

Chapter 6

Applications of Newton's Laws

First Edition

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Preface

Newton's laws provide a language for deciding which motions a physical system can sustain. A surface can prevent relative sliding only up to a limiting tangential force. A guide can push a carriage, but an ordinary one-sided contact cannot pull it back. A body falling through a fluid can approach a steady speed even though gravity continues to act. These examples are connected by one method: specify the motion, calculate the interaction forces it requires, and test whether those interactions can supply them.

The chapter begins with a package carried by an accelerating conveyor. Its acceleration comes from a contact force, so the central question is whether the package follows the belt or slips relative to it. The same reasoning develops into inclined and connected systems, banked turns, contact loss, shared traction demands, and linear and quadratic resistance. Calculus is introduced where it helps distinguish an instantaneous force balance from a complete time history. A final enrichment section translates the equations into the coordinates of an accelerating, nonrotating platform.

The core is calculus-based but gives local support for its mathematical steps. It is intended for technical and engineering students, including space-technology readers whose previous mechanics preparation may differ. Examples connect a force law with a diagram, a calculation, and a test of the proposed regime. The practice set includes conceptual distinctions, graph interpretation, multistep contact problems, synthetic data, numerical integration, and frame comparisons. Complete reasoned solutions are included in the same chapter.

The Guideline

Draw the chosen body's actual interactions before resolving components. Determine the force required by the proposed motion, then apply the contact or resistance law. An algebraic value outside the physical range is a reason to change the assumed regime.

Pedagogical boundary

Work, kinetic energy, and momentum are developed in the following chapters. They are not assumed here. Fluid-regime classification belongs to the fluids chapter; each drag law here is a stipulated model. The advanced frame transformation uses translating axes without rotation. Current vehicle or mission specifications are unnecessary for the idealized teaching applications.

The references supply complementary treatments of the governing mechanics. Exposition and exercise solutions are written for this chapter; the figure credits identify construction techniques adapted from TikZ.net. All experimental tables explicitly described as synthetic are teaching data, not reported measurements. The publication identity remains First Edition, 1 January 2026.

Conventions and notation

Vectors are bold, such as $\mathbf{F}$, whereas signed components such as F_x can be positive or negative. A magnitude is nonnegative. The equivalent symbols N and F_N denote the normal-contact magnitude; the choice is stated within each model. A signed friction component refers to a chosen tangent; its magnitude is nonnegative. Static and kinetic friction symbols are defined locally before use. The symbol T denotes tension, not temperature or a period, in this chapter.

Symbol	Meaning	SI unit
m	fixed mass of the chosen body	kg
$\mathbf{v}, \mathbf{a}$	inertial-frame velocity and acceleration	m s^{-1} , m s^{-2}
N, T, f	normal magnitude, tension magnitude, signed friction component	N
μ_s, μ_k	coefficients in the stipulated dry-friction model	1
R, R_c	circular radius and local radius of curvature	m
$\hat{\mathbf{t}}, \hat{\mathbf{n}}$	unit tangent and inward unit normal	1
v_{rel}	signed speed component relative to belt or medium	m s^{-1}
b, c	coefficients of linear and quadratic drag	kg s^{-1} , kg m^{-1}
v_t, τ	terminal-speed magnitude and a defined transient time scale	m s^{-1} , s
$\mathbf{A}, \mathbf{a}'$	frame acceleration and particle acceleration in that frame	m s^{-2}
$u(\theta), u(\mu)$	stated standard uncertainties	rad, 1

Numerical teaching examples use $g = 9.81 \text{ m s}^{-2}$. This is a stipulated local approximation, not an exact universal constant. Angles in displayed geometry may be degrees; derivatives and uncertainty propagation use radians. Parameters describing friction or resistance belong to the stated surfaces and model. A graph with normalized axes is quantitative; a setup or force diagram states when its distances or arrow lengths are schematic.

Note

Preparation: signed motion, vectors, trigonometric projections, Newton's laws, and free-body diagrams from Chapters 2–5. Inequalities and separable differential equations are explained at their first substantial use. The advanced section is optional; the core exercises identify their prerequisites independently.

Chapter 6

Applications of Newton's Laws

Motivation and learning goals

A package on a conveyor has no motor of its own. When the conveyor accelerates, friction may accelerate the package with it, or the belt may move ahead beneath the package. These outcomes cannot be decided from the word “friction” alone. We must calculate the contact-force demand and compare it with the contact’s capacity. That comparison is the organizing idea throughout this chapter.

Learning goals

After studying the chapter, you should be able to:

- distinguish a static-friction demand from its upper bound and determine friction from relative slip;
- resolve angled loading, inclined contacts, and connected bodies while testing all contact constraints;
- identify the actual radial and tangential force components in circular and general curved motion;
- calculate banked-turn intervals, contact-loss thresholds, and combined traction demands;
- compare sign-correct resistance laws, terminal equilibria, and complete transients;
- infer model parameters from synthetic observations and verify numerical integration;
- translate a nonrotating accelerating-frame description back to inertial Newtonian dynamics.

6.1 Static and kinetic friction

A belt can accelerate a package without a motor inside the package. A hand can press a crate harder against the floor and make it more difficult to start sliding. A block can remain motionless on an incline even though gravity has a component along the slope. These observations involve the same contact interaction, but they do not all require the same friction force. The first decision is whether the two contacting surfaces slide relative to one another. The second is whether the force required by the proposed motion is physically available.

The dry-friction model used here is empirical. At the microscopic level, deformation and adhesion occur at a collection of small contact regions. The visible area of a block’s base is therefore not the same as its actual microscopic contact area. We will not attempt a microscopic calculation of friction. Instead, we stipulate dimensionless coefficients and test what their macroscopic model predicts. Coefficients are properties of the contacting pair and its conditions, not immutable properties of one material. Moisture, surface preparation, temperature, wear and speed can change them. A constant coefficient is an assumption to be assessed, not a consequence of Newton’s laws.

6.1.1 Static friction is a response with a limit

Let $F_N \geq 0$ be the magnitude of the normal contact force. On a surface with one relevant tangential direction, let f_s denote a *signed* static-friction component. Positive and negative values correspond to opposite directions along our chosen tangent. The elementary Coulomb model permits

$$|f_s| \leq \mu_s F_N. \quad (6.1)$$

The coefficient μ_s is dimensionless because the two sides have force units. Equality describes a limiting equilibrium at impending slip; it is not the rule for every static contact. A book at rest on a level desk, with no horizontal applied force, needs $f_s = 0$. A horizontal push of 5 N on that book can require a friction force of 5 N in the opposite direction. Increasing the push increases the response until the permitted bound is reached.

Definition 6.1: Static-friction demand

The *required static friction* is the force found from Newton's laws after imposing no relative slip at the contact. The imposed motion is admissible only if this force lies within the static-friction bound and the normal contact force remains nonnegative.

Suppose a horizontal belt has prescribed acceleration a_b . If a package sticks to it, their accelerations are equal. The package has no other horizontal force, so Newton's second law requires $f_s = ma_b$. Its vertical acceleration is zero and the belt is level, giving $F_N = mg$. Thus the condition for this motion is

$$|ma_b| \leq \mu_s mg, \quad |a_b| \leq \mu_s g. \quad (6.2)$$

Mass cancels here because both inertia and the model's contact capacity scale with mass. This cancellation would not survive every geometry: an additional fixed external force, a changing vertical acceleration, or a non-Coulomb contact law can change the conclusion.

Figure 6.1 separates the force *demand* from the force *capacity*. The diagonal is the demand calculated from the hypothetical sticking motion, including its inadmissible continuation outside the band. It is not the actual force once slip starts. The distinction is useful whenever a constraint can fail: first solve the constrained model, then test it.

