

Chapter 5

Newton's Laws of Motion

First Edition

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Preface

The first four chapters described motion: position, displacement, velocity and acceleration. This chapter asks a different question: *why does motion change?* The answer in classical mechanics is expressed through interactions and Newton’s laws. A force is not a property that a moving body “uses up”; it is a vector description of an interaction acting on a selected system. The net external force determines the acceleration of that system in an inertial frame.

The central skill of the chapter is therefore not formula selection. It is disciplined modelling. Before writing an equation we must decide what object or collection of objects is the system, which frame is being used, which interactions cross the system boundary, and which kinematic constraints connect the motions of different bodies. A correct free-body diagram is part of the physics, not decoration added after the algebra.

The chapter is written for technical and engineering students. The core develops Newton’s three laws, common force models, free-body diagrams, connected bodies and apparent weight. Friction is mentioned only as a possible tangential contact interaction; its quantitative static/kinetic laws are intentionally postponed to Chapter 6. Energy and momentum methods are also postponed until their own chapters so that Newton’s laws remain visible here.

The worked examples and exercises are teaching constructions for this manuscript. Numerical values are stipulated model inputs unless explicitly identified as synthetic data. The references provide complementary treatments of Newtonian modelling; figure construction methods and source credits are identified in the reading guide at the end. A proposed measurement activity is not a report of a performed experiment.

The Guideline

*For every dynamics problem, first write four short statements: **system**, **frame**, **interactions**, and **constraints**. Then draw the free-body diagram and only afterward write Newton’s law. This order prevents many of the most persistent mistakes in mechanics.*

Pedagogical boundary

This chapter deliberately does *not* develop a coefficient-of-friction model, circular-motion force families, drag laws, or a general theory of accelerating frames. Those topics require Newton’s laws, but they should not obscure their first introduction. They are developed in Chapter 6, *Applications of Newton’s Laws*.

Conventions and notation

Vectors are bold, for example $\mathbf{F}$, $\mathbf{a}$, and $\mathbf{v}$. Cartesian unit vectors are $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$. The symbol g denotes the positive local magnitude of gravitational acceleration; the vector $\mathbf{g}$ points downward near Earth’s surface. Numerical examples use $g = 9.81 \text{ m s}^{-2}$ as a teaching value unless stated otherwise.

The notation $\mathbf{F}_{A \leftarrow B}$ means the force *on body A due to body B*. This target–source order is used consistently in the chapter. A normal-force magnitude is written F_N , and a tension magnitude is written F_T . For a linear spring, k_s is the spring constant and x_0 is the reference coordinate at which the spring is unstretched in the one-dimensional model.

The symbol $\sum \mathbf{F}_{\text{ext}}$ means the vector sum of forces exerted on the selected system by bodies outside that system. A force between two bodies that are both inside a chosen composite system is internal to that system. Whether a particular interaction is “internal” or “external” therefore depends on the chosen boundary.

Symbol	Meaning	SI unit
m	inertial mass	kg
$\mathbf{F}$	force	$\text{N} = \text{kg m s}^{-2}$
$\mathbf{a}$	acceleration	m s^{-2}
$\mathbf{W} = m\mathbf{g}$	gravitational force (weight vector in this chapter)	N
F_N	normal-force magnitude	N
F_T	tension magnitude	N
k_s	linear spring constant	N m^{-1}

A force magnitude is non-negative. A component such as F_x can have either sign. Keep one or two guard digits during calculations and round only the final reported result to the precision justified by the inputs. A negative value for a contact-force *magnitude* is not a special kind of contact force; it is a warning that the assumed contact state has failed.

Chapter 5

Newton's Laws of Motion

A precision instrument coasts along a nearly level air-bearing guide. Its measured velocity is constant for several seconds. Does a forward force have to be maintaining the motion? If the instrument were momentarily at rest, would the same interactions imply a different force balance? These questions expose the conceptual change from kinematics to dynamics. Motion itself is not evidence for a net force; *acceleration* is.

In this chapter we build a reusable method for answering such questions. The method begins with interactions and system boundaries, passes through Newton's three laws, and ends with constrained multi-body systems in which the force equations and the kinematic constraints must be kept logically separate.

Learning goals

After studying the core of this chapter, you should be able to:

- define a system and identify the external forces that act on it;
- distinguish an inertial frame from an accelerating frame and use Newton's first law as a diagnostic;
- apply $\sum \mathbf{F}_{\text{ext}} = m\mathbf{a}$ component by component for a constant-mass particle model;
- identify Newton's-third-law partners on different bodies without placing both on one free-body diagram;
- use weight, normal reaction, ideal tension and a linear spring model with their conditions stated;
- derive a kinematic constraint for bodies connected by a taut inextensible link rather than treating it as a force law;
- interpret a scale reading as a contact force and determine when assumed contact is lost;
- explain why equal tension is an approximation tied to a negligible-mass link and ideal connections.

Key message

Newton's laws become reliable problem-solving tools only after the system and frame have been chosen. The equation is short; the modelling decisions that make it applicable are the real work.

5.1 Interactions, systems and inertial frames

5.1.1 Interactions and selected systems

Dynamics studies how interactions change motion. In a macroscopic model an interaction between two bodies is represented by a force. Some interactions require visible contact, such as a table supporting a package. Others, such as Earth's gravitational interaction with the package, act

without mechanical contact. The useful distinction for Newton's laws is not primarily “contact” versus “non-contact”; it is whether the interaction crosses the boundary of the selected system.

Definition 5.1: System, environment and external force

A *system* is the body or collection of bodies chosen for analysis. Everything not in the system is its environment. A force exerted on the system by an object in the environment is an *external force*. Forces between bodies that are both inside a composite system are internal to that system.

System choice is an analytical decision. If two carts connected by a string are studied separately, the string force on each cart is external to that cart. If the two carts and the ideal string are enclosed inside one composite boundary, the cart-string interaction forces are internal and do not appear in the external force sum for the composite system.

A free-body diagram is made *after* the system is chosen. It contains only force vectors acting on that selected body or composite system. It does not contain the body's velocity, the body's acceleration, a “net-force arrow” in addition to the actual forces, or forces exerted by the body on something else.

5.1.2 Inertial frames and Newton's first law

Newton's laws have their simplest form in inertial reference frames. Operationally, an inertial frame is one in which an isolated particle can move with constant velocity. A frame moving with constant velocity relative to an inertial frame is also inertial in Newtonian mechanics. A frame that accelerates relative to an inertial frame is noninertial; its equations require additional frame terms.

Near Earth's surface, a laboratory attached to the ground is often an excellent approximate inertial frame for short-duration, modest-precision mechanics. It is not exactly inertial because Earth rotates and orbits. The approximation is a modelling statement, not an absolute fact.

Newton's first law can be written compactly as

$$\sum \mathbf{F}_{\text{ext}} = \mathbf{0} \implies \mathbf{a} = \mathbf{0} \implies \mathbf{v} = \text{constant}, \quad (5.1)$$

for a constant-mass particle described in an inertial frame. Constant velocity includes the special case $\mathbf{v} = \mathbf{0}$. Therefore zero resultant force is compatible both with rest and with uniform straight-line motion.

Caution

A zero velocity at one instant does *not* imply zero net force. A ball thrown vertically upward has zero instantaneous velocity at its highest point but still accelerates downward. Conversely, a nonzero velocity does not imply a nonzero net force.

Figure 5.1 separates two representations that are often confused. The interaction map answers “what bodies interact?” The free-body diagram answers “what forces act on the chosen system?” Once the system is fixed as the instrument, Earth and the guide are no longer drawn as bodies; their effects are represented by $m\mathbf{g}$ and $\mathbf{F}_N$.

Example 5.1: An instrument coasting on an ideal guide

An instrument glides on a level air-bearing guide. During a four-second interval its measured horizontal velocity remains 0.60 m s^{-1} to the resolution of the motion sensor. Compare this state with the same instrument resting on the guide. What can be concluded about the horizontal net force?

System and frame. The system is the instrument. Use the laboratory as an approximately

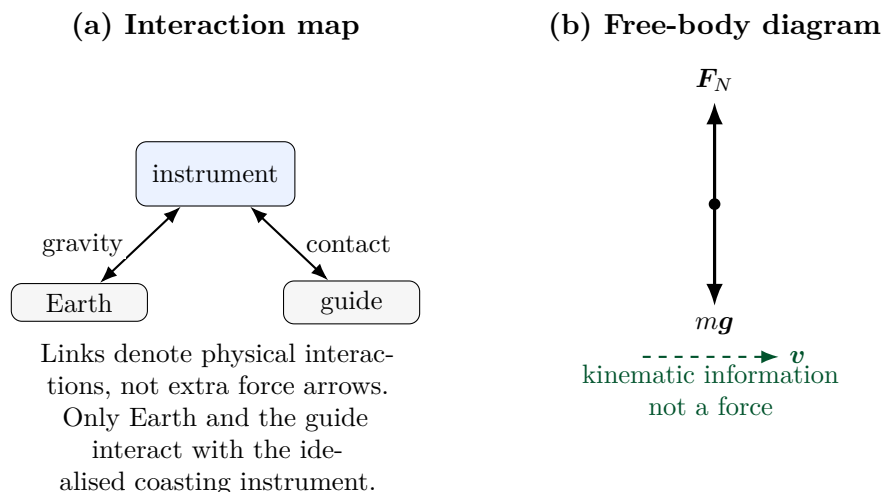


Figure 5.1: An instrument coasting on an ideal horizontal guide in an approximately inertial laboratory frame. The interaction map lists the physical partners; the free-body diagram contains only the two forces on the selected instrument. With negligible tangential resistance, the vertical forces balance and no horizontal force is required for constant horizontal velocity.

inertial frame over the short interval. The guide is idealised as having negligible tangential resistance.

Represent. In both states, the vertical interactions are the normal force and gravity, as in Fig. 5.1. No horizontal interaction is included in the ideal model.

Relate. Constant horizontal velocity means

$$a_x = \frac{dv_x}{dt} = 0.$$

Newton's first law, or equivalently the zero-acceleration case of the second law, gives

$$\sum F_x = 0.$$

The same conclusion holds when the instrument is at rest because rest is merely the constant-velocity case $v_x = 0$.

Interpret. Motion does not need a sustaining forward force. A real guide may have a small drag or residual slope; in that case a small drive could be required to maintain constant speed. The observed constant velocity by itself constrains the *net* force, not each individual interaction separately.

Checks. The result is unchanged if the instrument's speed is 0.20 m s^{-1} , 0.60 m s^{-1} , or zero. What matters is dv_x/dt , not v_x itself.

Guided checks

Check 1. A trolley moves east at constant speed. Must an eastward force act on it? **No.** Its horizontal force sum is zero in the inertial-frame model.

Check 2. A trolley is instantaneously at rest but has $a_x > 0$. Is it in equilibrium? **No.** Zero instantaneous velocity does not imply zero acceleration.

Check 3. May a velocity arrow be drawn on the same sketch? **Yes**, but label it as kinematic information and keep it visually distinct from the force arrows.

5.1.3 Changing observers without changing the force law

The word “constant” in the first law refers to a vector: both the speed and direction remain fixed. A vehicle that turns at constant speed is accelerating, and its attached frame is therefore not covered by the constant-velocity statement. Before deciding whether a moving observer is inertial, ask how the observer’s *velocity vector* changes relative to an already accepted inertial frame.

Two inertial observers can assign different velocities to one object while agreeing on its acceleration. The mathematical reason is short enough to establish here. Let frame S' translate at a constant velocity $\mathbf{V}$ relative to S , with parallel axes and origins coinciding at $t = 0$. Newtonian mechanics uses the same time t in both frames. The positions are related by

$$\mathbf{r}' = \mathbf{r} - \mathbf{V}t. \quad (5.2)$$

Differentiation means comparing the change of a quantity with a small change in time. Differentiating once and twice gives

$$\mathbf{v}' = \mathbf{v} - \mathbf{V}, \quad \mathbf{a}' = \mathbf{a}, \quad (5.3)$$

because the derivative of the constant vector $\mathbf{V}$ vanishes. A constant position offset between the origins could also be included; its derivative would vanish too. Consequently, a force-balanced object can be stationary for one inertial observer and moving for another. The distinction between rest and uniform motion cannot determine the force sum.

Figure 5.2 makes this point without an accelerating-frame calculation. Its position curves have different slopes, but both slopes are constant. A difference in measured velocity alone is therefore no evidence that an observer has become non-inertial.

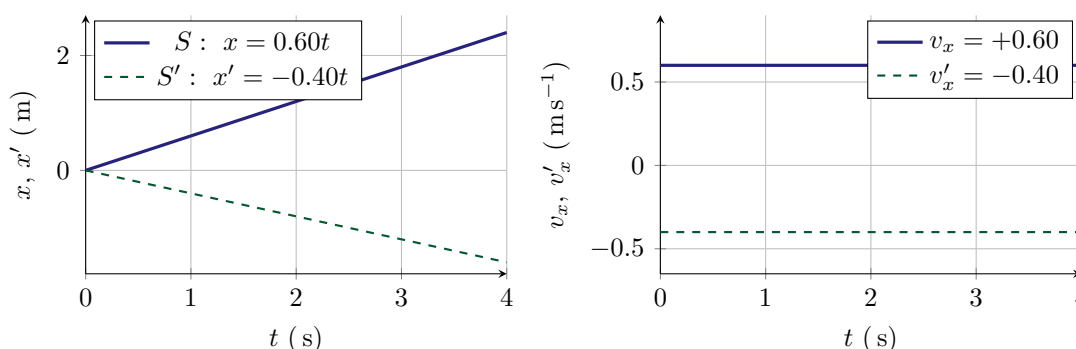


Figure 5.2: One coasting object described by two inertial observers. Frame S' moves along $+x$ at $V = 1.00 \text{ m s}^{-1}$ relative to S . The positions and velocities differ, while both accelerations are zero. Curves are exact analytical model predictions, not experimental records.

Example 5.2: Two observers of the same coasting carriage

A carriage moves on an ideal horizontal guide at $v_x = +0.60 \text{ m s}^{-1}$ relative to the laboratory. An observation platform moves parallel to the guide at $V = +1.00 \text{ m s}^{-1}$, with its origin passing the laboratory origin at $t = 0$. The carriage also passes that origin at $t = 0$. Find its velocity and position after 4.00 s for both observers. Decide whether the observation platform must introduce a new force to explain the carriage’s westward motion relative to it.

Model and strategy. The system is the carriage. Both frames translate without rotation, and the platform velocity is constant. Vertical support and gravity balance; tangential guide resistance is neglected. The frame transformation changes coordinates, so it is the suitable method for comparing observations. It does not change the physical carriage–guide or carriage–Earth interactions.

Calculation. In the laboratory,

$$x(4.00) = (0.60)(4.00) = 2.40 \text{ m}, \quad v_x = +0.60 \text{ m s}^{-1}.$$

For the platform observer,

$$v'_x = 0.60 - 1.00 = -0.40 \text{ m s}^{-1}, \quad x'(4.00) = 2.40 - (1.00)(4.00) = -1.60 \text{ m}.$$

The negative velocity means that the carriage moves toward decreasing platform coordinates; it does not mean that a westward resultant acts on it. Both observers find $a_x = a'_x = 0$, so both use the same zero horizontal force sum.

Checks and interpretation. The plotted position slopes in Fig. 5.2 reproduce the two velocities. Setting $V = v_x$ would make the carriage stationary in S' , still with no horizontal resultant. Setting $V = 0$ makes the two descriptions identical. A platform that speeds up or turns would require a different frame analysis; the constant-velocity transformation used here would no longer establish acceleration equality.

This comparison also clarifies what is meant by a test of inertial behaviour. The test is whether an isolated particle has constant velocity in the candidate frame, rather than whether the candidate observer happens to see a selected object at rest. A supported object is not isolated: its external forces may cancel. Identifying those forces remains necessary even when its acceleration is zero.

5.2 Mass and Newton's second law

5.2.1 Inertial mass and the force sum

The first law identifies what happens when the external resultant is zero. The second law quantifies what happens when it is not. For a constant-mass particle model in an inertial frame,

$$\boxed{\sum \mathbf{F}_{\text{ext}} = m\mathbf{a}.} \quad (5.4)$$

This is an empirical law of classical mechanics, not a result derived from geometry or dimensional analysis. It connects three independently meaningful ideas: interactions represented by force, resistance to acceleration represented by inertial mass, and the kinematic response represented by acceleration.

For the same net force, a larger mass has a smaller acceleration. For the same mass, doubling the entire force resultant doubles the acceleration. Because mass is positive in the models used here, $\sum \mathbf{F}_{\text{ext}}$ and $\mathbf{a}$ always point in the same direction.

The SI force unit follows from Eq. (5.4):

$$1 \text{ N} = 1 \text{ kg m s}^{-2}. \quad (5.5)$$

The unit statement checks dimensions; it does not prove the physical law.

5.2.2 Component equations and signed forces

Choose axes to make the geometry transparent. Equation (5.4) is equivalent to

$$\sum F_x = ma_x, \quad (5.6)$$

$$\sum F_y = ma_y, \quad (5.7)$$

$$\sum F_z = ma_z. \quad (5.8)$$

A force that points in the negative x direction contributes a negative x component; its magnitude is still positive. Components along one axis do not directly produce acceleration along a perpendicular axis.

Method

Model–Represent–Relate–Solve–Test–Interpret.

1. **Model:** name the system, frame, interactions and assumptions.
2. **Represent:** draw the geometry and a separate free-body diagram.
3. **Relate:** write Newton's law by components and write any constraints separately.
4. **Solve:** keep symbols until the model structure is clear; then substitute numbers with units.
5. **Test:** check units, signs, limiting cases and the original equations.
6. **Interpret:** say what the result means physically and where the model could fail.

5.2.3 Inferring an unknown force without adding magnitudes

The vector equation is especially useful when an interaction is unknown but the acceleration has been determined independently. Suppose that $\mathbf{F}_1$ and $\mathbf{F}_2$ are known forces on the selected body, and that exactly one further interaction $\mathbf{F}_3$ is included in the model. Rearranging the second law gives

$$\mathbf{F}_3 = m\mathbf{a} - \mathbf{F}_1 - \mathbf{F}_2. \quad (5.9)$$

The subtraction is performed by components. Adding force magnitudes would discard directions and can produce an answer incompatible with the measured motion. Similarly, subtracting only the known horizontal force does not determine an unknown vector: its vertical component needs a separate equation.

Two drawings serve different purposes in Fig. 5.3. All three actual force arrows start on the chosen particle in the free-body diagram. The head-to-tail construction translates the arrows without changing their lengths or directions so that their sum is visible. Its diagonal is a *resultant*, not a fourth physical interaction. These conventions help distinguish a force inventory from the vector arithmetic performed on that inventory.

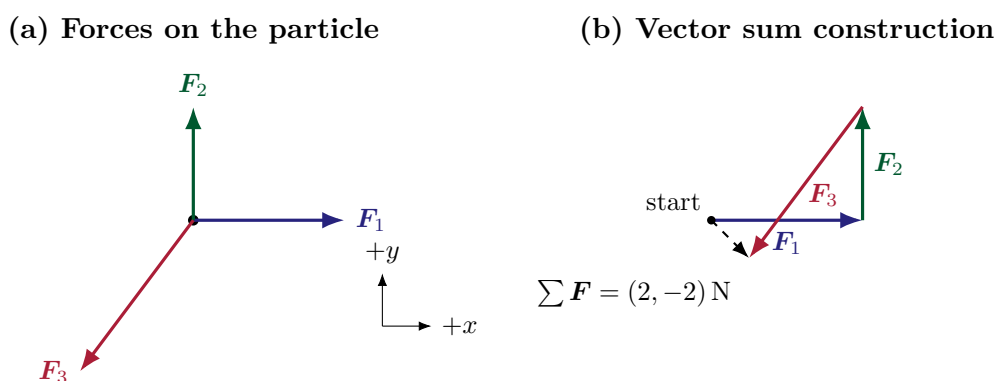


Figure 5.3: Recovering a missing interaction from a known acceleration. The left panel is a two-dimensional force diagram; support and gravity perpendicular to this plane cancel in the ideal horizontal-table model. The right panel performs vector addition and is not another free-body diagram. Arrow vectors share the stated scale 0.25 cm per newton.

Example 5.3: Recover a two-component actuator force

A 2.00 kg puck translates on an ideal horizontal air table. The x and y axes lie in the table plane. Two calibrated actuators apply $\mathbf{F}_1 = 8.00\hat{x}$ N and $\mathbf{F}_2 = 6.00\hat{y}$ N. A third actuator supplies an unknown vector force. Motion tracking finds $\mathbf{a} = (1.00\hat{x} - 1.00\hat{y}) \text{ m s}^{-2}$. Find both components and the magnitude of the third force. Verify the measured acceleration by reconstructing the force sum.

Model. The laboratory is an approximate inertial frame. The three actuator forces are the only in-plane forces. In the direction perpendicular to the table, the normal force balances gravity. Figure 5.3 therefore represents the complete *in-plane* force inventory.

Strategy and solution. The required resultant is

$$m\mathbf{a} = (2.00\hat{x} - 2.00\hat{y}) \text{ N}.$$

Subtract the two known interaction vectors:

$$\begin{aligned} F_{3x} &= ma_x - F_{1x} - F_{2x} = 2.00 - 8.00 - 0 = -6.00 \text{ N}, \\ F_{3y} &= ma_y - F_{1y} - F_{2y} = -2.00 - 0 - 6.00 = -8.00 \text{ N}. \end{aligned}$$

Thus $\mathbf{F}_3 = (-6.00\hat{x} - 8.00\hat{y})$ N and

$$|\mathbf{F}_3| = \sqrt{(-6.00)^2 + (-8.00)^2} = 10.0 \text{ N}.$$

It points southwest, $\tan^{-1}(8/6) = 53.1^\circ$ below the negative x axis. The angle is identified by its quadrant as well as by the inverse tangent.

Independent reconstruction. Adding the actual forces gives

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = (2.00, -2.00) \text{ N},$$

whose components divided by 2.00 kg are precisely the measured acceleration components. The resultant magnitude is 2.83 N, considerably smaller than any of several individual forces, because the components partly cancel. This is physically consistent and illustrates why magnitudes cannot be added as ordinary scalars.

Limitation. The inferred vector belongs to the stated three-interaction model. An unmodelled air-table slope or cable force would also enter the subtraction. Independent acceleration data determine the required total of all unknown forces, not their separate origins without additional evidence.

A useful inverse problem starts from measured or stipulated motion and asks what applied force is needed. Suppose a horizontal test carriage of mass m experiences a drive force F_{drive} toward $+x$ and a known resistance of magnitude R toward $-x$. Then

$$F_{\text{drive}} - R = ma_x, \quad F_{\text{drive}} = ma_x + R. \quad (5.10)$$

The drive force is not the net force. The net force is ma_x ; the actuator must also balance the specified resistance.

Example 5.4: Infer an actuator force from motion

A test carriage of mass $m = 18.0$ kg moves on a straight horizontal guide. A separately characterised resistance of magnitude $R = 12.0$ N acts opposite its motion. At one instant the measured acceleration is $a_x = 1.25 \text{ m s}^{-2}$. Determine the required drive force in the stated model.

(a) Test carriage model

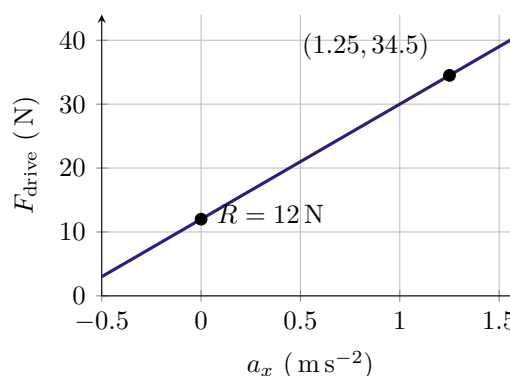
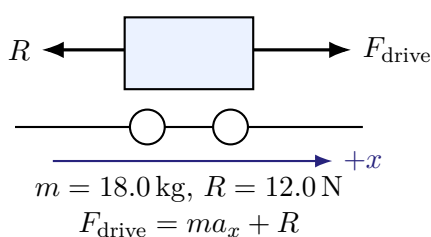


Figure 5.4: A prescribed-resistance force-inference model (horizontal forces only) and its exact analytical relation. For fixed resistance R , the intercept is R ; for fixed mass m , the slope is m . The plotted line uses $m = 18.0 \text{ kg}$ and $R = 12.0 \text{ N}$. These are stipulated model values, not measured device data.

Model. The laboratory is an approximate inertial frame. The carriage mass is constant. The resistance is a prescribed force of known magnitude; no coefficient-of-friction model is introduced.

Represent. Choose $+x$ along the motion. The horizontal forces are $+F_{\text{drive}}$ and $-R$. Vertical forces balance because the guide constrains vertical acceleration to zero.

Relate and solve.

$$\begin{aligned} F_{\text{drive}} - R &= ma_x, \\ F_{\text{drive}} &= (18.0)(1.25) + 12.0 \text{ N} \\ &= 34.5 \text{ N}. \end{aligned}$$

Thus the net horizontal force is only $ma_x = 22.5 \text{ N}$; the actuator supplies an additional 12.0 N to balance the stipulated resistance.

Tests. If $a_x \rightarrow 0$, the result becomes $F_{\text{drive}} = R = 12.0 \text{ N}$, as it must for constant velocity. Changing the mass changes the slope of the line in Fig. 5.4 but not its resistance intercept. Units are $\text{kg m s}^{-2} = \text{N}$.

Interpretation. This calculation infers the force required by the stated model. A real carriage may have speed-dependent drag, track inclination, bearing losses, actuator dynamics and measurement uncertainty. Those effects require additional evidence and model terms rather than hidden adjustments to Newton's law.

5.2.4 Acceleration need not point along the velocity

Because the second law relates force to acceleration rather than velocity, a force can change the direction of motion without being parallel to the instantaneous velocity. If, for example, $\mathbf{v}$ points east while $\sum \mathbf{F}_{\text{ext}}$ points north, the northward acceleration changes the north component of velocity. The detailed dynamics of circular and curved motion are deferred to Chapter 6, but the vector principle already follows from Eq. (5.4).

5.2.5 Initial velocity and subsequent force response

A force law and initial conditions play different roles. The force resultant determines the acceleration at each instant. The initial velocity determines how that acceleration is added to the motion already present. For a constant acceleration over a stated interval, integration of $d\mathbf{v}/dt = \mathbf{a}$ gives

$$\mathbf{v}(t) = \mathbf{v}_0 + \mathbf{a}t, \quad \mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0t + \frac{1}{2}\mathbf{a}t^2. \quad (5.11)$$

Each component of the acceleration changes the matching velocity component. There is no additional assumption that the resultant aligns with $\mathbf{v}_0$.

