbations cancel, and the quadratic has no central-difference velocity truncation error.

**Uncertainty under a separate statistical model.** If coordinates instead had independent equal standard uncertainty  $\sigma_y = 0.00500 \text{ m}$ , with exact times and an adequate quadratic model, linear propagation through the fit would give

$$\sigma_B = \frac{\sigma_y}{\sqrt{S_2}} = 0.00472 \text{ m s}^{-1}, \quad \sigma_{a_y} = 2\sigma_y \sqrt{\frac{N}{NS_4 - S_2^2}} = 0.0273 \text{ m s}^{-2}.$$

The three-point acceleration uncertainty for the same  $\sigma_y$  is  $\sqrt{6}\sigma_y/h^2 = 0.306 \text{ m s}^{-2}$ . The improvement uses information from a longer interval and assumes constant acceleration throughout it. These are conditional standard uncertainties, not bounds or statements that the deterministic perturbations are independent random noise.

**Residual check.** Subtracting the fit reproduces the seven specified perturbations. Their squared sum is  $0.000168 \text{ m}^2$ . Estimating an unknown position standard deviation from this sum would give  $\sqrt{0.000168/(7-3)} = 0.00648 \text{ m}$ , using four residual degrees of freedom. However, the alternating structure in Fig. 4.22 suggests a structured disturbance; a numerical residual standard deviation alone cannot establish independence or validate the camera calibration.

#### 4.7.7 Sampling balances truncation bias against noise

For the central velocity formula, Taylor expansion gives bias  $x^{(3)}h^2/6 + \mathcal{O}(h^4)$ . If  $x = (0.250 \text{ m s}^{-3})t^3$ , this bias is exactly  $ch^2$ , with  $c = 0.250 \text{ m s}^{-3}$ . For independent, zero-mean coordinate errors with standard uncertainty  $\sigma_x = 0.00200 \text{ m}$ , the noise standard uncertainty is  $\sigma_x/(\sqrt{2}h)$ . Treating these two effects separately gives a useful expected squared error about the true velocity,

$$E_v^2 = c^2h^4 + \frac{\sigma_x^2}{2h^2}. \quad (4.58)$$

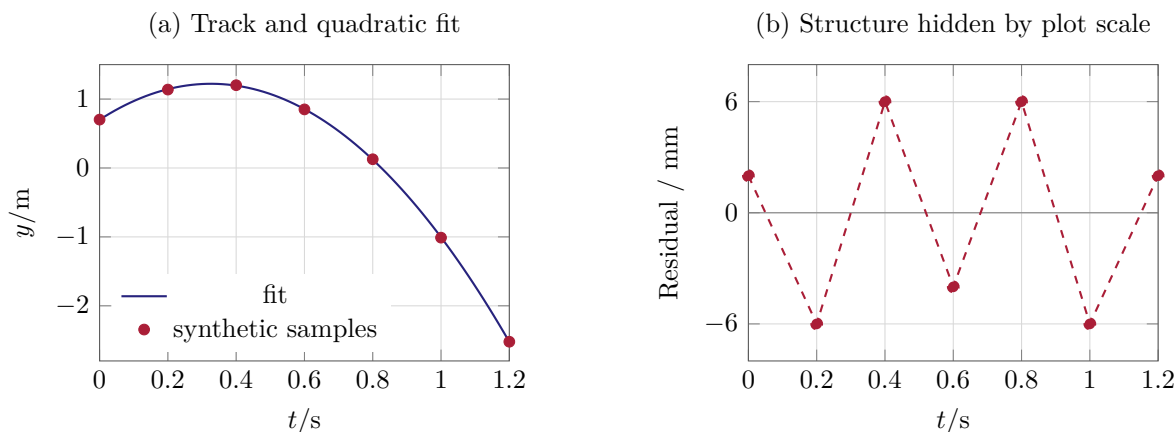


**Figure 4.22:** A fully specified synthetic track and its least-squares residuals. At the scale of the coordinate plot, the samples appear to lie almost exactly on the fit; the millimetre-scale residual panel reveals their alternating pattern. Joining residual markers guides the eye and does not assert a measured continuous disturbance.

The first term decreases with smaller spacing; the second increases. Differentiating with respect to positive  $h$  gives the minimum at

$$h_{\text{opt}} = \left( \frac{\sigma_x^2}{4c^2} \right)^{1/6} = 0.159 \text{ s.}$$

The units inside the root are  $\text{s}^6$ , as required. This result is specific to the cubic model, the central two-sided estimator, and independent equal errors. It illustrates why frame rate alone cannot determine a useful derivative uncertainty. One can retain a high-rate recording and estimate derivatives using a wider fitting window or wider sample separation; doing so changes temporal resolution and must be reported.

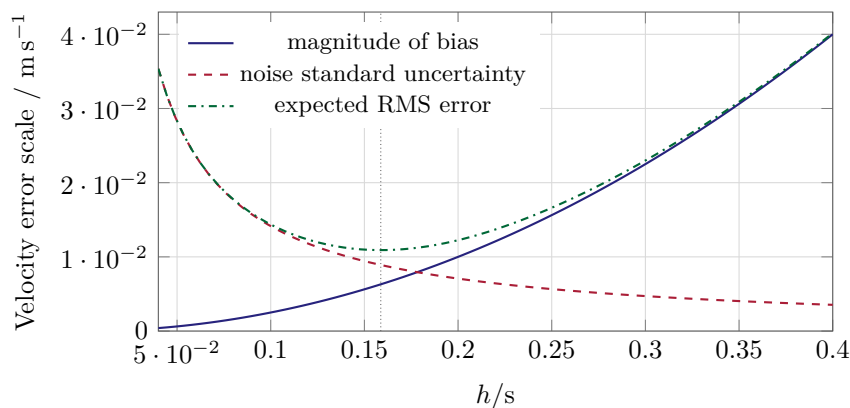


**Figure 4.23:** Calculated error balance for a central velocity estimate of a cubic coordinate with independent 2 mm standard position uncertainty. Bias and random uncertainty are distinct; the green curve combines their squared contributions. The optimum does not apply automatically to another path, estimator, or correlated disturbance.

#### 4.7.8 What a single projection cannot determine

A calibrated top view of a fixed plane can provide two Cartesian coordinates. A view of three-dimensional motion generally cannot. In the ideal orthographic model used here, a camera looking along  $z$  records only  $(x, y)$ , so an arbitrary function  $z(t)$  can be added without changing the recorded track. Perspective cameras require a different mapping; a scale calibration in one plane is not sufficient for unrestricted depth motion.

**Example 4.24: A straight image track with hidden acceleration**

A calibrated ideal top view records  $x = (1.00 \text{ m s}^{-1})t$ ,  $y = 0$ , from zero to two seconds. Compare the two possible spatial histories  $z_A = 0$  and  $z_B = \frac{1}{2}(1.00 \text{ m s}^{-2})t^2$ .

**Reconstruction.** Both produce identical image coordinates, identical projected velocity  $1.00\hat{\mathbf{i}} \text{ m s}^{-1}$ , and zero projected acceleration. History A has those same full vectors. History B instead has

$$\mathbf{v}_B = (1.00 \text{ m s}^{-1})\hat{\mathbf{i}} + (1.00 \text{ m s}^{-2})t\hat{\mathbf{k}}, \quad \mathbf{a}_B = 1.00\hat{\mathbf{k}} \text{ m s}^{-2}.$$

At two seconds its true speed is  $\sqrt{5} = 2.24 \text{ m s}^{-1}$ , although the projected speed remains  $1.00 \text{ m s}^{-1}$ .

**What is established?** Under the orthographic, fixed-axis assumptions, the recorded projected speed is a lower bound on the true speed because  $v^2 = v_x^2 + v_y^2 + v_z^2$ . The projected acceleration is likewise only part of the true vector. The recording cannot choose between these histories. One needs an independently verified constraint to a known plane, calibrated stereo observations, or an independent depth measurement. A smooth and apparently straight track does not supply that missing evidence.

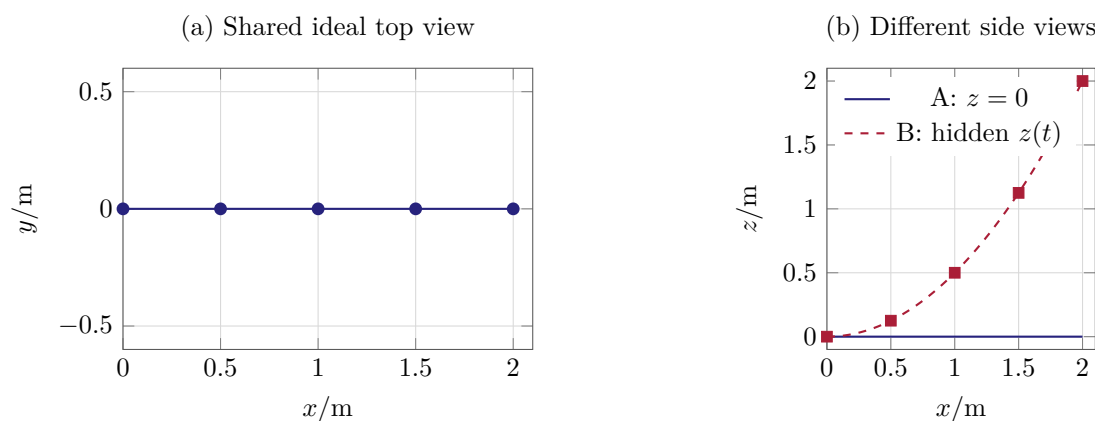


**Figure 4.24:** Two analytically prescribed spatial histories with the same recorded  $(x, y)$  projection. Markers are separated by 0.5 s. The side view reveals the different depth histories. Panel (a) uses a deliberately narrow vertical interval to display the straight projected track; panel (b) uses equal coordinate scales.

### 4.7.9 Event accuracy also depends on crossing speed

A position error and an event-time error have different units. For a transverse crossing of  $y = y_e$  at  $t_e$ , expand  $y(t_e + \delta t) \simeq y_e + v_y(t_e)\delta t$ . A small perturbation of the threshold or measured coordinate therefore produces the approximate event-time sensitivity

$$|\delta t_e| \simeq \frac{|\delta y|}{|v_y(t_e)|}, \quad v_y(t_e) \neq 0. \quad (4.59)$$

The sign depends on whether the perturbation changes the threshold or the measured coordinate. The magnitude relation isolates the sensitivity: a slow crossing turns a modest height error into a large time error.

**Example 4.25: A converged interpolation can still have uncertain timing**

An exact benchmark is  $y(t) = 1.00 \text{ m} - (1.00 \text{ m s}^{-2})t^2$ . Estimate the downward crossing of 0.300 m by linear interpolation of exact coordinate nodes, then compare with a 0.005 m height uncertainty.

**Exact event and interpolation.** The event is  $t_e = \sqrt{0.700} \text{ s} = 0.836660 \text{ s}$ , with  $v_y = -1.67332 \text{ m s}^{-1}$ . A spacing  $h = 0.100 \text{ s}$  brackets it between 0.8 and 0.9 s, where the heights are 0.36 and 0.19 m. Hence

$$\lambda = \frac{0.300 - 0.360}{0.190 - 0.360} = 0.352941, \quad t_e^{(h)} = 0.835294 \text{ s}.$$

For  $h = 0.050 \text{ s}$  the upper bracket ends at 0.85 s, giving  $t_e^{(h)} = 0.836364 \text{ s}$ . The absolute errors decrease from 0.001366 to 0.000296 s; they need not have an exact factor-four ratio because the bracket fraction changes.

**Measurement sensitivity.** The stated height uncertainty corresponds locally to  $\sigma_t \simeq 0.005/1.67332 = 0.00299 \text{ s}$ , larger than either interpolation error. Refining exact-node interpolation cannot remove this uncertainty in a real recorded event. The estimate assumes negligible timing errors and a locally linear crossing; it is not a complete uncertainty budget for a camera.

**Grazing limitation.** For  $y = 1.00 \text{ m} - (1.00 \text{ m s}^{-2})(t - 1.00 \text{ s})^2$ , the level 1.00 m is only touched at the apex. There is no sign-changing bracket there and  $v_y = 0$ , so Eq. (4.59) fails. Lowering the level by 0.005 m instead gives two intersections displaced by  $\sqrt{0.005} \text{ s} = 0.0707 \text{ s}$ ; raising it gives none. Event definition and crossing geometry matter as much as the number of samples.

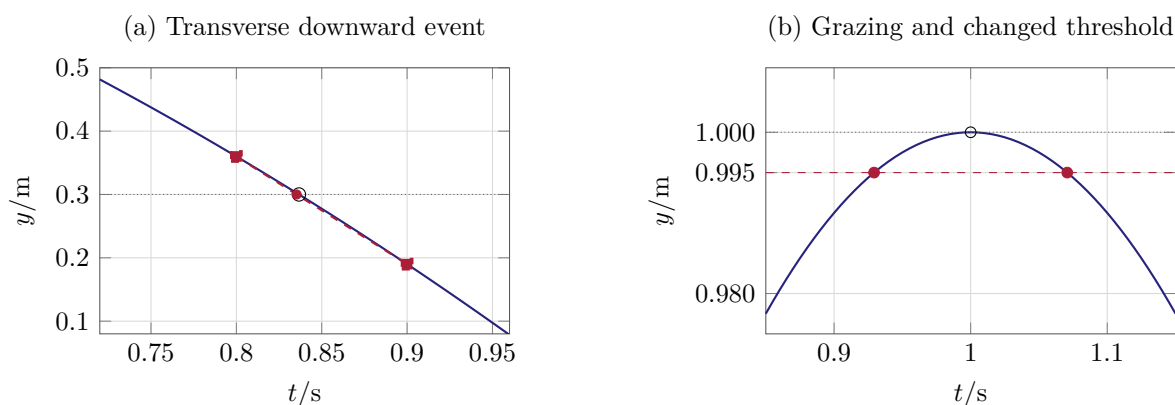


**Figure 4.25:** Verified event geometries. In (a) the open circle is the exact event, the filled dot the coarse interpolated event, and square markers the exact bracket nodes. In (b) the dotted level touches the trajectory, while a level only 5 mm lower creates two crossings approximately 0.141 s apart. The panels have different axis intervals to resolve their different sensitivities.