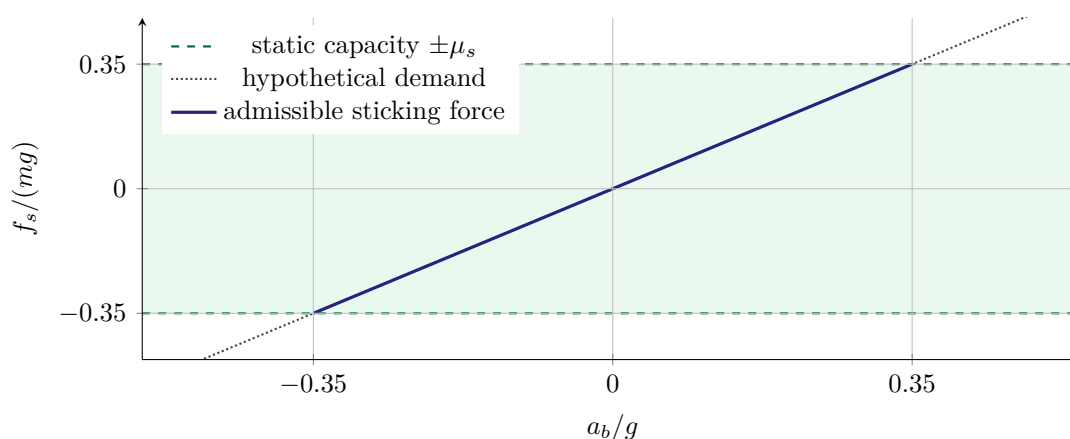


Figure 6.1: Static-friction demand for the horizontally accelerating package, with $\mu_s = 0.35$. The shaded band is a permitted force range. Only the solid segment describes admissible sticking. The dotted continuation diagnoses a failed assumption; it does not predict the sliding force.

Method

A demand test that respects the contact model. Identify the body and its external forces; assume the proposed no-slip motion; write the tangential and normal equations; solve

for the required signed friction and actual normal force; test $|f_{s,\text{req}}| \leq \mu_s F_N$ and $F_N \geq 0$. If a test fails, change the motion model before calculating the actual acceleration. Clipping an excessive friction demand while retaining the original acceleration violates Newton's second law.

An accelerating body can have static contact. The word “static” describes no sliding at the interface, not zero laboratory velocity or zero acceleration. Conversely, a block stationary in the laboratory can have kinetic contact with a belt sliding underneath it. A free-body diagram should therefore identify forces without using the object's visible laboratory motion as a shortcut for choosing the friction law.

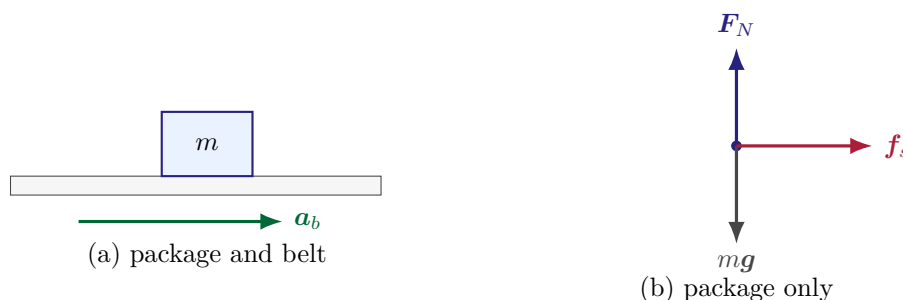


Figure 6.2: The physical setup and its separate package-only free-body diagram for rightward sticking acceleration. Belt acceleration is a kinematic quantity, so it appears in the setup rather than as an extra force in the free-body diagram. Arrow lengths are schematic.

Example 6.1: An accelerating conveyor: preserve or reject sticking?

A 12.0 kg package initially rests relative to a level belt. The stipulated coefficients are $\mu_s = 0.350$ and $\mu_k = 0.250$. Determine its friction force and acceleration when the belt begins accelerating rightward at 2.40 m s^{-2} . Repeat for 4.20 m s^{-2} .

System and strategy. Use the package-only diagram in Fig. 6.2. Gravity and the normal force balance vertically, so $F_N = 12.0(9.81) = 117.72 \text{ N}$. We first test sticking instead of presuming that the maximum friction acts.

For the smaller acceleration,

$$f_{s,\text{req}} = ma_b = 28.8 \text{ N}, \quad \mu_s F_N = 41.202 \text{ N}.$$

The demand is below the capacity, so the package sticks. Its acceleration is 2.40 m s^{-2} , and the actual friction is 28.8 N rightward. The unused capacity is not an additional force.

For the larger acceleration, sticking would require 50.4 N, which exceeds the same capacity. The belt initially moves ahead of the package; the package slips left relative to the belt. Kinetic friction therefore acts rightward, with

$$f_k = \mu_k mg = 29.43 \text{ N}, \quad a_p = \frac{f_k}{m} = 2.4525 \text{ m s}^{-2}.$$

The reported acceleration is 2.45 m s^{-2} . It is below the belt acceleration, so the assumed slip direction is dynamically consistent: the belt continues to gain velocity relative to the package.

Interpretation and check. The sticking threshold is $\mu_s g = 3.4335 \text{ m s}^{-2}$. Neither mass nor the belt's instantaneous speed alone determines the regime. The contact law changes at a failed inequality, and the acceleration must be recalculated with it. At the exact ideal threshold, sticking is marginal; a microscopic perturbation or a coefficient change can initiate sliding.

6.1.2 Kinetic friction opposes relative sliding

If the contact slides, define the relative tangential velocity of the body with respect to the contacting surface as $\mathbf{v}_{\text{rel}} = \mathbf{v}_{\text{body}} - \mathbf{v}_{\text{surface}}$. For nonzero relative velocity, the stipulated isotropic law is

$$\mathbf{F}_k = -\mu_k F_N \frac{\mathbf{v}_{\text{rel}}}{|\mathbf{v}_{\text{rel}}|}. \quad (6.3)$$

In one dimension it is convenient to write $f_k = -\mu_k F_N \text{sgn}(v_{\text{rel}})$. This signed expression is undefined at $v_{\text{rel}} = 0$ as a sliding prescription. At that event we must test the static model. Setting the sign function to zero and continuing blindly can miss an admissible sticking stage or an immediate reversal of slip.

Figure 6.3 gives two examples with the same positive laboratory velocity of the package. In the first, the belt is faster and friction accelerates the package to the right. In the second, the belt is slower and friction acts leftward. Relative velocity decides the direction. Kinetic friction resists sliding between surfaces; it need not oppose the body's laboratory velocity.

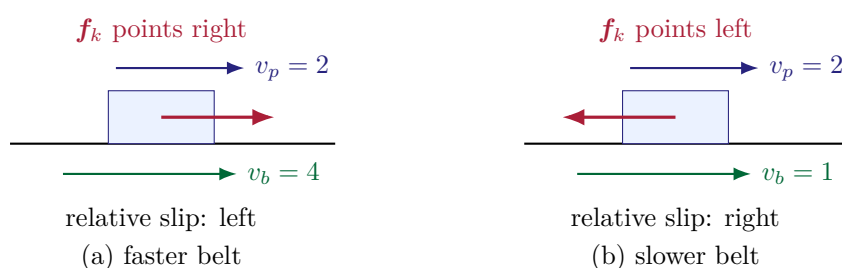


Figure 6.3: Velocities are prescribed in m s^{-1} , with rightward positive. In (a), $v_{\text{rel}} = 2 - 4 = -2$; in (b), $v_{\text{rel}} = 2 - 1 = +1$. Thus the friction force reverses even though the package moves rightward in both laboratory descriptions. Drawing lengths are schematic.