The distinction between changing velocity and changing speed can also be made precise. The squared speed is $v^2 = \mathbf{v} \cdot \mathbf{v}$. Differentiating yields

$$\frac{d(v^2)}{dt} = 2\mathbf{v} \cdot \mathbf{a}. \quad (5.12)$$

At a nonzero speed, division by $2v$ shows that the instantaneous speed change depends on the acceleration component along the current velocity. An acceleration perpendicular to that velocity gives zero *instantaneous* speed derivative. It can still change direction, and it need not remain perpendicular as the velocity develops. This derivative statement does not replace a full integration when finite-time changes are requested.

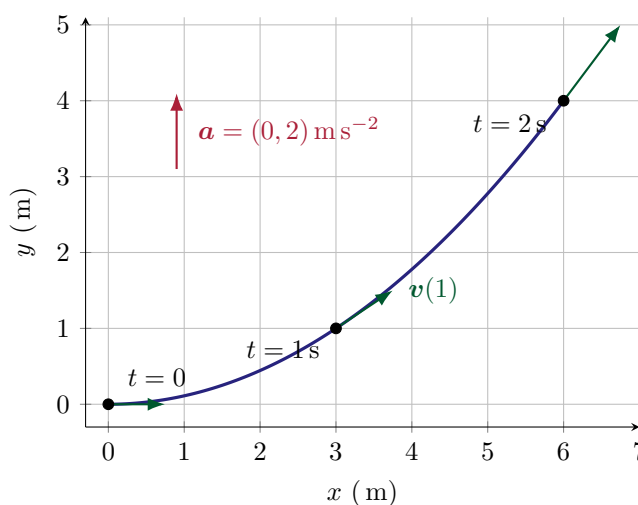


Figure 5.5: Motion under a constant northward in-plane force with an initially eastward velocity. The path is $x = 3t$, $y = t^2$ for $0 \leq t \leq 2$ s. Green tangent arrows are velocities, drawn at a common display multiplier of 0.25 s; the separate red arrow indicates acceleration direction and is not an added force. This is a motion diagram, not a free-body diagram.

Example 5.5: A northward force on an eastward-moving puck

A 2.00 kg puck moves on an ideal horizontal air table with initial velocity $\mathbf{v}_0 = 3.00\hat{\mathbf{x}} \text{ m s}^{-1}$. At $t = 0$ an actuator begins applying the constant in-plane force $\mathbf{F} = 4.00\hat{\mathbf{y}} \text{ N}$. Set the initial in-plane position to zero. Find the position, velocity, speed and direction at $t = 2.00$ s. Does the initially perpendicular force keep the speed fixed throughout the interval?

Model and representation. The system is the puck; the laboratory frame is approximately inertial. Vertical gravity and support cancel, with no other horizontal interaction. The actuator force acts along $+y$; $+x$ is the initial direction of motion. The trajectory and tangent vectors are shown in Fig. 5.5. The constant force establishes a constant acceleration, so the integrated kinematic equations apply.

Dynamics and integration. Newton's law gives

$$a_x = 0, \quad a_y = \frac{4.00}{2.00} = 2.00 \text{ m s}^{-2}.$$

Therefore $v_x(t) = 3.00 \text{ m s}^{-1}$, $v_y(t) = (2.00 \text{ m s}^{-2})t$, $x(t) = (3.00 \text{ m s}^{-1})t$, and $y(t) = (1.00 \text{ m s}^{-2})t^2$. At the requested time,

$$\mathbf{r} = (6.00\hat{\mathbf{x}} + 4.00\hat{\mathbf{y}}) \text{ m}, \quad \mathbf{v} = (3.00\hat{\mathbf{x}} + 4.00\hat{\mathbf{y}}) \text{ m s}^{-1}.$$

The speed is $\sqrt{3.00^2 + 4.00^2} = 5.00 \text{ m s}^{-1}$, and the velocity points $\tan^{-1}(4/3) = 53.1^\circ$ north of east.

Why the speed changes. The force is perpendicular to velocity only at the initial instant. For $t > 0$, the velocity acquires a northward component, so $\mathbf{v} \cdot \mathbf{a} = 4t$ in consistent SI units is positive. Explicitly, $v(t) = \sqrt{9.00 + 4.00t^2} \text{ m s}^{-1}$ when t is entered in seconds. Its initial derivative is zero, but its speed after two seconds is larger than its initial speed.

Checks. Differentiating the parametric position twice recovers $\mathbf{a} = (0, 2.00) \text{ m s}^{-2}$. The path slope at $t = 1 \text{ s}$ is $dy/dx = v_y/v_x = 2/3$, matching the drawn tangent. Turning a velocity through a finite angle at constant speed would require a force direction that changes with the motion; that application is developed in Chapter 6.

Guided checks

Check 1. Doubling every external force while holding the mass fixed doubles $\mathbf{a}$.

Check 2. Doubling the mass at fixed net force halves $\mathbf{a}$.

Check 3. In the carriage model, $F_{\text{drive}} = R$ corresponds to zero acceleration, not necessarily zero velocity.

5.3 Newton's third law

Newton's second law answers what the external force sum does to one selected system. Newton's third law describes a different idea: forces arise in interactions between pairs of bodies. If body B exerts a force on body A , then body A simultaneously exerts an equal-magnitude, opposite-direction force on body B :

$$\mathbf{F}_{A \leftarrow B} = -\mathbf{F}_{B \leftarrow A}. \quad (5.13)$$

The two vectors form one interaction pair. They are the same type of interaction, occur at the same time in the classical model, and act on *different bodies*.

5.3.1 Interaction pairs on different bodies

The most important consequence of Eq. (5.13) is graphical. If the free-body diagram is for body A , then $\mathbf{F}_{A \leftarrow B}$ belongs on it and $\mathbf{F}_{B \leftarrow A}$ does not. The partner force belongs on the free-body diagram of B .

This immediately diagnoses a common error. Suppose a package rests on a horizontal table. The downward gravitational force $m\mathbf{g}$ and the upward normal force $\mathbf{F}_N$ can have equal magnitudes, but they are not a third-law pair because both act on the package. The partner to the normal force on the package is the force of the package on the table. The partner to Earth's gravitational force on the package is the gravitational force of the package on Earth.

Caution

“Equal and opposite” is not enough to identify a third-law pair. Ask two questions: *Do the forces come from the same interaction?* and *Do they act on two different bodies?* If either answer is no, they are not Newton-third-law partners.

5.3.2 Internal versus external forces

A direct third-law pair cancels when equations for a system containing both interacting bodies are added. A connector introduces a third body, which must be named rather than hidden. Let C be a

straight link between carts A and B . The direct pairs are

$$\mathbf{F}_{A \leftarrow C} = -\mathbf{F}_{C \leftarrow A}, \quad \mathbf{F}_{B \leftarrow C} = -\mathbf{F}_{C \leftarrow B}. \quad (5.14)$$

The two forces *on the carts*, $\mathbf{F}_{A \leftarrow C}$ and $\mathbf{F}_{B \leftarrow C}$, have different target–source pairs. They are not each other's direct third-law partners. To establish whether their magnitudes are equal, isolate C and apply its own equation of motion.

Take rightward as positive, with A on the left, and use T_L, T_R for endpoint magnitudes. If external pulls P_A and P_B point outward, the horizontal equations are

$$T_L - P_A = m_A a_A, \quad (5.15)$$

$$P_B - T_R = m_B a_B, \quad (5.16)$$

$$T_R - T_L = m_C a_C. \quad (5.17)$$

The third line uses the two direct partners acting *on the link*. In the negligible-mass limit at finite acceleration it gives $T_R = T_L = T$. Equal tension is therefore a balance result for the connector model. It is not a new interaction-pair identity.

For a taut, straight, inextensible link, the geometrical constraint supplies $a_A = a_B = a_C = a$. Adding all three equations yields

$$P_B - P_A = (m_A + m_B + m_C)a. \quad (5.18)$$

Every direct cart–link pair is internal to $A + B + C$. As $m_C \rightarrow 0$, the combined mass becomes $m_A + m_B$. If the link has appreciable mass, summing only the cart equations leaves $T_L - T_R$: the link is outside that two-cart boundary and its two endpoint forces need not cancel.

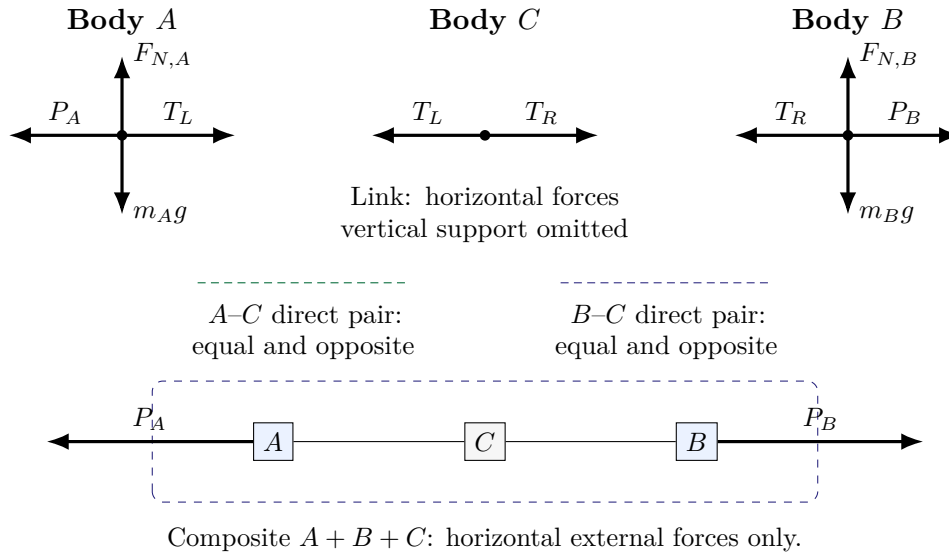


Figure 5.6: The carts and their connector are three distinct bodies. Dashed links identify direct interaction pairs between each cart and the link. The equal T_L labels are one pair; the equal T_R labels are another. Only a massless-link balance equates T_L with T_R . The bottom panel selects all three bodies; balanced vertical forces are omitted there. Arrow lengths are schematic.

Example 5.6: Carts, connector and system boundary

Carts A and B are joined by a straight taut connector C . External agents pull A leftward and B rightward. Identify the direct interaction pairs, explain when the endpoint tensions are equal, and state the force content of the complete system.

Identify bodies. The force on A due to C has its partner on C due to A . Likewise the force on B due to C has its partner on C due to B . These are two different interactions, as Figure 5.6 shows.

Check the connector model. The link equation is $T_R - T_L = m_C a_C$. If its mass is negligible and its acceleration finite, $T_R = T_L$. With appreciable mass and nonzero acceleration, equality fails although each direct third-law pair remains exactly paired in the Newtonian model.

Select the complete system. For $A + B + C$, both cart–link interactions are internal. The horizontal external resultant is $P_B - P_A$. The complete mass is $m_A + m_B + m_C$. A common acceleration follows only when the straight, taut, inextensible geometry enforces it; this condition is separate from every force equation.

Interpretation. Equal interaction forces on two otherwise unconstrained bodies do not imply equal accelerations if their masses differ. The ability to name both targets and both sources is a more reliable test of a third-law pair than a superficial equality of arrow lengths.

Guided checks

Check 1. Are the link's forces on A and on B a direct third-law pair? **No.** Each has its partner on the link, at its own endpoint.

Check 2. Why are end tensions equal in the ideal model? **The massless connector requires zero resultant at finite acceleration.**

Check 3. When are all cart–link forces internal? **When both carts and the connector are inside the system boundary.**

5.3.3 A direct interaction and two useful system boundaries

The distinction between an interaction pair and a force balance is particularly transparent for two bodies in direct contact. Consider carts A and B , with A to the left, touching on a level frictionless track. A horizontal external push P acts on A toward B . The direct contact between the carts exerts $\mathbf{F}_{A \leftarrow B}$ leftward on A and $\mathbf{F}_{B \leftarrow A}$ rightward on B . These two forces *are* a direct Newton-third-law pair. No assumption about a connector's mass is needed to establish their equal magnitudes.

If the contact remains closed and the carts translate without deformation or rotation relevant to the model, their separation remains fixed. This geometric fact gives a common acceleration. Writing $C \geq 0$ for the compressive contact-force magnitude, the two equations are

$$P - C = m_A a, \quad C = m_B a. \quad (5.19)$$

Adding the equations cancels the interaction pair and gives $P = (m_A + m_B)a$. This is exactly the equation obtained by drawing one boundary around both carts. The combined boundary is efficient for finding the common acceleration, while the boundary around B alone is efficient for finding the internal contact force.

The choice of boundary therefore determines which information an equation can reveal. The combined equation contains no C , so it cannot by itself determine C . Its absence is a useful elimination of an unknown, rather than evidence that the front cart accelerates without any horizontal interaction.

Example 5.7: Push two touching carts and find the contact force

Two touching carts of masses $m_A = 3.00 \text{ kg}$ and $m_B = 5.00 \text{ kg}$ stand on an ideal horizontal track. A 24.0 N external horizontal force pushes the rear cart A toward the front cart B . The carts maintain contact and translate without relevant rotation. Find the acceleration and the

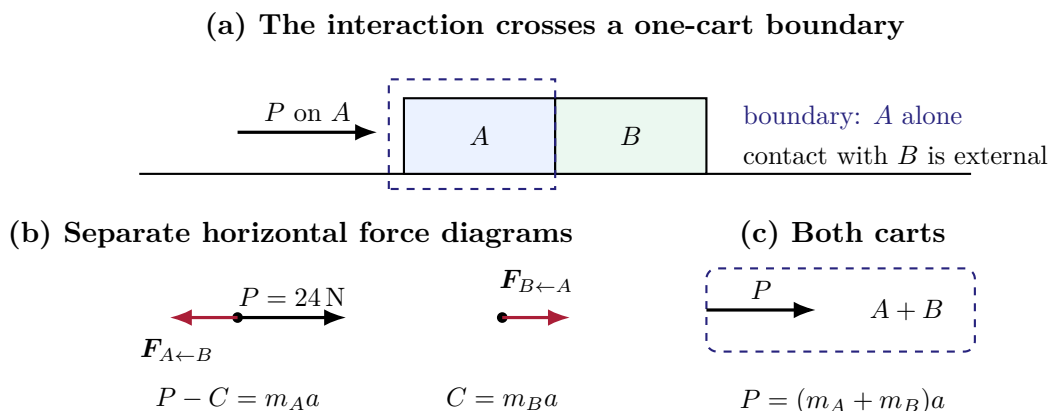


Figure 5.7: A direct cart–cart contact has a genuine third-law pair on different carts. For one cart the contact is external; for both carts it is internal. The lower panels show horizontal force content only: the omitted vertical normal forces and weights balance separately. Horizontal force lengths in panel (b) use 0.06 cm per newton; the apparatus and composite-system panel are schematic.

force each cart exerts on the other. Would applying the same push to the 5.00 kg cart, with that cart now at the rear, change these two results?

Systems and strategy. The approximately inertial laboratory is the frame. Gravity and track reactions cancel vertically. First enclose both carts, as in Fig. 5.7, to remove the unknown internal contact force. Then isolate the front cart, whose only horizontal force is the contact from the rear cart. The closed-contact condition supplies the common acceleration independently of the force laws.

Complete system. The total moving mass is 8.00 kg, so

$$a = \frac{P}{m_A + m_B} = \frac{24.0}{8.00} = 3.00 \text{ m s}^{-2}.$$

For front cart B, $C = m_B a = (5.00)(3.00) = 15.0 \text{ N}$. Thus the force on B due to A is $+15.0\hat{x} \text{ N}$, and the force on A due to B is $-15.0\hat{x} \text{ N}$. The rear-cart equation checks the result:

$$P - C = 24.0 - 15.0 = 9.00 \text{ N} = m_A a.$$

These equal contact forces coexist with different total force resultants on the two carts, namely 9.00 N and 15.0 N.

Reversing the mass order. The complete-system mass and external push are unchanged, so the common acceleration remains 3.00 m s^{-2} . The front cart is now the 3.00 kg cart, which requires only $C' = 3.00(3.00) = 9.00 \text{ N}$. The rear 5.00 kg cart satisfies $24.0 - 9.00 = 15.0 \text{ N} = 5.00(3.00)$. The internal contact force changes because it must accelerate a different front mass.

Physical test. Both computed contact magnitudes are positive, consistent with a pushing contact. The acceleration limit $m_B \rightarrow 0$ in the first arrangement makes $C \rightarrow 0$. The third law fixes each contact pair's equality, while geometry and the full force inventory determine the motion.

5.4 Common force models

Newton's laws do not tell us the detailed value of every force. We also need models for particular interactions. Some are simple and broadly useful; others depend strongly on materials and regime. This section introduces only the force models needed for the present chapter.

5.4.1 Weight: the local gravitational force

Near Earth's surface, over a region small enough that the gravitational field is approximately uniform, we write

$$\mathbf{W} = m\mathbf{g}, \quad W = mg. \quad (5.20)$$

Here $\mathbf{g}$ points downward and $g = |\mathbf{g}|$ is positive. The mass m measures inertia and is independent of the local value of g ; the gravitational force changes if the gravitational environment changes.

The word “weight” is used differently across contexts. In this chapter $\mathbf{W} = m\mathbf{g}$ denotes the gravitational force, while a scale reading is explicitly called the *normal force* or *apparent weight*. This removes the ambiguity that otherwise occurs in accelerating lifts.

5.4.2 Normal and tangential contact forces

A solid surface can prevent interpenetration by exerting a normal reaction approximately perpendicular to the local contact surface. Its magnitude F_N is not fixed by a universal formula. It is whatever value is required by the contact constraint, as long as the assumed contact state can physically be maintained.

A contact can also exert a force tangent to the surface. In this chapter we may determine the *tangential force required* by Newton's law, but we do not yet decide whether a dry surface can supply that amount. The static and kinetic friction laws, their inequalities and coefficients are introduced in Chapter 6.

Key message

A normal force is a response to a contact constraint. It is not automatically mg . The equality $F_N = mg$ is a special result for a body with no vertical acceleration and no other vertical force components on a horizontal support.

5.4.3 An inclined support: derive the normal reaction

Tilting the support changes both the normal direction and the gravitational component that the support must balance. Consider a smooth plane making an angle $0 < \theta < 90^\circ$ above horizontal. Let s increase uphill and let n increase perpendicular to the plane, outward from its surface. Relative to horizontal x and vertical y , the corresponding unit vectors are

$$\hat{\mathbf{t}} = \cos \theta \hat{\mathbf{x}} + \sin \theta \hat{\mathbf{y}}, \quad \hat{\mathbf{n}} = -\sin \theta \hat{\mathbf{x}} + \cos \theta \hat{\mathbf{y}}. \quad (5.21)$$

Their scalar product is zero and each has unit length, so they define perpendicular directions without changing the physical vector. A component along a unit vector is found by a scalar product. With $\mathbf{W} = -mg\hat{\mathbf{y}}$,

$$W_s = \mathbf{W} \cdot \hat{\mathbf{t}} = -mg \sin \theta, \quad W_n = \mathbf{W} \cdot \hat{\mathbf{n}} = -mg \cos \theta. \quad (5.22)$$

Both signs have a physical meaning: gravity points downhill and into the plane. Figure 5.8 shows the geometry that produces the sine and cosine factors, rather than asking the reader to memorize them from a particular orientation.

If the body stays in contact with a *fixed straight* plane, its acceleration perpendicular to that plane is zero. For a frictionless support, the only contact force is normal. Newton's equations then give

$$F_N - mg \cos \theta = 0, \quad -mg \sin \theta = ma_s, \quad (5.23)$$

and therefore $F_N = mg \cos \theta$ and $a_s = -g \sin \theta$. The result is not obtained by setting the vertical acceleration to zero. A body accelerating along an incline has a nonzero vertical acceleration. It is the *normal* acceleration that vanishes.

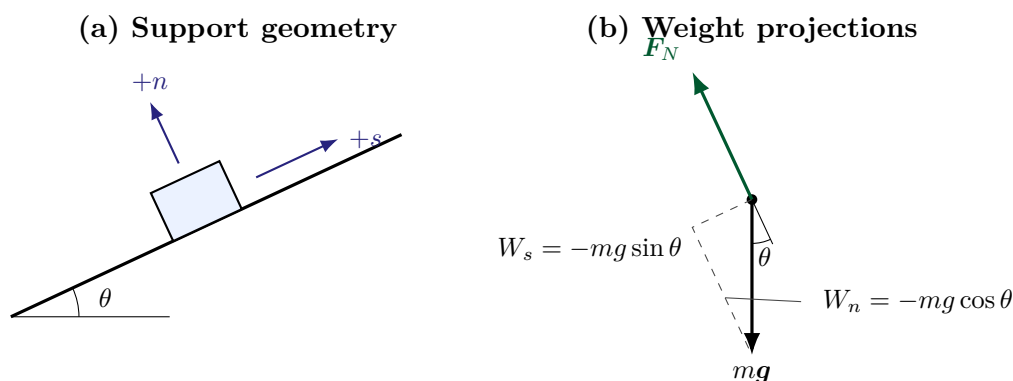


Figure 5.8: Weight components on a plane tilted by $\theta = 25.0^\circ$. The force diagram contains only gravity and the outward normal reaction. Dashed segments are projection guides for gravity, not extra interactions; they explain the signed components in the rotated basis. The shown force lengths use the 6.00 kg example and 0.035 cm per newton. Apparatus geometry is schematic.

Example 5.8: A carriage moves uphill before reversing

A 6.00 kg carriage is released on a fixed frictionless straight guide inclined at 25.0° above horizontal. At release it is moving uphill at 2.00 m s^{-1} . Find its normal force and tangential acceleration, the time until it is instantaneously at rest, and its maximum uphill displacement from release. Explain why the acceleration does not reverse when the velocity reverses.

Model and strategy. The carriage is the selected constant-mass particle; the laboratory is approximately inertial. The track supplies only a normal force. Use $+s$ uphill and $+n$ outward as in Fig. 5.8. The normal constraint is $a_n = 0$. Applying Newton's law in this rotated basis gives a constant tangential acceleration, after which the previously established constant-acceleration equations determine the turning point.

Forces and acceleration.

$$F_N = mg \cos 25.0^\circ = (6.00)(9.81) \cos 25.0^\circ = 53.3453 \text{ N} \approx 53.3 \text{ N},$$

$$a_s = -g \sin 25.0^\circ = -4.14589 \text{ m s}^{-2} \approx -4.15 \text{ m s}^{-2}.$$

The acceleration is downhill even while the velocity is uphill. From $v_s = v_{s0} + a_s t$, the turning time is

$$t_{\text{turn}} = \frac{v_{s0}}{g \sin \theta} = \frac{2.00}{4.14589} = 0.482 \text{ s}.$$

Substitute this time into $\Delta s = v_{s0} t - \frac{1}{2} g \sin \theta t^2$, or eliminate the time algebraically, to obtain

$$\Delta s_{\text{max}} = \frac{v_{s0}^2}{2g \sin \theta} = \frac{4.00}{8.29177} = 0.482 \text{ m}.$$

Checks and interpretation. At the turning point the velocity is zero but the force inventory is unchanged. The acceleration remains downhill; the carriage subsequently develops a downhill velocity. The positive normal force is consistent with contact. When $\theta \rightarrow 0$, $a_s \rightarrow 0$ and the finite turning time disappears, as expected for an ideal level guide. The mass cancels from the acceleration but not from the normal force, because gravitational force and inertial mass increase in the same proportion.

For an independent component check in horizontal and vertical coordinates, the two forces give $F_x = -F_N \sin \theta$ and $F_y = F_N \cos \theta - mg$. Dividing by m gives $a_x = -g \sin \theta \cos \theta$ and $a_y = -g \sin^2 \theta$, exactly the components of $a_s \hat{\mathbf{t}}$. Thus a nonzero downward vertical acceleration coexists with zero normal acceleration.

5.4.4 An angled tether changes two component equations

An applied force that is not parallel to a guide changes both the tangential acceleration and the normal reaction. Let a tether pull with tension magnitude $F_T \geq 0$ at an angle ϕ outward from the uphill direction. Its components are $F_T \cos \phi$ uphill and $F_T \sin \phi$ outward. Keeping the same axes as the preceding incline model gives

$$F_T \cos \phi - mg \sin \theta = ma_s, \quad F_N + F_T \sin \phi - mg \cos \theta = 0. \quad (5.24)$$

The second equation again uses $a_n = 0$ while contact is maintained. It follows that

$$a_s = \frac{F_T \cos \phi}{m} - g \sin \theta, \quad F_N = mg \cos \theta - F_T \sin \phi. \quad (5.25)$$

An outward tether component reduces the normal reaction. It cannot be ignored merely because the main question concerns motion along the guide.

For equilibrium along the guide, set $a_s = 0$ and solve for the required tension:

$$F_{T,\text{eq}} = \frac{mg \sin \theta}{\cos \phi}, \quad F_{N,\text{eq}} = mg(\cos \theta - \sin \theta \tan \phi). \quad (5.26)$$

These expressions assume $0 \leq \phi < 90^\circ$. A smooth nonadhesive support also requires $F_N \geq 0$. For $0 < \theta < 90^\circ$, the equilibrium contact branch is admissible only when $\phi \leq 90^\circ - \theta$. At the endpoint the tether is vertical in laboratory coordinates, its magnitude is mg , and the normal reaction is zero. Extrapolating the equilibrium formula beyond that angle produces a negative reaction and rejects the assumed supported state.

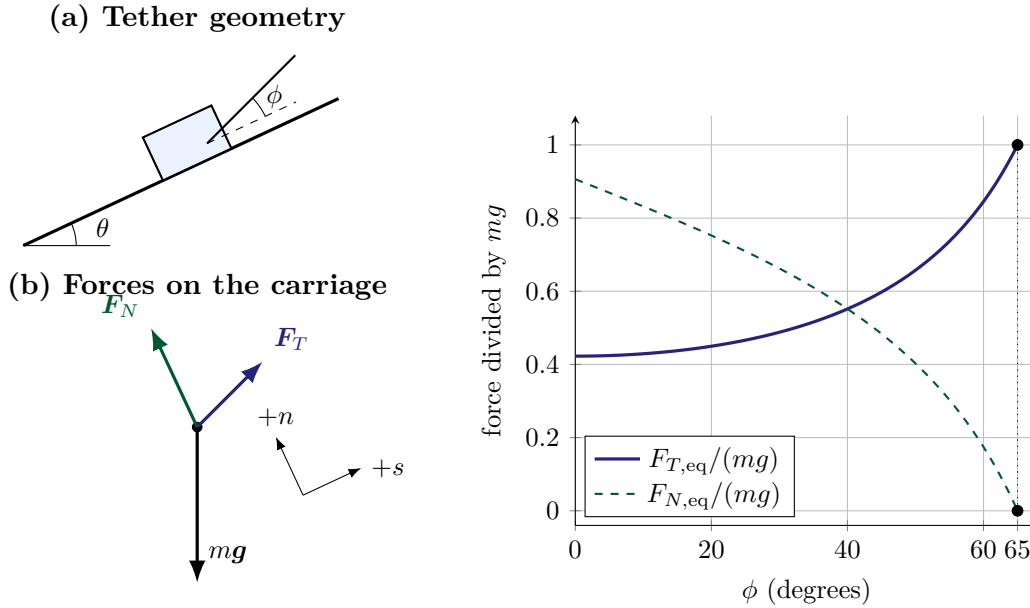


Figure 5.9: A tether angled away from a frictionless incline contributes to both component equations. Geometry uses $\theta = 25.0^\circ$, $\phi = 20.0^\circ$; the force diagram uses the accelerating example with $F_T = 35.0$ N. The analytical plot instead varies the tether angle and chooses the tension required for equilibrium at each angle. Its contact branch ends at $\phi = 65.0^\circ$, where the tether is vertical and $F_N = 0$.