## 4.8 Chapter summary and a working formula sheet

### Summary

A spatial trajectory supplies position as a function of a common time. In a fixed Cartesian basis, differentiation gives velocity and acceleration component by component. Velocity describes both speed and direction; acceleration can change either or both.

Constant vector acceleration permits direct integration, but every component must refer to the same event. A projectile's height roots must be tested against horizontal position, direction of motion and the available flight interval. Equal-height range formulas have narrower assumptions than the basic time equations.

Circular motion demonstrates nonzero acceleration at constant speed. For a general smooth moving path, acceleration separates into a signed tangential contribution and a non-negative principal-normal contribution. Relative velocities require named observers and vector addition,

not addition of speeds. Numerical convergence tests and measurement uncertainty estimates address different sources of error.

### Formula summary

#### Definitions in a fixed Cartesian basis

$$\mathbf{v} = \dot{\mathbf{r}}, \quad \mathbf{a} = \dot{\mathbf{v}}, \quad v = |\mathbf{v}|, \quad s = \int v \, dt.$$

#### Constant vector acceleration over elapsed time $\tau$

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{a}\tau, \quad \mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0\tau + \frac{1}{2}\mathbf{a}\tau^2.$$

#### Uniform-gravity, no-drag projectile

$$x = x_0 + v_{0x}t, \quad y = y_0 + v_{0y}t - \frac{1}{2}gt^2, \quad (v_x, v_y) = (v_{0x}, v_{0y} - gt).$$

A specified height has candidate roots

$$t_{\pm} = \frac{v_{0y} \pm \sqrt{v_{0y}^2 + 2g(y_0 - y_f)}}{g}.$$

Retain only physically available roots and test the horizontal event separately. For unobstructed return to launch height,  $0 < \theta < 90^\circ$ ,

$$T_f = 2v_0 \sin \theta / g, \quad R = v_0^2 \sin(2\theta) / g.$$

#### Curved motion, where the local basis is defined

$$\mathbf{a} = a_t \hat{\mathbf{t}} + a_n \hat{\mathbf{n}}, \quad a_t = dv/dt, \quad a_n = v^2/R_c, \quad |\mathbf{a}| = \sqrt{a_t^2 + a_n^2}.$$

For a circle with constant signed  $\omega$ ,  $v = R|\omega|$ ,  $a_n = R\omega^2$ , and  $T = 2\pi/|\omega|$  if  $\omega \neq 0$ .

#### Translating, nonrotating frames with a common time

$$\mathbf{v}_{A/C} = \mathbf{v}_{A/B} + \mathbf{v}_{B/C}, \quad \mathbf{a}_{A/C} = \mathbf{a}_{A/B} + \mathbf{a}_{B/C}.$$

Acceleration is invariant under *constant* relative translation. Rotating axes need additional terms.

## 4.9 Advanced Topics for Technical and Engineering Students

A scanner may move outward while its angular position changes. A robot's sensor may report range and bearing rather than Cartesian coordinates. In these cases polar coordinates can simplify the prescribed path but complicate differentiation. The complication has a precise origin: the radial and transverse unit vectors themselves turn.

This section develops that changing-basis calculation and uses it to analyse an expanding spiral. It requires the product rule, the chain rule and trigonometric differentiation, reviewed in the derivation. The core normal–tangential formulas remain valid, but their directions must not be confused with the polar basis. The results are optional for the core pathway and introduce neither a new force law nor a rotating-frame dynamics model.

### 4.9.1 Why coordinate rates are not always vector components

For a point in a fixed plane, let  $r > 0$  be its distance from the origin and  $\theta$  its counterclockwise polar angle. Express the polar unit vectors in the fixed Cartesian basis:

$$\hat{\mathbf{e}}_r = \cos \theta \hat{\mathbf{i}} + \sin \theta \hat{\mathbf{j}}, \quad \hat{\mathbf{e}}_\theta = -\sin \theta \hat{\mathbf{i}} + \cos \theta \hat{\mathbf{j}}. \quad (4.60)$$

Both have unit length and their dot product is zero. The vector  $\hat{\mathbf{e}}_r$  points outward;  $\hat{\mathbf{e}}_\theta$  points toward increasing  $\theta$ , at right angles to  $\hat{\mathbf{e}}_r$ . Their directions depend on the event's angular coordinate. Figure 4.26 displays the same change of angle in both basis vectors.

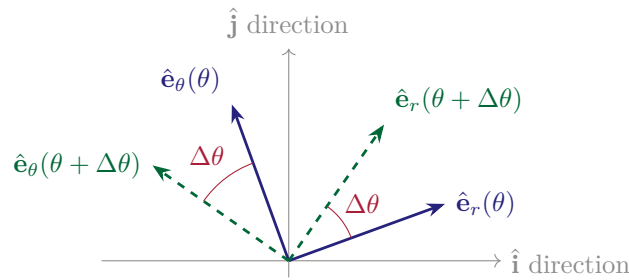


**Figure 4.26:** Both polar basis vectors rotate by the same angle. Equal arrow lengths represent dimensionless unit vectors; this is a basis-space construction, not a trajectory or a drawing of four velocities. The solid vectors are at  $20^\circ$ , the dashed vectors at  $55^\circ$ , with the same  $35^\circ$  increment.

For  $\theta = \theta(t)$ , the chain rule says  $d(\cos \theta)/dt = -\sin \theta \dot{\theta}$ , and similarly for sine. Therefore

$$\dot{\hat{\mathbf{e}}}_r = \dot{\theta}(-\sin \theta \hat{\mathbf{i}} + \cos \theta \hat{\mathbf{j}}) = \dot{\theta} \hat{\mathbf{e}}_\theta, \quad (4.61)$$

$$\dot{\hat{\mathbf{e}}}_\theta = -\dot{\theta}(\cos \theta \hat{\mathbf{i}} + \sin \theta \hat{\mathbf{j}}) = -\dot{\theta} \hat{\mathbf{e}}_r. \quad (4.62)$$

The signs are worth checking geometrically. At  $\theta = 0$ ,  $\hat{\mathbf{e}}_r = \hat{\mathbf{i}}$ . For positive angular rate it begins turning upward, so  $\dot{\hat{\mathbf{e}}}_r$  points toward  $+\hat{\mathbf{j}}$ . At the same event  $\hat{\mathbf{e}}_\theta = \hat{\mathbf{j}}$  begins turning toward  $-\hat{\mathbf{i}}$ . These observations independently check both signs.

A vector can change while its components in a moving basis remain constant. For example, a circle has constant polar radius but changing Cartesian position. Conversely, the derivative of a component is not generally the corresponding component of the derivative vector. The missing information is the basis rotation.

### 4.9.2 Deriving polar velocity and acceleration

The position vector is  $\mathbf{r} = r\hat{\mathbf{e}}_r$ . Apply the product rule to both factors:

$$\mathbf{v} = \dot{r}\hat{\mathbf{e}}_r + r\dot{\hat{\mathbf{e}}}_r = \dot{r}\hat{\mathbf{e}}_r + r\dot{\theta}\hat{\mathbf{e}}_\theta. \quad (4.63)$$

Thus the radial velocity component is  $v_r = \dot{r}$ , while the transverse velocity component is  $v_\theta = r\dot{\theta}$ , not  $\dot{\theta}$  alone. Their units must both be metres per second. Since the basis is orthonormal,

$$v = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}. \quad (4.64)$$

Differentiate Eq. (4.63) term by term before collecting components:

$$\begin{aligned} \frac{d}{dt}(\dot{r}\hat{\mathbf{e}}_r) &= \ddot{r}\hat{\mathbf{e}}_r + \dot{r}\dot{\theta}\hat{\mathbf{e}}_\theta, \\ \frac{d}{dt}(r\dot{\theta}\hat{\mathbf{e}}_\theta) &= (\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\mathbf{e}}_\theta - r\dot{\theta}^2\hat{\mathbf{e}}_r. \end{aligned}$$

The two transverse terms proportional to  $\dot{r}\dot{\theta}$  arise from two different differentiated factors. Their sum is essential:

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{e}}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\mathbf{e}}_\theta. \quad (4.65)$$

These are exact plane-kinematic relations wherever the coordinates are differentiable and  $r > 0$ . They are not approximations restricted to circular motion. At the coordinate origin the angle and polar basis are not uniquely defined; Cartesian coordinates avoid that singular description.

### Caution

The polar acceleration component  $a_r$  is  $\ddot{r} - r\dot{\theta}^2$ , not merely  $\ddot{r}$ . The polar transverse component  $a_\theta$  is  $r\ddot{\theta} + 2\dot{r}\dot{\theta}$ , not merely  $r\ddot{\theta}$ . A constant radial rate and a constant angular rate can therefore coexist with nonzero acceleration in both polar directions.

Three limiting cases test the formula. If  $\dot{\theta} = 0$  and  $\ddot{\theta} = 0$ , motion is along a fixed ray and  $\mathbf{a} = \ddot{r}\hat{\mathbf{e}}_r$ . If  $r = R$  is constant, then  $\mathbf{v} = R\dot{\theta}\hat{\mathbf{e}}_\theta$  and  $\mathbf{a} = -R\dot{\theta}^2\hat{\mathbf{e}}_r + R\ddot{\theta}\hat{\mathbf{e}}_\theta$ . If the angular rate is also constant, only the inward circular acceleration remains, agreeing with the independently derived core result. These tests probe different terms rather than repeating one special case three times.

### 4.9.3 A spiral scan: motion in two changing directions

A simple prescribed outward scan uses

$$r(t) = r_0 + ut, \quad \theta(t) = \Omega t, \quad r_0 > 0, \quad u > 0, \quad \Omega > 0. \quad (4.66)$$

Here  $u$  is the constant outward coordinate rate, not the total speed, and  $\Omega$  is the constant angular rate. The curve is an expanding spiral. A full turn increases radius by  $2\pi u/\Omega$ ; this radial separation of successive turns is a geometric property of the schedule, not a statement about exposure or sampling density along the path.

Since  $\ddot{r} = 0$  and  $\ddot{\theta} = 0$ ,

$$\mathbf{v} = u\hat{\mathbf{e}}_r + r\Omega\hat{\mathbf{e}}_\theta, \quad (4.67)$$

$$\mathbf{a} = -r\Omega^2\hat{\mathbf{e}}_r + 2u\Omega\hat{\mathbf{e}}_\theta, \quad (4.68)$$

$$v = \sqrt{u^2 + r^2\Omega^2}, \quad |\mathbf{a}| = \sqrt{r^2\Omega^4 + 4u^2\Omega^2}. \quad (4.69)$$

The growing circumference makes the transverse velocity increase even though the angular rate is fixed. The acceleration's transverse part also contains the change in direction of the outward velocity. Differentiating only the scalar rates would miss both mechanisms.

**Example 4.26: A prescribed expanding spiral**

A point-like scanner follows Eq. (4.66) for  $0 \leq t \leq 4.00$  s, with  $r_0 = 0.400$  m,  $u = 0.100$  m s<sup>-1</sup> and  $\Omega = 1.00$  rad s<sup>-1</sup>. Find its velocity and acceleration at  $t = \pi/2$  s. Is its actual direction of motion the polar transverse direction?

**Polar calculation.** At this event,  $\theta = \pi/2$ ,  $r = 0.400 + 0.100(\pi/2) = 0.557080$  m,  $\hat{\mathbf{e}}_r = \hat{\mathbf{j}}$ , and  $\hat{\mathbf{e}}_\theta = -\hat{\mathbf{i}}$ . Consequently,

$$\begin{aligned} \mathbf{v} &= (-0.557080\hat{\mathbf{i}} + 0.100\hat{\mathbf{j}}) \text{ m s}^{-1}, & v &= 0.565984 \text{ m s}^{-1}, \\ \mathbf{a} &= (-0.200\hat{\mathbf{i}} - 0.557080\hat{\mathbf{j}}) \text{ m s}^{-2}, & |\mathbf{a}| &= 0.591893 \text{ m s}^{-2}. \end{aligned}$$

**Actual tangent.** The tangent is  $\hat{\mathbf{t}} = \mathbf{v}/v$ , tilted outward from  $\hat{\mathbf{e}}_\theta$  by

$$\beta = \tan^{-1}\left(\frac{u}{r\Omega}\right) = 10.2^\circ.$$

Thus the transverse unit vector points left, while the actual motion is left and slightly upward. Figure 4.27 separates these directions.

**Independent Cartesian differentiation.** Starting instead from

$$x = (r_0 + ut) \cos \Omega t, \quad y = (r_0 + ut) \sin \Omega t,$$

ordinary product-rule differentiation gives

$$\begin{aligned} \dot{x} &= u \cos \Omega t - r\Omega \sin \Omega t, & \dot{y} &= u \sin \Omega t + r\Omega \cos \Omega t, \\ \ddot{x} &= -2u\Omega \sin \Omega t - r\Omega^2 \cos \Omega t, \\ \ddot{y} &= 2u\Omega \cos \Omega t - r\Omega^2 \sin \Omega t. \end{aligned}$$

Substitution at  $\Omega t = \pi/2$  reproduces all four Cartesian components above, without assuming the polar acceleration formula.

**Checks.** Every term in acceleration has units of length divided by time squared. Setting  $u = 0$  recovers a circle. Setting  $\Omega = 0$  gives straight outward motion at constant speed and zero acceleration. Setting only  $\ddot{r} = \ddot{\theta} = 0$  does not give zero acceleration: that is the actual spiral being analysed.

#### 4.9.4 Polar components are not normal–tangential components

In a circle about the origin,  $-\hat{\mathbf{e}}_r$  is the principal normal, and for counterclockwise motion  $\hat{\mathbf{e}}_\theta$  is the forward tangent. In a spiral neither identification is generally valid. The origin supplies a radial direction, but the local centre of curvature is elsewhere.