The complete contact model has a state-dependent structure. During sliding, integrate Newton's law with the kinetic force of the appropriate sign. Watch for a zero of relative velocity. At that time ask what friction would be required to keep the velocities equal. If its magnitude fits inside the static bound, continue with the sticking constraint. Otherwise continue sliding with the direction selected by the emerging relative motion. This is an example of an *event*: the equations governing the next interval are selected at a physically defined condition, rather than merely at a convenient time step.

Example 6.2: A belt stops accelerating: when does the package re-stick?

Use the sliding conveyor of the preceding example. From $t = 0$ to 1.00 s the belt accelerates at 4.20 m s^{-2} . It then keeps the velocity it has reached. The package starts from rest. Find the time at which sliding ends, the friction afterward, and the total lag of the package behind the belt.

First interval. The package acceleration is $a_k = \mu_k g = 2.4525\text{ m s}^{-2}$. Therefore

$$v_b = 4.20t, \quad v_p = a_k t, \quad v_{\text{rel}} = (a_k - 4.20)t < 0.$$

At $t_1 = 1.00\text{ s}$ the velocities are 4.20 and 2.4525 m s^{-1} , and the relative velocity is -1.7475 m s^{-1} . The belt has advanced 0.87375 m farther than the package.

Second interval. Put $s = t - t_1$. The belt velocity remains 4.20 m s^{-1} while the package still accelerates rightward, so

$$v_{\text{rel}} = -1.7475 + a_k s.$$

Its first zero occurs at $s_c = 1.7475/2.4525 = 0.712538\text{ s}$, or $t_c = 1.71254\text{ s}$. The sign remains negative before that event, so the assumed kinetic-friction direction remains correct throughout the integration.

Event test. To move with the now constant-speed belt, the package requires zero acceleration and hence $f_{s,\text{req}} = 0$. The static inequality is satisfied. The package re-sticks, its speed remains 4.20 m s^{-1} , and the friction force becomes zero. Continuing $+\mu_k mg$ after the event would accelerate it past the belt speed and violate the assumed slip direction.

Lag. Define $\ell = x_b - x_p$. During the first interval $\ell_1 = (4.20 - a_k)t_1^2/2$. During the second, integrate $\dot{\ell} = 1.7475 - a_k s$:

$$\ell_c = 0.87375 + 1.7475s_c - \frac{1}{2}a_k s_c^2 = 1.49633 \text{ m}.$$

Thus the package remains 1.50 m behind its corresponding initial point on the belt after re-sticking. Equal final velocities do not restore equal positions. The area between the velocity curves in Fig. 6.4 is precisely this lag, providing an independent graphical check.

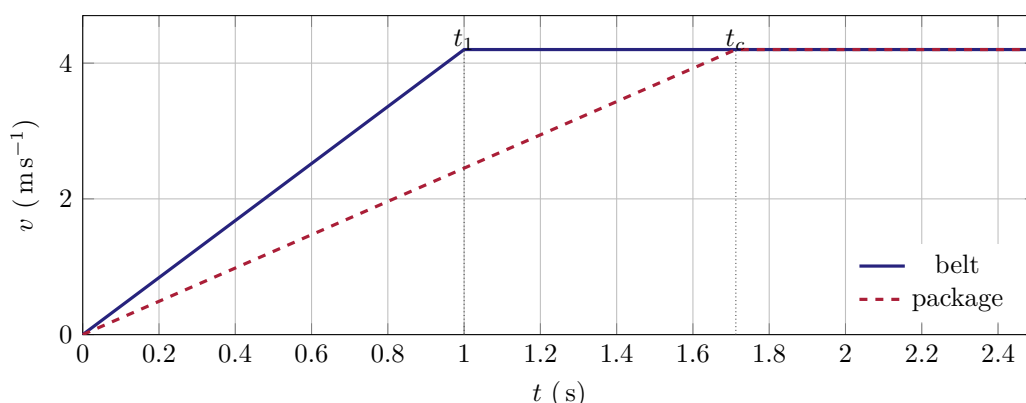


Figure 6.4: Analytical belt and package velocity histories for Example 6.2. The package slides until $t_c = 1.71254 \text{ s}$ and then moves at constant speed with zero friction. The area between the curves equals the final positional lag of 1.49633 m.

Guided checks

Diagnostic X06-01: demand, capacity and slip. A resting crate on a level floor with no horizontal force has zero friction: Newton's horizontal equation demands zero, regardless of the available capacity. A package moving right on a faster rightward belt receives rightward kinetic friction because it slips left relative to that belt. A larger normal load increases $\mu_s F_N$ in the stipulated model, but does not by itself prescribe the actual static-friction force. Each answer follows a distinct part of the model rather than a memorized arrow rule.

6.1.3 An angled force changes the normal load

An applied force can alter the friction capacity even when its main purpose is to move the body horizontally. Consider a level floor, a force magnitude P , and an angle α from horizontal. If a cable pulls upward, vertical equilibrium gives $F_N = mg - P \sin \alpha$. If a handle pushes downward, it gives $F_N = mg + P \sin \alpha$. The corresponding horizontal force demand is $P \cos \alpha$ in both cases. Consequently an upward pull reduces the capacity as it increases the demand, whereas a downward push increases both.

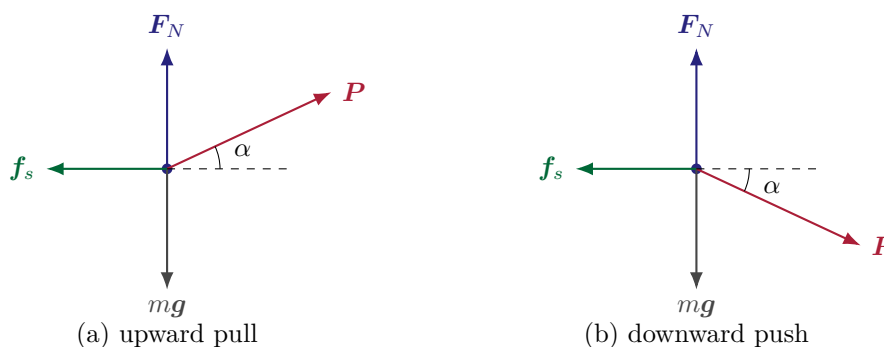


Figure 6.5: Separate body-only force diagrams for a crate that is just about to slide right. The applied force has the same magnitude and horizontal projection in the two cases, but its vertical projection changes sign. Normal-force magnitudes must be calculated; the arrows are not to scale.

Example 6.3: Pulling or pushing the same crate

A 15.0 kg crate rests on a level floor with $\mu_s = 0.400$. Find the force needed to initiate rightward sliding when the force is directed 25.0° above horizontal, and when it is directed 25.0° below horizontal. Ignore tipping by treating the crate as a particle.

Threshold equations. For the pull, $P \cos \alpha = \mu_s (mg - P \sin \alpha)$. For the push, $P \cos \alpha = \mu_s (mg + P \sin \alpha)$. Solving separately gives

$$P_{\text{pull}} = \frac{\mu_s mg}{\cos \alpha + \mu_s \sin \alpha}, \quad P_{\text{push}} = \frac{\mu_s mg}{\cos \alpha - \mu_s \sin \alpha}.$$

Substitution gives $P_{\text{pull}} = 54.7 \text{ N}$ and $P_{\text{push}} = 79.8 \text{ N}$. A horizontal force would require $\mu_s mg = 58.9 \text{ N}$. Thus the upward pull modestly reduces the starting force here, whereas the downward push increases it.

Admissibility. The pull threshold has $F_N = 124.0 \text{ N} > 0$, so contact remains. The push denominator is positive for these parameters. If $\cos \alpha - \mu_s \sin \alpha \leq 0$, this particle model predicts that increasing such a push cannot initiate sliding from rest: its added normal load strengthens the capacity at least as rapidly as its horizontal component raises the demand. Real tipping or deformation may intervene, so this mathematical limit is not a universal instruction for moving a crate.