Example 5.9: An inclined instrument guide with an angled tether

A 6.00 kg instrument moves along a frictionless fixed guide inclined at $\theta = 25.0^\circ$. A tether pulls uphill at $\phi = 20.0^\circ$ outward from the guide. (a) Find the acceleration and normal reaction when the tension is 35.0 N. (b) Find the tension required for equilibrium at the same angle,

and the corresponding normal reaction. (c) Determine the largest outward tether angle that permits equilibrium with nonnegative normal reaction.

Model and strategy. The selected body is the instrument, represented as a translating particle. The laboratory is approximately inertial. The three actual forces are the tether force, normal reaction and gravity, as drawn in Fig. 5.9. Use the same uphill and outward axes for every subpart. Resolve the tether before solving, and test $F_N \geq 0$ afterward.

(a) **Prescribed tension.** The two components give

$$a_s = \frac{35.0 \cos 20.0^\circ}{6.00} - 9.81 \sin 25.0^\circ = 1.33566 \text{ m s}^{-2} \approx 1.34 \text{ m s}^{-2},$$

$$F_N = (6.00)(9.81) \cos 25.0^\circ - 35.0 \sin 20.0^\circ = 41.3746 \text{ N} \approx 41.4 \text{ N}.$$

The acceleration is uphill, and the positive reaction supports the assumed contact. The tether decreases the reaction below the untethered value 53.3453 N.

(b) **Equilibrium tension.** Setting the uphill force sum to zero gives

$$F_{T,\text{eq}} = \frac{(6.00)(9.81) \sin 25.0^\circ}{\cos 20.0^\circ} = 26.4718 \text{ N} \approx 26.5 \text{ N}.$$

The outward equation then gives $F_{N,\text{eq}} = 44.2914 \text{ N} \approx 44.3 \text{ N}$. The equilibrium reaction is larger than in part (a), because the smaller tether tension pulls less strongly away from the surface. Equilibrium here means zero acceleration; it permits either rest or constant velocity along the guide.

(c) **Contact limit.** From Eq. (5.26), $F_N \geq 0$ requires $\tan \phi \leq \cot \theta$. Hence $\phi_{\text{max}} = 90.0^\circ - 25.0^\circ = 65.0^\circ$. At that limiting angle $F_{T,\text{eq}} = mg = 58.86 \text{ N}$, directed vertically, and $F_N = 0$. The equality is a marginal contact state; angles beyond it cannot be supported by a normal reaction that only pushes outward.

Independent vector check. For part (a), horizontal and vertical force sums are 7.26309 N and 3.38683 N. They equal $ma_s \cos 25.0^\circ$ and $ma_s \sin 25.0^\circ$, respectively. This checks the rotated-component result without relying on the normal-balance equation alone. The special case $\phi = 0$ also recovers an unchanged normal reaction $mg \cos \theta$.

5.4.5 Ideal tension

A taut flexible connector pulls along its own local direction, away from the body on which it acts. In the ideal model of a negligible-mass connector passing over ideal connections, one uses a common tension magnitude F_T along the connector. That equality is an approximation with conditions; Section 5.7 shows why it fails for a connector with appreciable mass. Pulley rotational inertia is deferred to Chapter 10.

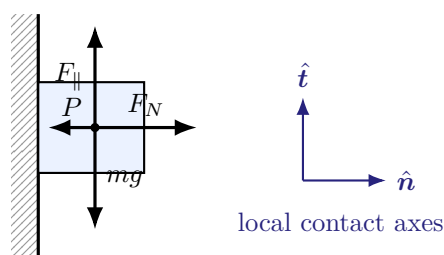
5.4.6 A linear spring model

For a one-dimensional spring operating in a regime where the force is approximately linear in displacement from a reference coordinate x_0 ,

$$F_{s,x} = -k_s(x - x_0), \quad k_s > 0. \quad (5.27)$$

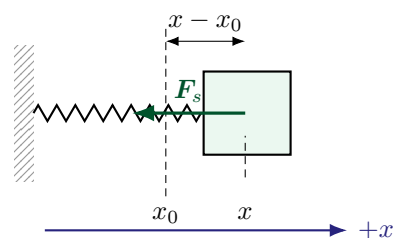
The minus sign expresses a restoring tendency: if $x > x_0$, the spring force points toward negative x ; if $x < x_0$, it points toward positive x . Equation (5.27) is a constitutive model for an ideal linear spring, not a universal law for arbitrary deformation.

(a) Package pressed against a wall



An upward tangential contact force is required. Whether the wall can supply it is tested only after a friction model is introduced.

(b) Linear spring model



$F_{s,x} = -k_s(x - x_0)$ within the stated linear-response regime.

Figure 5.10: Two common contact/elastic models. In the wall problem the normal and tangential contact components are determined from force balance; the chapter does not yet supply a friction-availability law. The spring inset distinguishes the unstretched reference coordinate from the current coordinate. Force-arrow lengths in these setup sketches are schematic.

Example 5.10: A package pressed against a vertical support

A package of mass $m = 3.20$ kg is pressed horizontally against a vertical wall by a constant force $P = 70.0$ N. It is observed to remain at rest. Determine the normal force and the tangential contact force *required* by equilibrium. Do not assume a coefficient-of-friction law.

System and frame. The system is the package, analysed in the laboratory frame. The package is at rest, so $a_x = a_y = 0$.

Horizontal balance. The push is toward the wall and the wall's normal reaction is away from the wall. Choosing $+x$ away from the wall,

$$F_N - P = 0 \quad \Rightarrow \quad F_N = 70.0 \text{ N}.$$

The value is set by horizontal balance, not by the package weight.

Vertical balance. Gravity is downward. Therefore the wall must supply an upward tangential contact force of required magnitude

$$F_{\parallel, \text{req}} = mg = (3.20)(9.81) \text{ N} = 31.392 \text{ N} \approx 31.4 \text{ N}.$$

Interpretation. The calculation tells us what tangential force is required *if the observed equilibrium is maintained*. It does not tell us whether a particular dry wall can supply that force from its material properties. That availability test belongs to the friction model of Chapter 6.

Third-law check. The 70.0 N normal force and the 31.4 N weight are not a third-law pair: they are perpendicular and both act on the package. Their respective partners act on the wall and on Earth.

Example 5.11: A normal force changed by an angled push

A 5.00 kg equipment case rests on a horizontal frictionless platform. A technician applies a 80.0 N force at 30.0° below the horizontal. The case remains in vertical contact with the platform. Find the normal force.

Vertical acceleration is zero. The downward component of the applied force is $P \sin 30^\circ = 40.0 \text{ N}$. Thus

$$F_N - mg - P \sin 30^\circ = 0,$$

so

$$F_N = (5.00)(9.81) + 40.0 = 89.05 \text{ N} \approx 89.1 \text{ N}.$$

The result exceeds $mg = 49.05 \text{ N}$ because the applied force has an additional downward component.

5.4.7 An upward angled pull and loss of contact

An upward angled pull is the complementary case to a downward angled push. Let a force of magnitude P pull a body on a smooth horizontal platform at angle α above horizontal. Choose $+x$ along the platform and $+y$ upward. While the platform contact is maintained, $a_y = 0$, so

$$P \cos \alpha = ma_x, \quad F_N + P \sin \alpha - mg = 0. \quad (5.28)$$

The normal reaction is consequently $F_N = mg - P \sin \alpha$. Changing the force direction can change the horizontal acceleration and the normal reaction in different ways: a larger upward angle reduces the horizontal component while also reducing the support load.

The support is modelled as a nonadhesive contact. It can push the body upward, but it cannot supply a downward normal force that holds the body against the surface. The inequality $F_N \geq 0$ is therefore part of the model, not an optional numerical check. If $P > mg$, the marginal contact angle is

$$\alpha_c = \sin^{-1}\left(\frac{mg}{P}\right), \quad 0 \leq \alpha_c < 90^\circ. \quad (5.29)$$

For angles beyond α_c , the body cannot share the platform's zero vertical acceleration. After separation, $F_N = 0$, and the remaining real forces determine $a_y = (P \sin \alpha - mg)/m$. A negative formal value from the contact equation indicates that this alternative physical state must be considered.

(a) Force diagram at $\alpha = 30^\circ$

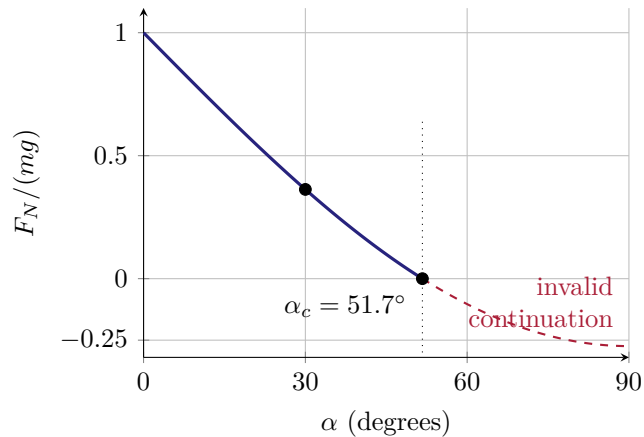
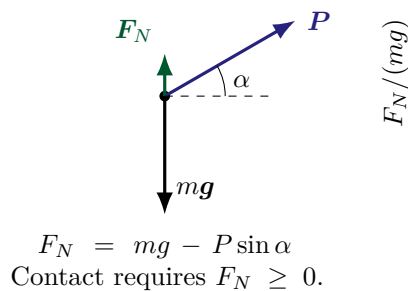


Figure 5.11: An upward angled pull unloads a horizontal support. The analytical curve uses $m = 4.00 \text{ kg}$, $P = 50.0 \text{ N}$, and $g = 9.81 \text{ m s}^{-2}$. The solid branch is the contact solution. The dashed negative continuation is an inadmissible required reaction, rather than the actual normal force after separation. Force arrows at $\alpha = 30.0^\circ$ use 0.04 cm per newton.

Example 5.12: Increase a tether angle until a case leaves its support

A 4.00 kg case is pulled by a 50.0 N force on a frictionless horizontal platform. The force direction is prescribed. (a) Determine the normal force and horizontal acceleration at $\alpha = 30.0^\circ$ above horizontal. (b) Find the contact-limit angle. (c) Determine the accelerations immediately after separation if the same force is maintained at $\alpha = 60.0^\circ$.

Model and strategy. The system is the case, treated as a translating constant-mass particle. The laboratory is approximately inertial; gravity is uniform. The platform contact is nonadhesive. Use the force diagram in Fig. 5.11 and test its contact assumption before applying its normal equation to a new angle.

(a) Contact at 30.0° . With $mg = 39.24$ N,

$$F_N = 39.24 - 50.0 \sin 30.0^\circ = 14.24 \text{ N} \approx 14.2 \text{ N},$$

which is positive. The horizontal acceleration is

$$a_x = \frac{50.0 \cos 30.0^\circ}{4.00} = 10.8253 \text{ m s}^{-2} \approx 10.8 \text{ m s}^{-2}, \quad a_y = 0.$$

The reaction is smaller than the gravitational force because the pull provides part of the upward force needed for zero vertical acceleration.

(b) Marginal contact. Set the required reaction to zero:

$$\sin \alpha_c = \frac{39.24}{50.0} = 0.7848, \quad \alpha_c = 51.7022^\circ \approx 51.7^\circ.$$

At this angle the upward pull component equals the weight. The ideal model is at the boundary of its contact domain; any further increase makes the zero-vertical-acceleration assumption fail.

(c) Separated state at 60.0° . The old contact expression would demand $F_N = 39.24 - 50.0 \sin 60.0^\circ = -4.06127$ N, which the platform cannot supply. Set the actual contact force to zero and use the remaining force components:

$$a_x = \frac{50.0 \cos 60.0^\circ}{4.00} = 6.25 \text{ m s}^{-2}, \quad a_y = \frac{50.0 \sin 60.0^\circ - 39.24}{4.00} = 1.02 \text{ m s}^{-2}.$$

The upward vertical acceleration is consistent with leaving the surface. These results apply immediately after separation provided the stated actuator force is maintained; a tether whose direction changes with body position would require an updated geometric model.

Checks. At $\alpha = 0$, the model returns $F_N = mg$. At $\alpha = \alpha_c$, the separated-state vertical acceleration is zero, matching the contact endpoint. The graph's dashed continuation communicates failure of an assumption, not a new pulling action by the support.

Guided checks

Check 1. Can an ideal flexible string push a body? **No.** It carries tension and pulls along its direction; a negative calculated tension signals failure of the assumed taut-string state.

Check 2. Is $F_N = mg$ a definition? **No.** It is a special result of particular geometry and acceleration conditions.

Check 3. If a spring is stretched twice as far within the linear regime, what happens to its force magnitude? **It doubles.**

5.4.8 A spring's unstretched coordinate is not always equilibrium

The zero of a spring force and the zero of a total force sum are different concepts. Write x_0 for the coordinate at which an ideal horizontal spring is unstretched, and define its signed extension

$e = x - x_0$. Within its stated linear range, $F_{s,x} = -k_s e$. This sign convention also permits compression: $e < 0$ gives a positive spring force, provided the spring is attached and designed to transmit the stated compressive force.

Now add a constant force P along $+x$ on the attached body. On a smooth horizontal guide, vertical support and gravity cancel, while the horizontal law becomes

$$ma_x = P - k_s(x - x_0). \quad (5.30)$$

The position at which $a_x = 0$ is

$$x_{\text{eq}} = x_0 + \frac{P}{k_s}. \quad (5.31)$$

At that position the spring force is $-P$, not zero. The term “equilibrium position” thus belongs to a complete force model. It coincides with the unstretched coordinate only when the other horizontal force sum vanishes. The same distinction becomes essential for a vertical spring, where gravity produces a loaded extension; a full oscillation theory will be developed in Chapter 15.

Figure 5.12 separates these two zeros. The spring-force line crosses zero at $e = 0$, while the net-force line crosses zero at $e = P/k_s$. Both lines have slope $-k_s$, so the stiffness can be obtained from a force–extension slope even when a constant additional force shifts the balance point.

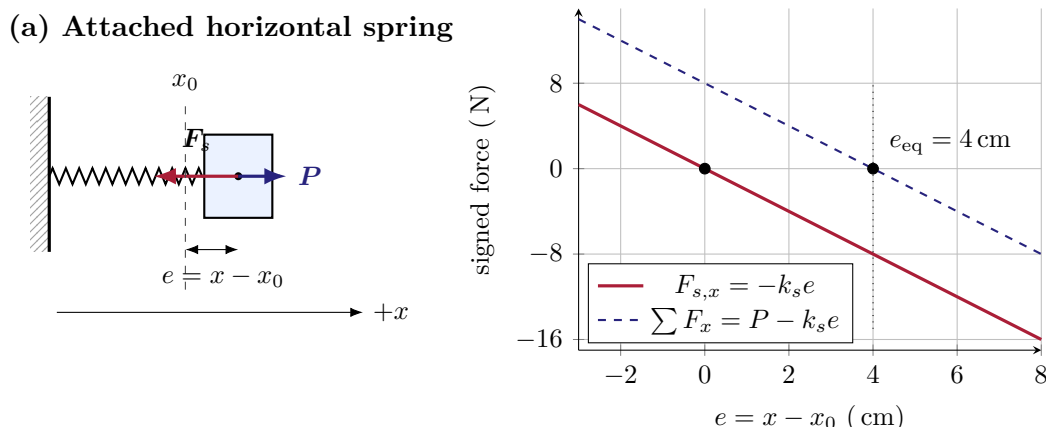


Figure 5.12: The zero of spring force differs from the zero of net force when another interaction is present. The analytical lines use $k_s = 200 \text{ N m}^{-1}$ and $P = 8.00 \text{ N}$. The horizontal axis is in centimetres, so each line’s displayed slope is -2.00 N cm^{-1} . The left schematic shows $x = 0.270 \text{ m}$, $x_0 = 0.200 \text{ m}$; its force arrows use 0.08 cm per newton and show horizontal force content only.

Example 5.13: Separate natural length, force balance and motion

A 1.50 kg instrument carriage is attached to a horizontal linear spring of stiffness $k_s = 200 \text{ N m}^{-1}$. The spring is unstretched when the carriage coordinate is $x_0 = 0.200 \text{ m}$. A constant actuator force $P = 8.00 \text{ N}$ acts along $+x$; all other horizontal resistance is neglected. The spring remains linear and can transmit both extension and compression over the range considered. (a) Find the loaded equilibrium coordinate. (b) Find the spring force, net force and instantaneous acceleration at $x = 0.270 \text{ m}$ and $x = 0.180 \text{ m}$. (c) Does a carriage passing the equilibrium coordinate necessarily come to rest there?

Model and method. The system is the carriage, and the laboratory is approximately inertial. The horizontal interactions are the actuator and spring; vertical support balances gravity. Use $e = x - x_0$, keeping its sign. Newton’s law determines the acceleration at a given coordinate without requiring an oscillation solution.

(a) **Loaded equilibrium.** Set the horizontal force sum to zero:

$$8.00 - 200(x_{\text{eq}} - 0.200) = 0, \quad x_{\text{eq}} = 0.200 + \frac{8.00}{200} = 0.240 \text{ m}.$$

The spring force there is -8.00 N ; it balances the actuator. At the unstretched coordinate x_0 , the spring force is zero but the net force is $+8.00 \text{ N}$, so that coordinate is not the loaded equilibrium.

(b) **Two signed extensions.** At $x = 0.270 \text{ m}$, $e = +0.0700 \text{ m}$, and

$$\begin{aligned} F_{s,x} &= -200(0.0700) = -14.0 \text{ N}, \\ \sum F_x &= 8.00 - 14.0 = -6.00 \text{ N}, \\ a_x &= -6.00/1.50 = -4.00 \text{ m s}^{-2}. \end{aligned}$$

At $x = 0.180 \text{ m}$, $e = -0.0200 \text{ m}$, and

$$\begin{aligned} F_{s,x} &= +4.00 \text{ N}, \\ \sum F_x &= 8.00 + 4.00 = 12.0 \text{ N}, \\ a_x &= +8.00 \text{ m s}^{-2}. \end{aligned}$$

In the first state the spring's leftward pull exceeds the actuator's rightward force. In the compressed state both forces act rightward. Figure 5.12 displays both signed force contributions without conflating them.

(c) **Position versus state of motion.** Passing x_{eq} gives zero instantaneous acceleration. It gives no condition on the instantaneous velocity. A carriage with a nonzero velocity can pass through this position and continue moving; a carriage placed there at rest remains there in the ideal model. Equilibrium as a maintained rest state requires both the force balance and the appropriate initial velocity.

Checks and limits. The displacement P/k_s has units of length. Removing the actuator gives $x_{\text{eq}} = x_0$. Doubling the stiffness at fixed actuator force halves the loaded extension. The acceleration points toward the loaded equilibrium on either side, but deriving the complete oscillation trajectory requires the later theory of oscillations.

5.5 Free-body diagrams and constraints

A multi-body problem contains two logically different kinds of equations:

1. **dynamical equations**, obtained by applying Newton's law to a chosen system; and
2. **constraint equations**, obtained from the geometry or construction that restricts the allowed motion.

Keeping them separate makes the assumptions visible. The statement “two connected carts have the same acceleration” is not Newton's second law. It is a kinematic consequence of a particular link model.

5.5.1 Drawing the selected system

A robust free-body-diagram procedure is:

1. choose one body or a clearly marked composite system;

2. replace every external interaction crossing that boundary by one force arrow;
3. place every arrow tail on the selected body (or its particle representation);
4. choose axes and resolve components only after the actual forces have been identified;
5. draw a separate free-body diagram for each additional body that requires its own equation;
6. write accelerations and geometric constraints outside the force diagram so they cannot be mistaken for forces.

If a force is split into components, the components replace that force for the chosen calculation; they are not extra physical interactions. Likewise, the net force is the vector sum of the actual force arrows and is not normally drawn as an additional independent force.

5.5.2 A taut inextensible straight link

Consider two carts at coordinates $x_1(t)$ and $x_2(t)$ connected by a straight, taut, inextensible link of fixed length L . If the connection points remain on the same straight line and the link does not wrap around a moving pulley, then

$$x_2 - x_1 = L. \quad (5.32)$$

Differentiate once:

$$v_{2x} - v_{1x} = 0, \quad (5.33)$$

and again:

$$a_{2x} = a_{1x} \equiv a. \quad (5.34)$$

Equation (5.34) is the connector constraint for this specific geometry. A different routing can produce a different relation. The equality also fails if the link becomes slack or stretches appreciably.

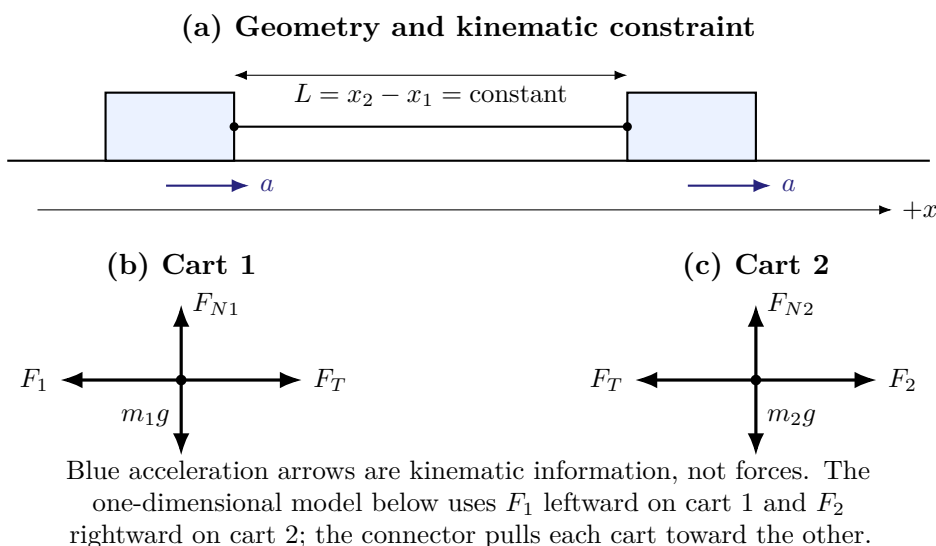


Figure 5.13: Two carts joined by a taut inextensible straight link. The fixed link length gives $a_{1x} = a_{2x}$; this constraint is written separately from each cart's Newton equation. The vertical reactions balance the weights in the level-track model.

For the sign convention shown in Fig. 5.13, suppose an external force F_1 acts leftward on cart 1 and an external force F_2 acts rightward on cart 2, with $+x$ to the right. Then

$$F_T - F_1 = m_1 a, \quad (5.35)$$

$$F_2 - F_T = m_2 a, \quad (5.36)$$

$$a_1 = a_2 = a. \quad (5.37)$$

Adding Eqs. (5.35) and (5.36) gives the combined-system check

$$F_2 - F_1 = (m_1 + m_2)a. \quad (5.38)$$

The two equal endpoint forces cancel algebraically in this massless-link limit. For a literal internal-force argument select both carts *and the link*; then each direct cart–link third-law pair lies inside the boundary.

Example 5.14: Two connected carts with unequal external pushes

Two carts move on a horizontal track with negligible resistance. Cart 1 has mass $m_1 = 6.00$ kg and is pushed leftward by $F_1 = 14.0$ N. Cart 2 has mass $m_2 = 4.00$ kg and is pushed rightward by $F_2 = 44.0$ N. They are connected by a taut massless inextensible link. Find the common acceleration and the tension magnitude.

Model. Constant masses, approximately inertial laboratory frame, straight taut inextensible link, negligible link mass, level track and negligible horizontal resistance. These conditions justify both a common acceleration and a common ideal tension magnitude.

Constraint.

$$a_1 = a_2 = a.$$

This is written before either force equation to emphasize its kinematic origin.

Newton equations. Taking $+x$ to the right,

$$\begin{aligned} F_T - 14.0 &= 6.00a, \\ 44.0 - F_T &= 4.00a. \end{aligned}$$

Add them:

$$30.0 = (10.00)a, \quad a = 3.00 \text{ m s}^{-2}.$$

Substitute into the first equation:

$$F_T = 14.0 + (6.00)(3.00) = 32.0 \text{ N}.$$

The second equation independently gives $F_T = 44.0 - (4.00)(3.00) = 32.0$ N.

Checks. The combined-system equation gives

$$44.0 - 14.0 = (6.00 + 4.00)(3.00) = 30.0 \text{ N}.$$

The tension is positive, so the assumed taut-string state is self-consistent. Substitution into both one-cart equations gives zero residual to the shown precision.

Diagnostic. A free-body diagram of the *combined* two-cart system that also contains two F_T arrows is not a valid external-force diagram. Those forces are internal to the selected composite system.

5.5.3 Why a negative tension is a model warning

A flexible string cannot push. Suppose an algebraic solution of an assumed taut-string problem gives $F_T < 0$. The correct interpretation is not “the string exerts a negative tension.” It is that the assumed state is inconsistent: the connector would have to push to satisfy those equations, so a real flexible string would go slack and the kinematic constraint would need to be reformulated.

Method

For a connected-body problem, perform three separate checks:

1. substitute the solution into every individual body's Newton equation;

2. substitute into the combined-system equation, where internal forces should cancel;
 3. test the physical domain of the constraint, such as $F_T \geq 0$ for a taut flexible link.
- A solution can be algebraically consistent yet physically incompatible with its assumed constraint.

5.5.4 A fixed pulley changes direction, not the length constraint

A connector can give equal *acceleration magnitudes* without giving equal acceleration vectors. This distinction becomes visible as soon as the connector bends around a fixed pulley. We use a flexible, taut, inextensible rope whose mass is negligible. The pulley rotates freely with negligible rotational inertia and axle friction. These explicit idealizations permit one tension magnitude T on both straight portions of the rope. They are a limiting model of the apparatus, rather than a general rule about pulleys. Rotation of a pulley with appreciable inertia requires the later rotational-dynamics chapter.