The distinction can be quantified without a full course in differential geometry. For the spiral, project the acceleration onto the actual velocity. Equations (4.67)–(4.68), together with orthonormality, give

$$\mathbf{v} \cdot \mathbf{a} = u(-r\Omega^2) + (r\Omega)(2u\Omega) = ur\Omega^2.$$

Hence, for the positive parameters in Eq. (4.66),

$$a_t = \frac{ur\Omega^2}{\sqrt{u^2 + r^2\Omega^2}}. \quad (4.70)$$

This is not the polar transverse component  $2u\Omega$ . It is also exactly the derivative of the speed in Eq. (4.69), which supplies an independent check.

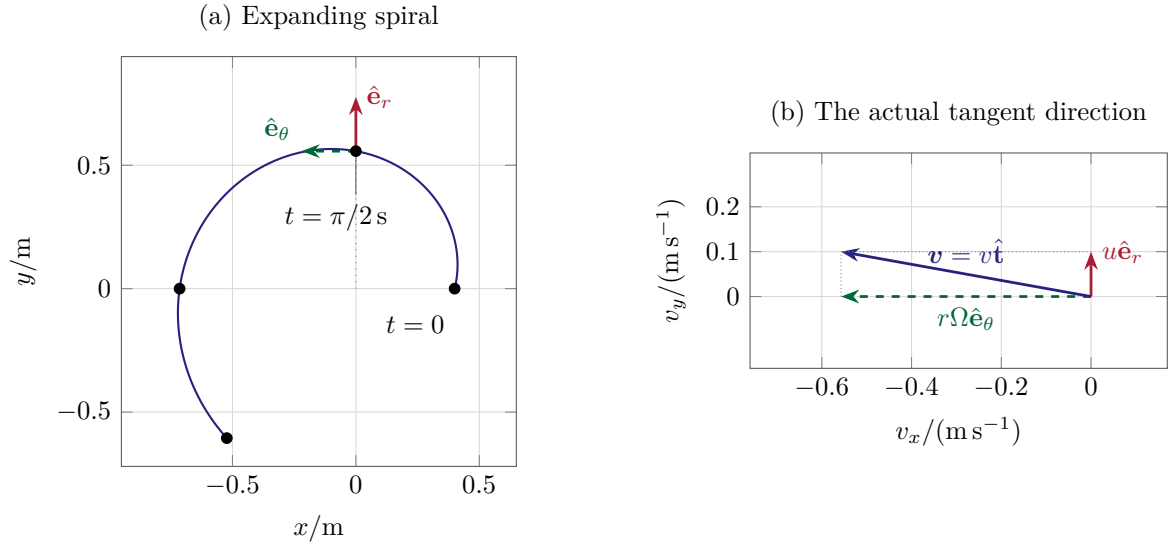


**Figure 4.27:** The calculated spiral and its velocity decomposition at the same event. Panel (a) has equal metre scales and direction-only unit-vector glyphs; these are not velocity arrows. Panel (b) uses a common velocity scale and shows the actual tangent  $\hat{t}$ , distinct from  $\hat{e}_\theta$  because  $\dot{r} \neq 0$ . The changing-basis geometry underlying this calculation is shown separately in Fig. 4.26.

For planar vectors, the magnitude of  $\mathbf{v} \times \mathbf{a}$  is  $va_n$ . In the ordered basis  $(\hat{e}_r, \hat{e}_\theta)$ , the signed perpendicular determinant is

$$v_r a_\theta - v_\theta a_r = 2u^2\Omega + r^2\Omega^3.$$

For  $\Omega > 0$ , this gives

$$a_n = \frac{\Omega(2u^2 + r^2\Omega^2)}{v}, \quad R_c = \frac{v^3}{\Omega(2u^2 + r^2\Omega^2)}. \quad (4.71)$$

For negative  $\Omega$ , replace the prefactor in these magnitude expressions by  $|\Omega|$ . The result assumes  $\Omega \neq 0$ ; the zero-rate path is straight and its curvature is zero rather than obtained by dividing by zero. At  $u = 0$ ,  $v = r|\Omega|$  and Eq. (4.71) reduces to  $R_c = r$ , as it should for a circle.

The two decompositions therefore answer different questions. Polar components describe acceleration relative to the line of sight from the origin. Normal–tangential components describe its effect on speed and turning. In engineering motion analysis, using the convenient measurement coordinates is useful only if the vector interpretation is retained.

#### 4.9.5 A bounded scan interval and an error-diagnosis investigation

Suppose a constructed scan specification requires  $|\mathbf{a}| \leq 0.700 \text{ ms}^{-2}$  for the parameters of Example 4.26. This is an imposed kinematic limit, not a tested device rating. Equation (4.69) gives

$$r^2\Omega^4 + 4u^2\Omega^2 \leq a_{\max}^2.$$

For the stated positive  $\Omega$ , the greatest permitted radius is

$$r_{\max} = \frac{\sqrt{a_{\max}^2 - 4u^2\Omega^2}}{\Omega^2} = \sqrt{0.45} \text{ m} = 0.670820 \text{ m}.$$

The specified interval must therefore end no later than

$$t_{\max} = \frac{r_{\max} - r_0}{u} = 2.70820 \text{ s},$$

not at four seconds. A real transition to a new schedule would itself need to satisfy the relevant continuity and acceleration constraints; truncating the formula does not design that transition. If  $a_{\max} < 2u\Omega$ , even the transverse contribution alone exceeds the imposed limit.

An informative computational check is to compare three expressions at several events: the Cartesian second derivative, the full polar formula, and the incorrect “coordinate-only” expression  $\ddot{r}\hat{\mathbf{e}}_r + r\ddot{\theta}\hat{\mathbf{e}}_\theta$ . For this spiral the incorrect expression is zero, whereas the other two agree and are nonzero. A subtler incorrect version retains  $-r\Omega^2\hat{\mathbf{e}}_r$  but omits  $2u\Omega\hat{\mathbf{e}}_\theta$ ; its vector error has the fixed magnitude  $2u\Omega = 0.200 \text{ m s}^{-2}$ . That error changes direction around the path, so comparing only one Cartesian component at one angle can conceal it.

The investigation should test several angles, the circular limit  $u \rightarrow 0$ , and the fixed-ray limit  $\Omega \rightarrow 0$ . It should also compare speed obtained from the Cartesian derivatives with  $\sqrt{\dot{r}^2 + r^2\dot{\theta}^2}$ . Agreement among independent representations is stronger evidence than a plausible spiral drawing. It still verifies only the prescribed kinematics, not the forces needed to produce the scan.

## 4.10 Exercises and problems

Use  $g = 9.81 \text{ m s}^{-2}$  for projectile models. Unless otherwise stated, coordinates are measured in a fixed laboratory frame, drag is neglected and objects are point-like. In coordinate functions explicitly described as “SI numerical units,” enter  $t$  in seconds; coordinate outputs are metres, and derivatives carry the corresponding SI units. State event domains, show units and explain vector directions. Problems 24–26 use the supporting data/computational pathway; Problems 27–32 use the advanced section.

### A. Concepts and representations

- Two planar paths have SI numerical coordinates  $\mathbf{r}_A(t) = (t, t^2)$  and  $\mathbf{r}_B(t) = (t^2, t)$ . Both points pass through  $(1, 1) \text{ m}$  at  $t = 1 \text{ s}$ . Compare their velocities, speeds and accelerations at that event. Explain what a single position observation cannot establish.
- Assess each statement: “Constant speed means zero acceleration” and “Constant acceleration magnitude permits  $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2$ .” Give a counterexample or the missing condition in each case.
- For  $x = 2t$  and  $y = 3t - t^2$  in SI numerical units, find the time when the  $y(t)$  graph has zero slope. At that event find  $\mathbf{v}$ ,  $\mathbf{a}$ , the speed and the slope of  $y(x)$ . Explain why neither motion nor acceleration has stopped.
- An observer replaces  $\mathbf{r}$  by  $\mathbf{r}' = \mathbf{r} - \mathbf{b}$ . Compare velocity and acceleration when (a)  $\mathbf{b}$  is constant and (b)  $\mathbf{b} = \mathbf{U}t$  with constant  $\mathbf{U}$ . Why is a rotation of the coordinate axes not covered by either subtraction alone?
- A point moves clockwise at nonzero constant speed around a circle centred at the origin. At the top of the circle, draw the velocity and acceleration directions. What reverses if the sense of rotation is reversed at the same position while speed is kept fixed?
- A point on a circle of radius  $2.00 \text{ m}$  has speed  $v(t) = 4.00 - 0.500t$  in SI numerical units for  $0 \leq t \leq 4 \text{ s}$ . Find  $a_t$ ,  $a_n$  and  $|\mathbf{a}|$  at two seconds. Translate the result into a convention that quotes a positive tangential deceleration magnitude rather than our signed  $a_t$ .

### B. Vector functions and event reconstruction

- A probe has  $\mathbf{r}(t) = (t^2, 2t, 0.5t^3)$  in SI numerical coordinates for  $0 \leq t \leq 3 \text{ s}$ . Find  $\mathbf{r}$ ,  $\mathbf{v}$ ,  $\mathbf{a}$  and their relevant magnitudes at two seconds. Find  $dy/dx$  and  $dz/dx$  there. Explain where the tails of position and local velocity arrows belong and why their drawn lengths need separate scales.
- For  $\mathbf{r}(t) = (2t, 1 - t, 0.5t^2)$  in SI numerical units, find the average velocity over  $0 \leq t \leq 2 \text{ s}$ . Compare it with  $\mathbf{v}(1 \text{ s})$ , and write the integral for average speed. Explain why equality of the

two velocity vectors does not make average speed equal to their magnitude.

9. A point has constant acceleration  $(0.500, -1.00) \text{ m s}^{-2}$ . Its initial position is  $(1.00, -2.00) \text{ m}$  and its position two seconds later is  $(5.00, 1.00) \text{ m}$ . Determine the required initial and final velocities and verify the endpoint.
10. A prescribed acceleration is  $\mathbf{a}(t) = (\cos t, \sin t)$  in SI numerical units, where the trigonometric argument means  $(1 \text{ s}^{-1})t$ . Initially  $\mathbf{r} = \mathbf{v} = 0$ . Integrate to obtain position and velocity, and evaluate them at  $t = \pi/2 \text{ s}$ . Why does the constant-vector-acceleration shortcut fail?
11. Revisit the bead and belt of Example 4.5. In case (a), reverse the belt velocity while leaving the bead's initial velocity unchanged. In a separate case (b), retain the original belt velocity but halve the bead's initial horizontal speed. Find the required initial marker coordinate in each case and explain which features of the component-time graphs change.
12. A point-like bead leaves a horizontal guide  $0.800 \text{ m}$  above a level floor with horizontal speed  $1.40 \text{ m s}^{-1}$ . Neglect drag. Find the first contact time, horizontal displacement and velocity immediately before contact, including the velocity angle from  $+x$ .

### C. Projectile events and model limits

13. A projectile starts at the origin with velocity  $(6.00, 8.00) \text{ m s}^{-1}$ . Find both times when it reaches  $y = 2.00 \text{ m}$ , their horizontal coordinates, and the apex height. Assume no intervening obstacle. Identify ascent and descent without inspecting a plotted curve.
14. Two no-drag launches from the same level have speed  $10.0 \text{ m s}^{-1}$  and angles  $30^\circ$  and  $60^\circ$ . Both return unobstructed to launch height. Compare their ranges, flight times and maximum height gains. State which equality would cease to be guaranteed if the landing heights differed.
15. Use the constructed descending interception of Example 4.6, with the target's initial coordinate defined by the unrounded expression  $x_{T0} = (3.00 \text{ m s}^{-1})t_+$ . At the earlier height root  $t_-$ , how far ahead of the bead is the target? At the actual interception, find the bead's velocity relative to the target. Explain why two height roots do not imply two interceptions.
16. A bead is launched from  $(0, 0.500) \text{ m}$  with velocity  $(3.00, 2.00) \text{ m s}^{-1}$ . A floor at  $y = 0$  extends to  $x = 0.600 \text{ m}$ , where a vertical riser reaches  $y = 0.650 \text{ m}$ . For all  $x \geq 0.600 \text{ m}$ , the platform top is at that height. Find the first contact. Check both the riser and the platform's restricted horizontal domain.

### D. Curved paths and relative motion

17. A marker moves on a circle of radius  $0.400 \text{ m}$  with initial angle  $\pi/2$  and constant angular rate  $-3.00 \text{ rad s}^{-1}$ . Find its position, velocity and acceleration at  $t = \pi/6 \text{ s}$ , and its period. Specify the viewing direction defining clockwise motion.
18. A circle is centred at  $\mathbf{r}_C = (1.00, -2.00) \text{ m}$ . A point at  $\mathbf{r}_P = (1.00, 0) \text{ m}$  moves clockwise at constant speed  $3.00 \text{ m s}^{-1}$ . Find  $\mathbf{v}$  and  $\mathbf{a}$ . Diagnose the direction error in applying  $\mathbf{a} = -\omega^2 \mathbf{r}_P$  in this laboratory coordinate system.
19. At one event a prescribed path must have speed  $5.00 \text{ m s}^{-1}$ , tangential acceleration magnitude  $1.00 \text{ m s}^{-2}$ , and total acceleration no greater than an illustrative limit  $3.00 \text{ m s}^{-2}$ . Find the minimum curvature radius. Explain why this is not a force-capacity certification.
20. At a moving event,  $\mathbf{v} = (3.00, 4.00) \text{ m s}^{-1}$  and  $\mathbf{a} = (-0.400, 2.20) \text{ m s}^{-2}$ . Find  $a_t$ ,  $\mathbf{a}_\perp$ ,  $a_n$ ,  $\hat{\mathbf{n}}$  and  $R_c$ . Check both orthogonality and the acceleration magnitude. What happens to  $a_t$  and the normal direction if the acceleration vector alone is reversed?

21. At one event, a marker has stage-relative velocity  $(-0.200, 0.500) \text{ m s}^{-1}$  and stage-relative acceleration  $(0, 0.200) \text{ m s}^{-2}$ . The stage translates without rotation with laboratory velocity  $(0.800, 0.300) \text{ m s}^{-1}$  and laboratory acceleration  $(0.100, 0) \text{ m s}^{-2}$ . Find the marker's laboratory velocity, speed and acceleration. Why do the observers not agree on acceleration?
22. For the channel in Example 4.16, a technician mistakenly programs the required shore velocity  $(-0.600, 1.200) \text{ m s}^{-1}$  as the craft's water-relative velocity. Where would the craft arrive after 50.0 s under that mistaken command? Does the command retain the required water-relative speed? Give the correct velocity triangle and assess an upstream crossing if the current is increased to  $2.00 \text{ m s}^{-1}$  while craft speed remains  $2.00 \text{ m s}^{-1}$ .