A below-threshold push. For the same crate pushed downward at $P = 40.0 \text{ N}$, $F_N = mg + P \sin 25^\circ = 164.055 \text{ N}$. The actual friction demand is $P \cos 25^\circ = 36.2523 \text{ N}$, whereas the capacity is 65.6219 N . The crate stays at rest with 36.3 N of leftward friction. Replacing it by the maximum would invent a leftward resultant instead of the required equilibrium.

Check. With $\alpha = 0$, both expressions reduce to $\mu_s mg$. This limit and the opposite signs of the vertical contributions verify the algebra and its physical interpretation.

6.1.4 Stopping motion requires an event as well

For a block sliding right on a stationary horizontal floor, kinetic friction is leftward and $a = -\mu_k g$. While $v > 0$, constant-acceleration kinematics gives $v = v_0 - \mu_k g t$. When this expression reaches zero, the kinetic prescription has completed its valid interval. If no other horizontal force acts, static equilibrium is then admissible with zero friction and the block stays at rest. Extending the linear velocity through zero would invent leftward motion while leaving the friction directed leftward; it would turn resistance into an accelerating force in the direction of slip.

Example 6.4: A sliding case comes to rest

A case initially slides at 6.00 m s^{-1} on a stationary level floor. With $\mu_k = 0.250$, determine its stopping time and distance. Use Newton's law and kinematics, without assuming an energy method from the next chapter.

During sliding $a = -2.4525 \text{ m s}^{-2}$. The first zero of v occurs at

$$t_s = \frac{v_0}{\mu_k g} = 2.44648 \text{ s}.$$

Integration of the velocity over that interval yields

$$d = v_0 t_s - \frac{1}{2} \mu_k g t_s^2 = \frac{v_0^2}{2 \mu_k g} = 7.33945 \text{ m}.$$

The answers are 2.45 s and 7.34 m. Afterward $v = 0$ and $f_s = 0$. The area under the triangular positive-velocity part of Fig. 6.6 is $v_0 t_s / 2$, reproducing the distance independently. Doubling the initial speed would double the stopping time but quadruple the distance in this constant-coefficient model.

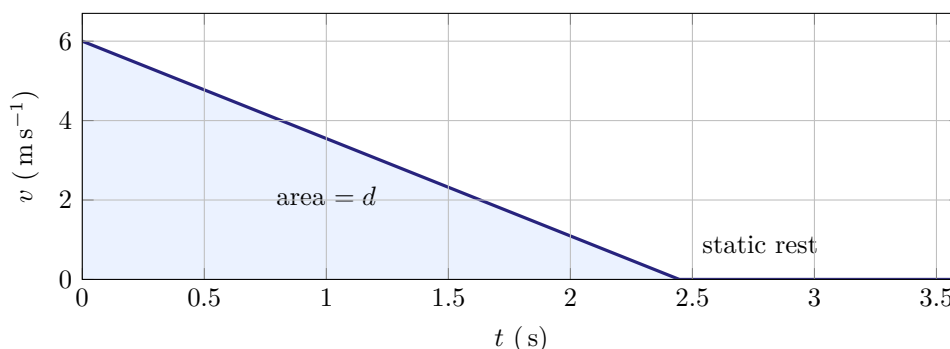


Figure 6.6: The sliding solution ends at its first zero of velocity. The admissible continuation is rest. The shaded area is the positive displacement accumulated before stopping, not an area continued into an invented reverse-motion interval.

6.2 Inclined and connected contact systems

6.2.1 Resolve geometry before choosing a friction value

Choose $+t$ up an incline of angle θ and $+n$ outward normal to it. These are orthogonal axes, so a force can be resolved along each independently. Gravity is vertically downward. Its tangential component is $-mg \sin \theta$ and its normal component is $-mg \cos \theta$. Figure 6.7 shows this projection as a mathematical construction, separate from the actual-force diagram. The two components are representations of one force, not two additional interactions.

On a bare, fixed, straight incline a block that remains in contact has no normal acceleration. If its only forces are gravity, normal contact and tangential friction, Newton's normal equation gives $F_N = mg \cos \theta$. This familiar expression is conditional. A cable lifting away from the plane, a push into it, a curved path, or acceleration of the plane can alter the normal equation. A correct formula with the wrong assumptions remains a wrong model of the situation.

For a block at rest on the bare incline, the tangential equation requires $f_s = mg \sin \theta$. Testing the static bound gives

$$mg \sin \theta \leq \mu_s mg \cos \theta, \quad \tan \theta \leq \mu_s, \quad 0 \leq \theta < 90^\circ. \quad (6.4)$$

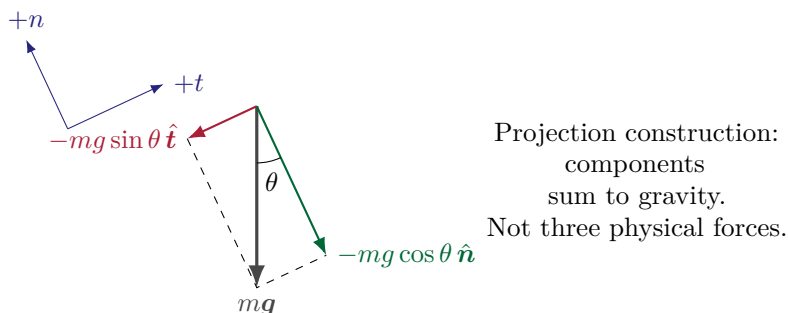


Figure 6.7: Gravity resolved in incline-aligned axes, drawn at $\theta = 25^\circ$. The dashed parallelogram verifies the vector sum. This is a component construction rather than a free-body diagram. The rotated-angle construction method is adapted from Izaak Neutelings, *Friction*, TikZ.net, CC BY-SA 4.0.

The critical angle is $\theta_c = \arctan \mu_s$. Below it the actual friction increases with angle while the capacity decreases with angle. The point at which they meet is the threshold, not a sudden change in gravity. Above it a previously resting block begins to slip downhill; once sliding is established, the downhill acceleration in the constant-coefficient model is $g(\sin \theta - \mu_k \cos \theta)$.

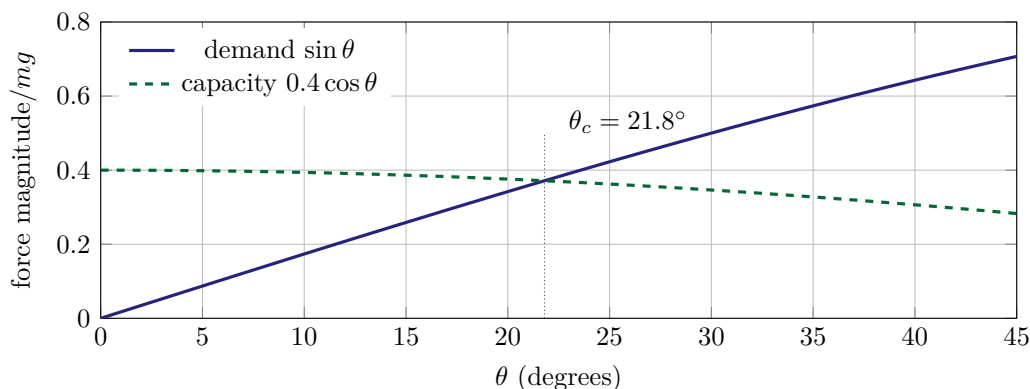


Figure 6.8: Competing force magnitudes on a bare incline with $\mu_s = 0.400$. The demand and capacity cross at $\arctan(0.400) = 21.8^\circ$. To the right, the plotted demand is a failed equilibrium requirement, rather than the actual kinetic friction.