Figure 5.14 separates the apparatus from the force diagrams. Only rope and gravitational forces act on each freely hanging body. The pulley axle supports the pulley; its support force does not act directly on either hanging mass. Rope tensions on the two masses are not a third-law pair: each acts on a different mass due to the rope, and its partner acts on the rope due to that particular mass.

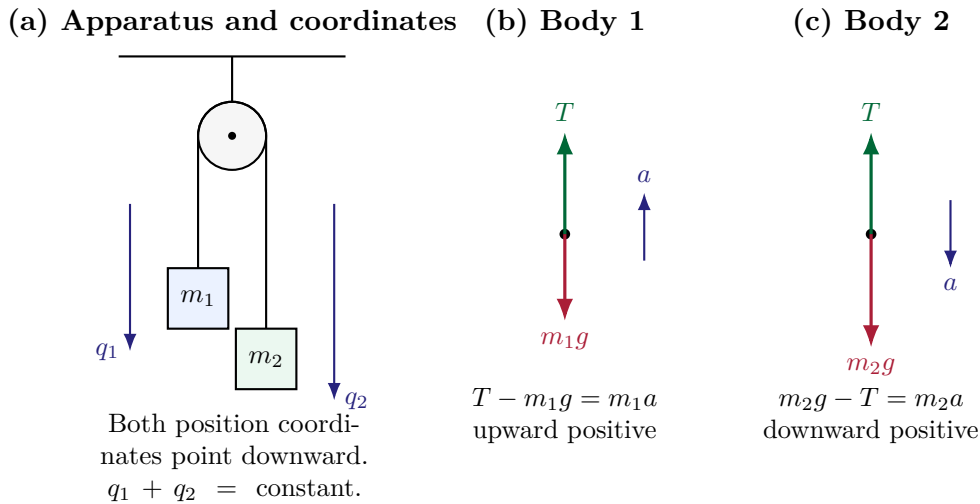


Figure 5.14: An ideal fixed-pulley system, often called an Atwood machine. Panels (b) and (c) are separate free-body diagrams; the blue acceleration arrows are outside them. For $m_2 > m_1$, the masses accelerate in opposite vertical directions with equal magnitudes. Force-arrow lengths indicate direction and qualitative ordering, not a calibrated force scale.

Let q_1 and q_2 measure the downward positions of the two connection points from fixed levels. The rope's curved portion has constant length because the pulley radius and position are fixed. The total length therefore gives

$$q_1 + q_2 + C = L, \quad \dot{q}_1 + \dot{q}_2 = 0, \quad \ddot{q}_1 + \ddot{q}_2 = 0. \quad (5.39)$$

Here C contains the constant arc and fixed offsets. For $m_2 > m_1$, define $a = \ddot{q}_2 = -\ddot{q}_1$, positive when body 2 accelerates downward. Using positive axes along these assumed motions gives

$$T - m_1g = m_1a, \quad (5.40)$$

$$m_2g - T = m_2a. \quad (5.41)$$

Adding these scalar equations and then substituting back yields

$$a = \frac{m_2 - m_1}{m_1 + m_2}g, \quad T = \frac{2m_1m_2}{m_1 + m_2}g. \quad (5.42)$$

The addition eliminated an unknown tension after choosing different positive directions on the two bodies. It is not a vector force sum for a composite body whose entire mass shares one vertical acceleration. In a common downward axis, the vertical equation for the two-mass system, with the rope outside its boundary, is $(m_1 + m_2)g - 2T = (m_2 - m_1)a$; it agrees with the individual equations. Recognizing which addition is being made prevents misuse of the shortcut $F_{\text{net}} = (m_1 + m_2)a$.

Example 5.15: Unequal hanging masses

Two bodies of masses $m_1 = 2.00 \text{ kg}$ and $m_2 = 3.50 \text{ kg}$ are joined as in Fig. 5.14. The rope is taut, massless and inextensible; the fixed pulley has negligible rotational inertia and axle friction. The bodies are released from rest without striking a stop. Find their accelerations, the tension, and their speeds after 0.600 s .

Strategy and model. Analyse each body in the inertial laboratory frame. The length constraint gives opposite vertical accelerations, while the ideal-rope/pulley model gives a common tension. Newton's law supplies the force balances; constant inputs make the acceleration constant over the stated interval.

Calculation. The heavier body accelerates downward with

$$a = \frac{3.50 - 2.00}{3.50 + 2.00}(9.81) = 2.67545 \text{ m s}^{-2} \approx 2.68 \text{ m s}^{-2}.$$

Body 1 has the same magnitude upward. Its force balance gives

$$T = 2.00(9.81 + 2.67545) = 24.9709 \text{ N} \approx 25.0 \text{ N}.$$

Both speed magnitudes after 0.600 s are $at = 1.60527 \text{ m s}^{-1} \approx 1.61 \text{ m s}^{-1}$. Their displacements have magnitude $at^2/2 = 0.481582 \text{ m}$, so the model requires at least that much unobstructed travel in the relevant directions.

Independent checks. Body 2 gives $3.50(9.81) - 24.9709 = 9.36409 \text{ N} = 3.50a$. In a single downward axis the external vertical resultant is $53.955 - 2(24.9709) = 4.01318 \text{ N}$, which equals $(3.50 - 2.00)a$. The tension lies between the two weight magnitudes, $19.62 < T < 34.335 \text{ N}$, as expected: body 1 needs an upward resultant, body 2 a downward resultant.

Interpretation. The bodies begin with zero velocity but nonzero acceleration. Their equal speed magnitudes follow from the rope geometry; their unequal masses do not alter that constraint. Equal tensions follow from the ideal connector model, rather than from a third-law relationship between the two bodies.

The mass-ratio plot in Fig. 5.15 turns these expressions into a model check. At equal masses, the acceleration vanishes although the nonzero tension supports each body. If the second mass is made much larger while m_1 stays fixed, its acceleration approaches g from below; simultaneously T approaches $2m_1g$, because body 1 approaches upward acceleration g . The rope never makes the heavier body fall faster than an unsupported body in this ideal model.

5.5.5 A cart connected to a hanging body

Changing the rope route changes the coordinate interpretation. Figure 5.16 uses a level track and a fixed pulley. Let x be the cart connection point's rightward position and y the hanging body's downward position. If the fixed pulley has horizontal reference coordinate D , the variable straight rope lengths are $D - x$ and y . Thus

$$(D - x) + y + C = L \quad \Rightarrow \quad -\dot{x} + \dot{y} = 0 \quad \Rightarrow \quad a_y = a_x = a. \quad (5.43)$$

The accelerations have equal magnitudes along perpendicular directions. Writing $\mathbf{a}_{\text{cart}} = \mathbf{a}_{\text{hanger}}$ would therefore be incorrect.

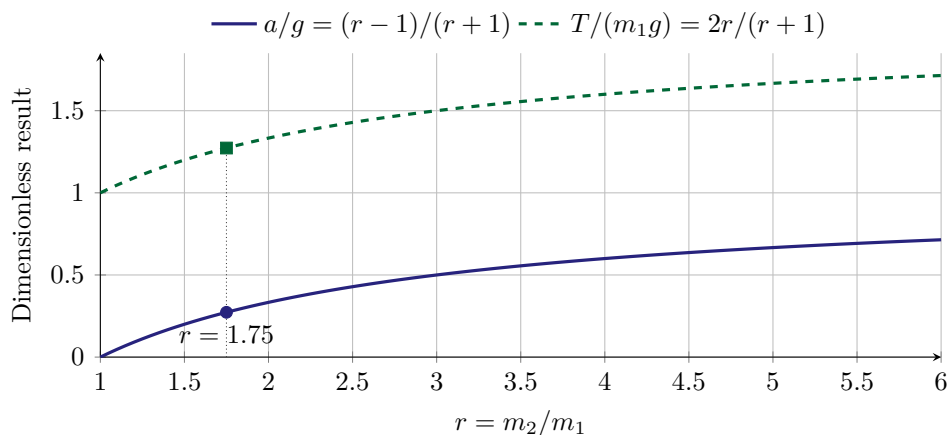


Figure 5.15: Analytical mass-ratio dependence for the ideal Atwood model. The markers use Example 5.15. The two curves are normalized differently, as indicated in the legend; their vertical separation is not a difference between dimensional forces and accelerations.

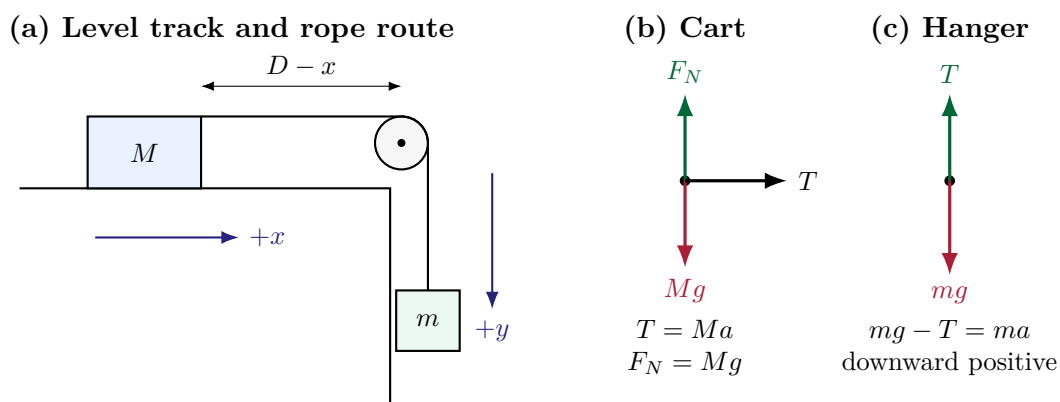


Figure 5.16: A frictionless cart and hanging body connected by an ideal rope over a fixed pulley. The route gives $a_y = a_x$ for the indicated axes. The cart's normal force and weight are included even though they cancel in the vertical equation. The common tension requires negligible rope mass, pulley inertia and axle friction.

Newton's equations along the permitted motions are $T = Ma$ and $mg - T = ma$. Solving them gives

$$a = \frac{m}{M+m}g, \quad T = \frac{Mm}{M+m}g. \quad (5.44)$$

The track's vertical support balances Mg , while the hanging body's weight drives the coupled motion. It would be incorrect to divide mg by the cart mass alone, because the hanging body must also accelerate.

Example 5.16: Predicting a cart's motion from a hanging drive

A test cart and all its attached fixtures have total mass $M = 1.80$ kg. A hanging body has mass $m = 0.600$ kg. Use the ideal arrangement in Fig. 5.16, with negligible track resistance, level motion and no initial velocity. Find the acceleration, rope tension, cart normal force, and time to move 0.500 m. Assume sufficient hanging-body clearance.

Model and strategy. Use an inertial laboratory frame, separate cart and hanger systems, and the length constraint before solving the two force equations. The rope pulls horizontally on the cart, so its tension adds no vertical component.

Force solution.

$$a = \frac{.600}{1.80 + .600}(9.81) = 2.45250 \text{ m s}^{-2}, \quad T = (1.80)(2.45250) = 4.41450 \text{ N}.$$

The cart has no vertical acceleration, hence $F_N = Mg = 17.6580$ N. Since $\Delta x = at^2/2$,

$$t = \sqrt{\frac{2(.500)}{2.45250}} = 0.63855 \text{ s}, \quad v = at = 1.56605 \text{ m s}^{-1}.$$

The hanger moves downward by the same 0.500 m, so that clearance is required.

Independent checks. The hanger equation gives $5.88600 - 4.41450 = 1.47150 \text{ N} = (.600)a$. The normal force equals the cart weight because neither the rope nor the level track generates vertical acceleration in this model. Also $0 < a < g$: the hanging body cannot have free-fall acceleration while its rope is accelerating the cart.

Interpretation. A motion measurement on the cart also predicts the hanger's acceleration, but not its direction as a vector. If the total cart mass were doubled at fixed hanging mass, the denominator in Eq. (5.44) would change, rather than the acceleration simply halving. This distinction follows because only one part of the total moving mass has been doubled.

5.5.6 A moving pulley gives a two-to-one constraint

Equal acceleration magnitudes are not a universal rope rule. In Fig. 5.17, one rope end is fixed, the rope passes around a pulley attached to a load, and the free end passes over a fixed pulley to a second body. Both pulleys have negligible mass and rotational inertia; rope mass, stretch and axle friction are negligible. The suspended load mass M includes all non-negligible translating attachments. The lower pulley turns with the rope, but its inertia is neglected. The small rod joining it to the load is part of the load assembly.

Measure the lower pulley centre's downward displacement by z , and the free body's downward displacement by y , from fixed reference levels. Each lower vertical rope leg contributes one copy of z ; the rope arcs and upper support offsets have constant length. Consequently

$$2z + y + C = L \quad \Rightarrow \quad 2\dot{z} + \dot{y} = 0 \quad \Rightarrow \quad \boxed{a_y = -2a_z.} \quad (5.45)$$

This derivation shows why neither equal velocities nor equal accelerations can be assumed here. If the load descends, the free end rises twice as far in the same time.

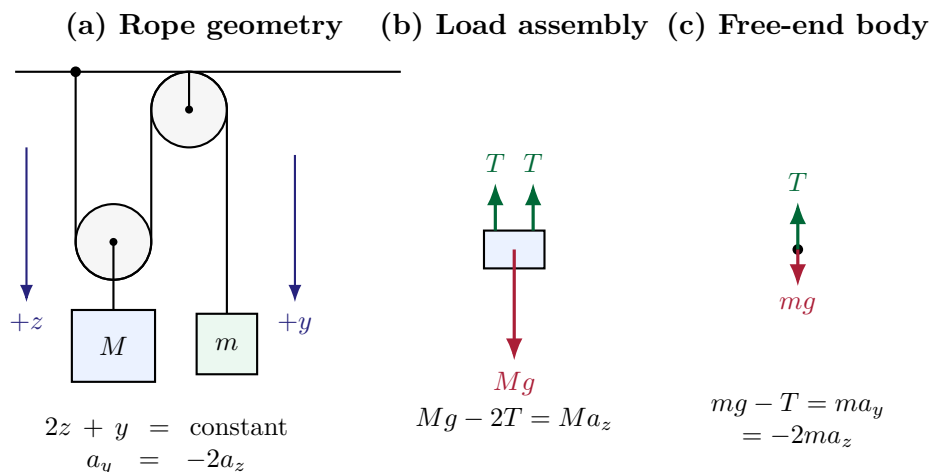


Figure 5.17: An ideal movable-pulley arrangement and two subsystem force diagrams. Each of the two vertical rope legs bordering the lower pulley lengthens by the load's downward displacement Δz . The free-end body must therefore move upward by $2\Delta z$. Both y and z are defined downward; a negative a_y denotes an upward acceleration. The force diagrams use the numerical example below with a common force scale of 0.025 cm N^{-1} .

There are two upward tensions on the load assembly because the rope contacts it along two legs. With both accelerations expressed in downward-positive coordinates,

$$Mg - 2T = Ma_z, \quad mg - T = ma_y = -2ma_z. \quad (5.46)$$

The second equation gives $T = mg + 2ma_z$. Substituting into the first gives

$$a_z = \frac{M - 2m}{M + 4m}g, \quad a_y = -2a_z, \quad T = \frac{3Mm}{M + 4m}g. \quad (5.47)$$

The denominator $M + 4m$ is the result of eliminating a geometrical constraint; it is not a total physical mass. The stationary case is $M = 2m$, where the two upward rope tensions balance the load weight. For positive masses the tension is positive, and the signs of the calculated accelerations determine which body actually accelerates downward. The equations also describe a body moving one way while accelerating the other, provided the rope remains taut.

Example 5.17: Solving a two-to-one pulley constraint

The apparatus of Fig. 5.17 has load-assembly mass $M = 6.00 \text{ kg}$ and free-end body mass $m = 2.00 \text{ kg}$. It is released from rest. Find both signed accelerations, the tension, and the displacements after 0.400 s . Verify each body's Newton equation and the rope-length change.

Model and strategy. Neglect rope and pulley inertia, use constant g , and allow enough unobstructed travel to keep the stated geometry. Analyse the complete load assembly rather than its pulley and attached load separately. Its support force is $2T$; the independently derived constraint is $a_y = -2a_z$.

Calculation.

$$a_z = \frac{6.00 - 2(2.00)}{6.00 + 4(2.00)}(9.81) = 1.40143 \text{ m s}^{-2},$$

so the load accelerates downward. The free body has $a_y = -2.80286 \text{ m s}^{-2}$, upward. The tension is

$$T = 2.00[9.81 + 2(1.40143)] = 25.2257 \text{ N}.$$

From release at rest, $\Delta z = a_z t^2 / 2 = 0.112114 \text{ m}$ and $\Delta y = a_y t^2 / 2 = -0.224229 \text{ m}$. The free body's upward travel is twice the load's downward travel.

Checks. For the load, $58.8600 - 2(25.2257) = 8.40857 \text{ N} = 6.00a_z$. For the free body, $19.6200 - 25.2257 = -5.60571 \text{ N} = 2.00a_y$. Finally $2\Delta z + \Delta y = 0$ to rounding precision, as the constant-length condition requires.

Interpretation. Replacing $a_y = -2a_z$ by equal acceleration magnitudes would solve a different apparatus. A negative signed acceleration here is physically meaningful because it describes direction; it is not a negative tension or failure of the connector model. At $M = 2m$, Eq. (5.47) gives zero acceleration and $T = mg = Mg/2$, providing an independent equilibrium limit.

5.6 Connected bodies and apparent weight

A scale does not directly measure mass and it does not necessarily read mg . A typical platform scale responds to the contact force between the supported body and the platform. That force is the normal force. When the support accelerates, the value required by Newton's law changes.

5.6.1 The lift equation

Take upward as positive. A body of mass m remains in contact with a horizontal lift floor and shares the floor's vertical acceleration a_y . The forces on the body are the upward normal force F_N and the downward gravitational force mg . Newton's second law gives

$$F_N - mg = ma_y, \quad (5.48)$$

so, while contact is maintained,

$$F_N = m(g + a_y). \quad (5.49)$$

The quantity F_N is the ideal scale reading in force units.

Several limits are immediate:

$$a_y = 0 \quad \Rightarrow \quad F_N = mg, \quad (5.50)$$

$$a_y > 0 \quad \Rightarrow \quad F_N > mg, \quad (5.51)$$

$$-g < a_y < 0 \quad \Rightarrow \quad 0 < F_N < mg, \quad (5.52)$$

$$a_y = -g \quad \Rightarrow \quad F_N = 0. \quad (5.53)$$

A unilateral contact surface can push but cannot pull the body downward through itself. Therefore the contact model requires

$$F_N \geq 0. \quad (5.54)$$

If Eq. (5.49) would predict $F_N < 0$, the assumed contact state is impossible; the body loses contact and a new dynamical model is required.

Example 5.18: Instrument scale in a starting and stopping lift

A 68.0 kg instrument stands on a force-calibrated platform scale in a lift. Use $g = 9.81 \text{ m s}^{-2}$. Determine the scale reading for: (a) upward acceleration $a_y = +1.40 \text{ m s}^{-2}$; (b) constant-speed motion; (c) downward acceleration $a_y = -1.40 \text{ m s}^{-2}$; and (d) the free-fall limit $a_y = -g$. What would it mean if someone inserted $a_y = -12.0 \text{ m s}^{-2}$ into the contact formula?

Model. The instrument remains upright; the platform is horizontal; the lift frame is not used as the dynamical frame. We analyse the instrument in the approximately inertial ground frame and use the lift acceleration as kinematic data.

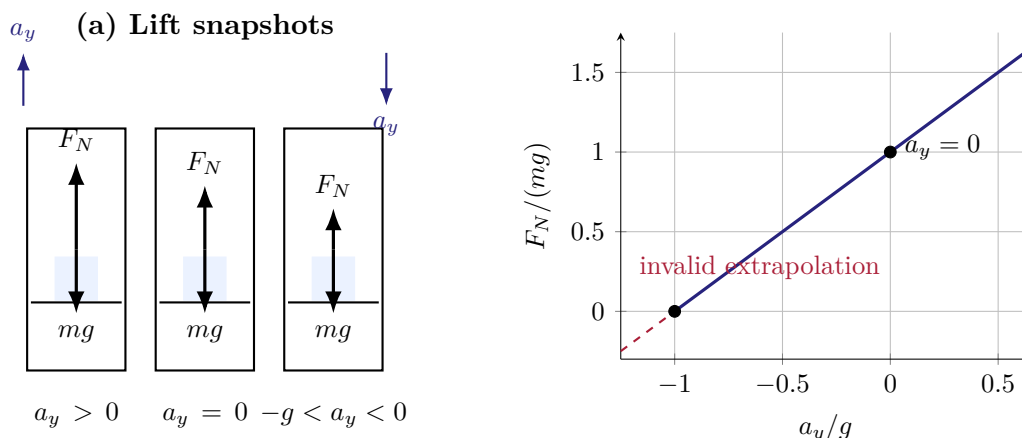


Figure 5.18: Apparent weight in a vertically accelerating lift. The exact contact-branch relation is $F_N/(mg) = 1 + a_y/g$. The branch is physically admissible only for $a_y \geq -g$ while the body remains in contact with the floor; the dashed extrapolation below $-g$ is invalid for floor contact. The actual force after separation is zero, not this negative continuation. Snapshot arrow lengths distinguish the qualitative load cases; velocity is unspecified.

Using Eq. (5.49):

- (a) $F_N = 68.0(9.81 + 1.40) = 762.28 \text{ N} \approx 762 \text{ N}$,
- (b) $F_N = 68.0(9.81) = 667.08 \text{ N} \approx 667 \text{ N}$,
- (c) $F_N = 68.0(9.81 - 1.40) = 571.88 \text{ N} \approx 572 \text{ N}$,
- (d) $F_N = 68.0(9.81 - 9.81) = 0$.

The constant-speed result is the same whether the lift is moving upward or downward because constant speed implies zero acceleration.

If $a_y = -12.0 \text{ m s}^{-2}$ is substituted blindly, Eq. (5.49) returns $F_N < 0$. The platform cannot pull the instrument downward through contact. The negative value therefore rejects the contact assumption; once contact is lost, the instrument's acceleration is determined by the forces that actually remain, principally gravity in the ideal model.

Limit check. The graph in Fig. 5.18 passes through $(a_y/g, F_N/mg) = (0, 1)$ and $(-1, 0)$, exactly as the equations require.

5.6.2 Starting, stopping and direction of travel

Apparent weight depends on acceleration, not on the direction of velocity. A lift moving upward can have $a_y < 0$ while slowing. A lift moving downward can also have $a_y < 0$ while speeding up. Both situations give $F_N < mg$. Conversely, a lift moving downward while slowing has $a_y > 0$ and gives $F_N > mg$.

This is a useful diagnostic because students often associate “moving upward” with a larger scale reading. The correct question is whether the velocity is changing upward or downward.

Guided checks

Check 1. A lift moves upward at constant speed. Is the scale reading larger than mg ? **No.** Acceleration is zero.

Check 2. A lift moves downward but slows. Is a_y positive or negative? **Positive**, so $F_N > mg$.

Check 3. What does $F_N = 0$ mean? **The support exerts no contact force**; in the ideal vertical model this occurs at the free-fall limit.

5.6.3 Reading a complete lift journey from synchronized graphs

One snapshot can hide the distinction between velocity and acceleration. A synchronized record makes it explicit: first read the slope of the velocity graph, then use that acceleration in $F_N = m(g + a_y)$. Figure 5.19 compares an upward journey with a downward journey that has the same speed history. The laboratory vertical axis is upward in both cases; changing direction of travel does not change the definition of a positive force or acceleration.

For the upward journey, an ideal control programme gives

$$a_y(t) = \begin{cases} +a_0, & 0 < t < \tau, \\ 0, & \tau < t < \tau + t_c, \\ -a_0, & \tau + t_c < t < 2\tau + t_c. \end{cases} \quad (5.55)$$

Starting from rest, integration gives a rise to $v_{max} = a_0\tau$, a constant-speed interval, and a fall back to zero velocity. The return to zero velocity is produced by a downward acceleration while the lift is still moving upward. The downward journey is obtained by reversing both v_y and a_y . The force reading then obeys $F_{N,\downarrow}(t) = m[g - a_{y,\uparrow}(t)]$, and generally differs from the upward journey at the same elapsed time.

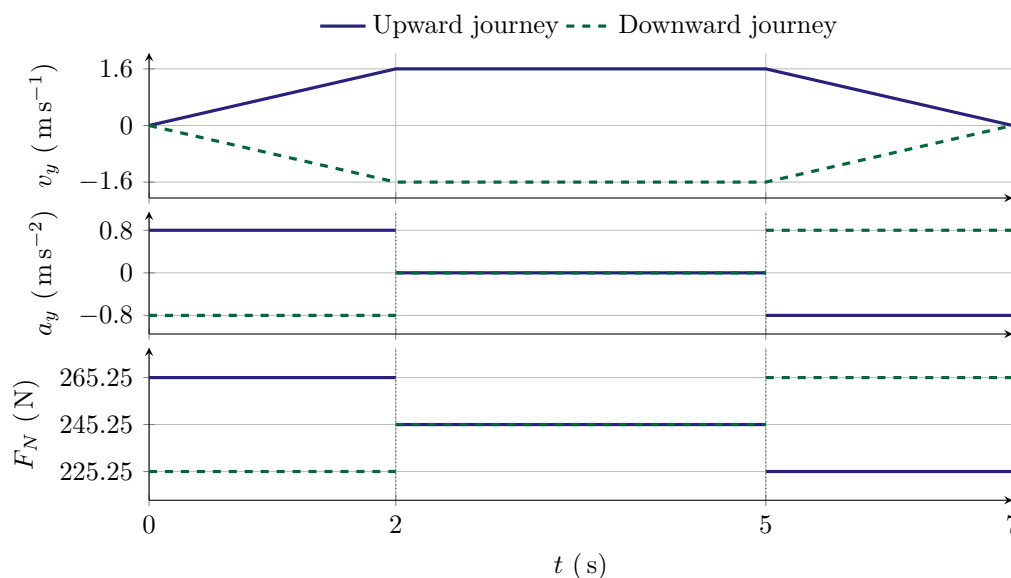


Figure 5.19: Synchronized ideal lift records for a 25.0 kg supported cargo, $a_0 = 0.800 \text{ m s}^{-2}$, $\tau = 2.00 \text{ s}$, and $t_c = 3.00 \text{ s}$. Within each open time interval, the middle curve is the derivative of the upper curve, and the lower curve is $m(g + a_y)$. Dotted guides mark changes of acceleration at 2 and 5 s; the ideal jumps have no unique instantaneous acceleration. Real controls smooth those transitions. At zero acceleration both journeys have the same $mg = 245.25 \text{ N}$ reading.

Example 5.19: A scale record for upward and downward travel

An instrument of mass 25.0 kg remains on a horizontal scale inside a lift. Starting from rest, the lift accelerates upward at 0.800 m s^{-2} for 2.00 s, coasts for 3.00 s, and accelerates downward at 0.800 m s^{-2} until it stops. Find the duration of the final interval, the total displacement, and the reading in every interval. Compare with a downward journey using the same speed programme.