### E. Data, computation and investigations

23. Two points have relative position  $\mathbf{R}_0 = (2.00, 5.00) \text{ m}$  and constant relative velocity  $\mathbf{V} = (1.00, -1.00) \text{ m s}^{-1}$ . Find their closest approach for  $t \geq 0$ . Repeat when the model is valid only for  $0 \leq t \leq 1.00 \text{ s}$ .
24. Use the synthetic coordinate table in the sampled-coordinate example to estimate  $\mathbf{v}$  and  $\mathbf{a}$  at  $t = 0.400 \text{ s}$  by central differences. Compare with the exact path. Explain why the acceleration bias has the opposite sign to that at 0.600 s.
25. For the prescribed-acceleration model of Example 4.21, use the closed-form node identity to find  $x(1.2 \text{ s})$  numerically with  $h = 0.150 \text{ s}$ , and reproduce the interpolated upward event from its bracket. Separately, derive the forward Euler position error after  $N$  equal steps for a constant vector acceleration, with exact initial state. Compare this error with midpoint and discuss whether exact nodes guarantee an exact linearly interpolated event.
26. Design a short planar video-tracking investigation that distinguishes genuine turning from coordinate noise. State the observations, calibration, timing checks, derivative method, comparison of analysis settings and a defensible conclusion criterion. No particular apparatus or measured data are supplied; give a reasoning plan rather than invented results.

### F. Advanced challenges

27. For the expanding spiral, a proposed formula is  $\mathbf{a}_{\text{bad}} = -r\Omega^2\hat{\mathbf{e}}_r + u\Omega\hat{\mathbf{e}}_\theta$ . Identify its missing product-rule contribution and compute  $\mathbf{a}_{\text{bad}} - \mathbf{a}$  at  $t = \pi/2 \text{ s}$  for the chapter's spiral parameters. Why might a test using only circular motion fail to expose the error?
28. A different prescribed planar path has  $r(t) = 0.300 + 0.0200t^2$  in metres and  $\theta(t) = 2.00t + 0.100t^2$  in radians, with  $t$  entered in seconds and  $0 \leq t \leq 2 \text{ s}$ . Find  $v_r$ ,  $v_\theta$ ,  $a_r$ ,  $a_\theta$  and speed at one second. Keep radial coordinate acceleration distinct from the radial component of vector acceleration.
29. Evaluate the spiral's normal-tangential acceleration components and radius of curvature at the event of Example 4.26. Compare  $a_t$  with  $a_\theta$  and  $R_c$  with  $r$ . Explain geometrically why these pairs are not equal in general.
30. Retain the full 0–4.00 s spiral interval,  $r_0 = 0.400 \text{ m}$  and  $u = 0.100 \text{ m s}^{-1}$ , but allow the positive constant angular rate  $\Omega$  to change. Find the largest  $\Omega$  consistent throughout the interval with the illustrative acceleration bound  $|\mathbf{a}| \leq 0.700 \text{ m s}^{-2}$ . Explain why checking the final radius is sufficient for this model.
31. A point moves at constant Cartesian velocity along  $x = b$ ,  $y = Vt$ , with  $b > 0$  and  $V > 0$ . Use  $r = \sqrt{b^2 + V^2t^2}$  and  $\theta = \tan^{-1}(Vt/b)$  to show that the full polar acceleration is zero even though both coordinate second derivatives can be nonzero. State the role of  $b > 0$ .

32. Independently estimate the spiral acceleration at  $t_* = \pi/2$  s from Cartesian positions only, using

$$\mathbf{a}_h = \frac{\mathbf{r}(t_* + h) - 2\mathbf{r}(t_*) + \mathbf{r}(t_* - h)}{h^2}.$$

Use the parameters of Example 4.26 and  $h = 0.100, 0.0500, 0.0250$  s. Report vector errors against the analytical acceleration and the ratios of their norms. Explain what convergence does and does not establish.

## G. Synthesis and visual inference

These final tasks combine the chapter's representations. Problems 38–39 use the supporting measurement/numerical pathway; Problem 40 uses the advanced polar section.

33. Two points follow the same parabola with SI coordinate functions  $\mathbf{r}_A(t) = (t, t^2)$  and  $\mathbf{r}_B(t) = (2t, 4t^2)$  for  $t \geq 0$ . (a) Find when each passes through (1, 1) m. (b) Compare their velocity and acceleration vectors at that spatial point. (c) Explain why path geometry alone does not determine acceleration.
34. A point starts at (1, 2, 3) m with  $\mathbf{v}_0 = (1, 2, 0) \text{ m s}^{-1}$  and constant  $\mathbf{a} = (0, 1, 1) \text{ m s}^{-2}$ . (a) Find its position at 2 s. (b) Use  $\mathbf{v}_0 \times \mathbf{a}$  to find a normal to the plane of motion. (c) Give a Cartesian equation of that plane and check the endpoint. Distinguish three varying coordinates from a path that explores a three-dimensional volume.
35. Figure 4.28 gives velocity components of a projectile launched at  $y_0 = 0.200$  m. (a) Use signed graph area to find its height at 0.400 s. (b) Find the apex time and total speed there. (c) Find the first contact with a level floor at  $y = 0$ , and state the physical graph domain.

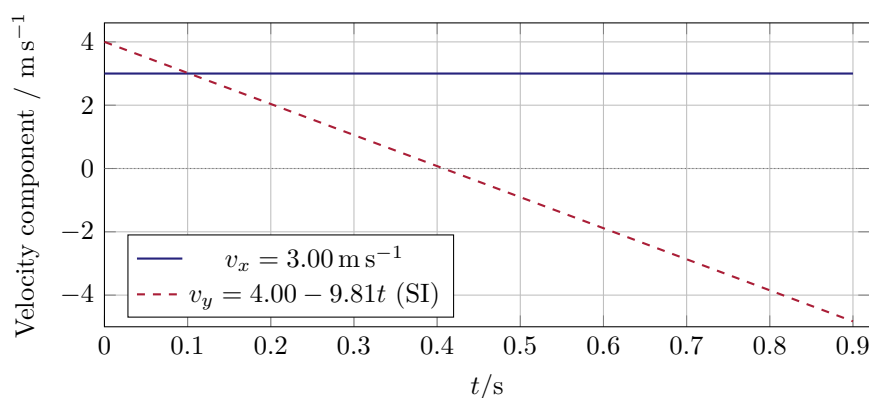


**Figure 4.28:** Velocity data for Problem 35. The lines are the ideal no-contact continuation; the floor event determines which interval represents free flight. Signed area under  $v_y(t)$  is vertical displacement, not distance.

36. The curve in Fig. 4.29 represents a point following a bend of constant radius with signed  $a_t = -0.600 \text{ m s}^{-2}$ . (a) Infer the bend radius from the stated normal acceleration at  $v = 2.00 \text{ m s}^{-1}$ . (b) Find the greatest permitted speed if  $|\mathbf{a}| \leq 1.50 \text{ m s}^{-2}$ . (c) Explain why deceleration and inward acceleration cannot be added as scalar magnitudes.
37. A nonrotating translating stage has laboratory origin  $\mathbf{b}(t) = (0.100 \text{ m s}^{-2})t^2\hat{\mathbf{i}}$ . A marker's stage-relative position is  $\mathbf{r}_{M/S} = (0.500 \text{ m})[\cos(\Omega t)\hat{\mathbf{i}} + \sin(\Omega t)\hat{\mathbf{j}}]$ , with  $\Omega = 2.00 \text{ rad s}^{-1}$ . (a) Find laboratory position, velocity and acceleration at  $t = 0$ . (b) Which acceleration terms come from translation and from circular motion? (c) Explain why subtracting the stage velocity does not make the two measured accelerations identical.

38. Five synthetic samples are

$$t/s = (-0.4, -0.2, 0, 0.2, 0.4),$$

$$x/m = (0.162, 0.042, -0.003, 0.038, 0.161).$$

Fit  $x = c + bt + At^2$  by unweighted least squares. (a) Find the fitted  $b$  and acceleration  $2A$ . (b) Compare with the three-point central acceleration at  $t = 0$ . (c) Explain why the smoother fit need not be closer to the true motion. Negative times here denote observations before a selected middle frame, not extrapolation before a launch.

39. A prescribed coordinate is  $y(t) = [1 - (t - 1)^2]$  m when  $t$  is entered in seconds, on  $0 \leq t \leq 2$  s. (a) Find its event times at  $y_e = 1.000$  m and  $y_e = 0.990$  m. (b) Explain why samples at 0.8 and 1.2 s miss both crossings of the lower event in an endpoint-sign test. (c) Explain why an event near a maximum is sensitive to the chosen height, and give a method that detects the maximum.

40. An optional polar scan has  $r(t) = 0.500 \text{ m} + (0.0500 \text{ m}) \cos(\omega_r t)$  and  $\theta(t) = \Omega t$ , with  $\omega_r = 2.00 \text{ rad s}^{-1}$ ,  $\Omega = 1.00 \text{ rad s}^{-1}$ , and  $t \geq 0$ . (a) Find its Cartesian velocity and acceleration at  $t = 0$ . (b) Compare with an assumed circle of the instantaneous radius. (c) Identify the contribution that the circle model omits, and verify the answer by differentiating Cartesian position directly.

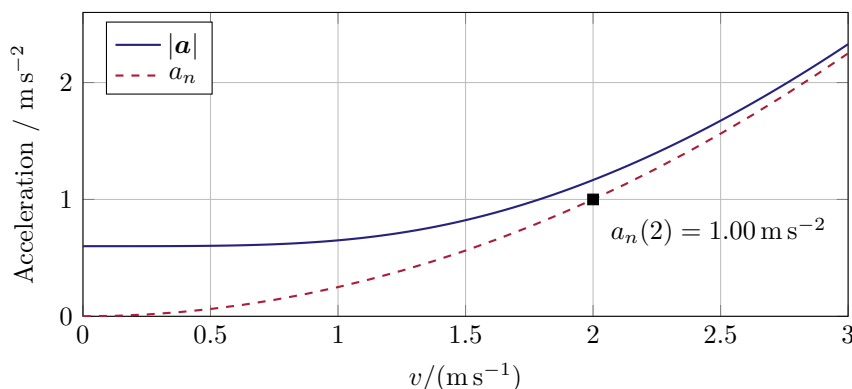


**Figure 4.29:** Acceleration versus speed for Problem 36. The circle radius is constant. The nonzero vertical intercept of the solid curve reflects the stipulated tangential component; its extension to  $v = 0$  is a limiting magnitude, where a velocity-defined forward tangent needs separate interpretation.

## 4.11 Complete reasoned solutions

Every retained problem and subpart is addressed below. The working assumptions are those of the corresponding statement.

These solutions use the assumptions and event domains stated with the problems. A derivative is taken before an event time is inserted; a root is accepted only after its physical domain is checked. Coordinate formulas described as SI numerical formulas have the dimensions supplied in their statements. The four solution figures display verified geometry or computed error data, and their captions distinguish vectors from mere direction guides.

### Problem 1: one position does not specify a motion

Differentiate in the fixed Cartesian basis:

$$\mathbf{v}_A = (1, 2t), \quad \mathbf{v}_B = (2t, 1), \quad \mathbf{a}_A = (0, 2), \quad \mathbf{a}_B = (2, 0).$$

At  $t = 1$  s, the velocities are  $(1, 2)$  and  $(2, 1) \text{ m s}^{-1}$ . Their common speed is  $\sqrt{5} = 2.23607 \text{ m s}^{-1}$ , but their directions differ: their tangent slopes are 2 and  $1/2$ . The accelerations are respectively  $(0, 2)$

and  $(2, 0) \text{ m s}^{-2}$ , both of magnitude  $2 \text{ m s}^{-2}$ . Thus a position snapshot fixes neither the tangent, speed, nor acceleration. Here even identical position, event time and speed leave velocity and acceleration different. Their values require the surrounding time dependence, not the coordinates of one event alone.

### Problem 2: constant magnitudes and constant vectors

Both assertions are false as stated. Consider uniform circular motion with fixed  $R > 0$  and angular rate  $\omega \neq 0$ . Its speed is  $R|\omega|$ , constant, but its velocity direction turns. Differentiation gives the nonzero inward acceleration  $\mathbf{a} = -\omega^2 \mathbf{r}$ , of constant magnitude  $R\omega^2$ . This disproves the first assertion.

The same circle disproves the second. Although  $|\mathbf{a}|$  is constant, its Cartesian components change as its direction turns. The expression  $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2} \mathbf{a} t^2$  requires one acceleration *vector* to remain constant throughout the interval in the chosen fixed basis. It follows by integrating that constant vector twice; a constant norm does not allow it to be taken outside either integral. For a changing direction, integrate  $\mathbf{a}(t)$ , even if its norm is known exactly.

### Problem 3: a horizontal tangent is not rest

The coordinate-time slope is  $\dot{y} = 3 - 2t$ , so it vanishes at  $t = 1.50 \text{ s}$ . The full derivatives are

$$\mathbf{v} = (2, 3 - 2t) \text{ m s}^{-1}, \quad \mathbf{a} = (0, -2) \text{ m s}^{-2}.$$

At that event  $\mathbf{v} = (2, 0) \text{ m s}^{-1}$ , giving speed  $2 \text{ m s}^{-1}$ . Since  $\dot{x} = 2 \neq 0$ , the path slope is  $dy/dx = \dot{y}/\dot{x} = 0$ . Eliminating time checks this result:  $y = 1.5x - 0.25x^2$ , whose slope vanishes at  $x = 3 \text{ m}$ . The point moves horizontally through its maximum  $y$ ; only its vertical velocity is zero. The downward acceleration is still nonzero and will make the vertical velocity negative immediately afterward.