Example 6.5: A case at rest below the critical angle

A 20.0 kg case rests on a 18.0° incline, with $\mu_s = 0.400$. Determine whether equilibrium is possible, and find the actual friction. Explain why using the maximum friction would give the wrong acceleration.

The normal force is $F_N = mg \cos 18^\circ = 186.597 \text{ N}$. Equilibrium demands $f_s = mg \sin 18^\circ = 60.6291 \text{ N}$ uphill. The available maximum is 74.6389 N, so the case can remain at rest and its actual friction is 60.6 N. Equivalently, $\tan 18^\circ = 0.324920 < 0.400$.

Inserting the maximum into the tangential equation would give a spurious uphill resultant of 14.0 N. No physical cause forces static friction to take its maximum. The demand solution gives zero resultant, exactly as required by the assumed rest. The mass affects both forces but not the critical angle in this model.

6.2.2 A cable can reverse the friction direction

Now a cable exerts tension $T \geq 0$ uphill at angle β above the plane. Its components are $T \cos \beta$ along $+t$ and $T \sin \beta$ along $+n$. For static equilibrium,

$$T \cos \beta + f_s - mg \sin \theta = 0, \quad (6.5)$$

$$F_N + T \sin \beta - mg \cos \theta = 0. \quad (6.6)$$

Thus

$$f_{s,\text{req}} = mg \sin \theta - T \cos \beta, \quad F_N = mg \cos \theta - T \sin \beta. \quad (6.7)$$

Increasing T simultaneously lowers the demand for uphill friction and reduces its capacity by lifting partly away from the plane. Eventually the demand changes sign: the cable tends to pull the body uphill, and friction must act downhill. There is no rule that friction on an incline always points uphill.

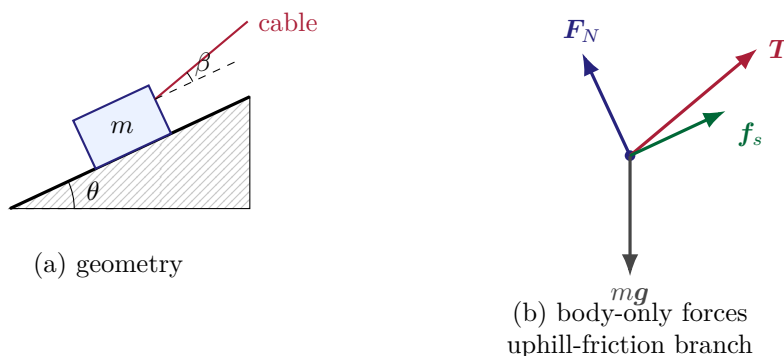


Figure 6.9: A cable on an incline with geometry generated from $\theta = 25^\circ$ and $\beta = 15^\circ$. The separate free-body diagram displays the uphill-friction branch; a sufficiently large tension requires the opposite friction direction. The normal is perpendicular to the plane. Rotated-angle construction adapted from Izaak Neutelings, *Friction*, TikZ.net, CC BY-SA 4.0.

Example 6.6: The admissible tension interval on an incline

An 18.0 kg equipment case rests on a 25.0° incline. A cable pulls uphill at 15.0° above the plane, and $\mu_s = 0.350$. Find the entire tension range consistent with static contact, the tension at which friction vanishes, and the friction at $T = 70.0$ N.

Demand and admissibility. Use Eq. (6.7), require $F_N \geq 0$, and impose

$$-\mu_s F_N \leq mg \sin \theta - T \cos \beta \leq \mu_s F_N.$$

At the lower tension threshold the tendency is downhill and $f_s = +\mu_s F_N$. Solving gives

$$T_{\min} = mg \frac{\sin \theta - \mu_s \cos \theta}{\cos \beta - \mu_s \sin \beta} = 21.2642 \text{ N}.$$

At the upper threshold the tendency is uphill and $f_s = -\mu_s F_N$, giving

$$T_{\max} = mg \frac{\sin \theta + \mu_s \cos \theta}{\cos \beta + \mu_s \sin \beta} = 123.651 \text{ N}.$$

Both denominators are positive for the stipulated angles and coefficient. The algebra should not be transplanted to arbitrary angles without checking their signs. At $T_{\max}$ the normal force is $128.0 \text{ N} > 0$. The formal zero-normal threshold is $mg \cos \theta / \sin \beta = 618.3 \text{ N}$, far above the upper static limit. Thus loss of static equilibrium occurs before contact separation here, and the answer is

$$21.3 \text{ N} \leq T \leq 123.7 \text{ N}.$$

Friction reversal. The demand vanishes at $T_0 = mg \sin \theta / \cos \beta = 77.2585 \text{ N}$. Below T_0 friction acts uphill; above it friction acts downhill while equilibrium can still persist. At $T = 70.0 \text{ N}$,

$$F_N = 141.918 \text{ N}, \quad f_s = +7.01112 \text{ N}, \quad \mu_s F_N = 49.6715 \text{ N}.$$

The actual friction is therefore 7.01 N uphill. Figure 6.10 makes the two thresholds and the interior reversal distinct.

Checks. At $T = 0$, $\tan 25^\circ = 0.4663 > 0.350$: a cable is indeed needed to prevent downhill slip. At each computed endpoint the demand equals the appropriate signed bound. Between them, increasing tension changes the required friction continuously, not by forcing it to remain at its maximum.

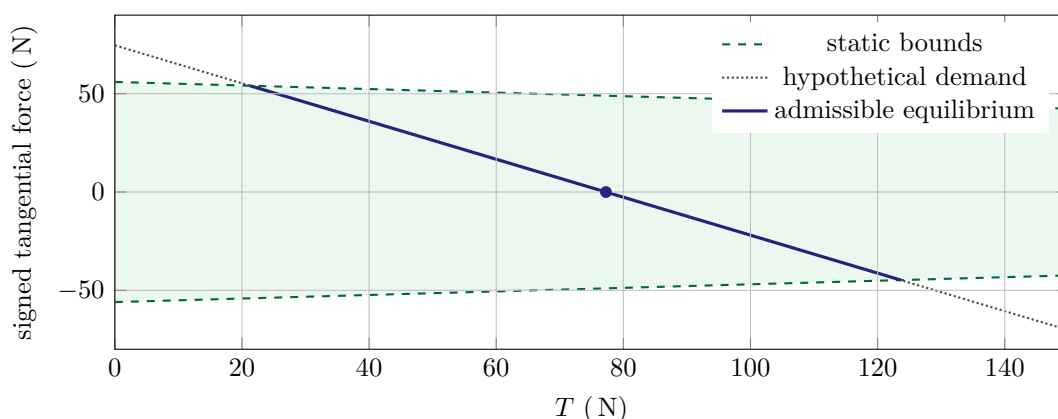


Figure 6.10: The cable example's decreasing friction demand and narrowing static-force band. The solid segment lies inside the band from $T_{\min}$ to $T_{\max}$. Its zero marks a friction reversal within equilibrium, rather than the onset of slip. Positive friction is uphill.

Guided checks

Diagnostic X06-02: the zero-cable limit. Setting $T = 0$ in the demand and normal equations gives $f_s = mg \sin \theta$ and $F_N = mg \cos \theta$. Their inequality reduces to $\tan \theta \leq \mu_s$, as required. A candidate tension with $T < 0$ would require a cable to push; one with $F_N < 0$ would require the plane to pull the body toward it. Both are outside this model, regardless of whether their algebra satisfies a remaining equation.