System and method. The instrument is the dynamical system; the laboratory is the inertial-frame approximation. Since contact is maintained, its acceleration is the specified lift acceleration. Use the velocity slopes for kinematics and the instrument's vertical force balance for the reading; no fictitious force is needed in this frame.

Kinematics. The maximum speed is $(.800)(2.00) = 1.60 \text{ m s}^{-1}$. A downward acceleration of $-.800 \text{ m s}^{-2}$ takes $1.60/.800 = 2.00 \text{ s}$ to remove this upward velocity. The total upward displacement is the area under the upper blue curve:

$$\Delta y = \frac{1}{2}(2.00)(1.60) + (3.00)(1.60) + \frac{1}{2}(2.00)(1.60) = 8.00 \text{ m}.$$

Scale readings. Applying $F_N = 25.0(9.81 + a_y)$ gives 265.25 N while accelerating upward, 245.25 N while coasting, and 225.25 N while stopping the upward journey. For the downward journey the displacements and velocities change sign, its total displacement is -8.00 m , and the readings occur in the reverse order: 225.25, 245.25, and 265.25 N.

Checks and interpretation. Every reading is positive, so the assumed support contact is admissible. The higher reading appears when slowing a downward-moving lift, because the acceleration is upward. The final velocity is zero, and after stopping the reading returns to 245.25 N. The ideal abrupt changes in the graph are model transitions; interpreting a measured transient requires the actual smooth acceleration record and the scale's response time. This example specifies an ideal control programme and makes no claim about a performed lift experiment.

5.7 Limits of ideal force models

Ideal models are useful because they expose structure. They are dangerous only when their assumptions become invisible. The “same tension everywhere” rule provides a good example. It is exact neither for an arbitrary massive connector nor for an arbitrary pulley; it follows from a limiting model.

5.7.1 A straight translating link with finite mass

Isolate a straight link of mass m_ℓ translating along its own axis with acceleration a_ℓ . Let the right endpoint pull the link to the right with magnitude $F_{T,R}$, and let the left endpoint pull it to the left with magnitude $F_{T,L}$. Taking rightward as positive,

$$F_{T,R} - F_{T,L} = m_\ell a_\ell. \quad (5.56)$$

Thus an accelerating link with appreciable mass generally requires different endpoint tensions. If $m_\ell \rightarrow 0$ while a_ℓ remains finite, Eq. (5.56) reduces to

$$F_{T,R} = F_{T,L}, \quad (5.57)$$

which is the familiar ideal-string result.

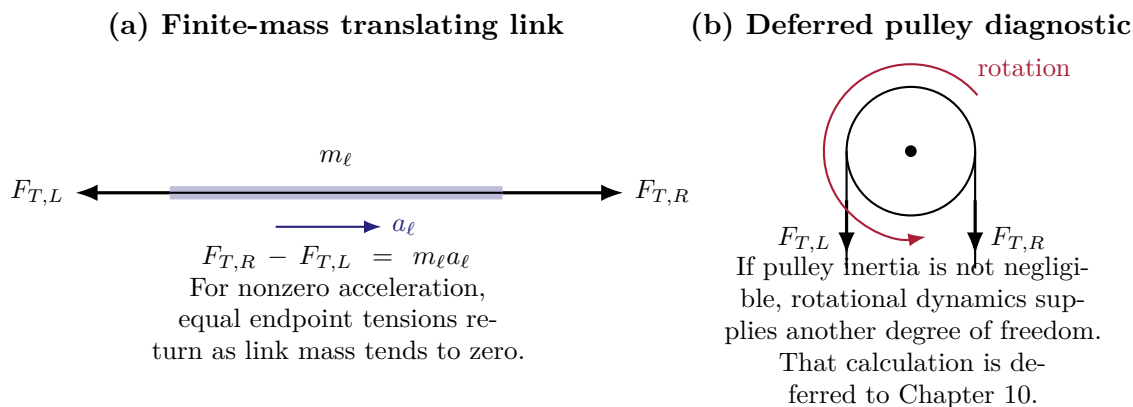


Figure 5.20: The equal-tension rule is a model limit. A finite-mass straight link that accelerates requires a net force and therefore unequal endpoint tensions. A pulley with appreciable rotational inertia introduces a rotational equation not yet available in this chapter; the silhouette is a boundary marker, not a solved pulley model.

Example 5.20: When equal tension ceases to work

A straight connector of mass $m_\ell = 1.50 \text{ kg}$ translates to the right with acceleration $a_\ell = 2.40 \text{ m s}^{-2}$. The force at its right endpoint has magnitude $F_{T,R} = 18.0 \text{ N}$ and pulls rightward. Find the force magnitude required at its left endpoint.

Equation (5.56) gives

$$18.0 - F_{T,L} = (1.50)(2.40) = 3.60 \text{ N},$$

so

$$F_{T,L} = 14.4 \text{ N}.$$

The right endpoint tension is larger because the link itself needs a rightward resultant force.

Limit check. If the same acceleration were prescribed while $m_\ell \rightarrow 0$, the difference $F_{T,R} - F_{T,L} = m_\ell a_\ell$ tends to zero and the ideal equal-tension result is recovered.

Scope check. This calculation assumes a straight translating link. It does not solve the tension difference across a finite-inertia pulley, where a rotational equation is required.

5.7.2 Model hierarchy

It is useful to state the hierarchy explicitly:

- **Particle model:** neglect size and rotation when they do not affect the question.
- **Rigidly constrained translation:** bodies share the required translational kinematics.
- **Massless taut connector:** the connector transmits tension without needing a net force for its own translation.
- **Finite-mass connector:** tension generally varies along the connector.
- **Finite-inertia pulley or rotating body:** translation alone is insufficient; rotational dynamics enters.

Moving to a more detailed model should be motivated by a failed assumption, an accuracy requirement, or a new observable. More detail is not automatically better if it introduces parameters that are neither known nor needed.

Pedagogical boundary

A uniformly accelerating coordinate frame can be useful, but its inertial-force term is not a Newton-third-law interaction partner. This chapter uses inertial-frame equations for its core examples. Translating accelerated frames are developed systematically in Chapter 6.

5.8 Engineering, computational, and measurement connections

Newton's laws become especially useful when the result is treated as an inference inside a stated engineering model rather than as an isolated textbook number. The same equation can support forward prediction, inverse force estimation, sensor cross-checks and model validation.

5.8.1 Engineering model: force inference on a test carriage

The thrust-inference model of Eq. (5.10) illustrates a common technical task. If acceleration is obtained from an independent motion measurement and the carriage mass is known, then the net force is ma_x . If a separately characterised opposing force R is included, the actuator force is inferred as

$$F_{\text{drive}} = ma_x + R.$$

The inferred actuator force and the net force are different physical quantities. Reporting only “force = ma ” without naming which force is being inferred loses that distinction.

A real test system can depart from this model through track inclination, rolling resistance, air drag, cable forces, actuator transients, sensor calibration, structural compliance and mass variation. These effects do not invalidate Newton's second law; they change the external-force model or the interpretation of the measured acceleration.

Method**Force-inference checklist.**

1. Define the system boundary: does the quoted mass include the actuator carriage, payload, cable and sensor?
2. Establish the observation frame and axis orientation.
3. Determine whether the resistance term is independently known, fitted, or merely assumed.
4. Keep the measured acceleration and the ideal force model as separate evidence streams.
5. Test the zero-acceleration intercept and at least one change of mass or direction.

5.8.2 Computational connection: solve, substitute, and report residuals

For the two-cart model of Section 5.5, write the unknowns as a and F_T . Equations (5.35)–(5.36) can be written as

$$\begin{bmatrix} m_1 & -1 \\ m_2 & 1 \end{bmatrix} \begin{bmatrix} a \\ F_T \end{bmatrix} = \begin{bmatrix} -F_1 \\ F_2 \end{bmatrix}. \quad (5.58)$$

A numerical program should not stop after producing a and F_T . It should substitute them back into the original equations and report residuals, for example

$$\mathcal{R}_1 = F_T - F_1 - m_1 a, \quad (5.59)$$

$$\mathcal{R}_2 = F_2 - F_T - m_2 a. \quad (5.60)$$

For exact arithmetic both residuals vanish. With floating-point arithmetic they should be small compared with the force scale of the problem.

The following short standard-Python implementation mirrors the mathematics and deliberately prints the residuals.

Listing 5.1: Two-cart Newton-equation solver with residual checks. The code is printed for reproducibility; no code execution is required to compile this chapter.

```
def two_cart(m1, m2, F1, F2):
    if m1 <= 0 or m2 <= 0:
        raise ValueError("masses must be positive")
    a = (F2 - F1)/(m1 + m2)
    T = F1 + m1*a
    r1 = T - F1 - m1*a
    r2 = F2 - T - m2*a
    return a, T, r1, r2

cases = [
    ("baseline", 6.0, 4.0, 14.0, 44.0),
    ("equal masses", 5.0, 5.0, 14.0, 44.0),
    ("no left load", 6.0, 4.0, 0.0, 44.0),
    ("zero acceleration", 6.0, 4.0, 14.0, 14.0),
]
for name, m1, m2, F1, F2 in cases:
    print(name, two_cart(m1, m2, F1, F2))
```

The analytically checked outputs are summarised in Table 5.1. The table uses unrounded expressions internally and displays three significant figures where rounding is needed.

Table 5.1: Model checks for the two-cart computation. Forces are in newtons, masses in kilograms, acceleration in ms^{-2} .

Case	m_1	m_2	F_1	F_2	a	F_T
Baseline	6.0	4.0	14.0	44.0	3.00	32.0
Equal masses	5.0	5.0	14.0	44.0	3.00	29.0
No left load	6.0	4.0	0.0	44.0	4.40	26.4
Zero acceleration	6.0	4.0	14.0	14.0	0.00	14.0

These cases test different aspects of the model. Equal masses preserve the combined acceleration because the total mass is unchanged, but change the internal tension. Removing the opposing left load increases the external resultant. Equal and opposite external loads produce zero acceleration while the connector can still remain under nonzero tension.

5.8.3 Measurement connection: a low-force cart experiment

A safe introductory measurement can use a low-mass cart on a horizontal track, a spring scale or force sensor, and video or position-sensor data. The purpose is not to “prove Newton’s law” from one run; it is to compare independently obtained force and acceleration estimates under controlled assumptions.

A defensible workflow is:

1. zero the force sensor with no applied load and record the zero check;
2. check track level independently, because a small slope creates a force component along the track;
3. determine the total moving mass, including attached sensor fixtures that accelerate with the cart;
4. acquire force and position/velocity data over the same time interval;

5. estimate acceleration from the kinematic data without using the force measurement;
6. compare the measured force resultant with ma_x , including uncertainty and any independently characterised rolling resistance;
7. repeat with a changed mass while keeping the force protocol as similar as practical.

If a constant rolling-resistance estimate R is used, the comparison can be phrased as

$$F_{\text{sensor}} - R \stackrel{?}{=} ma_x. \quad (5.61)$$

A discrepancy may indicate calibration error, time misalignment, a nonconstant resistance, a small track slope, or an omitted force. It is not scientifically useful to absorb every discrepancy into an unnamed “experimental error.”

Caution

Use low-force carts or an instructor-provided dataset. Occupied lifts, heavy suspended loads, high accelerations and improvised human-loading experiments are unnecessary for learning the model and introduce avoidable safety risks.

5.8.4 Independent observations and a fitted force model

An experiment becomes circular if acceleration is calculated from the same force reading used to test $F = ma$. A useful arrangement therefore has two observation channels. A zeroed force sensor records the applied drive force; independently, position or velocity measurements determine acceleration. For example, a fitted quadratic $x(t) = x_0 + v_0t + \frac{1}{2}at^2$ estimates a constant acceleration over a stated interval. Differentiating noisy position values twice without fitting can amplify scatter and obscure the comparison.

Figure 5.21 shows the physical meaning of each inferred quantity. The force sensor measures an interaction force, rather than the full resultant unless opposing forces are negligible. For a cart that always moves in the same direction and a stipulated constant opposing force R ,

$$F_i = ma_i + R + \epsilon_i. \quad (5.62)$$

Here ϵ_i describes the differences between individual readings and the chosen linear model. A constant sensor offset or a small unaccounted track inclination could also contribute to the fitted intercept. Consequently interpreting that intercept as resistance requires independent calibration and levelling evidence.

What a straight-line fit calculates

For equal force uncertainties and acceleration uncertainties small enough to neglect in the fit, ordinary least squares minimizes $\sum_i (F_i - \hat{m}a_i - \hat{R})^2$. A hat denotes an estimate rather than a new physical quantity. Let $\bar{a}$ and $\bar{F}$ denote the arithmetic means and define $S_{aa} = \sum_i (a_i - \bar{a})^2$. Differentiating the squared-residual sum with respect to its two unknown parameters gives

$$\hat{m} = \frac{\sum_i (a_i - \bar{a})(F_i - \bar{F})}{S_{aa}}, \quad \hat{R} = \bar{F} - \hat{m}\bar{a}. \quad (5.63)$$

The slope has units $\text{N}/\text{m s}^{-2} = \text{kg}$; the intercept has units of force. The fitted residuals are $r_i = F_i - \hat{m}a_i - \hat{R}$. For $n > 2$, an illustrative estimate of their common standard deviation is

$$s_F = \sqrt{\frac{\sum_i r_i^2}{n-2}}, \quad u_m = \frac{s_F}{\sqrt{S_{aa}}}, \quad u_R = s_F \sqrt{\frac{1}{n} + \frac{\bar{a}^2}{S_{aa}}}. \quad (5.64)$$

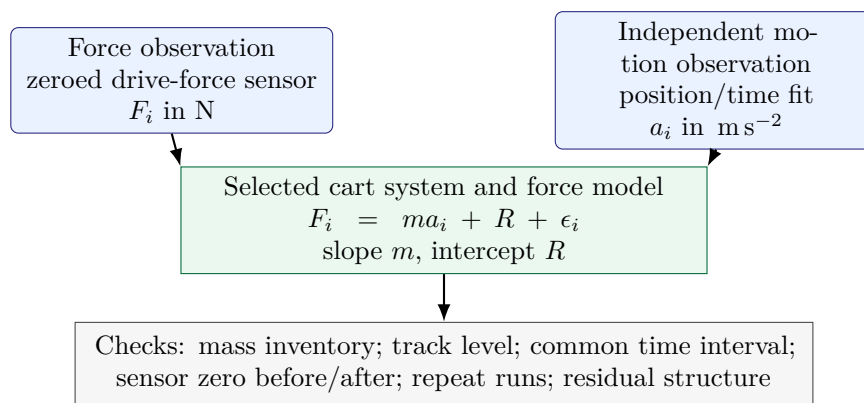


Figure 5.21: Two independent observation channels for a cart dynamics comparison. The model converts paired observations into estimated parameters. Calibration and residual checks support interpretation of the parameters; agreement with a fitted line alone does not establish the apparatus model. This is a proposed measurement architecture, not a reported experiment.

Two fitted parameters account for the $n - 2$ denominator. The uncertainty formulas are conditional on the linear model, independent equal-variance force errors, and adequately known accelerations. They describe scatter within that model; they do not include force-sensor calibration, uncertain total mass, timing errors, or a systematically tilted track. If acceleration uncertainty is appreciable, the ordinary fit is no longer the appropriate uncertainty model.

Example 5.21: Mass and resistance from synthetic paired observations

Table 5.2 supplies six *synthetic teaching pairs*, rather than measured laboratory results. The acceleration values are treated as exact for this exercise. Fit $F = ma + R$, calculate every residual, estimate the residual-based parameter uncertainties, and infer the mean drive force at $a_* = 1.10 \text{ m s}^{-2}$.

Model and method. Use a fixed-mass cart in a level inertial laboratory frame, with a constant prescribed resistance opposing the chosen direction of travel. A straight-line fit uses all six independent force/acceleration pairs. The stipulated offsets in these synthetic data are illustrative scatter, not claims about random errors in an actual instrument.

Numerical fit. The means are $\bar{a} = .875 \text{ m s}^{-2}$, $\bar{F} = 1.580 \text{ N}$, and $S_{aa} = 1.09375 \text{ m}^2 \text{ s}^{-4}$. The numerator in Eq. (5.63) is $1.75000 \text{ N m s}^{-2}$. Therefore

$$\hat{m} = 1.60000 \text{ kg}, \quad \hat{R} = 1.58000 - 1.60000(.875) = .18000 \text{ N}.$$

Subtracting this fitted line from each force gives the residuals listed in the table and plotted in Fig. 5.22. Their squared sum is $.001200 \text{ N}^2$, so

$$s_F = \sqrt{\frac{.001200}{4}} = .0173205 \text{ N}, \quad u_m = .0165616 \text{ kg}, \quad u_R = .0161245 \text{ N}.$$

At one residual-based standard uncertainty, the conditional results can be reported as $\hat{m} = (1.600 \pm .017) \text{ kg}$ and $\hat{R} = (.180 \pm .016) \text{ N}$.

Inference at a new acceleration. The fitted mean drive force at $a_* = 1.10 \text{ m s}^{-2}$ is $\hat{F}_* = 1.600(1.10) + .180 = 1.940 \text{ N}$. Its conditional standard uncertainty is

$$u_{\hat{F}_*} = s_F \sqrt{\frac{1}{n} + \frac{(a_* - \bar{a})^2}{S_{aa}}} = .007993 \text{ N}.$$

This estimates the fitted mean at an exactly known acceleration. An individual future force reading includes additional scatter and has larger uncertainty. The inferred *net* force is $\hat{m}a_* = 1.760 \text{ N}$, distinct from the drive force because the model includes $.180 \text{ N}$ resistance.

Checks and interpretation. The residuals obey $\sum r_i = 0$ and $\sum (a_i - \bar{a})r_i = 0$, the two fit equations, to the displayed precision. The graph shows small positive and negative differences, not an exact identity between each data point and the model line. Because the data are synthetic and the fit uncertainty is conditional, the recovered slope does not demonstrate a real mass measurement. With actual data, curvature or direction-dependent residuals would motivate a changed external-force model; a displaced intercept would also require a zero-offset and levelling check before assigning it to resistance.

Table 5.2: Synthetic paired data and the actual fitted-line residuals for Example 5.21. All inputs and results are contained in this standalone source.

$a_i \text{ (m s}^{-2}\text{)}$	$F_i \text{ (N)}$	$\hat{F}_i \text{ (N)}$	$r_i \text{ (N)}$
.250	.590	.580	+.010
.500	.960	.980	-.020
.750	1.390	1.380	+.010
1.000	1.790	1.780	+.010
1.250	2.160	2.180	-.020
1.500	2.590	2.580	+.010

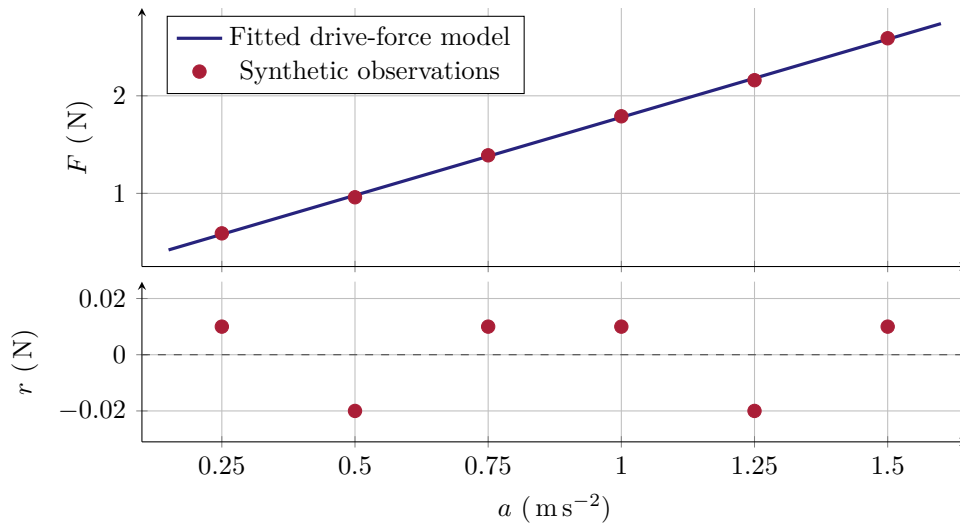


Figure 5.22: The fitted force model and its residuals on a shared acceleration coordinate. The lower panel magnifies deviations concealed by the upper panel's force scale. The residuals use the actual fitted slope 1.60000 kg and intercept .18000 N; they are not an unrelated decorative scatter sequence.

The uncertainty of the fitted mean cannot be obtained correctly by simply treating the fitted slope and intercept as independent. They are correlated because $\hat{R} = \bar{F} - \hat{m}\bar{a}$. Equation (5.64) and the prediction formula above account for that relation. Keeping this distinction visible is a useful early connection between dynamics, parameter inference and measurement practice.

5.9 Advanced Topics for Technical and Engineering Students

The core treated an ideal connector as either negligible in mass or, in Section 5.7, as a single finite-mass body with two endpoint tensions. Engineering models often need one further question: *how does tension vary along a distributed connector?* This section develops that result and uses it to quantify the error made by a massless-link approximation. The additional mathematical prerequisite is elementary differentiation applied to a small material segment.

The analysis remains one-dimensional and translational. It does not introduce pulley rotational inertia, bending stiffness, wave propagation, elastic stretch or a general non-inertial-frame formulation. Those require additional mechanics.

5.9.1 Distributed tension in a uniformly accelerating link

Consider a straight connector of length L , uniform linear mass density λ , and total mass

$$m_\ell = \lambda L. \quad (5.65)$$

Let s measure position from the left end, and suppose every material point has the same rightward acceleration a . Denote the local tension magnitude by $T(s)$.

Isolate a short segment from s to $s + \Delta s$. The material to the right pulls the segment to the right with $T(s + \Delta s)$; the material to the left pulls it to the left with $T(s)$. The segment mass is $\lambda \Delta s$. Newton's second law gives

$$T(s + \Delta s) - T(s) = \lambda \Delta s a. \quad (5.66)$$

Divide by Δs and take the limit $\Delta s \rightarrow 0$:

$$\boxed{\frac{dT}{ds} = \lambda a.} \quad (5.67)$$

For uniform λ and constant a , integration yields

$$T(s) = T_L + \lambda a s, \quad (5.68)$$

where $T_L = T(0)$. At $s = L$,

$$T_R - T_L = \lambda L a = m_\ell a, \quad (5.69)$$

recovering Eq. (5.56). The endpoint relation is therefore the integral of a local force-balance statement.

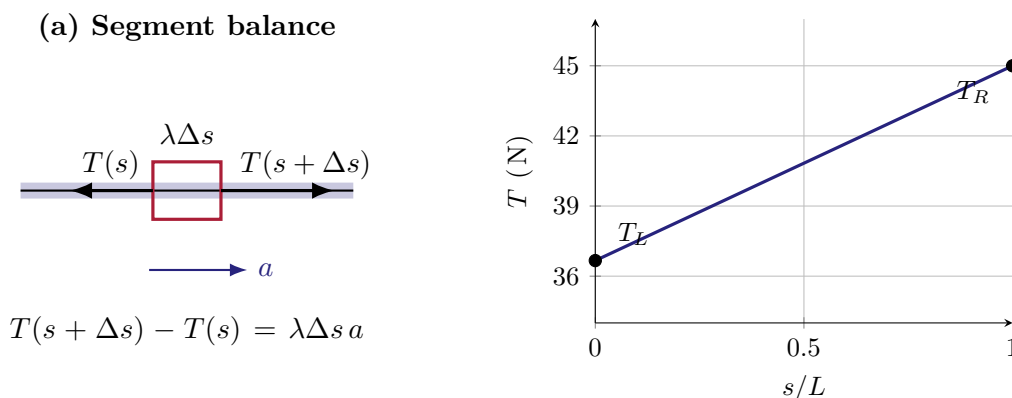


Figure 5.23: Advanced connector model. A local segment of a uniformly accelerating link requires a tension gradient $dT/ds = \lambda a$. The left segment diagram uses schematic force-arrow lengths; the right panel is quantitative. Its analytical profile uses the worked investigation below: $T_L = 36.67$ N, $T_R = 45.00$ N.

5.9.2 Worked investigation: two carts joined by a massive link

Example 5.22: Quantifying the massless-link approximation

A 4.00 kg left cart and a 6.00 kg right cart are joined by a straight uniform link of mass 2.00 kg. The track resistance is negligible. An external force of 20.0 N acts leftward on the left cart and an external force of 70.0 N acts rightward on the right cart. The link remains straight, taut and inextensible. Find the common acceleration and the tension at each end. Then compare with the massless-link approximation.

Complete system. For the two carts plus link, the external horizontal resultant is

$$F_{\text{ext}} = 70.0 - 20.0 = 50.0 \text{ N},$$

and the total mass is 12.00 kg. Therefore

$$a = \frac{50.0}{12.00} = 4.166\bar{6} \text{ m s}^{-2}.$$

Left cart. If T_L is the tension at the left endpoint,

$$T_L - 20.0 = (4.00)a,$$

so

$$T_L = 36.666\bar{6} \text{ N} \approx 36.7 \text{ N}.$$

Right cart. If T_R is the tension at the right endpoint,

$$70.0 - T_R = (6.00)a,$$

so

$$T_R = 45.000 \text{ N}.$$

Link check.

$$T_R - T_L = 8.333\bar{3} \text{ N} = (2.00)(4.166\bar{6}) \text{ N} = m_{\ell}a.$$

For a uniform link, Eq. (5.68) predicts the linear profile plotted in Fig. 5.23.

Massless-link approximation. If the 2.00 kg link mass is wrongly neglected, the model uses only 10.00 kg of moving mass and predicts

$$a_0 = \frac{50.0}{10.00} = 5.00 \text{ m s}^{-2},$$

with a single tension $T_0 = 40.0 \text{ N}$. The acceleration is overpredicted by

$$\frac{a_0 - a}{a} = \frac{5.00 - 4.166\bar{6}}{4.166\bar{6}} = 0.200,$$

or 20.0%. The ideal tension 40.0 N also hides the real endpoint difference.

Interpretation. The massless model is not “wrong” in the abstract; it is inaccurate for this parameter ratio. Whether it is acceptable depends on a stated tolerance and on the observable of interest.

5.9.3 A nonuniform connector: cumulative mass sets tension

The uniform-link model can be extended without changing its mechanics. The new prerequisite is a definite integral. Keep the same straight, inextensible translating connector and assume no unbalanced distributed force along its axis. If the local linear density is $\lambda(s)$, a short material segment has mass $\lambda(s)ds$, so the local equation becomes $dT/ds = \lambda(s)a$. Integration from the left

endpoint gives

$$T(s) = T_L + a \int_0^s \lambda(u) du, \quad T_R - T_L = a \int_0^L \lambda(u) du = m_\ell a. \quad (5.70)$$

Thus tension increases in proportion to the cumulative mass accelerated to the left of a cut. The endpoint difference depends on total mass, while the interior shape also depends on its distribution. A density that increases toward the right creates an increasing tension gradient and an upward-curving profile for $a > 0$.