### Problem 4: translating origins and rotating bases

The subtraction assumes that both coordinate systems use parallel, nonrotating axes and the same clock. In case (a), differentiating  $\mathbf{r}' = \mathbf{r} - \mathbf{b}$  with constant  $\mathbf{b}$  gives  $\mathbf{v}' = \mathbf{v}$  and  $\mathbf{a}' = \mathbf{a}$ . The position components change because the origin changes; their derivatives do not.

In case (b),  $\mathbf{b} = \mathbf{U}t$ , with constant  $\mathbf{U}$ , gives

$$\mathbf{v}' = \mathbf{v} - \mathbf{U}, \quad \mathbf{a}' = \mathbf{a}.$$

The observers disagree on velocity but agree on acceleration because the relative origin velocity is constant. A fixed rotation needs an additional matrix transformation of all vector components; subtraction alone cannot rotate an axis. If the axes rotate with time, differentiating the changing basis adds further terms. Those terms are absent from these pure-translation relations, so these relations must not be applied to a rotating observer without a separate derivation.

### Problem 5: clockwise motion at the top

View the  $xy$  plane from  $+z$ , with  $+x$  to the right and  $+y$  upward. At the top, the radius points upward. Clockwise motion has a rightward tangent, so draw  $\mathbf{v}$  from the moving point toward  $+x$ , and draw  $\mathbf{a}$  downward toward the centre. Acceleration magnitude is  $v^2/R$ , even though speed is constant.

Comparing the two senses of *uniform* rotation at the same position reverses  $\mathbf{v}$  but leaves  $\mathbf{a}$  inward and unchanged in magnitude. This compares two allowed motions. A physical instantaneous reversal would require an additional transient acceleration and cannot itself be represented by uninterrupted uniform circular motion.

**Problem 6: signed tangential acceleration**

The speed remains positive over the stated interval. At  $t = 2.00$  s, it is  $v = 4.00 - 0.500(2.00) = 3.00 \text{ m s}^{-1}$ . The forward tangent convention gives

$$a_t = \frac{dv}{dt} = -0.500 \text{ m s}^{-2}, \quad a_n = \frac{v^2}{R} = \frac{3.00^2}{2.00} = 4.50 \text{ m s}^{-2}.$$

Because the two components are perpendicular,

$$|\mathbf{a}| = \sqrt{(-0.500)^2 + 4.50^2} = 4.52769 \text{ m s}^{-2}.$$

A convention using a positive deceleration magnitude writes  $d = 0.500 \text{ m s}^{-2}$  and  $\mathbf{a} = -d\hat{\mathbf{t}} + a_n\hat{\mathbf{n}}$ . It changes the scalar convention, not the vector. Taking  $a_t = +0.500$  along  $\hat{\mathbf{t}}$  would instead predict increasing speed and would be wrong here.

**Problem 7: a spatial path and three distinct arrows**

Component differentiation gives  $\mathbf{v} = (2t, 2, 1.5t^2)$  and  $\mathbf{a} = (2, 0, 3t)$ , with the derivative units implied by the SI numerical coordinates. At  $t = 2.00$  s,

$$\mathbf{r} = (4, 4, 4) \text{ m}, \quad \mathbf{v} = (4, 2, 6) \text{ m s}^{-1}, \quad \mathbf{a} = (2, 0, 6) \text{ m s}^{-2}.$$

Their magnitudes are respectively  $\sqrt{48} = 6.92820 \text{ m}$ ,  $\sqrt{56} = 7.48331 \text{ m s}^{-1}$ , and  $\sqrt{40} = 6.32456 \text{ m s}^{-2}$ . Since  $v_x = 4 \neq 0$ ,

$$\frac{dy}{dx} = \frac{v_y}{v_x} = \frac{1}{2}, \quad \frac{dz}{dx} = \frac{v_z}{v_x} = \frac{3}{2}.$$

The position arrow starts at the fixed origin and ends at the point. A velocity arrow starts locally at the point and follows the tangent; an acceleration arrow can also start there but generally does not follow the tangent. On a position plot, multiplying velocity by a stated time scale converts its drawn components to lengths; acceleration needs a time-squared scale. Equal graphical arrow lengths do not compare the physical magnitudes of vectors with different dimensions. The constant nonzero  $v_y$  and zero  $a_y$  provide a useful component check.

**Problem 8: average velocity and average speed**

The endpoint positions are  $\mathbf{r}(0) = (0, 1, 0) \text{ m}$  and  $\mathbf{r}(2) = (4, -1, 2) \text{ m}$ . Therefore

$$\mathbf{v}_{\text{avg}} = \frac{(4, -2, 2) \text{ m}}{2 \text{ s}} = (2, -1, 1) \text{ m s}^{-1}.$$

The instantaneous velocity is  $\mathbf{v}(t) = (2, -1, t)$  in numerical SI units, so  $\mathbf{v}(1 \text{ s})$  equals that average. Here every position coordinate is at most quadratic, for which the symmetric secant equals the midpoint derivative.

Distance is instead the integral of the speed. In SI numerical values,

$$\bar{v} = \frac{1}{2} \int_0^2 \sqrt{5 + t^2} dt = \frac{1}{4} \left[ t\sqrt{5 + t^2} + 5 \operatorname{asinh}\left(\frac{t}{\sqrt{5}}\right) \right]_0^2 = 2.50590 \text{ m s}^{-1}.$$

It exceeds  $|\mathbf{v}_{\text{avg}}| = \sqrt{6} = 2.44949 \text{ m s}^{-1}$ . The velocity direction varies because its  $z$  component varies, so the path length exceeds the endpoint chord. Equality of one instantaneous vector and the average vector does not turn the path into that chord or remove the variation during the interval.

**Problem 9: infer the launch from an endpoint**

Constant acceleration permits an inverse endpoint calculation:

$$\mathbf{v}_0 = \frac{\mathbf{r}_f - \mathbf{r}_0}{T} - \frac{1}{2}\mathbf{a}T = \frac{(4, 3) \text{ m}}{2 \text{ s}} - (0.500, -1.00) \text{ m s}^{-1} = (1.50, 2.50) \text{ m s}^{-1}.$$

After  $T = 2.00 \text{ s}$ ,  $\mathbf{v}_f = \mathbf{v}_0 + \mathbf{a}T = (2.50, 0.500) \text{ m s}^{-1}$ . Checking the position independently gives

$$(1, -2) + (1.5, 2.5)(2) + \frac{1}{2}(0.5, -1)(2)^2 = (1, -2) + (3, 5) + (1, -2) = (5, 1)$$

in metres. The acceleration raises  $v_x$  and lowers  $v_y$ , matching the two final components. This finds a kinematically required initial velocity; it does not establish how a launcher supplies it.

**Problem 10: integrate a rotating acceleration**

Write the frequency as  $\omega = 1 \text{ s}^{-1}$ , so the arguments  $\omega t$  are dimensionless. In the SI numerical notation of the statement, integration from the zero initial state gives

$$\begin{aligned}\mathbf{v}(t) &= \left( \int_0^t \cos s \, ds, \int_0^t \sin s \, ds \right) = (\sin t, 1 - \cos t), \\ \mathbf{r}(t) &= \left( \int_0^t \sin s \, ds, \int_0^t (1 - \cos s) \, ds \right) = (1 - \cos t, t - \sin t).\end{aligned}$$

The velocity outputs are in  $\text{m s}^{-1}$ , and positions in metres. At  $t = \pi/2 \text{ s}$ ,

$$\mathbf{v} = (1, 1) \text{ m s}^{-1}, \quad \mathbf{r} = (1, 0.570796) \text{ m}.$$

Both functions vanish initially, and differentiating them recovers the given acceleration, checking the integration constants. Although  $|\mathbf{a}| = 1 \text{ m s}^{-2}$ , its direction turns. Inserting its initial vector into  $\frac{1}{2}\mathbf{a}t^2$  would suppress its later  $y$  contribution and cannot reproduce this motion.

**Problem 11: change the horizontal schedule**

In Example 4.5, the bead falls 1.80 m with zero initial vertical velocity. Its contact time is unchanged by either horizontal modification:

$$t_f = \sqrt{\frac{2(1.80)}{9.81}} = 0.605783 \text{ s}.$$

The bead position is  $x_B = v_{0x}t$ , and the marker is  $x_M = x_{M0} + wt$ . Coincidence at that time requires  $x_{M0} = (v_{0x} - w)t_f$ .

In case (a),  $v_{0x} = 2.40 \text{ m s}^{-1}$  and  $w = -0.600 \text{ m s}^{-1}$ , giving

$$x_{M0} = 3.00t_f = 1.81735 \text{ m}.$$

The marker's coordinate-time line now has negative slope; the bead's horizontal line and vertical parabola are unchanged. Checking gives  $x_M(t_f) = (3.00 - 0.600)t_f = 2.40t_f = x_B(t_f)$ .

In case (b),  $v_{0x} = 1.20 \text{ m s}^{-1}$  and  $w = 0.600 \text{ m s}^{-1}$ , so

$$x_{M0} = 0.600t_f = 0.363470 \text{ m}, \quad x_B(t_f) = 1.20t_f = 0.726939 \text{ m}.$$

The bead's horizontal slope is halved; the marker retains its original slope but starts closer. The vertical parabola and the shared event time are identical in the two cases. Each graph must use that same clock: changing a horizontal velocity cannot alter the fall time in this model.

**Problem 12: contact after a horizontal release**

Before first contact,  $x = 1.40t$  and  $y = 0.800 - \frac{1}{2}gt^2$ . Setting  $y = 0$  and accepting the positive root gives

$$t_f = \sqrt{\frac{1.600}{9.81}} = 0.403855 \text{ s}, \quad x_f = 1.40t_f = 0.565397 \text{ m}.$$

The negative time root lies before release. Immediately before contact,

$$\mathbf{v}_f = (1.40, -gt_f) = (1.40, -3.96182) \text{ m s}^{-1}, \quad |\mathbf{v}_f| = 4.20190 \text{ m s}^{-1}.$$

With rightward  $v_x > 0$  and downward  $v_y < 0$ , the angle from  $+x$  is  $\text{atan2}(v_y, v_x) = -70.538^\circ$ , or  $289.462^\circ$  if an angle in  $[0, 360^\circ)$  is preferred. The sign identifies a descending rightward impact. The free-flight equations specify the pre-contact velocity only; subsequent contact motion requires a new model.

**Problem 13: identify ascent and descent algebraically**

The height condition is  $2.00 = 8.00t - 4.905t^2$ , giving

$$t_{\pm} = \frac{8.00 \pm \sqrt{8.00^2 - 2(9.81)(2.00)}}{9.81} = \begin{cases} 0.308263 \text{ s}, \\ 1.322726 \text{ s}. \end{cases}$$

Both times are positive, and no earlier obstacle is stipulated. The horizontal coordinates are  $x_- = 6t_- = 1.84958 \text{ m}$  and  $x_+ = 6t_+ = 7.93636 \text{ m}$ .

At these roots,  $v_y(t_-) = +\sqrt{24.76} \text{ m s}^{-1}$  and  $v_y(t_+) = -\sqrt{24.76} \text{ m s}^{-1}$ . The signs identify ascent and descent without a graph. The apex satisfies  $v_y = 0$ , so

$$t_A = \frac{8}{9.81} = 0.815494 \text{ s}, \quad y_A = \frac{8^2}{2(9.81)} = 3.26198 \text{ m}.$$

The target height lies below this apex and the two times straddle it, providing independent consistency checks. Only vertical velocity vanishes at the apex; horizontal velocity remains  $6 \text{ m s}^{-1}$ .

**Problem 14: complementary angles and equal heights**

For unobstructed return to launch height, the nonzero height root is  $T = 2v_0 \sin \theta / g$ . Multiplying by  $v_0 \cos \theta$  gives  $R = v_0^2 \sin 2\theta / g$ , while  $H = v_0^2 \sin^2 \theta / (2g)$ . Substitution yields

| $\theta$   | $R/\text{m}$ | $T/\text{s}$ | $H/\text{m}$ |
|------------|--------------|--------------|--------------|
| $30^\circ$ | 8.82799      | 1.01937      | 1.27421      |
| $60^\circ$ | 8.82799      | 1.76560      | 3.82263      |

The ranges agree because  $\sin 60^\circ = \sin 120^\circ$ . The high-angle launch stays aloft  $\sqrt{3}$  times as long and rises three times as high. If the landing heights differ from the respective launch heights, the equal-range conclusion is no longer guaranteed: the selected positive root depends on the height difference as well as on  $v_{0y}$ . Even a common unequal landing level generally breaks the complementary-angle equality.

**Problem 15: two height roots, one interception**

Example 4.6 has  $x_B = 4t$ ,  $x_T = x_{T0} + t$ , and height roots  $t_{\pm} = (5 \pm \sqrt{5.38})/9.81$ . With the unrounded construction  $x_{T0} = 3t_+$ , the target-minus-bead separation is

$$x_T - x_B = 3(t_+ - t).$$

At the earlier height event,

$$x_T(t_-) - x_B(t_-) = 3(t_+ - t_-) = \frac{6\sqrt{5.38}}{9.81} = 1.41864 \text{ m}.$$

The target is ahead, so there is no interception then. At  $t_+$ , the bead has  $\mathbf{v}_B = (4, -\sqrt{5.38}) \text{ m s}^{-1}$ , while the target has  $\mathbf{v}_T = (1, 0) \text{ m s}^{-1}$ . Their relative velocity is

$$\mathbf{v}_{B/T} = (3.00, -2.31948) \text{ m s}^{-1}, \quad |\mathbf{v}_{B/T}| = \sqrt{14.38} = 3.79210 \text{ m s}^{-1}.$$

A height root meets only the vertical condition. Interception requires equality of the complete positions at the same time; only the later root meets the horizontal condition. Using a rounded target coordinate would introduce a small artificial miss and obscure the exact construction.

### Problem 16: the first contact belongs to a surface domain

Free flight gives  $x = 3t$ ,  $y = 0.500 + 2t - 4.905t^2$ . The bead reaches the riser's horizontal coordinate at  $t_R = 0.600/3.00 = 0.200 \text{ s}$ , when

$$y(t_R) = 0.500 + 2(0.200) - 4.905(0.200)^2 = 0.703800 \text{ m}.$$

This lies above the riser's top  $0.650 \text{ m}$ , so the vertical face is cleared. Before this event the bead is above the floor: its height is positive on  $[0, 0.200] \text{ s}$ .