6.2.3 Connected bodies require a contact test on each body

A common acceleration is a constraint, not evidence that every interface can maintain it. Consider a small block of mass m on a lower block of mass M , with a smooth floor beneath the lower block. A horizontal force P pulls the lower block rightward. If the upper block sticks, its only horizontal force is interface friction. Hence $f_s = ma$. For the two-block system, that internal friction cancels and $P = (M + m)a$. Combining these equations and the upper block's normal balance gives

$$a = \frac{P}{M + m}, \quad f_{s,\text{req}} = \frac{mP}{M + m}, \quad |P| \leq (M + m)\mu_s g. \quad (6.8)$$

The combined-system equation efficiently gives acceleration, but it hides the interface force. We need the upper-block equation to test the constraint. The floor normal is $(M + m)g$, whereas the normal at the upper interface is mg . Using the floor normal in the upper contact's bound would overestimate its capacity.

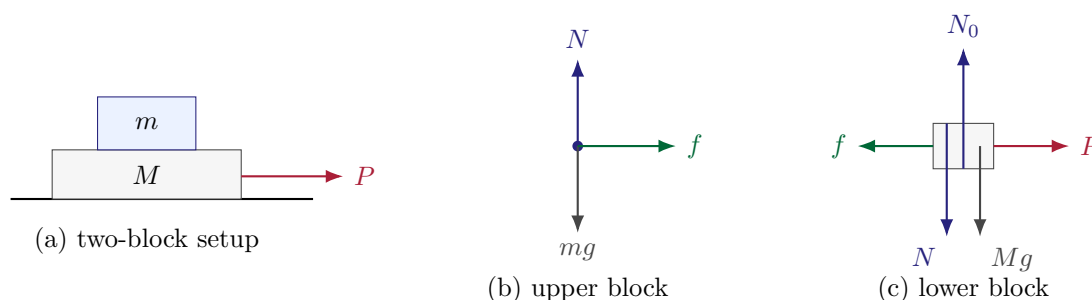


Figure 6.11: The upper and lower bodies require separate contact equations. Their interface forces are opposite third-law partners; the floor force N_0 is a different interaction. The isolated lower-block diagram offsets the origins of its parallel vertical forces to distinguish their interactions; their directions remain vertical. Force lengths are schematic.

Example 6.7: A two-block stack: common acceleration or relative slip?

A 4.00 kg upper block rests on a 10.0 kg lower block. The floor is frictionless, and the block–block coefficients are $\mu_s = 0.300$ and $\mu_k = 0.220$. Initially both are at rest. A force pulls the lower block rightward. Find both accelerations for $P = 40.0$ N and $P = 50.0$ N.

For the smaller force, the candidate common acceleration is $a = 40.0/14.0 = 2.85714 \text{ m s}^{-2}$. The required interface friction is $ma = 11.4286$ N, and its capacity is $\mu_s mg = 11.772$ N. The test passes, so both accelerations are 2.86 m s^{-2} . The largest force permitting this motion is $(M + m)\mu_s g = 41.202$ N.

For 50.0 N, the hypothetical common acceleration would demand 14.2857 N at the interface. The constraint fails. The lower block begins moving ahead, so the upper block slips left relative to it. The kinetic force on the upper block is rightward, of magnitude $f_k = \mu_k mg = 8.6328$ N. Newton's law on each body gives

$$a_m = \frac{f_k}{m} = 2.1582 \text{ m s}^{-2}, \quad a_M = \frac{P - f_k}{M} = 4.13672 \text{ m s}^{-2}.$$

The lower acceleration exceeds the upper acceleration, verifying the emerging slip direction. Adding the two independent equations gives $ma_m + Ma_M = 50.0$ N, which checks the combined-system force balance even though there is no longer one common acceleration.

If the force is applied to the *upper* block instead, the common acceleration is still $P/(M + m)$ but the lower body needs friction Ma . Its bound involves the same interface normal mg , so the permitted force is now $|P| \leq \mu_s mg(M + m)/M$. The point where the external force enters the system matters. The combined resultant alone cannot identify which contact transmits it or which contact fails first.

6.3 Measuring and testing a friction model

An introductory friction experiment should distinguish the observed threshold from the coefficient inferred by a model. Slowly tilting an incline supplies a convenient static test. Record the angle at the onset of sustained sliding rather than the first small vibration. Under the bare-incline assumptions, $\mu_s = \tan \theta_c$. This inference presumes that the block does not tip, that the plane is fixed, and that the onset represents a reproducible limiting state. A block with a large height relative to its base can tip first; in that case the observed angle is not a sliding-friction measurement.

Repeated trials reveal variation in the threshold, while an instrument calibration error can shift all trials together. Reducing the scatter through repetition does not remove a common zero offset. For

an uncertainty u_θ small enough for a local linear approximation, differentiation gives

$$u_{\mu_s} \simeq \left| \frac{d(\tan \theta)}{d\theta} \right| u_\theta = \sec^2 \theta u_\theta, \quad (6.9)$$

where the angle uncertainty must be in *radians*. If it is entered in degrees, convert by $\pi/180$ before multiplication. The growing derivative at large angles means the same angular uncertainty can produce very different coefficient uncertainties.

Example 6.8: Synthetic threshold trials and a kinetic-force calibration

Five synthetic critical-angle readings are 18.0° , 18.4° , 18.2° , 18.6° and 18.3° . An independently characterized angle-zero uncertainty is 0.20° , interpreted as a standard uncertainty. Infer μ_s and its uncertainty assuming independent random and zero-offset contributions. A separate synthetic kinetic test records force magnitudes 5.1, 10.1, 15.2 and 20.1 N at normal loads 20, 40, 60 and 80 N. Infer a linear slope and discuss the intercept.

Static analysis. The mean angle is 18.3° . The sample standard deviation,

$$s_\theta = \sqrt{\frac{\sum_i (\theta_i - \bar{\theta})^2}{5 - 1}},$$

is 0.223607° . The standard uncertainty of the mean from independent trial scatter is $s_\theta/\sqrt{5} = 0.100^\circ$. Combining it with the common zero contribution gives

$$u_\theta = \sqrt{(0.100^\circ)^2 + (0.20^\circ)^2} = 0.223607^\circ = 0.00390267 \text{ rad}.$$

Equation (6.9) then gives $\mu_s = \tan 18.3^\circ = 0.330718$ and $u_{\mu_s} = 0.00432953$. Report the model-conditioned result as $\mu_s = 0.331 \pm 0.004$ with a stated *standard* uncertainty. It is not an automatic confidence interval and does not quantify all model imperfections.

Kinetic calibration. Fit $f_k = f_0 + \mu_k F_N$ with an intercept rather than enforcing a line through zero. For equal-weight synthetic observations the slope is

$$\mu_k = \frac{\sum_i (F_{N,i} - \bar{F}_N)(f_i - \bar{f})}{\sum_i (F_{N,i} - \bar{F}_N)^2} = 0.2505, \quad f_0 = \bar{f} - \mu_k \bar{F}_N = 0.100 \text{ N}.$$

The observed-minus-fitted residuals are -0.010 , -0.020 , $+0.070$ and -0.040 N. They sum to zero as an intercept fit requires. Their small sizes support, but do not prove, the adequacy of the straight-line model over this load range.

Interpretation. The ideal law predicts a zero intercept; a fitted offset can reflect sensor tare, parasitic forces or inadequacy of the contact law. These four synthetic readings contain no instrument uncertainty specification, so they cannot establish that the 0.100 N intercept differs significantly from zero. We have inferred a useful slope and identified a question for calibration, without inventing a precision claim.

For a single low-load incline reading reported as $\theta_c = (21.0 \pm 0.5)^\circ$, the inference is $\mu_s = \tan 21.0^\circ = 0.383864$. If the stated 0.5° is interpreted as a standard angular uncertainty, first-order propagation gives $u_{\mu_s} = \sec^2(21.0^\circ)(0.5\pi/180) = 0.0100125$. Thus $\mu_s = 0.384 \pm 0.010$ under the stated model and uncertainty interpretation. The same plus-or-minus symbol would not determine this statistical meaning without an explicit specification. This proposed measurement is distinct from the repeated synthetic trial example above.