For an illustrative graded connector, let

$$\lambda(s) = \lambda_0 \left(1 + \beta \frac{s}{L}\right), \quad T(s) = T_L + \lambda_0 a \left(s + \frac{\beta s^2}{2L}\right), \quad m_\ell = \lambda_0 L \left(1 + \frac{\beta}{2}\right). \quad (5.71)$$

Require $\lambda_0 > 0$ and $1 + \beta s/L > 0$ throughout the connector. These are model inputs, not a claim that a particular rope or cable has this density profile. In a real compliant cable, changes propagate at finite speed; this uniformly translating idealization applies only when those effects are negligible for the question.

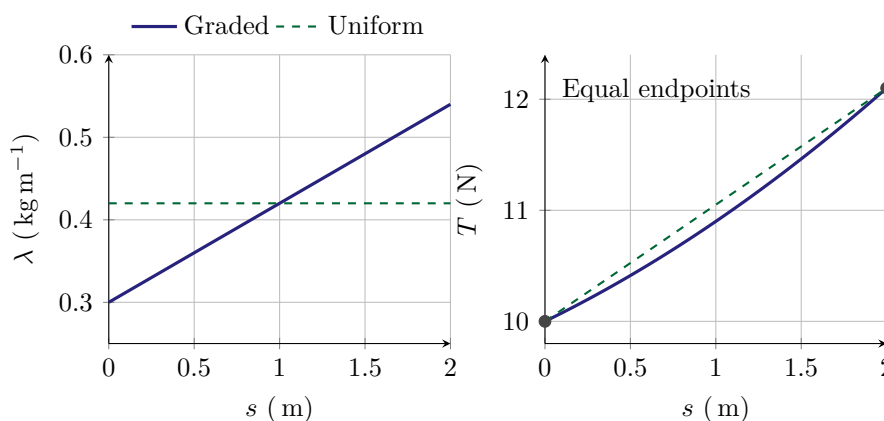


Figure 5.24: Graded (solid) and uniform (dashed) connectors with the same length, total mass, left tension and acceleration. The uniform comparison uses $\lambda = m_\ell/L = .420 \text{ kg m}^{-1}$. Equal endpoint tensions do not establish equal interior tension profiles; the cumulative mass distribution determines the difference.

Example 5.23: A graded link with a verified tension profile

A straight translating connector has $L = 2.00 \text{ m}$, $\lambda_0 = .300 \text{ kg m}^{-1}$, and $\beta = .800$. Its prescribed common acceleration is $a = 2.50 \text{ m s}^{-2}$, rightward, and its left-end tension is $T_L = 10.0 \text{ N}$. Find its mass, tension profile, right-end tension and midpoint tension. Compare the midpoint with a uniform connector having the same total mass and length.

System and strategy. Work in the inertial laboratory frame and select each short material segment in turn. Since every segment has the same acceleration, integrate $\lambda(s)a$, rather than substituting a single density into the uniform result. The stipulated left tension fixes the integration constant.

Calculation. The total mass is $.300(2.00)(1 + .800/2) = .840 \text{ kg}$. With s measured in metres,

$$T(s) = 10.0 \text{ N} + (.750 \text{ N m}^{-1})s + (.150 \text{ N m}^{-2})s^2.$$

Thus $T(2.00 \text{ m}) = 12.100 \text{ N}$ and $T(1.00 \text{ m}) = 10.900 \text{ N}$. The uniform comparison has $\lambda = .840/2.00 = .420 \text{ kg m}^{-1}$, so its profile is $T_U(s) = 10.0 \text{ N} + (1.050 \text{ N m}^{-1})s$. Its midpoint tension is 11.050 N , $.150 \text{ N}$ higher than the graded connector's.

Independent checks. The endpoint difference $12.100 - 10.000 = 2.100 \text{ N}$ equals $m_\ell a =$

.840(2.50). Differentiating the profile gives $.750 + .300s$ in N m^{-1} , equal to $\lambda(s)a$ at every point. The two profiles share both endpoints, yet $T - T_U = .150s(s - 2)$ in newtons when s is in metres, which is negative inside $0 < s < 2$. Its most negative value occurs at the midpoint. **Interpretation.** A total-mass measurement determines the net endpoint difference for this translating model, but an interior load estimate also needs the mass distribution. When $\beta \rightarrow 0$, the quadratic term vanishes and the previously derived uniform profile returns. All tensions in the stated example are positive, consistent with a taut connector.

5.9.4 A model-error criterion for link mass

The preceding comparison has a simple general form. Let the two carts have combined mass $M = m_1 + m_2$, let the link have mass m_ℓ , and let the net external horizontal force on the complete translating system be F . The finite-link model gives

$$a = \frac{F}{M + m_\ell}, \quad (5.72)$$

whereas the massless-link model gives

$$a_0 = \frac{F}{M}. \quad (5.73)$$

For a nonzero external resultant, their relative acceleration difference, measured against the finite-link result, is

$$\frac{a_0 - a}{a} = \frac{m_\ell}{M}. \quad (5.74)$$

This exact result for the stated one-dimensional model is a useful engineering estimate. If the resultant is zero, both accelerations are zero and the relative ratio is undefined; no percentage error is assigned. For a negative resultant the same ratio measures the relative overprediction of acceleration magnitude. If the link mass is 1% of the cart mass, the massless-link acceleration is high by 1%. If the link mass is 20% of the cart mass, as in the worked investigation, the error is 20%.

The tension error is more situation-dependent because the endpoint tensions also depend on where the external forces are applied. Equation (5.56), however, always sets the endpoint difference for this translating-link model.

5.9.5 Linear algebra and residuals for constrained systems

Equation (5.58) illustrates a broader idea. Once the systems, force models and constraints have been chosen, many small Newtonian models reduce to a linear system

$$\mathbf{A}\mathbf{q} = \mathbf{b}, \quad (5.75)$$

where $\mathbf{q}$ collects unknown accelerations and constraint forces. The matrix is not the physics by itself: its rows come from specific Newton equations and kinematic constraints.

After a numerical solver produces $\mathbf{q}_{\text{num}}$, compute the residual vector

$$\mathbf{r} = \mathbf{A}\mathbf{q}_{\text{num}} - \mathbf{b}. \quad (5.76)$$

A small residual verifies that the numerical solution satisfies the equations that were supplied. It does not verify that the equations represent the physical apparatus correctly. That requires independent model checks: signs, missing forces, constraint validity, parameter provenance and limiting cases.

Because accelerations and tensions have different units, a normalized matrix residual requires explicit dimensionless scaling. The next subsection constructs such a diagnostic; the dimensional force residuals remain the most direct tests of individual body equations.

5.9.6 Residuals with explicit physical scales

The acceleration and the tension in a two-cart solution have different units. It is therefore meaningless to form a Euclidean norm of a column containing their dimensional numerical values without specifying scales. A change from metres to centimetres would otherwise change the apparent quality of the same solution.

For the two-cart equations choose a positive acceleration scale a_* and force scale F_* . Define

$$\alpha = \frac{a}{a_*}, \quad \tau = \frac{F_T}{F_*}, \quad \mu_i = \frac{m_i a_*}{F_*}, \quad p_i = \frac{F_i}{F_*}.$$

The dimensionless system is

$$\tilde{\mathbf{A}}\tilde{\mathbf{q}} = \tilde{\mathbf{b}}, \quad \tilde{\mathbf{A}} = \begin{bmatrix} \mu_1 & -1 \\ \mu_2 & 1 \end{bmatrix}, \quad \tilde{\mathbf{q}} = \begin{bmatrix} \alpha \\ \tau \end{bmatrix}, \quad \tilde{\mathbf{b}} = \begin{bmatrix} -p_1 \\ p_2 \end{bmatrix}. \quad (5.77)$$

Its determinant is $\mu_1 + \mu_2 > 0$ for positive masses and scales, so these two independent equations determine a unique candidate. Physical admissibility, such as nonnegative tension, remains a separate check.

For a numerical candidate use $\tilde{\mathbf{r}} = \tilde{\mathbf{A}}\tilde{\mathbf{q}} - \tilde{\mathbf{b}}$. A dimensionless measure is

$$\epsilon_{\text{res}} = \frac{\|\tilde{\mathbf{r}}\|_{\infty}}{\|\tilde{\mathbf{A}}\|_{\infty}\|\tilde{\mathbf{q}}\|_{\infty} + \|\tilde{\mathbf{b}}\|_{\infty}}, \quad (5.78)$$

where the vector infinity norm is the largest absolute entry and the matrix norm is the largest absolute row sum. The denominator is positive for the nontrivial examples here. Dimensional force residuals should also be reported because they are immediately interpretable.

Example 5.24: A unit-independent residual diagnostic

For the baseline cart data $m_1 = 6.00$ kg, $m_2 = 4.00$ kg, $F_1 = 14.0$ N and $F_2 = 44.0$ N, evaluate the candidate $a = 3.00$ m s⁻², $F_T = 31.0$ N. Use $a_* = 1.00$ m s⁻² and $F_* = 10.0$ N.

The dimensionless matrix is $\begin{bmatrix} 0.6 & -1 \\ 0.4 & 1 \end{bmatrix}$, the candidate is $(3.00, 3.10)^T$, and the right side is $(-1.40, 4.40)^T$. Multiplication gives residual $(0.10, -0.10)^T$. The corresponding dimensional Newton-equation residuals, defined in the opposite row-sign convention earlier, are $\mathcal{R}_1 = -1.00$ N and $\mathcal{R}_2 = +1.00$ N.

The norms are $\|\tilde{\mathbf{A}}\|_{\infty} = 1.60$, $\|\tilde{\mathbf{q}}\|_{\infty} = 3.10$, $\|\tilde{\mathbf{b}}\|_{\infty} = 4.40$. Hence $\epsilon_{\text{res}} = 0.10/(1.60 \cdot 3.10 + 4.40) = 0.0106838$, about 1.07%. This exceeds ordinary floating-point roundoff by many orders of magnitude. Replacing 31.0 N with the correct 32.0 N makes both residuals zero. A small residual verifies the stated equations; it cannot discover a force that was omitted before those equations were written.

5.9.7 Connection to the next dynamics chapter

A translating accelerating frame can rewrite the same physical motion using an additional frame term. That term is not an interaction force and has no Newton-third-law partner. Deriving and applying the accelerated-frame formulation, together with friction, circular motion and drag, belongs to Chapter 6. The present chapter's core remains anchored in inertial-frame dynamics.

Take-home message

Advanced modelling should expose the size and consequence of an approximation. The finite-mass connector provides a clean example: local Newton equations generate a tension gradient, the integrated relation recovers the endpoint force difference, and the massless limit is obtained

transparently rather than assumed by habit.

5.10 Chapter summary and working formula sheet

Summary

A force represents an interaction on a selected system. Newton's laws are applied in an inertial frame, or in a frame that is being treated as inertial within a stated approximation. Zero net external force means zero acceleration, not zero velocity. For a constant-mass particle, $\sum \mathbf{F}_{\text{ext}} = m\mathbf{a}$. Third-law partners are equal and opposite forces on different bodies.

Weight, normal reaction, ideal tension and spring force are different models. The normal force is set by a contact constraint and is not universally mg . A flexible ideal string pulls; a negative calculated tension signals failure of the assumed taut state. Constraint equations come from geometry and must be written separately from Newton's laws.

A scale reads a contact force. In a vertical lift, $F_N = m(g + a_y)$ while contact exists; $F_N = 0$ at the ideal free-fall limit. Equal tension along a connector is a negligible-mass idealisation; for a finite-mass translating link, $F_{T,R} - F_{T,L} = m_\ell a_\ell$.

Formula summary

Inertial-frame dynamics for a constant-mass particle

$$\sum \mathbf{F}_{\text{ext}} = m\mathbf{a}, \quad \sum F_x = ma_x, \quad \sum F_y = ma_y, \quad \sum F_z = ma_z.$$

Newton's first-law case

$$\sum \mathbf{F}_{\text{ext}} = \mathbf{0} \quad \Rightarrow \quad \mathbf{v} = \text{constant}.$$

Newton's third law

$$\mathbf{F}_{A \leftarrow B} = -\mathbf{F}_{B \leftarrow A}.$$

Common force models

$$\mathbf{W} = m\mathbf{g}, \quad F_{s,x} = -k_s(x - x_0).$$

Taut straight inextensible link

$$x_2 - x_1 = L = \text{constant} \quad \Rightarrow \quad v_{2x} = v_{1x}, \quad a_{2x} = a_{1x}.$$

Lift contact branch, upward positive

$$F_N - mg = ma_y, \quad F_N = m(g + a_y), \quad F_N \geq 0.$$

Finite-mass translating link

$$F_{T,R} - F_{T,L} = m_\ell a_\ell.$$

Before accepting a dynamics result, ask: What is the system? Which frame is being used? Are all and only the external forces included? Are the axes and signs consistent? Is each constraint a separate geometrical statement? Does the solution remain inside the physical domain of the assumed contact or connector model?

5.11 Exercises and problems

Unless stated otherwise, use $g = 9.81 \text{ m s}^{-2}$, treat the laboratory as approximately inertial, and use constant-mass particle models. For linked carts, a “taut massless inextensible string” means that the stated kinematic constraint is active and the string can carry tension but not compression. No coefficient-of-friction law is to be introduced in these Chapter 5 problems.

A. Systems, frames and Newton's first law

1. [X05-01] A student draws a free-body diagram for a puck coasting east on a horizontal air table. The diagram shows F_N upward, mg downward, and a third arrow labelled “velocity force” eastward. Diagnose the diagram and redraw its force content in words. What can be inferred if the measured velocity is constant?
2. A trolley moves west at 2.0 m s^{-1} on a straight low-resistance track. Its velocity remains constant for 5.0 s. State the horizontal acceleration and the horizontal net force. Does the result depend on whether the velocity is eastward or westward?
3. A passenger observes a loose object inside a train car. In trial A the train moves in a straight line at constant speed; in trial B the train accelerates forward. Which train frame can be treated as inertial under the idealised Newtonian test, and why?
4. Two carts and the light connector joining them are selected as one system. Classify the following as internal or external to that system: (a) the connector force on cart 1, (b) the connector force on cart 2, (c) Earth's gravitational force on cart 1, and (d) a technician's push on cart 2.
5. A ball thrown vertically upward is instantaneously at rest at the highest point of its trajectory. Neglect air resistance. Is the ball in equilibrium at that instant? State its acceleration and explain which common misconception this tests.

B. Newton's second law and inverse force inference

6. [X05-02] A 12.0 kg test carriage moves along $+x$ and has a prescribed resistance $R = 9.0 \text{ N}$ opposite its motion. The measured acceleration is $+1.80 \text{ m s}^{-2}$. Find the required drive force. Then set the acceleration to zero and interpret the result.
7. A 20.0 kg carriage moves along the positive axis and experiences a 15.0 N resistance. The actuator applies 55.0 N forward. Determine the acceleration. If the mass is doubled while the two force magnitudes are unchanged, what is the new acceleration?
8. A 2.50 kg particle has acceleration $\mathbf{a} = (1.20\hat{x} - 0.80\hat{y}) \text{ m s}^{-2}$. Two known forces are $\mathbf{F}_1 = (4.0\hat{x} + 1.0\hat{y}) \text{ N}$ and $\mathbf{F}_2 = (-1.0\hat{x} + 3.0\hat{y}) \text{ N}$. Find the third force $\mathbf{F}_3$.
9. The net force on a particle is held fixed while its mass is changed from m to $3m$. By what factor does the acceleration vector change? What happens if instead every external force is multiplied by -2 while the mass is unchanged?
10. A linear graph of required drive force versus acceleration has slope 18.0 kg and vertical intercept 12.0 N. Identify the physical meaning of the slope and intercept in the model $F_{\text{drive}} = ma + R$. What force is required at $a = 1.50 \text{ m s}^{-2}$?
11. A 4.00 kg cart moves along $+x$ throughout the interval and has prescribed resistance $R = 2.00 \text{ N}$. Its acceleration history is $a_x(t) = (0.50 \text{ m s}^{-3})t$ for $0 \leq t \leq 4 \text{ s}$. Find the required drive force as a function of time and evaluate it at $t = 3.00 \text{ s}$.

C. Third-law pairs and common force models

12. [X05-03] A book rests on a table. Name the Newton-third-law partner of (a) the gravitational force of Earth on the book and (b) the normal force of the table on the book. Explain why the book's weight and the table's normal force are not a third-law pair even when their magnitudes are equal.
13. Two skaters push off one another on nearly frictionless ice. During the push, the force magnitude between them is 40.0 N. Their masses are 50.0 kg and 80.0 kg. Find the acceleration magnitude of each during the interaction. How can the forces be equal while the accelerations differ?
14. Two unequal carts connected by a taut light string experience equal and opposite forces from a massless connector. Are those two forces a direct third-law pair? Is their equality alone enough to conclude that their accelerations are equal? State the extra assumption that can make their accelerations equal.
15. [X05-04] A 2.50 kg package is pressed horizontally against a vertical wall by 60.0 N and remains at rest. Find the normal force and the upward tangential contact force required by equilibrium. Explain why this chapter cannot yet decide from these numbers alone whether the wall can provide that tangential force.
16. An 8.00 kg case remains in contact with a horizontal platform while a 50.0 N force is applied 37.0° below the horizontal. Find the normal force. Use the exact trigonometric value in the calculation and round only the final result.
17. A spring with $k_s = 180 \text{ N m}^{-1}$ has $x_0 = 0.120 \text{ m}$. Find $F_{s,x}$ when $x = 0.160 \text{ m}$, with $+x$ in the direction of increasing coordinate. Repeat for $x = 0.090 \text{ m}$.
18. A student writes "the normal force is always mg ." Give two physically distinct counterexamples from this chapter and write the corresponding vertical force equation for each.

D. Free-body diagrams and connected systems

19. [X05-05] Two carts on a horizontal low-resistance track are joined by a taut massless inextensible link. Cart 1 has $m_1 = 3.00 \text{ kg}$ and an external 10.0 N force to the left. Cart 2 has $m_2 = 5.00 \text{ kg}$ and an external 42.0 N force to the right. Find the common acceleration and tension. Verify both one-cart equations and the combined-system equation.
20. Two 2.00 kg carts are separated by a flexible string that is initially straight. External forces of 50.0 N and 10.0 N both act to the right on the left and right carts, respectively. Assume, temporarily, that the string remains taut and that its tension direction is toward the other cart. Solve the common-acceleration equations and interpret the sign of the calculated tension. What physical state should replace the failed assumption?
21. Starting from $x_2 - x_1 = L$ for a straight taut inextensible link, derive the velocity and acceleration constraints. State one physical change that would invalidate this simple relation.
22. Three carts of masses m_1, m_2, m_3 are connected in a line by two ideal massless taut strings and move together on a low-resistance horizontal track. If the whole three-cart system is selected, which string tensions appear in the external horizontal force sum? If only the middle cart is selected, which connector forces appear?
23. Return to the data of Problem 19, but double both cart masses while keeping the external forces fixed. Determine the new acceleration and tension. Explain why uniform scaling of both masses changes the acceleration but leaves the ideal tension unchanged in this model.

E. Apparent weight and contact loss

24. [X05-06] A 55.0 kg passenger stands on a force scale in a lift accelerating upward at 2.00 m s^{-2} . Find the scale reading. Compare it with mg .
25. A 70.0 kg instrument is moving downward in a lift but slowing at 1.50 m s^{-2} . Find the direction of the acceleration and the scale reading. Why does the direction of velocity not determine the reading?
26. A 60.0 kg package stands on a platform scale that reads 500 N. Assuming contact and using upward as positive, infer the vertical acceleration of the package.
27. A model calculation for a lift gives $a_y = -11.0 \text{ m s}^{-2}$ while a body is assumed to remain in contact with the floor. What normal force does Eq. (5.49) formally produce for $m = 10.0 \text{ kg}$? Why is that result not a physically valid contact force, and what conclusion should be drawn?
28. In the ideal free-fall limit, a platform scale reads zero. Does the gravitational force on the supported object vanish? Explain using a free-body diagram after contact has been lost.

F. Limits of ideal links

29. [X05-07] A straight connector of mass 3.00 kg accelerates rightward at 1.50 m s^{-2} . Its left endpoint tension is 20.0 N. Find the right endpoint tension and state which one is larger.
30. List the assumptions required before you use one uniform tension magnitude for a flexible connector in this chapter. Which additional degree of freedom appears if a pulley has appreciable rotational inertia?
31. A 5.00 kg left cart, a 1.00 kg straight link and a 4.00 kg right cart translate together on a horizontal low-resistance track. A 45.0 N external force pulls the right cart to the right; there is no other horizontal external force. Find the common acceleration, the left-end link tension and the right-end link tension. Verify the link equation.
32. A diagram shows a rope passing over a pulley and labels unequal tensions on the two sides. Can Chapter 5 determine those tensions if the pulley has appreciable rotational inertia but its moment of inertia is not supplied? Explain exactly what piece of dynamics is missing.

G. Computation and measurement

33. For the baseline two-cart case in Table 5.1, a program reports $a = 3.00 \text{ m s}^{-2}$ and $F_T = 32.0 \text{ N}$. Compute $\mathcal{R}_1$ and $\mathcal{R}_2$ explicitly. What would residuals of order 10^{-12} N mean in a floating-point implementation with force scales of tens of newtons?
34. Two exact points on a drive-force line are $(a, F_{\text{drive}}) = (0, 8.0 \text{ N})$ and $(2.00 \text{ m s}^{-2}, 38.0 \text{ N})$. Assuming $F_{\text{drive}} = ma + R$, infer m and R .
35. A synthetic cart dataset is generated by a model with $m = 2.00 \text{ kg}$. Three force/acceleration pairs are $(2.00 \text{ N}, 0.90 \text{ m s}^{-2})$, $(4.00 \text{ N}, 1.90 \text{ m s}^{-2})$, and $(6.00 \text{ N}, 2.90 \text{ m s}^{-2})$. For each pair compute $R_i = F_i - ma_i$. Are the values consistent with a constant resistance? These are synthetic values, not measurements.
36. A force sensor has an unnoticed constant zero offset of $+0.40 \text{ N}$. In a force-versus-acceleration fit $F_{\text{sensor}} = ma + R$, which fitted quantity is biased most directly: the slope or the intercept? What simple calibration check detects the problem?

37. Design a low-force cart experiment that tests $\sum F_x = ma_x$ without using the same sensor to infer both force and acceleration. Give at least four controls or recorded quantities and one limiting-case check.

H. Advanced challenges

38. A uniform straight link has $\lambda = 0.400 \text{ kg m}^{-1}$, length $L = 2.50 \text{ m}$, and rightward acceleration $a = 3.00 \text{ m s}^{-2}$. If $T(0) = 12.0 \text{ N}$, derive and evaluate $T(s)$, and find $T(L)$.
39. For two carts with combined mass M joined by a link of mass m_ℓ , prove Eq. (5.74). What maximum ratio m_ℓ/M keeps the massless-link acceleration error below 2.0% when the error is measured relative to the finite-link result?
40. In the baseline two-cart equations $m_1 = 6.00$, $m_2 = 4.00$, $F_1 = 14.0$, $F_2 = 44.0$ in SI units, a numerical routine proposes $a = 3.00 \text{ m s}^{-2}$ and $F_T = 31.0 \text{ N}$. Compute the two equation residuals. Is the candidate a solution of the supplied equations?
41. The complete translating system in the advanced worked investigation has cart mass $M = 10.00 \text{ kg}$. What link mass would make the massless-link approximation overpredict the acceleration by 5.0%, measured relative to the finite-link result? If the actual link mass is 0.20 kg , what is that relative acceleration error?
42. Explain why a frame term introduced in a uniformly accelerating coordinate frame is not the Newton-third-law partner of any interaction force. What must remain invariant when the same physical motion is described from different frames?

I. Graph, geometry and model investigations

43. The exact motion graph in Figure 5.5 represents $x = 3t$ and $y = t^2$ when distances are in metres and time is in seconds. A 2.00 kg puck follows that motion on a horizontal air table. (a) Recover the in-plane velocity, acceleration and net force. (b) Find the speed and its direction at $t = 2.00 \text{ s}$. (c) Explain why the initial eastward velocity does not require an eastward resultant force.
44. A 4.00 kg case is pulled on a level frictionless platform by $P = 60.0 \text{ N}$ at angle θ above the horizontal. (a) Derive the horizontal acceleration and normal force while contact exists. (b) Determine the largest admissible angle between 0° and 90° . (c) Evaluate both quantities at 30.0° , then explain what changes at 60.0° . Sketch or describe the force geometry before calculating.
45. Use the mass-ratio graph in Figure 5.15 for an ideal Atwood machine. Its measured acceleration magnitude is $0.200g$ and the total hanging mass is 10.0 kg . (a) Infer the larger-to-smaller mass ratio and both masses. (b) Determine the common tension without rounding intermediate results. (c) Explain why equating the two laboratory-frame acceleration *vectors* would be an incorrect length constraint.
46. For the movable-pulley geometry in Figure 5.17, both load coordinate z and free-end coordinate y increase downward. (a) If the lower pulley moves downward by 0.120 m , find the free-end displacement. (b) Derive the load acceleration for load mass M and free-end mass m . (c) Find the mass ratio for zero acceleration and the acceleration directions when $M = 6.00 \text{ kg}$, $m = 2.00 \text{ kg}$. Neglect rope and pulley mass.
47. A 25.0 kg cargo travels with an upward lift: $a_y = +0.800 \text{ m s}^{-2}$ for 2.00 s , zero for the next 3.00 s , and -0.800 m s^{-2} for the last 2.00 s . It starts from rest. (a) Construct the velocity and scale-force histories. (b) Determine the total upward displacement. (c) A second trip reverses every velocity and acceleration sign. Which trip stages have scale readings above mg ? Does the scale alone identify travel direction?