Next solve for the platform height:

$$t_{\pm} = \frac{2 \pm \sqrt{4 - 2(9.81)(0.150)}}{9.81} = 0.0990719, 0.3086753 \text{ s}.$$

The earlier root has  $x_- = 0.297216 \text{ m} < 0.600 \text{ m}$ , where no platform top exists. It is a height crossing, not contact. The later root has  $x_+ = 0.926026 \text{ m}$ , inside the platform domain and after clearing the face. Thus first contact is

$$t_f = 0.308675 \text{ s}, \quad (x_f, y_f) = (0.926026, 0.650) \text{ m}.$$

There is no valid floor contact beforehand: for  $x < 0.600 \text{ m}$  the height remains positive, and beyond that coordinate the platform top is the relevant first surface. The restricted geometry in Fig. 4.30 makes the root-selection reasoning visible.

### Problem 17: signed angular rate

The angle is  $\theta = \pi/2 - 3t$ , so  $\theta(\pi/6) = 0$ . With  $\omega = -3.00 \text{ rad s}^{-1}$ , differentiation gives

$$\mathbf{r} = R(\cos \theta, \sin \theta), \quad \mathbf{v} = R\omega(-\sin \theta, \cos \theta), \quad \mathbf{a} = -R\omega^2(\cos \theta, \sin \theta).$$

At the stated event,

$$\mathbf{r} = (0.400, 0) \text{ m}, \quad \mathbf{v} = (0, -1.20) \text{ m s}^{-1}, \quad \mathbf{a} = (-3.60, 0) \text{ m s}^{-2}.$$

The period is a positive elapsed time,  $T = 2\pi/|\omega| = 2.09440 \text{ s}$ , not  $2\pi/\omega$ . Clockwise is defined when viewing from  $+z$  toward the  $xy$  plane. Viewed from the opposite side, the apparent sense reverses. The acceleration still points inward from the event on the positive  $x$  axis; its dependence on  $\omega^2$  checks the direction independently.

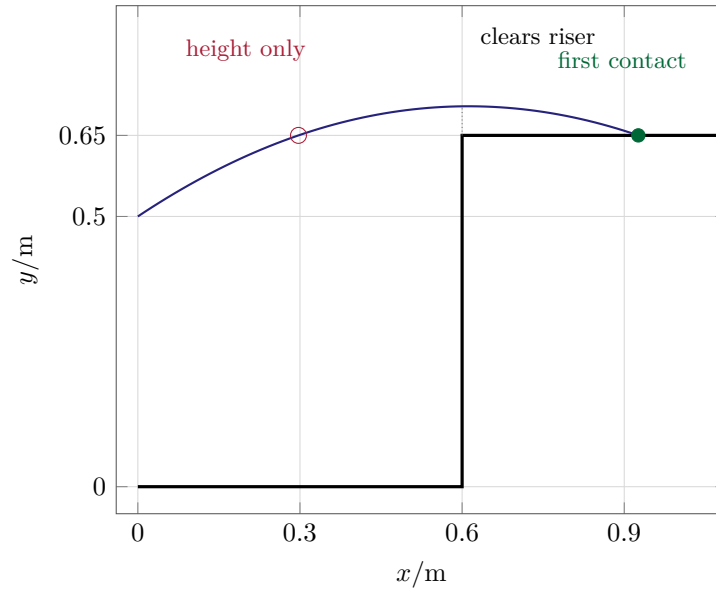


**Figure 4.30:** Problem 16: the earlier height root is rejected by the platform's horizontal domain. Both axes use the same length scale. The solid trajectory ends at the first physical contact; the vertical dotted segment shows the riser clearance.

### Problem 18: the radius begins at the circle centre

The centre-to-point vector is  $\mathbf{r}_P - \mathbf{r}_C = (0, 2) \text{ m}$ , hence  $R = 2.00 \text{ m}$ . The point is at the top of this circle, so clockwise motion has  $\mathbf{v} = (3.00, 0) \text{ m s}^{-1}$ . Its inward acceleration is

$$\mathbf{a} = -\frac{v^2}{R} \hat{\mathbf{j}} = (0, -4.50) \text{ m s}^{-2}.$$

The angular-rate magnitude is  $v/R = 1.50 \text{ s}^{-1}$ . Thus the correct vector formula is  $\mathbf{a} = -\omega^2(\mathbf{r}_P - \mathbf{r}_C)$ . Using  $-\omega^2 \mathbf{r}_P$  instead gives  $(-2.25, 0) \text{ m s}^{-2}$ , pointing toward the laboratory origin rather than toward the centre. It has both a wrong direction and a wrong magnitude. A fixed shift of coordinate origin cannot change the physical acceleration, which is why the centre-relative radius must be used.

### Problem 19: a minimum curvature radius

At a moving event,  $|\mathbf{a}|^2 = a_t^2 + a_n^2$ , with  $a_n = v^2/R_c$ . Either sign of  $a_t$  has the stipulated magnitude  $1.00 \text{ m s}^{-2}$ . The total bound therefore requires

$$1.00^2 + \left(\frac{5.00^2}{R_c}\right)^2 \leq 3.00^2, \quad a_n \leq \sqrt{8} \text{ m s}^{-2}.$$

Consequently,

$$R_c \geq \frac{25}{\sqrt{8}} = 8.83883 \text{ m}.$$

At equality, the component norm is exactly  $3.00 \text{ m s}^{-2}$ ; increasing radius reduces the normal part. This is a necessary check on the prescribed local motion. It supplies no mass, force law, friction condition, motor torque, transient behaviour, or engineering margin, so it does not certify a device's force capacity.

### Problem 20: project onto the actual tangent

The speed is  $v = 5.00 \text{ m s}^{-1}$  and the unit tangent is  $\hat{\mathbf{t}} = (0.6, 0.8)$ . The signed tangential projection is

$$a_t = \mathbf{a} \cdot \hat{\mathbf{t}} = (-0.4)(0.6) + (2.2)(0.8) = 1.52 \text{ m s}^{-2}.$$

Subtract its *vector* contribution:

$$\mathbf{a}_\perp = \mathbf{a} - a_t \hat{\mathbf{t}} = (-0.4, 2.2) - (0.912, 1.216) = (-1.312, 0.984) \text{ m s}^{-2}.$$

Hence

$$a_n = |\mathbf{a}_\perp| = 1.64 \text{ m s}^{-2}, \quad \hat{\mathbf{n}} = (-0.8, 0.6), \quad R_c = \frac{v^2}{a_n} = 15.2439 \text{ m}.$$

Orthogonality is checked by  $(-1.312, 0.984) \cdot (3, 4) = 0$ . The magnitude check is  $1.52^2 + 1.64^2 = 5 = (-0.4)^2 + 2.2^2$ , giving  $|\mathbf{a}| = \sqrt{5} \text{ m s}^{-2}$ . Figure 4.31 represents that vector subtraction geometrically.

If acceleration alone is reversed at the same velocity, linearity of projection gives  $a'_t = -1.52 \text{ m s}^{-2}$  and  $\mathbf{a}'_\perp = -\mathbf{a}_\perp$ . Thus  $\hat{\mathbf{n}}' = (0.8, -0.6)$ ;  $a_n$  and  $R_c$  retain their magnitudes. The new normal direction describes an opposite local bending direction for the changed motion. Speed would now be decreasing rather than increasing.

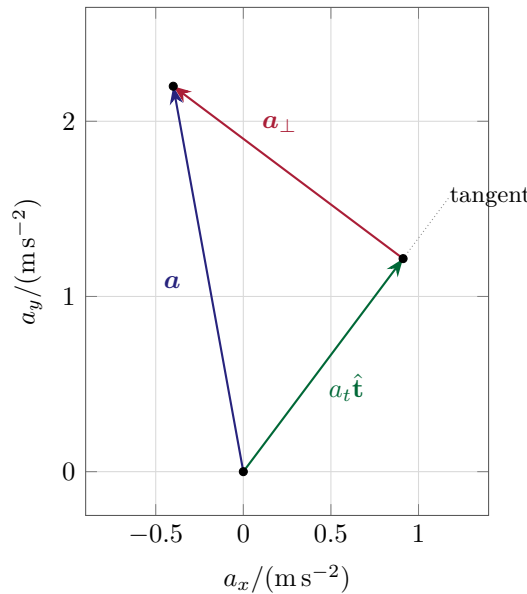


**Figure 4.31:** Problem 20: the acceleration triangle satisfies  $\mathbf{a} = a_t \hat{\mathbf{t}} + \mathbf{a}_\perp$ . Equal component scales preserve its perpendicular geometry. The dotted ray identifies the tangent direction; it is not a velocity vector plotted in acceleration units.

### Problem 21: acceleration in a translating stage

Because the stage translates without rotation, laboratory position is the sum of stage-origin position and stage-relative marker position. Differentiating once and twice gives

$$\mathbf{v}_{M/L} = \mathbf{v}_{M/S} + \mathbf{v}_{S/L} = (-0.200, 0.500) + (0.800, 0.300) = (0.600, 0.800) \text{ m s}^{-1},$$

$$|\mathbf{v}_{M/L}| = \sqrt{0.600^2 + 0.800^2} = 1.00 \text{ m s}^{-1}, \quad \mathbf{a}_{M/L} = (0, 0.200) + (0.100, 0) = (0.100, 0.200) \text{ m s}^{-2}.$$

The origin accelerates, so the two acceleration reports differ by  $\mathbf{a}_{S/L}$ . Agreement of accelerations applies to a constant-velocity translation, not to every nonrotating translation. There is no contradiction: the observers are differentiating coordinates whose origins have different second derivatives.

### Problem 22: a command specified in the wrong frame

The actual current is  $\mathbf{v}_{W/S} = (1.00, 0) \text{ m s}^{-1}$ . Under the mistaken water-relative command,

$$\mathbf{v}_{C/S} = (-0.600, 1.200) + (1.00, 0) = (0.400, 1.200) \text{ m s}^{-1}.$$

Starting from the original origin, after 50.0 s the craft arrives at (20.0, 60.0) m, downstream of its start. The programmed water-relative speed is  $\sqrt{0.600^2 + 1.200^2} = 1.34164 \text{ m s}^{-1}$ , not  $2.00 \text{ m s}^{-1}$ .

The correct velocity triangle, with all components in  $\text{m s}^{-1}$ , is

$$\underbrace{(-0.600, 1.200)}_{v_{C/S}} = \underbrace{(-1.600, 1.200)}_{v_{C/W}} + \underbrace{(1.00, 0)}_{v_{W/S}}.$$

The water-relative vector has norm 2.00 and produces the desired shore displacement  $(-30.0, 60.0) \text{ m}$  in fifty seconds.

With the increased current  $w = 2.00 \text{ m s}^{-1}$  and craft speed  $U = 2.00 \text{ m s}^{-1}$ , any genuine crossing requires  $v_{C/W,y} > 0$ . The most upstream permitted component is then  $v_{C/W,x} = -\sqrt{U^2 - v_{C/W,y}^2} > -U$ , so  $v_{C/S,x} > 0$ . Setting the water component to  $-U$  exactly leaves zero crossing component and cannot reach the opposite bank. An upstream destination across the channel is therefore impossible under the fixed-speed, uniform-current model.

### Problem 23: closest approach within the allowed time interval

Relative position is  $\mathbf{R}(t) = (2 + t, 5 - t)$  in numerical SI units. Minimize its squared norm, avoiding an unnecessary square root:

$$d^2(t) = (2 + t)^2 + (5 - t)^2 = 29 - 6t + 2t^2.$$

Its derivative vanishes at  $-6 + 4t = 0$ , or  $t_* = 1.50 \text{ s}$ . Since the second derivative is positive, this is the global unconstrained minimum. It belongs to  $t \geq 0$ , giving

$$\mathbf{R}(t_*) = (3.50, 3.50) \text{ m}, \quad d_{\min} = \sqrt{24.5} = 4.94975 \text{ m}.$$

The independent geometric check is  $\mathbf{R}(t_*) \cdot \mathbf{V} = 0$ : closest separation is perpendicular to the relative velocity.

On  $0 \leq t \leq 1.00 \text{ s}$ , the stationary point is unavailable. The derivative  $-6 + 4t$  is negative throughout that interval, so the minimum lies at its final endpoint. There  $\mathbf{R} = (3, 4) \text{ m}$ , hence  $d_{\min} = 5.00 \text{ m}$ . The initial distance  $\sqrt{29} \text{ m}$  is larger. Figure 4.32 shows why extending a model's time domain changes the answer.

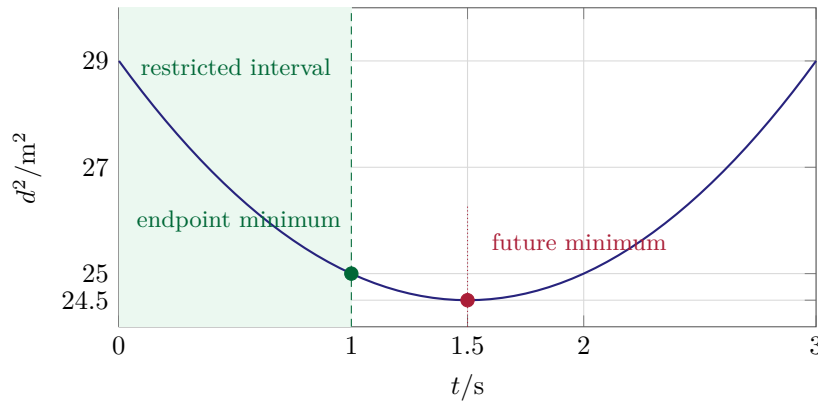


**Figure 4.32:** Problem 23: the stationary minimum at 1.50 s is unavailable when the motion model ends at 1.00 s. Shading identifies the restricted interval. Squared distance has the same minimizing time as distance because both are non-negative.