In a proposed laboratory investigation, measure or estimate the normal load independently, move the body at approximately constant relative speed, and determine the tangential force after initial transients. Use several loads, inspect residuals, and repeat the test at more than one speed. A persistent curve in the residuals suggests that a straight line is inadequate; a change with speed

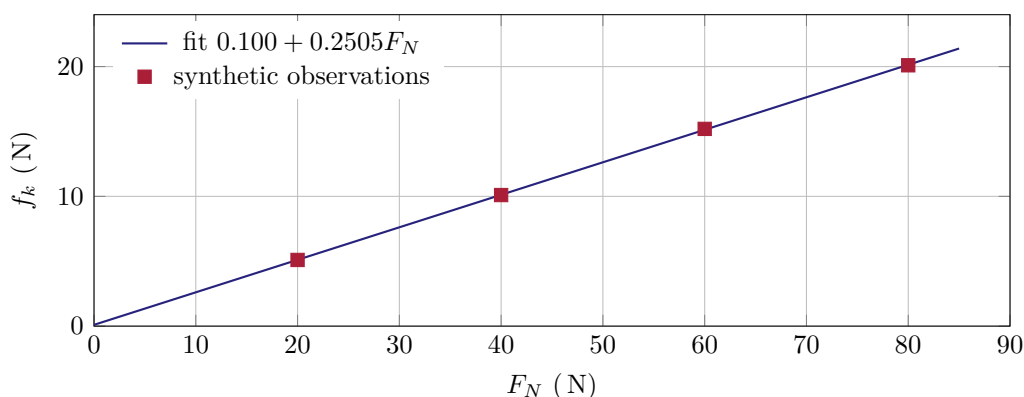


Figure 6.12: Synthetic kinetic-friction calibration data, explicitly distinct from measurements performed for this book. The slope estimates the stipulated model's coefficient over the tested load range. The fitted offset raises a sensor/model question; without uncertainty information its statistical significance is undetermined.

questions the constant- μ_k assumption. A slope extracted from only one load cannot distinguish a proportional contact law from an additive force offset. These checks connect the force model to evidence while keeping laboratory safety, apparatus limits and full tribological design outside this chapter's scope.

6.4 Circular-motion dynamics

A straight-path acceleration changes the speed, the direction, or both. Circular motion makes the distinction particularly clear: a body can travel with constant speed while its velocity continuously changes direction. Newton's second law therefore requires a nonzero resultant even when a speedometer reading is steady. The useful question is which actual interactions provide that resultant. A string supplies tension; a horizontal road supplies tyre friction; a banked surface supplies a tilted normal force; gravity and a guide can act together. The expression mv^2/R identifies the required inward *resultant*, not a further force to add to the free-body diagram.

We first reconnect the acceleration with its geometry. This short derivation also prepares the nonuniform cases later in the section. Let a particle move anticlockwise on a circle of radius R , centred at the origin. Its angular position $\phi(t)$ is measured in radians, so that

$$\mathbf{r} = R(\cos \phi \hat{\mathbf{x}} + \sin \phi \hat{\mathbf{y}}), \quad (6.10)$$

$$\hat{\mathbf{t}} = -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}}, \quad \hat{\mathbf{n}} = -\cos \phi \hat{\mathbf{x}} - \sin \phi \hat{\mathbf{y}}. \quad (6.11)$$

Here $\hat{\mathbf{t}}$ points along the direction of travel and $\hat{\mathbf{n}}$ points inward. For motion in this chosen direction, $v = R\dot{\phi}$ and $\mathbf{v} = v\hat{\mathbf{t}}$. Differentiating the tangent gives $d\hat{\mathbf{t}}/dt = \dot{\phi}\hat{\mathbf{n}}$; applying the product rule to velocity then gives

$$\mathbf{a} = \dot{v}\hat{\mathbf{t}} + \frac{v^2}{R}\hat{\mathbf{n}}. \quad (6.12)$$

The first term changes speed; the second rotates the velocity. If travel is clockwise, define the tangent along that travel instead: the inward term remains v^2/R . No negative speed is needed to describe the reversal of direction.

Figure 6.13 separates position geometry from acceleration. In the left panel, the tangent is perpendicular to the radius. In the right panel, the two acceleration components are resolved at one instant; their drawn lengths are schematic. They must not be read as additional forces.

With inward positive, the normal equation is consequently

$$\sum F_n = m \frac{v^2}{R}. \quad (6.13)$$

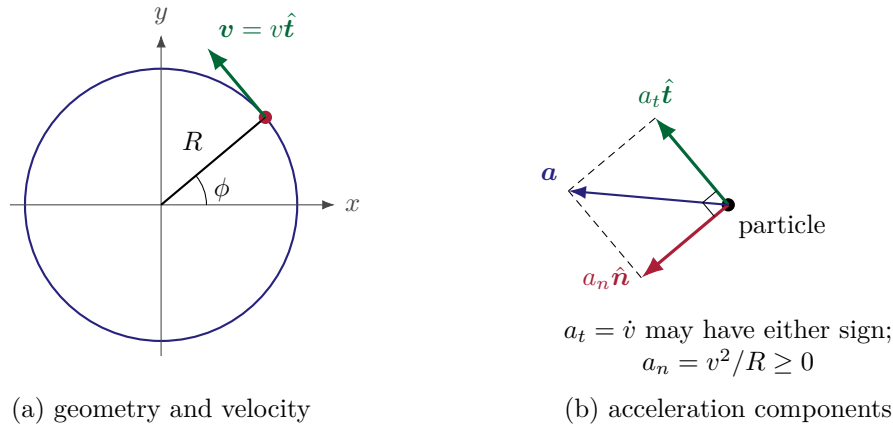


Figure 6.13: Circular geometry and the acceleration decomposition. Both panels use equal spatial units on the two axes. The acceleration sketch represents increasing speed; for decreasing speed the tangential arrow reverses while the normal component remains inward. Polar point placement and angle offsets follow the construction method studied in Neutelings' *Uniform circular motion* example; the present diagram and physical model are independently constructed.

The notation $\sum F_n$ means the signed inward projections of *all* external forces. An outward actual force contributes negatively. At constant speed the separate tangential equation is $\sum F_t = 0$; it does not imply that the complete resultant vanishes. Since v^2/R has units of acceleration, doubling speed at fixed radius requires four times the normal resultant. Halving the radius at fixed speed requires twice as much.

6.4.1 A flat turn: friction supplies the horizontal resultant

Consider a particle model of a vehicle on a level circular path. Assume negligible vertical acceleration, no additional horizontal force, and enough contact to apply a single effective static-friction bound. The vertical equation gives $N = mg$. Horizontal static friction points inward and must have magnitude $f_s = mv^2/R$. It is a constraint force: it takes the required value until the inequality $f_s \leq \mu_s N$ fails. Thus

$$v \leq \sqrt{\mu_s g R}. \quad (6.14)$$

The point-particle approximation leaves vehicle attitude, load sharing between tyres and wheel torques unresolved. Its value is the transparent force balance, rather than a complete vehicle-performance prediction.

In Figure 6.14, the plan view establishes inward direction, while the body-only diagram separately shows the actual forces. Weight and the normal force balance vertically, but they are not a third-law pair: both act on the chosen vehicle. The matching road and Earth reaction forces act on other bodies.