48. The following stipulated synthetic pairs describe acceleration a in m s^{-2} and sensor force F in newtons:

$$(0.25, 0.59), (0.50, 0.96), (0.75, 1.39), (1.00, 1.79), (1.25, 2.16), (1.50, 2.59).$$

- (a) Fit $F = ma + b$ by equal-weight least squares, using the method developed in the measurement section. (b) Compute all residuals and explain what the intercept represents if a constant opposing force and sensor zero offset are both possible. (c) The unloaded sensor subsequently reads $+0.050 \text{ N}$. Correct the inferred resistance and explain whether the fitted mass changes.
49. A horizontal straight inextensible link has $L = 2.00 \text{ m}$ and nonuniform density $\lambda(s) = \lambda_0(1 + \beta s/L)$, with $\lambda_0 = 0.300 \text{ kg m}^{-1}$, $\beta = 0.800$. It translates rightward at $a = 2.50 \text{ m s}^{-2}$ and has $T(0) = 10.0 \text{ N}$. (a) Derive its total mass and tension profile. (b) Find the midpoint tension and right-end tension. (c) Explain why a straight line between the endpoint tensions predicts the wrong local slope even though it matches the endpoint force difference.
50. A diagram for a frictionless incline includes mg downward, its two components $mg \sin \theta$ along the slope and $mg \cos \theta$ into the slope, and F_N outward. A student sums all four arrows as four independent forces. (a) Diagnose the error and construct a corrected actual-force diagram. (b) Derive the downhill acceleration and normal force for $m = 3.00 \text{ kg}$, $\theta = 25.0^\circ$. (c) How would the normal equation change if an additional upward-normal tether component $T \sin \phi$ were applied?

5.12 Complete solutions to the exercises and problems

The following solutions cover every numbered problem and every requested subpart. Unless the problem specifies otherwise, the laboratory is treated as approximately inertial and $g = 9.81 \text{ m s}^{-2}$. Force magnitudes, including normal forces and tensions, must be nonnegative; signed components may be negative. The diagrams distinguish force arrows from kinematic information. A numerical answer is accepted only after the model, equations, and physical domain have been checked.

Problem 1: repairing the puck's free-body diagram

System and interactions. Select the puck. Earth exerts the downward force mg ; the horizontal air table exerts an upward normal force F_N . Tangential resistance is negligible in the stipulated ideal model. The eastward “velocity force” must be deleted: velocity describes motion and does not identify an interacting source body.

The corrected force diagram therefore contains two vertical arrows. They balance if the puck stays at fixed height. If the measured *velocity vector* is constant, then $\mathbf{a} = d\mathbf{v}/dt = \mathbf{0}$, so

$$\sum \mathbf{F}_{\text{ext}} = m\mathbf{a} = \mathbf{0}.$$

In particular, $\sum F_x = 0$. This inference determines the resultant; a real drive and resistance could cancel and would both belong on a nonideal force diagram. Constant motion alone does not prove that each individual force vanishes.

Problem 2: constant velocity toward the west

Take east as $+x$. The trolley has $v_x = -2.0 \text{ m s}^{-1}$ throughout the five-second interval, so

$$a_x = \frac{\Delta v_x}{\Delta t} = \frac{-2.0 - (-2.0)}{5.0} = 0, \quad \sum F_x = ma_x = 0.$$

The same conclusion holds for constant eastward velocity: the derivative of either constant is zero. Direction and speed affect the displacement, but the force resultant is linked to the change in velocity. Individual forward and backward forces could be nonzero if they cancel; “low resistance” is an approximation about their sizes.

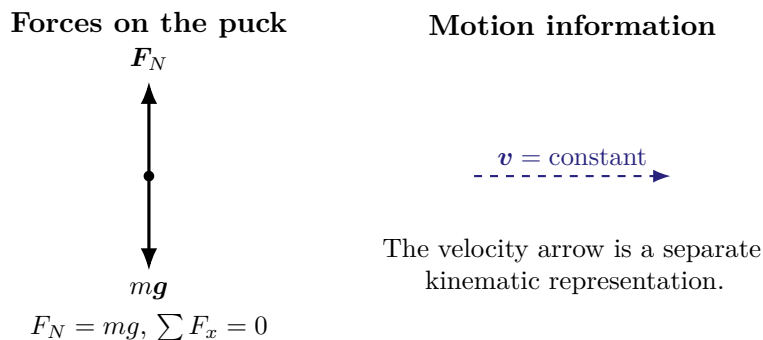


Figure 5.25: Solution to Problem 1. The actual force arrows act on the selected puck. A separate dashed velocity arrow conveys the direction of motion without adding an interaction. The force magnitudes balance in the ideal horizontal-table model; arrow lengths are schematic.

Problem 3: the two train frames

Assume the ground frame is approximately inertial and both motions are straight. In trial A the train velocity relative to the ground is constant. If $x' = x - Vt - x_0$ with constant V , differentiating twice gives $a' = a$. A particle with no resultant force therefore has zero acceleration in either frame. Trial A can be treated as inertial under this model.

In trial B the train has forward acceleration A . A loose ideal particle can have $a = 0$ in the ground frame while its train-frame acceleration is $a' = a - A = -A$. It appears to accelerate backward without a corresponding backward interaction. The accelerating train frame is non-inertial; using it for dynamics requires the appropriate frame term. This does not imply that an ordinary object touching the carriage is force-free: contact interactions must still be identified separately.

Problem 4: moving the system boundary

The selected system contains both carts *and* their connector. (a) The connector force on cart 1 is internal because source and target lie inside this boundary. (b) The connector force on cart 2 is internal for the same reason. Each cart–connector force has its own third-law partner on the connector; the two cart forces are not automatically a direct cart–cart third-law pair.

(c) Earth's gravitational force on cart 1 is external because Earth is outside the system. (d) The technician's push on cart 2 is external because the technician is outside it. If cart 1 alone were selected, its connector force would instead be external. "Internal" describes a relation to a chosen boundary rather than a permanent category of force.

Problem 5: zero velocity at a turning point

Select the ball and take upward as positive. With air resistance neglected, its only force after release is gravity. Thus

$$-mg = ma_y \quad \Rightarrow \quad a_y = -g = -9.81 \text{ m s}^{-2}.$$

This remains true at the highest point, even though $v_y = 0$ at that instant. The ball is not in equilibrium: its nonzero force resultant causes the velocity to change from upward to downward. Equilibrium is diagnosed by zero acceleration and zero resultant, not by an instantaneous zero velocity. The sign agrees with the downward force and does not depend on whether the ball is rising or falling.

Problem 6: required carriage drive

Choose $+x$ along the motion; the prescribed resistance opposes it. For the carriage,

$$F_{\text{drive}} - R = ma_x, \quad F_{\text{drive}} = ma_x + R.$$

Substituting the stated values gives

$$F_{\text{drive}} = (12.0 \text{ kg})(1.80 \text{ m s}^{-2}) + 9.0 \text{ N} = 30.6 \text{ N}.$$

The net force is 21.6 N, so reporting ma_x as the drive would omit the resistance. At zero acceleration the required drive is 9.0 N: it balances the stipulated resistance and allows constant velocity, including rest only when the resistance model remains applicable. Units reduce to newtons, and the zero-acceleration intercept provides a limiting check.

Problem 7: acceleration and doubled mass

For the carriage, the forward resultant is $55.0 - 15.0 = 40.0 \text{ N}$. Hence

$$a_x = \frac{40.0 \text{ N}}{20.0 \text{ kg}} = 2.00 \text{ m s}^{-2}.$$

Doubling the moving mass to 40.0 kg while retaining both force magnitudes gives

$$a'_x = \frac{40.0 \text{ N}}{40.0 \text{ kg}} = 1.00 \text{ m s}^{-2} = \frac{a_x}{2}.$$

This comparison holds because the problem explicitly fixes the forces. A resistance law that changed with the mass would require a new force model. Both accelerations point forward, as the positive resultant requires.

Problem 8: reconstructing the missing force vector

Newton's law is a vector equation for the particle:

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = m\mathbf{a}.$$

Compute the required resultant first:

$$m\mathbf{a} = (3.00\hat{x} - 2.00\hat{y}) \text{ N}.$$

The two known forces sum to $(3.0\hat{x} + 4.0\hat{y}) \text{ N}$. Subtracting component by component therefore yields

$$\boxed{\mathbf{F}_3 = (0\hat{x} - 6.0\hat{y}) \text{ N}.$$

As a check, the x sum is $4.0 - 1.0 + 0 = 3.0 \text{ N}$, while the y sum is $1.0 + 3.0 - 6.0 = -2.0 \text{ N}$. Dividing by 2.50 kg recovers both specified acceleration components. The known resultant and the missing vector need not point along the final resultant.

Problem 9: mass and force scaling

Let $\mathbf{F}_{\text{net}}$ be the original fixed resultant and $\mathbf{a} = \mathbf{F}_{\text{net}}/m$. Replacing m by $3m$ gives

$$\mathbf{a}' = \frac{\mathbf{F}_{\text{net}}}{3m} = \frac{\mathbf{a}}{3}.$$

Its magnitude falls by a factor of three and its direction is unchanged. If instead every force vector is multiplied by -2 , their sum is also multiplied by -2 . With the original mass,

$$\mathbf{a}'' = \frac{-2\mathbf{F}_{\text{net}}}{m} = -2\mathbf{a}.$$

For a nonzero resultant the acceleration reverses direction and doubles in magnitude. If the original resultant is zero, both new accelerations remain zero; a direction for the zero vector is not defined. These are mathematical comparisons of prescribed forces, not assertions that every contact state would survive such a change.

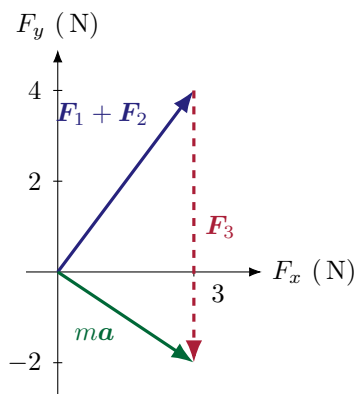


Figure 5.26: Solution to Problem 8. A quantitative head-to-tail force-vector construction, with components expressed in newtons. The downward vector $\mathbf{F}_3$ changes the known resultant $(3, 4)$ N into $(3, -2)$ N. Both axes use the same physical scale, so vector slopes also represent their component ratios. This is vector addition, not a body-only force diagram.

Problem 10: reading a force–acceleration line

In $F_{\text{drive}} = ma + R$, plotting force vertically and acceleration horizontally gives slope $\Delta F / \Delta a = m$ and intercept $F(a = 0) = R$. The slope's units are $\text{N} / \text{m s}^{-2} = \text{kg}$, so the given line describes $m = 18.0 \text{ kg}$ and prescribed resistance $R = 12.0 \text{ N}$.

At $a = 1.50 \text{ m s}^{-2}$,

$$F_{\text{drive}} = 18.0(1.50) + 12.0 = 39.0 \text{ N}.$$

Subtracting the intercept gives 27.0 N ; dividing that resultant by 18.0 kg recovers the specified acceleration. The intercept is an opposing-force parameter within this model, and should not automatically be interpreted as a friction coefficient or a sensor offset.

Problem 11: a time-dependent required drive

Write the acceleration dimensionally as $a_x(t) = \beta t$, where $\beta = 0.50 \text{ m s}^{-3}$ and t has units of seconds. With resistance always opposing the stated motion,

$$F_{\text{drive}}(t) = m\beta t + R = (2.00 \text{ N s}^{-1})t + 2.00 \text{ N}, \quad 0 \leq t \leq 4 \text{ s}.$$

Thus $F_{\text{drive}}(3.00 \text{ s}) = 8.00 \text{ N}$. The acceleration there is 1.50 m s^{-2} , so the net force is 6.00 N , with another 2.00 N required for resistance. At $t = 0$, the drive equals the resistance and the instantaneous acceleration vanishes. Writing $(2.00t + 2.00) \text{ N}$ is shorthand valid only when the numerical value of t is entered in seconds; the dimensional expression above avoids that ambiguity.

Problem 12: the book, Earth, and table interactions

(a) The partner of Earth's gravitational force on the book is the book's gravitational force on Earth:

$$\mathbf{F}_{\text{book} \leftarrow \text{Earth}} = -\mathbf{F}_{\text{Earth} \leftarrow \text{book}}.$$

(b) The partner of the table's normal force on the book is the book's contact force on the table:

$$\mathbf{F}_{\text{book} \leftarrow \text{table}} = -\mathbf{F}_{\text{table} \leftarrow \text{book}}.$$

Each pair belongs to one interaction and acts on two different bodies. The book's weight and normal force both act on the book and arise from different source bodies. Their equality, when the book has zero vertical acceleration and no other vertical forces, follows from $F_N - mg = 0$. It is an equilibrium result, not a third-law relation. The partner gravitational force on Earth exists even though Earth's resulting acceleration is extremely small.

Problem 13: equal forces on unequal skaters

Analyse each skater separately along the horizontal push direction. Assume the mutual push dominates other horizontal interactions. Newton's second law gives acceleration magnitudes

$$a_{50} = \frac{40.0}{50.0} = 0.800 \text{ m s}^{-2}, \quad a_{80} = \frac{40.0}{80.0} = 0.500 \text{ m s}^{-2}.$$

The accelerations point away from one another because the third-law force vectors are opposite. Their ratio is $a_{50}/a_{80} = 80.0/50.0 = 1.60$, the inverse mass ratio. Equal interaction forces therefore coexist with unequal acceleration magnitudes. A check on the two-skater system is $50.0\mathbf{a}_{50} + 80.0\mathbf{a}_{80} = \mathbf{0}$ for opposite signed accelerations: the mutual forces cancel internally.

Problem 14: force equality and a separate constraint

Equal and opposite ideal-string forces on two carts are not sufficient by themselves to prove equal acceleration. Each cart obeys its own equation, $\sum \mathbf{F}_i = m_i \mathbf{a}_i$, and unequal masses or different other forces can give different accelerations.

For a straight, taut, inextensible connector with fixed length and collinear attachment motion,

$$x_2 - x_1 = L = \text{constant} \quad \Rightarrow \quad v_{2x} = v_{1x}, \quad a_{2x} = a_{1x}.$$

The equal acceleration is a geometrical constraint. Equal endpoint tensions follow instead from force balance on a negligible-mass connector. The actual third-law pairs are the force on each cart due to its connector and the opposite force on that connector due to the cart. Treating the two endpoint forces as direct cart–cart partners would hide the massive-link limitation developed later.

Problem 15: equilibrium against a vertical wall

Take $+x$ away from the wall and $+y$ upward. The selected package is at rest. Its horizontal equation is $F_N - 60.0 = 0$, so

$$F_N = 60.0 \text{ N}.$$

Vertically, gravity must be balanced by tangential wall contact:

$$F_{\parallel, \text{req}} - mg = 0, \quad F_{\parallel, \text{req}} = 2.50(9.81) = 24.525 \text{ N} \approx 24.5 \text{ N},$$

directed upward. The normal force is not mg , since it acts horizontally. These are required forces under the observed-equilibrium assumption; neither number specifies the wall's available tangential force. A constitutive contact model and its regime test, introduced with friction in Chapter 6, are needed to decide whether a particular surface can maintain this equilibrium. Both calculated magnitudes are nonnegative.

Problem 16: a downward angled push

The horizontal platform constrains the case to zero vertical acceleration while contact persists. The applied force has downward vertical component $P \sin \theta$, with the angle measured below horizontal. Newton's vertical equation is

$$F_N - mg - P \sin \theta = 0.$$

Using the unrounded trigonometric value,

$$\begin{aligned} F_N &= (8.00)(9.81) + (50.0) \sin 37.0^\circ \text{ N} \\ &= 78.48 \text{ N} + 30.090751 \dots \text{ N} \\ &= 108.570751 \dots \text{ N} \approx 108.6 \text{ N}. \end{aligned}$$

The reading exceeds mg because the platform must oppose both downward force components. In the limit $P \rightarrow 0$, the expression returns mg ; at $\theta = 0$, a horizontal push makes no direct change to the vertical balance. A separate horizontal equation determines horizontal motion and is not needed to calculate this normal force.

Problem 17: spring-force signs

Use the supplied unstretched reference coordinate in $F_{s,x} = -k_s(x - x_0)$. For $x = 0.160$ m, the displacement from that reference is $+0.040$ m, hence

$$F_{s,x} = -(180)(0.040) = -7.20 \text{ N}.$$

The force points toward decreasing x . For $x = 0.090$ m, the displacement is -0.030 m, giving

$$F_{s,x} = -(180)(-0.030) = +5.40 \text{ N},$$

toward increasing x . Each force opposes the signed deformation and has units $\text{N m}^{-1} \text{m} = \text{N}$. This result assumes a spring that supports the stated linear deformation, including compression. Its unstretched reference need not be the equilibrium position of a spring-loaded system with additional forces.

Problem 18: two counterexamples to $F_N = mg$

One counterexample changes the applied forces while preserving zero vertical acceleration: a downward push at angle θ on a horizontally supported case gives

$$F_N - mg - P \sin \theta = 0 \quad \Rightarrow \quad F_N = mg + P \sin \theta.$$

For $P \sin \theta > 0$, this exceeds mg . A second counterexample changes the kinematic constraint: a body following the vertical acceleration a_y of a lift floor obeys

$$F_N - mg = ma_y \quad \Rightarrow \quad F_N = m(g + a_y).$$

With upward acceleration the reading exceeds mg ; with moderate downward acceleration it is smaller. In both cases the normal force is obtained by applying Newton's law and testing contact, rather than being assigned by definition. Other correctly specified cases, such as a body supported on an inclined surface with no other normal components, also provide valid counterexamples.

Problem 19: two carts, equations, and checks

Take rightward as positive. The straight taut inextensible link imposes $a_1 = a_2 = a$. The negligible link mass justifies one common tension T . The individual equations are

$$T - 10.0 = 3.00a, \quad 42.0 - T = 5.00a.$$

Add them to remove the internal connector forces:

$$42.0 - 10.0 = (3.00 + 5.00)a \quad \Rightarrow \quad a = 4.00 \text{ m s}^{-2}.$$

Then $T = 10.0 + 3.00(4.00) = 22.0 \text{ N}$. Both one-body checks are explicit:

$$22.0 - 10.0 = 12.0 = 3.00(4.00), \quad 42.0 - 22.0 = 20.0 = 5.00(4.00),$$

with each term in newtons. Their sum is $32.0 \text{ N} = 8.00 \text{ kg} \times 4.00 \text{ m s}^{-2}$, as the composite-system equation requires. The positive tension is physically admissible. A connector can exert equal end tensions while the separate net forces 12.0 and 20.0 N differ, allowing the unequal masses to share one acceleration.

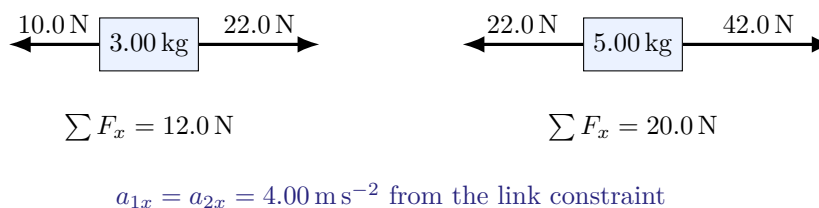
Cart 1: horizontal equation**Cart 2: horizontal equation**

Figure 5.27: Solution to Problem 19. Horizontal projections of the separate cart force diagrams; the vertically balanced normal and gravitational forces are omitted from these projections. The end tensions are equal by negligible-mass connector balance, while the cart resultants differ. Arrow lengths are schematic, and the displayed acceleration is kinematic information.

Problem 20: an impossible taut-string force

Let cart 1 be on the left and take rightward as positive. Under the temporary taut-string assumption, its tension force points right; that on cart 2 points left. The equations are

$$50.0 + T = 2.00a, \quad 10.0 - T = 2.00a.$$

Their sum gives $a = 60.0/4.00 = 15.0 \text{ m s}^{-2}$. Back-substitution yields

$$T = 2.00(15.0) - 50.0 = -20.0 \text{ N}.$$

This algebraic value would require a connector to push the carts apart rather than pull them together. A flexible string cannot do so; the assumed common-acceleration state fails. The string slackens as the faster-accelerating left cart closes the separation. Immediately after slackening, with the stated forces still acting and before any collision,

$$T = 0, \quad a_1 = \frac{50.0}{2.00} = 25.0 \text{ m s}^{-2}, \quad a_2 = \frac{10.0}{2.00} = 5.00 \text{ m s}^{-2}.$$

If the initial relative velocity is zero, then $x_2 - x_1 = L - \frac{1}{2}(20.0 \text{ m s}^{-2})t^2$ for a sufficiently short interval after release. The fixed-separation constraint is no longer valid. Later slack removal or collision needs additional information and a new model.

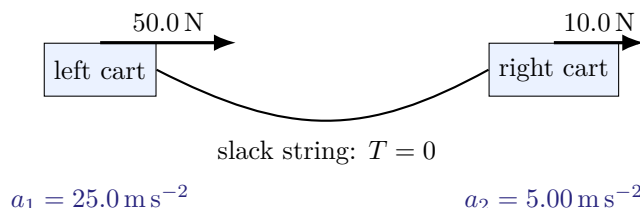


Figure 5.28: Solution to Problem 20. The admissible immediate state has no string tension; the carts accelerate independently. The curved slack string is a qualitative geometry sketch, with gravity on the negligible-mass string omitted. The forces shown are horizontal projections; the accelerations are separately labelled.

Problem 21: differentiating the link constraint

The coordinates of two collinear attachment points satisfy $x_2(t) - x_1(t) = L$. Because the link is inextensible, $dL/dt = 0$. One differentiation gives

$$\frac{dx_2}{dt} - \frac{dx_1}{dt} = v_{2x} - v_{1x} = 0.$$

A second gives $a_{2x} - a_{1x} = 0$. The velocities and accelerations therefore have equal *signed* components along the chosen common axis. This is not a force law. Stretch would replace constant L by a time-dependent length; slack would remove the fixed-separation condition; a pulley routing could preserve total rope length while giving a different relation between the cart coordinates. Any one of these is a valid change that invalidates this simple equation.

Problem 22: three carts and the middle-cart boundary

For the whole three-cart system together with the negligible-mass strings, neither string tension appears in the external horizontal force sum. Their direct cart-string interaction pairs are internal to that complete boundary. The system equation contains only horizontal forces exerted by outside bodies:

$$\sum F_{x,\text{ext}} = (m_1 + m_2 + m_3)a$$

for the common constrained acceleration and negligible connector masses.

For the middle cart alone, the left string pulls left with tension T_{12} , and the right string pulls right with tension T_{23} . Both are external to this one-cart system. Thus, if there is no other horizontal interaction on the middle cart,

$$T_{23} - T_{12} = m_2 a.$$

There is no requirement that tensions in *different* strings be equal. Their difference supplies the middle cart's acceleration. Equal tension along each ideal string is a separate approximation.

Problem 23: uniformly scaling both cart masses

The new masses are 6.00 and 10.00 kg, while the resultant remains $42.0 - 10.0 = 32.0$ N. Hence

$$a' = \frac{32.0}{16.00} = 2.00 \text{ m s}^{-2}, \quad T' = 10.0 + 6.00(2.00) = 22.0 \text{ N}.$$

The acceleration is half the Problem 19 value, while the tension is unchanged. This is exact for the stated uniform scaling, rather than merely an accidental numerical result. In general,

$$a = \frac{F_2 - F_1}{m_1 + m_2}, \quad T = \frac{m_2 F_1 + m_1 F_2}{m_1 + m_2}.$$

Multiplying both masses by a factor $c > 0$ divides a by c and cancels out of T . Changing the mass ratio instead can change the tension. The masses here affect acceleration inversely through the total mass; they do not produce a proportional increase of the internal force. Checking the second cart gives $42.0 - 22.0 = 10.00(2.00) = 20.0$ N.

Problem 24: a passenger accelerating upward

Select the passenger in the ground frame and take upward as positive. While contact is maintained,

$$F_N - mg = ma_y \quad \Rightarrow \quad F_N = m(g + a_y).$$

Thus

$$F_N = 55.0(9.81 + 2.00) = 649.55 \text{ N} \approx 650 \text{ N}.$$

The gravitational force is $mg = 55.0(9.81) = 539.55 \text{ N} \approx 540 \text{ N}$. The extra 110.0 N is the upward resultant ma_y , not an extra independent force. The positive normal force confirms that the assumed floor contact is admissible. The ratio $F_N/(mg) = 1 + 2.00/9.81 = 1.20387 \dots$ shows a reading about 20.4% above the ordinary gravitational-force magnitude.

Problem 25: descending while slowing

With upward positive, the downward velocity is negative. Slowing means that velocity becomes less negative, so $a_y = +1.50 \text{ m s}^{-2}$: the acceleration is upward. The contact-branch equation gives

$$F_N = 70.0(9.81 + 1.50) = 791.7 \text{ N} \approx 792 \text{ N}.$$

The gravitational force is 686.7 N, so the upward resultant is $791.7 - 686.7 = 105.0 \text{ N} = 70.0(1.50) \text{ N}$. This substitution verifies both magnitude and sign. The direction of velocity describes the present motion; the acceleration describes its change and determines the force imbalance. A lift moving upward and speeding up could have the same acceleration and scale reading.

Problem 26: inferring acceleration from a scale

Select the package and use $F_N - mg = ma_y$. Solving for acceleration before substituting gives

$$a_y = \frac{F_N}{m} - g = \frac{500}{60.0} - 9.81 = -1.476666 \dots \text{ m s}^{-2} \approx -1.48 \text{ m s}^{-2}.$$

The acceleration is downward. This does not reveal whether the platform is descending faster or ascending more slowly; a velocity measurement is needed to distinguish those motions. The reading is positive and the inferred acceleration exceeds $-g$, so the assumed contact branch is self-consistent. Substitution gives $60.0(9.81 - 1.476666 \dots) = 500 \text{ N}$.

Problem 27: rejecting negative normal force

Blindly imposing floor acceleration on the body gives

$$F_N = 10.0(9.81 - 11.0) = -11.9 \text{ N}.$$

This is an algebraic diagnostic, not an allowed floor force. A unilateral supporting floor can push upward but cannot pull the body downward through the contact. The equation's assumption that the body shares the floor acceleration therefore fails.

After contact is lost, the ideal body has $F_N = 0$ and $a_{y,\text{body}} = -g = -9.81 \text{ m s}^{-2}$, while the prescribed floor can accelerate at -11.0 m s^{-2} . Their relative acceleration is $a_{y,\text{body}} - a_{y,\text{floor}} = +1.19 \text{ m s}^{-2}$, so the body separates upward relative to the floor. This independently confirms the contact-loss interpretation. The missing downward force cannot be supplied by inventing a negative normal force.

Problem 28: free fall and the remaining force

A zero scale reading means zero supporting contact force, not zero gravitational force. After contact is lost the body-only free-body diagram contains a downward mg arrow and no floor-contact arrow. Its Newton equation is

$$mg = ma \quad \Rightarrow \quad a = g.$$

If the body and platform are both freely falling with matching initial conditions, they can also remain geometrically adjacent with zero normal force. In either case the scale reads zero because it responds to support, while gravity continues to accelerate the body. This distinguishes gravitational weight, defined here as mg , from apparent weight.

Problem 29: the two endpoint tensions

Isolate the straight connector and choose rightward as positive. Its endpoint forces are $+T_R$ and $-T_L$, so

$$T_R - T_L = m_\ell a.$$

For the given data,

$$T_R = 20.0 + 3.00(1.50) = 24.5 \text{ N}.$$

The right tension exceeds the left by 4.50 N; this is the force required to accelerate the 3.00 kg connector. Both tensions are positive. Equal endpoint tensions would give zero horizontal resultant and would contradict the prescribed nonzero connector acceleration. As $m_\ell \rightarrow 0$ at finite a , their difference tends to zero, recovering the ideal limit.