### Problem 24: distinguish truncation from alternating errors

Use the samples at 0.200, 0.400, 0.600 s, with  $h = 0.200 \text{ s}$ . Central differences give

$$v_x^{\text{est}} = \frac{0.6206 - 0.1998}{0.400} = 1.052 \text{ m s}^{-1}, \quad a_x^{\text{est}} = \frac{0.6206 - 2(0.4074) + 0.1998}{0.0400} = 0.140 \text{ m s}^{-2}.$$

For  $y$ , the samples 0.0200, 0.0800, 0.1800 m give

$$v_y^{\text{est}} = \frac{0.1800 - 0.0200}{0.400} = 0.400 \text{ m s}^{-1}, \quad a_y^{\text{est}} = \frac{0.1800 - 2(0.0800) + 0.0200}{0.0400} = 1.00 \text{ m s}^{-2}.$$

The exact path has  $\mathbf{v} = (1 + 0.3t^2, t)$  and  $\mathbf{a} = (0.6t, 1)$  in SI numerical units. At this event they are  $(1.048, 0.400) \text{ m s}^{-1}$  and  $(0.240, 1.00) \text{ m s}^{-2}$ . The estimated-minus-exact errors are thus  $(0.004, 0) \text{ m s}^{-1}$  and  $(-0.100, 0) \text{ m s}^{-2}$ .

At the central even-index sample, the position perturbation is  $+0.001 \text{ m}$ , while both neighbouring perturbations are  $-0.001 \text{ m}$ . Their acceleration combination is  $(-0.001) - 2(+0.001) + (-0.001) = -0.004 \text{ m}$ , divided by  $h^2$ . At  $0.600 \text{ s}$  the signs reverse, giving  $+0.100 \text{ m s}^{-2}$ . Equal outer perturbations cancel in the velocity difference; its remaining error is the cubic central-difference truncation term  $0.1h^2 = 0.004$  in the appropriate SI units. The acceleration alternation is caused by supplied synthetic coordinate errors, not by genuine oscillatory acceleration.

### Problem 25: numerical states and interpolated events

For the chapter's explicit midpoint scheme, the closed-form horizontal node identity is  $x_n = t_n + 0.1t_n^3 - 0.1t_nh^2$  in SI numerical units. At  $t_n = 1.20 \text{ s}$ ,  $h = 0.150 \text{ s}$ ,

$$x_n = 1.20 + 0.1(1.20)^3 - 0.1(1.20)(0.150)^2 = 1.370100 \text{ m}.$$

The exact value is  $1.372800 \text{ m}$ , so the node error is  $-0.002700 \text{ m}$ .

The upward  $y = 0.500 \text{ m}$  event lies between  $0.900$  and  $1.050 \text{ s}$ , whose exact vertical nodes are  $0.405000$  and  $0.551250 \text{ m}$ . The interpolation fraction is

$$\lambda = \frac{0.500000 - 0.405000}{0.551250 - 0.405000} = 0.64957265.$$

The horizontal node identity gives  $x_L = 0.970875 \text{ m}$  and  $x_U = 1.163400 \text{ m}$ . Interpolating both coordinates using the same fraction produces

$$t_e^{(h)} = 0.900 + \lambda(0.150) = 0.99743590 \text{ s}, \quad x_e^{(h)} = x_L + \lambda(x_U - x_L) = 1.09593397 \text{ m}.$$

Compared with the exact event  $(1.000 \text{ s}, 1.100 \text{ m})$ , the errors are  $-0.00256410 \text{ s}$  and  $-0.00406603 \text{ m}$ . The bracket and  $0 < \lambda < 1$  check the event selection.

For the separate constant-acceleration benchmark, Euler velocities are exact:  $\mathbf{v}_n = \mathbf{v}_0 + n\mathbf{a}h$ . Summing its position increments yields

$$\mathbf{r}_N^E = \mathbf{r}_0 + Nh\mathbf{v}_0 + \mathbf{a}h^2 \sum_{n=0}^{N-1} n = \mathbf{r}_0 + T\mathbf{v}_0 + \frac{1}{2}\mathbf{a}h^2N(N-1), \quad T = Nh.$$

Subtracting  $\mathbf{r}(T) = \mathbf{r}_0 + T\mathbf{v}_0 + \frac{1}{2}\mathbf{a}T^2$  gives

$$\mathbf{r}_N^E - \mathbf{r}(T) = -\frac{1}{2}\mathbf{a}Th.$$

At fixed  $T$ , the Euler position error is first order in  $h$ . Explicit midpoint adds the missing  $\frac{1}{2}\mathbf{a}h^2$  each step, so constant-acceleration states are exact at nodes in exact arithmetic. This exactness does not extend to a linearly interpolated event on a quadratic coordinate-time curve. In this example the exact  $y$  nodes still generate the nonzero event-time error above. State integration and event location must therefore be checked separately.

**Problem 26: a defensible video-tracking investigation**

A complete response is a plan and a criterion, because no measured data are supplied. One possible investigation tracks a high-contrast marker along a smooth planar bend, recording an additional straight segment as a control. Keep the camera fixed with its optical axis approximately normal to that plane. State the plane and the laboratory coordinate axes; a single camera cannot reconstruct unrestricted three-dimensional turning.

**Observations and calibration.** Retain the original frames, marker centres, and the estimated uncertainty of each centre. Convert pixel coordinates to metres using several known in-plane separations, checking that the scale is consistent across the field. A tilted view needs a justified perspective correction; a reference outside the motion plane is inadequate. Check timestamp spacing from the actual recording rather than assuming the nominal frame rate, and flag dropped or duplicated frames. Estimate timing uncertainty as well as position uncertainty.

**Derivatives and comparisons.** Plot  $x(t)$  and  $y(t)$  first, including flagged points. On a uniformly timed subset, estimate velocity and acceleration by central differences. Repeat with a larger spacing and with a local polynomial fit over more than one reasonable window. Differentiate the fitted coordinates analytically; inspect coordinate residuals so a smooth-looking derivative does not conceal a poor fit. Endpoint derivatives require separate one-sided estimates or should be omitted. Validate the analysis on a known synthetic smooth path and inspect the straight-segment control for false normal acceleration.

**Conclusion criterion.** At events with speed resolved above zero, compute  $\mathbf{a}_\perp = \mathbf{a} - (\mathbf{a} \cdot \hat{\mathbf{t}})\hat{\mathbf{t}}$  and compare its direction with the observed change of tangent. Evidence for turning should persist over multiple events, exceed propagated derivative uncertainty under a stated criterion, and retain its direction under reasonable spacing/window changes. If independent coordinate errors have standard deviation  $\sigma_r$ , central differences give scales  $\sigma_v = \sigma_r/(\sqrt{2}h)$  and  $\sigma_a = \sqrt{6}\sigma_r/h^2$  for each component; correlated errors or irregular timing need a different uncertainty model. An isolated large acceleration at a mistracked frame is insufficient evidence.

A sound report either identifies resolved turning with its uncertainties and settings, or states that turning is unresolved at the available precision. Credit depends on six elements: observed quantities, in-plane calibration, timing checks, a specified derivative method, a justified comparison of settings, and an uncertainty-based conclusion. It must also state projection and model limitations. No numerical outcome or force law should be invented from this proposed experiment.

**Problem 27: the second transverse product-rule term**

For  $r = r_0 + ut$  and  $\theta = \Omega t$ , velocity is  $\mathbf{v} = u\hat{\mathbf{e}}_r + r\Omega\hat{\mathbf{e}}_\theta$ . Differentiating the first contribution gives  $u\dot{\hat{\mathbf{e}}}_\theta = u\Omega\hat{\mathbf{e}}_\theta$ ; differentiating the scalar factor in the second gives another  $\dot{r}\Omega\hat{\mathbf{e}}_\theta = u\Omega\hat{\mathbf{e}}_\theta$ . The rotation of  $\hat{\mathbf{e}}_\theta$  gives  $-r\Omega^2\hat{\mathbf{e}}_r$ . The full acceleration is therefore

$$\mathbf{a} = -r\Omega^2\hat{\mathbf{e}}_r + 2u\Omega\hat{\mathbf{e}}_\theta, \quad \mathbf{a}_{\text{bad}} - \mathbf{a} = -u\Omega\hat{\mathbf{e}}_\theta.$$

The proposed formula misses one of the two transverse product-rule contributions. At  $t = \pi/2$  s, the stated parameters give  $\hat{\mathbf{e}}_\theta = -\hat{\mathbf{i}}$ , so

$$\mathbf{a}_{\text{bad}} - \mathbf{a} = +0.100\hat{\mathbf{i}} \text{ m s}^{-2}.$$

Specifically,  $\mathbf{a} = (-0.200, -0.557080) \text{ m s}^{-2}$  and  $\mathbf{a}_{\text{bad}} = (-0.100, -0.557080) \text{ m s}^{-2}$ . A circular test sets  $u = 0$ , making the missing term vanish; it therefore cannot expose this error. Testing a term requires a model in which that term is nonzero.

**Problem 28: coordinate second derivatives are not vector components**

Differentiate the supplied schedules:

$$\dot{r} = 0.0400t, \quad \ddot{r} = 0.0400, \quad \dot{\theta} = 2.00 + 0.200t, \quad \ddot{\theta} = 0.200,$$

with dimensions inherited from the coefficients. At  $t = 1.00 \text{ s}$ ,  $r = 0.320 \text{ m}$ ,  $\theta = 2.10 \text{ rad}$ ,  $\dot{r} = 0.0400 \text{ m s}^{-1}$  and  $\dot{\theta} = 2.20 \text{ s}^{-1}$ . Thus

$$\begin{aligned} v_r &= \dot{r} = 0.0400 \text{ m s}^{-1}, & v_\theta &= r\dot{\theta} = 0.704 \text{ m s}^{-1}, \\ a_r &= \ddot{r} - r\dot{\theta}^2 = 0.0400 - 0.320(2.20)^2 = -1.5088 \text{ m s}^{-2}, \\ a_\theta &= r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0.320(0.200) + 2(0.0400)(2.20) = 0.240 \text{ m s}^{-2}. \end{aligned}$$

The speed is  $\sqrt{0.0400^2 + 0.704^2} = 0.705135 \text{ m s}^{-1}$ . Although the radial coordinate has positive second derivative  $\ddot{r}$ , the radial *vector* acceleration is inward because the turning contribution  $-r\dot{\theta}^2$  dominates. Both quantities have acceleration units but describe different operations. Differentiating  $x = r \cos \theta$ ,  $y = r \sin \theta$  twice and resolving along the event basis independently reproduces these components.

### Problem 29: the spiral's tangent is tilted from the polar direction

At the event of Example 4.26, write  $r_* = 0.557079633 \text{ m}$ . The velocity and acceleration are  $\mathbf{v} = (-r_*, 0.100) \text{ m s}^{-1}$  and  $\mathbf{a} = (-0.200, -r_*) \text{ m s}^{-2}$ , interpreting the displayed numerical components with their stated units. Hence  $v = 0.56598385 \text{ m s}^{-1}$ . Their scalar product gives

$$a_t = \frac{\mathbf{v} \cdot \mathbf{a}}{v} = \frac{0.100r_*}{\sqrt{0.100^2 + r_*^2}} = 0.0984268 \text{ m s}^{-2}.$$

For planar motion the magnitude of the perpendicular acceleration is also

$$a_n = \frac{|v_x a_y - v_y a_x|}{v} = \frac{r_*^2 + 0.0200}{\sqrt{0.0100 + r_*^2}} = 0.5836522 \text{ m s}^{-2}.$$

Thus

$$R_c = \frac{v^2}{a_n} = 0.5488504 \text{ m}.$$

The identity  $a_t^2 + a_n^2 = 0.200^2 + r_*^2$  checks the decomposition. Here  $a_t \neq a_\theta = 0.200 \text{ m s}^{-2}$ : the polar transverse direction is leftward, while the actual tangent is leftward and slightly outward. Projection onto these different directions gives different components. Also  $R_c \neq r_*$ :  $r_*$  is distance from the chosen polar origin, whereas  $R_c$  measures local tangent turning and refers to the osculating circle's centre. They coincide for the constant-radius circular limit, not for a general spiral.

### Problem 30: choose a rate valid throughout the interval

For the full prescribed interval, radius increases from  $0.400$  to  $0.800 \text{ m}$ . At positive constant  $\Omega$ ,

$$|\mathbf{a}|^2 = r(t)^2 \Omega^4 + 4u^2 \Omega^2.$$

Its derivative is  $2ru\Omega^4 > 0$ , so checking the final radius suffices. In SI numerical units, let  $q = \Omega^2$ . Saturating the stipulated bound gives

$$0.640q^2 + 0.0400q - 0.490 = 0.$$

The physically allowed root is

$$q = \frac{-0.0400 + \sqrt{0.0400^2 + 4(0.640)(0.490)}}{2(0.640)} = 0.84430786 \text{ s}^{-2},$$

$$\Omega_{\max} = 0.91886226 \text{ rad s}^{-1}.$$

The other root is negative and cannot be the square of a real angular rate. Acceleration magnitude grows with positive  $q$ , so this root is the largest allowable rate. Substitution at  $r = 0.800 \text{ m}$  gives  $0.700 \text{ m s}^{-2}$ ; all earlier values are smaller. The conclusion applies to this monotone-radius constant-rate schedule and the imposed kinematic limit, not to an unspecified scanner's mechanical capacity or transition into the schedule.

**Problem 31: a straight line expressed in polar coordinates**

With  $r = \sqrt{b^2 + V^2 t^2}$ , direct differentiation gives

$$\dot{r} = \frac{V^2 t}{r}, \quad \ddot{r} = \frac{V^2}{r} - \frac{V^4 t^2}{r^3} = \frac{V^2 b^2}{r^3}.$$

For  $\theta = \tan^{-1}(Vt/b)$ ,

$$\dot{\theta} = \frac{bV}{r^2}, \quad \ddot{\theta} = -\frac{2bV^3 t}{r^4}.$$

The full radial acceleration cancels:

$$a_r = \ddot{r} - r\dot{\theta}^2 = \frac{V^2 b^2}{r^3} - \frac{b^2 V^2}{r^3} = 0.$$

So does the full transverse acceleration:

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = -\frac{2bV^3 t}{r^3} + \frac{2bV^3 t}{r^3} = 0.$$

This agrees with the independent Cartesian result  $\mathbf{v} = (0, V)$ ,  $\mathbf{a} = 0$ . The nonzero coordinate derivatives describe changes of distance and direction relative to the polar origin; moving-basis terms compensate them. The assumption  $b > 0$  keeps  $r > 0$ , places the entire line in the right half-plane, and makes the stated arctangent a continuous correct angle branch. It avoids a polar-origin singularity and the quadrant ambiguity that would require a different angle convention for  $b < 0$ .