Problem 30: assumptions behind a single tension

The connector must be taut to transmit a pull, and its mass must be negligible compared with the required precision so that finite acceleration does not demand an appreciable tension difference. The straight connector should have no intervening tangential external loading; where it changes direction, the routing must be ideal. A pulley transmitting a common tension is taken to have negligible rotational inertia and negligible axle friction. Those conditions concern *force transmission*.

Negligible stretch is additionally required for the simple inextensible kinematic constraints used in this chapter. It is not the independent reason that endpoint tensions are equal: a massive inextensible rope can have unequal tensions. If a pulley has appreciable inertia, its angular acceleration is an additional dynamical variable; a torque equation and a no-slip routing constraint connect its rotation to the bodies' translation. Chapter 10 develops that calculation.

Problem 31: three translating subsystems

The complete system contains both carts and the finite-mass link. Its total mass is $5.00 + 1.00 + 4.00 = 10.00$ kg, and the sole horizontal external force is 45.0 N. Thus

$$a = \frac{45.0}{10.00} = 4.50 \text{ m s}^{-2}.$$

For the left cart, its only horizontal force is the link pull:

$$T_L = 5.00a = 22.5 \text{ N}.$$

For the right cart, $45.0 - T_R = 4.00a$, giving $T_R = 27.0$ N. The isolated link obeys

$$T_R - T_L = 27.0 - 22.5 = 4.50 \text{ N} = (1.00 \text{ kg})(4.50 \text{ m s}^{-2}).$$

This checks all three subsystem equations and explains why one common tension would be inappropriate. Their sum recovers $45.0 = 10.00(4.50)$ in SI units. Positive tensions support the stated taut translating-link model.

Problem 32: what the finite-inertia pulley requires

Unequal tensions alone do not determine their values. A pulley with appreciable rotational inertia requires a rotational equation, schematically

$$(T_R - T_L)r - \tau_{\text{axle}} = I\alpha,$$

with a declared rotational sign convention. Here r is the pulley radius, I its moment of inertia, α its angular acceleration, and τ_{axle} any resisting axle torque. A no-slip condition such as $a = r\alpha$ links the connector's motion to its rotation. The supported bodies also need their own translational equations and input masses/loads.

Without the inertia information and those constraints, there is no closed model for the general apparatus. Chapter 5 can identify the missing physics, but its translational connector equation does not replace pulley torque balance. A known special equilibrium with zero angular acceleration would be a different, explicitly stated case.

Problem 33: residuals for a correct candidate

Using the original one-cart equations, not only their sum,

$$\begin{aligned}\mathcal{R}_1 &= T - F_1 - m_1 a = 32.0 - 14.0 - 6.00(3.00) = 0 \text{ N}, \\ \mathcal{R}_2 &= F_2 - T - m_2 a = 44.0 - 32.0 - 4.00(3.00) = 0 \text{ N}.\end{aligned}$$

Residuals of order 10^{-12} N against forces of order 10–40 N have relative size of roughly 10^{-13} – 10^{-14} . Such tiny values would usually be numerically negligible for this teaching model and are consistent with floating-point evaluation or solver tolerance. They need not equal the theoretical machine-roundoff bound of a particular implementation. Check the units, scaling, and conditioning before interpreting them. Small residuals establish satisfaction of the supplied equations; they do not establish correct apparatus modelling or correct sensor inputs.

Problem 34: inverse mass and resistance

The zero-acceleration point directly fixes the intercept:

$$R = 8.0 \text{ N}.$$

Subtracting the line equations for the two points eliminates R :

$$38.0 - 8.0 = m(2.00 - 0), \quad m = 15.0 \text{ kg}.$$

The slope's units are $\text{N}/\text{ms}^{-2} = \text{kg}$. Back-substitution predicts $F(0) = 8.0 \text{ N}$ and $F(2.00) = 15.0(2.00) + 8.0 = 38.0 \text{ N}$, so both exact points are reproduced. Two exact points determine a straight line, but real measured points would require uncertainty analysis and further runs to test whether a constant-resistance line is an adequate model.

Problem 35: the synthetic resistance values

For each supplied pair, compute the opposing-force parameter from $R_i = F_i - ma_i$:

$$\begin{aligned}R_1 &= 2.00 - 2.00(0.90) = 0.20 \text{ N}, \\ R_2 &= 4.00 - 2.00(1.90) = 0.20 \text{ N}, \\ R_3 &= 6.00 - 2.00(2.90) = 0.20 \text{ N}.\end{aligned}$$

The three results are identical at the supplied precision, so the synthetic values are exactly consistent with constant resistance. Substitution into $a = (F - 0.20)/2.00$ returns every listed acceleration. This is a check of a stipulated dataset, not an experimental determination of resistance. For actual data, variation of R_i beyond estimated uncertainties could reveal a speed-dependent force, zero offset, slope, or another omitted term.

Problem 36: a constant force-sensor zero offset

Let the true relation be $F_{\text{true}} = ma + R_{\text{true}}$. With the constant offset $\delta = +0.40 \text{ N}$, the recorded force is

$$F_{\text{sensor}} = F_{\text{true}} + \delta = ma + (R_{\text{true}} + 0.40 \text{ N}).$$

The fitted intercept is therefore biased upward by 0.40 N. In the ideal exact-line comparison the slope is unchanged because subtracting two observations cancels the same offset: $\Delta F_{\text{sensor}} = \Delta F_{\text{true}}$. Unload the sensor in its installed measurement orientation and check that it reads zero; repeat that check after the run to detect drift. Correcting the measured offset before fitting prevents a sensor artifact from being misidentified as physical resistance.

Problem 37: an independent low-force experiment

Physical question. Does the horizontal force resultant inferred from a force sensor and independently characterised resistance agree with the cart mass times its acceleration inferred from motion? Use a light cart on a level track, a force sensor pulling along the track, and a position camera or motion sensor with an independently calibrated distance scale. The force sensor must not be used to compute the acceleration being tested.

Model and worked exemplar. Assume fixed total moving mass, negligible vertical acceleration, and an approximately constant independently determined resistance R during a short run with positive velocity. As *stipulated example values*, take $m = 0.500$ kg, sensor force $F = 0.300$ N, and $R = 0.050$ N. The force-model prediction is

$$a_{\text{pred}} = \frac{F - R}{m} = \frac{0.250}{0.500} = 0.500 \text{ m s}^{-2}.$$

Fit the measured position independently to $x(t) = x_0 + v_0 t + \frac{1}{2} a_{\text{kin}} t^2$ over the same interval. Comparing a_{kin} with a_{pred} , rather than feeding one into the other, provides the required independent evidence. Alternatively compare $F - R$ with ma_{kin} directly. These illustrative values are not measured results.

Controls and records. A satisfactory design records at least four concrete controls; a developed version includes the following.

1. Measure the total moving mass, including fixtures and any sensor mass that travels with the cart.
2. Check and record the force-sensor unloaded zero before and after the run, and retain its calibration information.
3. Level the track independently and record the direction of any residual slope; gravity projected along the track is otherwise an omitted external force.
4. Calibrate the motion distance scale and time stamps, and synchronize the force and motion records.
5. Estimate resistance in a separate coasting run using the independent deceleration, or use an independently established resistance model with its uncertainty.
6. Repeat runs and inspect the position-fit residuals; changing systematic curvature can indicate a nonconstant acceleration rather than random scatter.

Limiting check. For a run with drive force approaching the independently determined resistance, the model predicts $a \rightarrow 0$ while the cart can continue at nonzero velocity. Another valid check holds the force protocol and resistance fixed while doubling the mass, predicting half the acceleration. If added payload changes resistance, that change must be measured before applying this second comparison.

Evaluation and plausible outcomes. Report units, model assumptions, uncertainty in both sides of the comparison, and the difference $\Delta F = (F - R) - ma_{\text{kin}}$. Agreement means that this difference is compatible with the independently estimated uncertainty over the selected interval; perfect equality is not an experimental expectation. A repeatable nonzero difference prompts investigation of sensor zero, timing, slope, resistance variation, and omitted cable forces. Full-credit work uses distinct force and motion evidence, at least four explicit controls, a valid limiting check, and an explanation of how discrepancy would be interpreted. Different apparatus choices and numerical scales are acceptable if they satisfy these criteria.

Problem 38: the distributed-link tension profile

For a short segment $[s, s + ds]$, its horizontal forces give

$$T(s + ds) - T(s) = \lambda ds a.$$

Taking the limit yields $dT/ds = \lambda a$. The uniform parameters give

$$\lambda a = (0.400)(3.00) = 1.20 \text{ N m}^{-1},$$

and integrating with $T(0) = 12.0 \text{ N}$ fixes the integration constant:

$$T(s) = 12.0 \text{ N} + (1.20 \text{ N m}^{-1})s, \quad 0 \leq s \leq 2.50 \text{ m}.$$

The right endpoint has $T(L) = 15.0 \text{ N}$. The total link mass is $\lambda L = 1.00 \text{ kg}$, so $T(L) - T(0) = 3.00 \text{ N} = m_\ell a$, an independent whole-link check. The tension stays positive over the domain and increases rightward because every segment needs a rightward resultant.

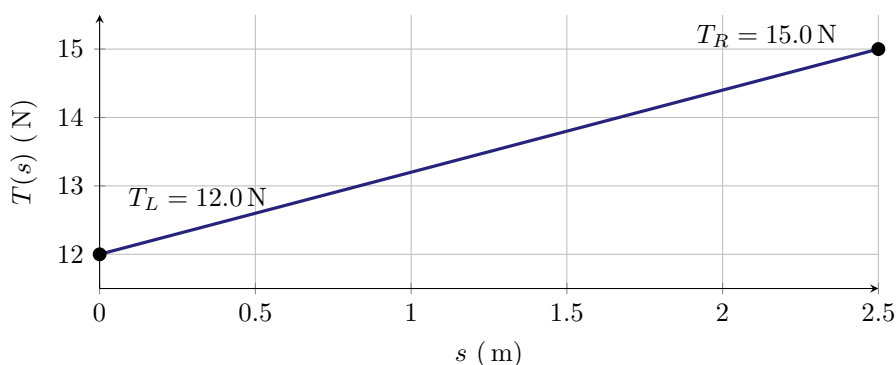


Figure 5.29: Solution to Problem 38. The quantitative analytical tension profile has slope $\lambda a = 1.20 \text{ N m}^{-1}$ and endpoint increase $m_\ell a = 3.00 \text{ N}$. The vertical axis is truncated below 11.5 N to make the finite tension difference readable; this does not indicate a zero-tension endpoint.

Problem 39: a massless-link accuracy criterion

Assume the same nonzero external resultant F acts in both models and the complete system remains in the same straight translational regime. The finite-link model and massless-link approximation give

$$a = \frac{F}{M + m_\ell}, \quad a_0 = \frac{F}{M}.$$

Taking their ratio, which removes the unknown resultant,

$$\frac{a_0 - a}{a} = \frac{a_0}{a} - 1 = \frac{M + m_\ell}{M} - 1 = \frac{m_\ell}{M}.$$

This is also the fractional overprediction of the *acceleration magnitude* if F is negative. To keep the error strictly below 2.0%, require $m_\ell/M < 0.020$. If “at most 2.0%” were required, equality would be allowed. At zero resultant both models give zero acceleration and this relative-error quotient is undefined; the displayed criterion applies to nonzero acceleration. The formula controls acceleration error, not each endpoint-tension error.

Problem 40: a plausible acceleration with wrong tension

Evaluate each supplied equation at $a = 3.00 \text{ m s}^{-2}$, $T = 31.0 \text{ N}$:

$$\mathcal{R}_1 = 31.0 - 14.0 - 6.00(3.00) = -1.0 \text{ N},$$

$$\mathcal{R}_2 = 44.0 - 31.0 - 4.00(3.00) = +1.0 \text{ N}.$$

These are substantial relative to the displayed input precision and vastly exceed plausible rounding from these simple calculations. The candidate is not a solution. Correct tension at this acceleration is $T = 14.0 + 6.00(3.00) = 32.0 \text{ N}$.

Notice that the residuals sum to zero. The wrong candidate satisfies the *combined* equation $44.0 - 14.0 = (6.00 + 4.00)(3.00)$, because tension cancels from that equation. This shows why a whole-system check alone cannot validate an internal force: the individual body equations must also be tested.

Problem 41: converting an error tolerance to link mass

The relative acceleration overprediction is $\epsilon = m_\ell/M$, measured against the finite-link result. With $M = 10.00 \text{ kg}$, a 5.0% error means $\epsilon = 0.050$, so

$$m_\ell = \epsilon M = 0.050(10.00) = 0.500 \text{ kg}.$$

For an actual link mass 0.20 kg ,

$$\epsilon = \frac{0.20}{10.00} = 0.020 = 2.0\%.$$

As a direct check using the investigation's $F = 50.0 \text{ N}$, the massless result is $a_0 = 5.00 \text{ m s}^{-2}$, while a 0.500 kg link gives $a = 50.0/10.50 = 4.76190 \dots \text{ m s}^{-2}$. Their fractional difference is exactly 0.050 . The criterion depends on the mass ratio, not on the magnitude of the nonzero resultant in this model.

Problem 42: the origin of an accelerating-frame term

For a translating frame whose origin is $\mathbf{R}(t)$, write $\mathbf{r}' = \mathbf{r} - \mathbf{R}$. Two differentiations give $\mathbf{a}' = \mathbf{a} - \mathbf{A}$, where $\mathbf{A} = d^2\mathbf{R}/dt^2$. Substitution into inertial-frame Newton dynamics yields

$$m\mathbf{a}' = \sum \mathbf{F}_{\text{interactions}} - m\mathbf{A}.$$

The term $-m\mathbf{A}$ originates in the acceleration of the coordinate frame. It has no second interacting body that receives an opposite force, so it is not a Newton-third-law partner. It vanishes when $\mathbf{A} = \mathbf{0}$, restoring the inertial translating-frame form.

Both descriptions must identify the same physical events, contacts, separations, and interaction forces within the Newtonian model after applying the coordinate transformation. Relative positions between bodies remain the same for this common translation, and predicted trajectories agree when transformed back to a common frame. Coordinate velocities and accelerations need *not* remain numerically invariant: $\mathbf{v}' = \mathbf{v} - \dot{\mathbf{R}}$ and $\mathbf{a}' = \mathbf{a} - \mathbf{A}$. Confusing physical agreement with equal coordinate accelerations is precisely the error the transformation avoids. This short derivation answers the question without replacing the fuller accelerated-frame applications reserved for Chapter 6.

Problem 43: extracting dynamics from a path

(a) **Differentiate with units.** Write $x = (3.00 \text{ m s}^{-1})t$ and $y = (1.00 \text{ m s}^{-2})t^2$. Then

$$\mathbf{v} = (3.00\hat{x} + 2.00(t/\text{s})\hat{y}) \text{ m s}^{-1}, \quad \mathbf{a} = 2.00\hat{y} \text{ m s}^{-2}.$$

For the stipulated 2.00 kg mass, the in-plane resultant is $\mathbf{F}_{\text{net}} = 4.00\hat{y} \text{ N}$. The vertical support balances gravity outside this horizontal plane. This is the force *sum*; the motion alone does not identify every separate in-plane interaction.

(b) **Recover direction.** At 2.00 s , $\mathbf{v} = (3.00, 4.00) \text{ m s}^{-1}$, so its magnitude is 5.00 m s^{-1} and its angle is $\tan^{-1}(4/3) = 53.13^\circ$ north of east. Both components are positive, so the first-quadrant

angle is appropriate. The tangent in the path plot has slope $dy/dx = v_y/v_x = 4/3$, an independent geometrical check.

(c) Interpret. An initial velocity is an initial condition, not a force. The eastward component remains constant because the eastward resultant is zero; the northward component increases because the northward resultant is positive.

Problem 44: the admissible contact range

(a) Select the case. The actual forces are gravity downward, platform normal upward, and the applied pull inclined above horizontal. With $a_y = 0$,

$$a_x = \frac{P \cos \theta}{m}, \quad F_N = mg - P \sin \theta.$$

The force components are replacements for the pull in component equations, rather than additional interactions.

(b) Enforce contact. A unilateral floor requires $F_N \geq 0$. Since $mg/P = 39.24/60.0 = 0.654$, the limiting angle is

$$\theta_c = \sin^{-1}(0.654) = 40.8439^\circ \approx 40.8^\circ.$$

At the boundary, the normal is zero. Below it the platform can maintain contact; beyond it the assumed zero vertical acceleration fails.

(c) Compare two regimes. At 30.0° ,

$$F_N = 39.24 - 30.00 = 9.24 \text{ N}, \quad a_x = 15.0 \cos 30.0^\circ = 12.9904 \text{ m s}^{-2} \approx 13.0 \text{ m s}^{-2}.$$

At 60.0° the contact formula would give $F_N = -12.7215 \text{ N}$, which is inadmissible. Immediately after separation, with the same force direction maintained, the actual normal is zero and

$$a_y = \frac{60.0 \sin 60.0^\circ}{4.00} - 9.81 = 3.18038 \text{ m s}^{-2}, \quad a_x = 7.50 \text{ m s}^{-2}.$$

The positive upward acceleration confirms separation rather than a pulling force from the floor. This immediate model is valid only while the applied pull remains prescribed.

Problem 45: an inverse Atwood model

(a) Invert the graph. If $r = m_2/m_1 \geq 1$, then $a/g = (r - 1)/(r + 1)$. Therefore

$$r = \frac{1 + 0.200}{1 - 0.200} = 1.50, \quad m_1 = \frac{10.0}{1 + 1.50} = 4.00 \text{ kg}, \quad m_2 = 6.00 \text{ kg}.$$

The acceleration magnitude is 1.962 m s^{-2} .

(b) Check both bodies. For the lighter rising mass, $T = m_1(g + a) = 47.088 \text{ N} \approx 47.1 \text{ N}$. For the heavier falling mass, $T = m_2(g - a) = 47.088 \text{ N}$. Agreement independently checks the mass ratio and sign choices. The common tension is equal at the two ends because the rope and pulley are ideal.

(c) State the actual constraint. With both laboratory coordinates upward, $a_{1y} + a_{2y} = 0$. The acceleration vectors are opposite, while their magnitudes are equal. A claim $\mathbf{a}_1 = \mathbf{a}_2$ would require both bodies to move in the same vector direction and contradict the routing.

Problem 46: geometry creates a factor of two

(a) Count changing lengths. Two rope legs change with the load position. The taut-length condition is $2z + y = \text{constant}$; hence $\Delta y = -2\Delta z = -0.240 \text{ m}$, or 0.240 m upward.

(b) Combine dynamics and constraint. Differentiation twice gives $a_y = -2a_z$. For the load and free-end mass, respectively,

$$Mg - 2T = Ma_z, \quad mg - T = ma_y = -2ma_z.$$

Thus $T = m(g + 2a_z)$ and

$$a_z = \frac{(M - 2m)g}{M + 4m}.$$

The doubled rope force on the load and doubled acceleration of the free end have separate origins; neither is obtained by assigning a different tension to each leg.

(c) Test equilibrium and signs. Zero acceleration requires $M = 2m$. At $M = 6.00 \text{ kg}$ and $m = 2.00 \text{ kg}$,

$$a_z = \frac{2}{14}9.81 = 1.40143 \text{ m s}^{-2} \text{ downward}, \quad a_y = -2.80286 \text{ m s}^{-2} \text{ upward}.$$

The tension is 25.2257 N . Substituting gives $58.86 - 2(25.2257) = 6(1.40143)$ and $19.62 - 25.2257 = 2(-2.80286)$ to rounding precision. Positive tension confirms the stipulated taut state.

Problem 47: paired lift histories

(a) Integrate the given acceleration. The upward trip has

$$v_y(t) = \begin{cases} 0.800t, & 0 \leq t \leq 2, \\ 1.600, & 2 \leq t \leq 5, \\ 1.600 - 0.800(t - 5), & 5 \leq t \leq 7, \end{cases}$$

where numerical t is in seconds and velocities are in m s^{-1} . The velocity is continuous and ends at zero. The scale force $25.0(g + a_y)$ is 265.25 N , 245.25 N , and 225.25 N over the three successive open intervals. Ideal acceleration jumps produce ideal force jumps; a real drive rounds those transitions.

(b) Use the velocity areas. The first and last triangles each contribute $\frac{1}{2}(2.00)(1.600) = 1.600 \text{ m}$ and the middle rectangle contributes $(3.00)(1.600) = 4.800 \text{ m}$. Total displacement is 8.000 m upward. Direct piecewise integration gives the same result.

(c) Reverse the trip. On the downward trip the velocities are negative and the acceleration signs reverse. Its first stage has $F_N = 225.25 \text{ N}$ and its stopping stage $F_N = 265.25 \text{ N}$. Larger apparent weight occurs during upward-trip starting and downward-trip stopping. The scale diagnoses acceleration in this model; it cannot by itself determine whether velocity is upward, downward or instantaneously zero.

Problem 48: synthetic fit and calibration

(a) Compute the slope from centred data. Here $\bar{a} = 0.875 \text{ m s}^{-2}$, $\bar{F} = 1.580 \text{ N}$, $S_{aa} = \sum(a_i - \bar{a})^2 = 1.09375 \text{ m}^2 \text{ s}^{-4}$, and $S_{aF} = 1.75000 \text{ N m s}^{-2}$. Consequently

$$m = \frac{S_{aF}}{S_{aa}} = 1.600 \text{ kg}, \quad b = \bar{F} - m\bar{a} = 0.180 \text{ N}.$$

The equal-weight model treats the acceleration values as exact teaching inputs; uncertainty in both axes would require a different fit model.

(b) Inspect all residuals. Subtract $1.600a_i + 0.180$ from each force. In order, the residuals are

$$(0.010, -0.020, 0.010, 0.010, -0.020, 0.010) \text{ N}.$$

They satisfy both $\sum r_i = 0$ and $\sum a_i r_i = 0$, the independent normal-equation checks for an intercept fit. They do not prove that a real apparatus has constant resistance: these values are synthetic. If sensor zero is c , then $b = R + c$; resistance and zero offset cannot be distinguished from this one line alone.

(c) Correct using independent evidence. The unloaded reading supplies $c = +0.050 \text{ N}$, so $R = b - c = 0.130 \text{ N}$. Subtracting a constant from every force shifts only the intercept. Differences, centred values, fitted mass and residuals are unchanged.

Problem 49: a density gradient makes a curved tension profile

(a) Integrate mass before force. The link mass is

$$m_\ell = \int_0^L \lambda_0(1 + \beta s/L) ds = \lambda_0 L(1 + \beta/2) = 0.840 \text{ kg}.$$

A local segment obeys $dT/ds = \lambda(s)a$. Integrating with $T(0) = T_L$ gives

$$T(s) = T_L + \lambda_0 a \left(s + \frac{\beta s^2}{2L} \right) = 10.0 \text{ N} + (0.750 \text{ N m}^{-1})s + (0.150 \text{ N m}^{-2})s^2.$$

The quadratic coefficient carries units; the density-gradient parameter is dimensionless.

(b) Evaluate and test. At $s = 1.00 \text{ m}$ the tension is 10.900 N . At $s = 2.00 \text{ m}$ it is 12.100 N . Its endpoint difference, 2.100 N , equals $(0.840)(2.50) \text{ N}$ independently. Both endpoint tensions and the entire increasing profile are positive.

(c) Distinguish local and integrated evidence. A straight endpoint interpolation has slope $2.100/2.00 = 1.050 \text{ N m}^{-1}$ everywhere. The actual local slope increases from 0.750 to 1.350 N m^{-1} , following the increasing density. An endpoint balance alone does not verify the internal distribution.

Problem 50: components are a representation, not new forces

(a) Correct the force inventory. Only Earth and the frictionless surface interact with the body. The actual-force diagram therefore has two arrows: $m\mathbf{g}$ downward and $\mathbf{F}_N$ perpendicular outward. The weight-component construction is a separate projection triangle; its two components may replace the weight in calculations, but are not added to the weight again.

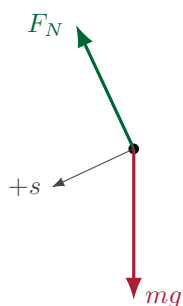
(b) Choose downhill and outward axes. The normal acceleration is zero, so

$$ma_s = mg \sin \theta, \quad F_N - mg \cos \theta = 0.$$

Thus $a_s = 9.81 \sin 25.0^\circ = 4.14589 \text{ m s}^{-2} \approx 4.15 \text{ m s}^{-2}$ and $F_N = 3.00(9.81) \cos 25.0^\circ = 26.6726 \text{ N} \approx 26.7 \text{ N}$. The limit $\theta \rightarrow 0$ yields zero downhill acceleration and $F_N \rightarrow mg$; approaching vertical makes the surface reaction vanish.

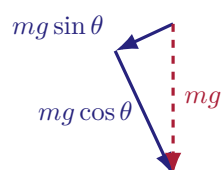
(c) Add one actual interaction. If a tether has outward-normal component $T \sin \phi$, the contact equation becomes $F_N + T \sin \phi - mg \cos \theta = 0$, hence $F_N = mg \cos \theta - T \sin \phi$. It must remain nonnegative. The tangential equation also acquires the tether's tangential component; its sign depends on the stated direction.

(a) Actual forces



Two physical interactions.

(b) Weight decomposition



Components replace the weight.

Figure 5.30: Solution to Problem 50 at $\theta = 25^\circ$. The actual-force diagram and the auxiliary weight projection serve different purposes. The head-to-tail component arrows sum to the dashed weight; adding all three would count gravity twice. Directions and relative lengths follow the projection geometry, with the weight assigned a display length of 2 cm.

Bridge to Chapter 6

Newton's laws do not change when the force model becomes more complicated. What changes is the work needed to identify the correct interaction law and the correct physical regime. In this chapter a wall could supply a required tangential force, but we deliberately did not ask whether a particular dry interface can supply it. A cart could be constrained to a straight path, but we did not analyse the force required to bend a trajectory. A moving body could experience resistance, but the resistance was prescribed rather than derived from a drag model.

Chapter 6 develops those applications. Static and kinetic friction require a regime decision; circular motion requires a radial force sum built from real interactions rather than an invented “centripetal force”; drag requires a stated velocity-dependent model; and accelerating frames require explicit frame terms. The modelling discipline established here remains unchanged: choose the system and frame, identify interactions, write the free-body diagram, add the constraint or constitutive model, and then test the solution against its physical domain.

Take-home message

The shortest correct statement of Newtonian dynamics is $\sum \mathbf{F}_{\text{ext}} = m\mathbf{a}$. The most transferable skill is knowing exactly what belongs in that sum and why.

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