**Problem 32: verify acceleration using positions only**

Evaluate the original positions  $x = (0.400 + 0.100t) \cos t$  and  $y = (0.400 + 0.100t) \sin t$ , in their SI numerical convention, at  $t_* - h, t_*, t_* + h$ . Subtract them as stated in the problem. The exact product-rule acceleration at  $t_* = \pi/2$  s is

$$\mathbf{a}_* = (-0.200000, -0.557079633) \text{ m s}^{-2}.$$

The position-only estimates and estimated-minus-exact vector errors are

| $h/\text{s}$ | $a_{h,x}$    | $a_{h,y}$    | $(a_h - a_*)_x$ | $(a_h - a_*)_y$ |
|--------------|--------------|--------------|-----------------|-----------------|
| 0.100        | -0.199666833 | -0.556615554 | 0.000333167     | 0.000464078     |
| 0.0500       | -0.199916677 | -0.556963584 | 0.000083323     | 0.000116049     |
| 0.0250       | -0.199979167 | -0.557050619 | 0.000020833     | 0.000029014     |

All four acceleration/error columns are in  $\text{m s}^{-2}$ . The error norms are respectively  $5.7128691 \times 10^{-4}$ ,  $1.4286351 \times 10^{-4}$ , and  $3.5718490 \times 10^{-5} \text{ m s}^{-2}$ . Consecutive norm ratios are

$$\frac{E_{0.100}}{E_{0.0500}} = 3.99883, \quad \frac{E_{0.0500}}{E_{0.0250}} = 3.99971.$$

They approach four as expected for second-order error. In Fig. 4.33, both actual estimates and an  $h^2$  guide display that prediction quantitatively.

There is an additional independent check on the stencil. Trigonometric identities at the quarter-turn event give

$$a_{h,x} = -2u\Omega \frac{\sin(\Omega h)}{\Omega h}, \quad a_{h,y} = \frac{2r_*[\cos(\Omega h) - 1]}{h^2},$$

which reproduce the direct position computations. Taylor expansion predicts

$$\mathbf{a}_h - \mathbf{a}_* = \frac{h^2}{12} \mathbf{r}^{(4)}(t_*) + \mathcal{O}(h^4).$$

The leading errors are positive in both components here. This verifies consistency between a prescribed Cartesian path, its analytical acceleration and numerical differentiation. It does not show that real measured coordinates follow the model, that a scanner has sufficient force capacity, or that noisy data will converge similarly. At fixed position uncertainty the  $h^{-2}$  division amplifies noise; extremely small steps can also reveal floating-point cancellation.

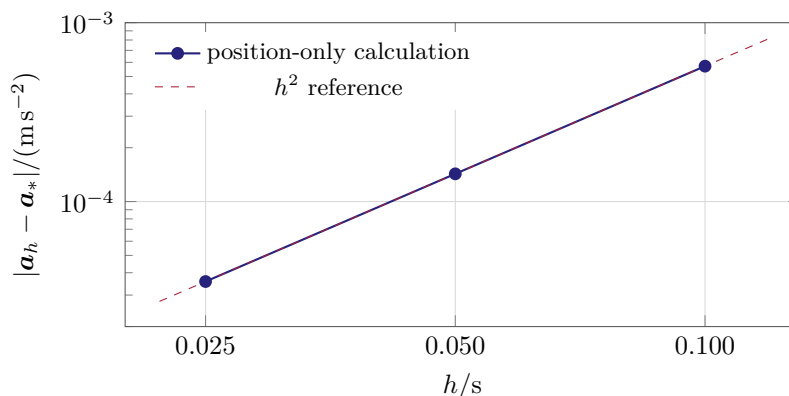


**Figure 4.33:** Problem 32: halving the step reduces the acceleration error norm by approximately four. The dashed reference uses the leading Taylor-error coefficient 0.05715098 in numerical SI units; the solid points come from the specified Cartesian-position stencil. These are discretization errors for a smooth exact model, not experimental uncertainties.

### Problem 33: same curve, different clocks

(a) The first coordinate gives  $t_A = 1$  s and  $t_B = 0.500$  s; each second coordinate is then 1 m. (b) Differentiation before evaluation gives  $\mathbf{v}_A = (1, 2t)$  and  $\mathbf{a}_A = (0, 2)$ , while  $\mathbf{v}_B = (2, 8t)$  and  $\mathbf{a}_B = (0, 8)$  in their SI units. At the shared spatial point,  $\mathbf{v}_B = 2\mathbf{v}_A$  and  $\mathbf{a}_B = 4\mathbf{a}_A$ . (c) Both eliminate time to  $y = x^2$  in the stipulated numerical units. The geometry fixes a tangent and curvature; the schedule fixes speed and its rate of change. Traversing the curve twice as fast quadruples the normal acceleration. The chain rule gives this factor without assuming any force model.

### Problem 34: a planar path in space

(a) Constant vector acceleration gives

$$\mathbf{r}(2) = (1, 2, 3) + (2, 4, 0) + (0, 2, 2) = (3, 8, 5) \text{ m}.$$

(b) The cross product is  $(2, -1, 1)$  multiplied by the units of  $\mathbf{v}_0 \times \mathbf{a}$ . Only its direction matters for the plane. This vector is perpendicular to both  $\mathbf{v}_0$  and  $\mathbf{a}$ , so it is perpendicular to every displacement  $\mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2$ . (c) The plane equation is  $2(x - 1 \text{ m}) - (y - 2 \text{ m}) + (z - 3 \text{ m}) = 0$ . At the endpoint its left side is  $4 - 6 + 2 = 0$  in metres. Three nonconstant coordinates therefore do not establish genuinely nonplanar motion. If  $\mathbf{v}_0$  and  $\mathbf{a}$  were parallel, the cross product would vanish and the path would instead lie on a line.

### Problem 35: areas and events in a velocity graph

(a) The trapezoidal signed area under  $v_y$  up to 0.400 s is

$$\Delta y = (4.00)(0.400) - \frac{1}{2}(9.81)(0.400)^2 = 0.8152 \text{ m},$$

so  $y = 1.0152$  m. The area has units of velocity times time. (b) The vertical velocity is zero at  $t_A = 4.00/9.81 = 0.407747$  s. The horizontal velocity remains  $3.00 \text{ m s}^{-1}$ , so the total speed at the apex is  $3.00 \text{ m s}^{-1}$  and the acceleration is still  $-9.81\hat{\mathbf{j}} \text{ m s}^{-2}$ . (c) Floor contact solves  $0 = 0.200 + 4.00t - 4.905t^2$ . Its roots are  $[4.00 \pm \sqrt{16.00 + 3.924}]/9.81$  seconds. Only  $t_f = 0.862755$  s is a later event; the other root is negative. Thus the free-flight domain is  $0 \leq t \leq t_f$ . Integrating the plotted line beyond that time describes a hypothetical continuation through the floor, not the stated physical flight.

**Problem 36: an acceleration budget**

(a) The marked value gives  $R_c = v^2/a_n = 4.00$  m. (b) Orthogonal components require

$$0.600^2 + \frac{v^4}{4.00^2} \leq 1.50^2, \quad v_{\max} = \sqrt{4.00\sqrt{1.50^2 - 0.600^2}} = 2.345 \text{ m s}^{-1}.$$

At that speed  $a_n = \sqrt{1.89} = 1.3748 \text{ m s}^{-2}$ ; combining it in quadrature with 0.600 returns  $1.50 \text{ m s}^{-2}$ .

(c) The negative  $a_t$  points backward along the tangent;  $a_n$  points toward the bend's local centre. These directions are perpendicular. A scalar sum discards their geometry and overestimates the magnitude. The bound is stipulated kinematics, with no claim about actual motor capacity or contact forces.

**Problem 37: acceleration in two translating frames**

(a) Add relative position to the stage origin before differentiating. At zero time,  $\mathbf{r}_{M/L} = 0.500\hat{\mathbf{i}}$  m,  $\mathbf{v}_{M/L} = 1.00\hat{\mathbf{j}}$  m s<sup>-1</sup>, and

$$\mathbf{a}_{M/L} = (0.200 - 0.500 \times 2.00^2)\hat{\mathbf{i}} \text{ m s}^{-2} = -1.80\hat{\mathbf{i}} \text{ m s}^{-2}.$$

(b) The stage contributes  $+0.200\hat{\mathbf{i}}$  m s<sup>-2</sup>; the circular relative motion contributes  $-2.00\hat{\mathbf{i}}$  m s<sup>-2</sup>.

(c) Although the stage velocity is zero at this event, its acceleration is not. Differentiating  $\mathbf{v}_{M/L} = \mathbf{v}_{M/S} + \dot{\mathbf{b}}$  gives  $\mathbf{a}_{M/L} = \mathbf{a}_{M/S} + \ddot{\mathbf{b}}$ . Acceleration agreement requires constant relative translation over an interval, not merely a momentarily zero origin velocity.

**Problem 38: fitting and differencing synthetic positions**

(a) Least squares minimizes  $\sum_i [x_i - c - bt_i - At_i^2]^2$ . Setting its three coefficient derivatives to zero gives the normal equations. Symmetry supplies  $\sum t_i = \sum t_i^3 = 0$ , so  $b = \sum t_i x_i / \sum t_i^2 = -0.00300 \text{ m s}^{-1}$ . The other sums are  $\sum x_i = 0.400$  m,  $\sum t_i^2 = 0.400 \text{ s}^2$ ,  $\sum t_i^4 = 0.0544 \text{ s}^4$ , and  $\sum t_i^2 x_i = 0.05488 \text{ m s}^2$ . Solving

$$5c + 0.400A = 0.400, \quad 0.400c + 0.0544A = 0.05488$$

with dimensional coefficients understood gives  $c = -0.00171429$  m,  $A = 1.02142857 \text{ m s}^{-2}$ . The fitted acceleration is  $2.042857 \text{ m s}^{-2}$ . (b) The three-point value is  $[0.038 - 2(-0.003) + 0.042]/0.200^2 = 2.15 \text{ m s}^{-2}$ . (c) The fit averages information over a wider window under a constant-acceleration hypothesis. It can suppress random noise but bias a true rapidly changing acceleration. These stipulated numbers do not specify a true trajectory or an error distribution, so neither estimate can be declared closer to truth from smoothness alone. A calibrated uncertainty model and tests with different windows would support that assessment.

**Problem 39: a grazing event**

(a) At  $y_e = 1.000$  m,  $t = 1.000$  s is a double root and a tangency. At 0.990 m,  $(t - 1)^2 = 0.0100$  in numerical SI units, so  $t = 0.900$  and  $1.100$  s. (b) The two endpoint samples have  $y = 0.960$  m; both are below 0.990 m. There are nevertheless two crossings between them. Endpoint signs alone cannot resolve an excursion whose extremum lies inside one step. (c) Lowering the event from the maximum by  $\delta y > 0$  splits its time by  $\sqrt{\delta y/(1 \text{ m s}^{-2})}$  on each side, so linear division by a nonzero crossing speed is unavailable at the maximum. Detect a change in sign of  $v_y = -2(t - 1)$ , locate its zero, and evaluate the coordinate there; refine the interval around the extremum before searching its two monotone sides. This checks the event geometry as well as the integration error.

**Problem 40: a radially breathing scan**

(a) At  $t = 0$ ,  $r = 0.550$  m,  $\dot{r} = 0$ ,  $\ddot{r} = -0.200 \text{ m s}^{-2}$ ,  $\hat{\mathbf{e}}_r = \hat{\mathbf{i}}$ , and  $\hat{\mathbf{e}}_\theta = \hat{\mathbf{j}}$ . Thus  $\mathbf{v} = 0.550\hat{\mathbf{j}}$  m s<sup>-1</sup> and  $\mathbf{a} = (-0.200 - 0.550)\hat{\mathbf{i}}$  m s<sup>-2</sup> =  $-0.750\hat{\mathbf{i}}$  m s<sup>-2</sup>. (b) A fixed circle of radius 0.550 m at  $\Omega = 1.00 \text{ rad s}^{-1}$

gives only  $-0.550\hat{\mathbf{i}}\text{ m s}^{-2}$ . (c) The omitted part is  $\ddot{r}\hat{\mathbf{e}}_r$ , even though radial velocity is momentarily zero. Direct differentiation of  $x = r \cos \Omega t$ ,  $y = r \sin \Omega t$  gives  $\ddot{x} = \ddot{r} \cos \Omega t - 2\dot{r}\Omega \sin \Omega t - r\Omega^2 \cos \Omega t$  and the analogous  $\ddot{y}$ . At zero these are  $-0.750$  and  $0$  in acceleration units, independently confirming the polar result.

## 4.12 Bridge to Newton's laws

Return to the inspection probe at the opening of this chapter. A coordinate observation alone did not reveal its motion. A timed trajectory supplied its velocity and acceleration; projection separated increasing speed from turning; a choice of observer determined the reported components; and numerical or experimental samples required explicit error analysis. The spiral extension showed that even the direction vectors used to express a path can change with time.

None of these calculations explained what causes the probe to accelerate. The same kinematic trajectory might be maintained by different arrangements of supports, motors or interactions. The next chapter, *Newton's Laws of Motion*, introduces the relation between acceleration and the forces on a chosen object. The subsequent applications chapter uses that relation to test whether a proposed trajectory is dynamically consistent with its physical constraints.

A useful question to carry forward is therefore not “Which force points along the motion?” but “What net force would be needed to produce this acceleration, and which actual interactions can supply it?” In a circular path the acceleration is inward even though the velocity is tangent. In a speeding-up bend it has two perpendicular contributions. Accurate kinematics prepares a clear force analysis; it does not replace one.

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