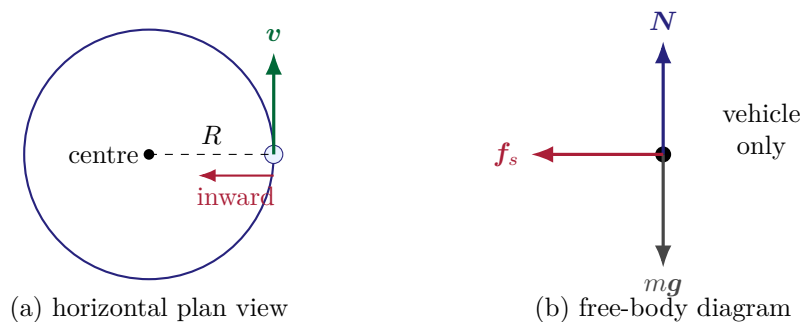


Figure 6.14: A flat circular turn requires inward static friction. The velocity appears in the geometry panel, and only real external forces appear in the free-body diagram. Arrow lengths are schematic.

Example 6.9: A flat-turn limit and a friction demand

A vehicle of model mass 900 kg follows a level turn of radius 40.0 m. The stipulated coefficient is $\mu_s = 0.650$. Find the largest constant speed admitted by the model, then test a speed of 12.0 m s^{-1} .

Model and strategy. Figure 6.14 applies, so $N = mg = 8829 \text{ N}$. Compare the required friction with its capacity, rather than assuming friction is always at its maximum.

$$v_{\max} = \sqrt{(0.650)(9.81)(40.0)} = 16.0 \text{ m s}^{-1}.$$

At the proposed speed,

$$f_s = \frac{(900)(12.0)^2}{40.0} = 3240 \text{ N}, \quad \mu_s N = 5739 \text{ N}.$$

The demand is only 0.565 of capacity, so no slip is compatible with the stated assumptions. The actual static friction is 3240 N, not 5739 N.

Interpretation and check. Mass cancels from the limiting-speed formula because both the required resultant and the available bound scale with mass. At $\mu_s = 0$, this two-force vertical model cannot maintain any nonzero-speed turn; at very large R , the directional acceleration becomes small. These limits agree with the geometry.

6.4.2 Banking supplies part of the turning force

Banking tilts the contact normal toward the centre of the turn. Let the bank angle be β above the horizontal and R the horizontal radius of the circular trajectory. The surface rises toward the outside. Use horizontal inward coordinate x and upward coordinate z . The upward normal has components $(N \sin \beta, N \cos \beta)$. Its magnitude in the frictionless case is $N = mg / \cos \beta > mg$ for a nonzero bank angle: the vertical component, rather than the full normal force, balances weight. Figure 6.15 gives the geometry and an isolated-body diagram separately.

For a frictionless bank, weight and the tilted normal are the only forces. Vertical acceleration is zero, so $N \cos \beta = mg$. The horizontal equation is $N \sin \beta = mv^2/R$. Dividing eliminates the unknown normal force:

$$\tan \beta = \frac{v_0^2}{Rg}, \quad v_0 = \sqrt{Rg \tan \beta}. \quad (6.15)$$

This is a single ideal speed for a specified bank angle. It is not a maximum speed when friction is absent: a different speed fails the two equations simultaneously. At lower speed the normal's inward component is too large; at higher speed it is too small. Friction can correct the mismatch in either direction.

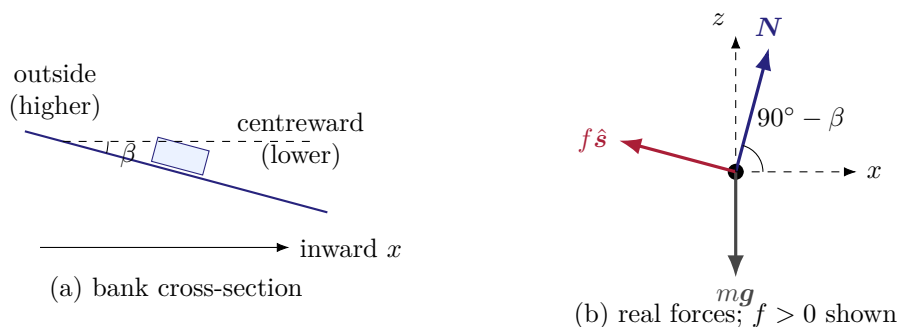


Figure 6.15: Bank geometry and actual forces, drawn at $\beta = 15^\circ$ with equal spatial scales. Positive signed friction f points up the bank, outward and upward. If the calculation gives $f < 0$, its actual direction is down the bank. The axis arrows are auxiliary directions and do not represent interactions.

6.4.3 The full static-friction speed interval on a bank

Define $\hat{s}$ up the bank and let the scalar f be positive in that direction. This signed convention avoids guessing friction direction. With $a = v^2/R$, Newton's law becomes

$$N \sin \beta - f \cos \beta = ma, \quad (6.16)$$

$$N \cos \beta + f \sin \beta = mg. \quad (6.17)$$

Multiplying and adding to eliminate one unknown at a time gives

$$N = m(g \cos \beta + a \sin \beta), \quad f = m(g \sin \beta - a \cos \beta). \quad (6.18)$$

For $0 \leq \beta < 90^\circ$ and $a \geq 0$, N is positive. Contact is therefore admissible in this ideal cross-section. Static friction must also satisfy $|f| \leq \mu_s N$. At low speed $f > 0$: friction acts uphill, preventing the tendency to slide inward and down the bank. At high speed $f < 0$: friction acts downhill, opposing the outward/uphill slip tendency. At v_0 it vanishes.

Substitution into the two-sided inequality is more reliable than inserting $f = \mu_s N$ with a guessed sign. The upper condition $f \leq \mu_s N$ implies

$$a(\cos \beta + \mu_s \sin \beta) \geq g(\sin \beta - \mu_s \cos \beta).$$

The coefficient of a is positive. If the right side is negative, even $a = 0$ satisfies this condition. The lower condition $f \geq -\mu_s N$ implies

$$a(\cos \beta - \mu_s \sin \beta) \leq g(\sin \beta + \mu_s \cos \beta).$$

When its left coefficient is positive, the speed interval is

$$v_{\min}^2 = Rg \max \left[0, \frac{\sin \beta - \mu_s \cos \beta}{\cos \beta + \mu_s \sin \beta} \right], \quad (6.19)$$

$$v_{\max}^2 = Rg \frac{\sin \beta + \mu_s \cos \beta}{\cos \beta - \mu_s \sin \beta}, \quad \cos \beta > \mu_s \sin \beta. \quad (6.20)$$

The square roots give speeds. If $\cos \beta - \mu_s \sin \beta \leq 0$, the lower-friction inequality supplies *no finite upper speed* in this particular model: its left side is nonpositive and its right side positive. That mathematical conclusion reflects the increasing normal load with speed and the unrestricted Coulomb model; it does not establish unlimited physical vehicle capability. If $\mu_s \geq \tan \beta$, the minimum speed is zero because static friction can support a parked particle on the bank. For $\mu_s = 0$ the interval collapses to v_0 , and for $\beta = 0$ it becomes $0 \leq v \leq \sqrt{\mu_s g R}$. Both are useful independent checks.

Figure 6.16 displays the signed demand alongside both friction limits for one numerical bank. The intersections define the endpoints. The graph makes the reversal of friction at the ideal speed visible without adding a spurious force.

Example 6.10: A banked turn with two limiting speeds

For the bank in Figure 6.16, find the ideal speed, full no-slip speed interval and the signed friction and normal force at $v = 10.0 \text{ m s}^{-1}$ for a 1000 kg model vehicle.

Strategy. Resolve the two real force components first, then test both friction endpoints. The ideal speed is

$$v_0 = \sqrt{(60.0)(9.81) \tan 15.0^\circ} = 12.6 \text{ m s}^{-1}.$$

Since $\cos 15^\circ - 0.200 \sin 15^\circ > 0$, both bounds are finite. Equation (6.20) gives

$$v_{\min} = 6.16 \text{ m s}^{-1}, \quad v_{\max} = 17.1 \text{ m s}^{-1}.$$

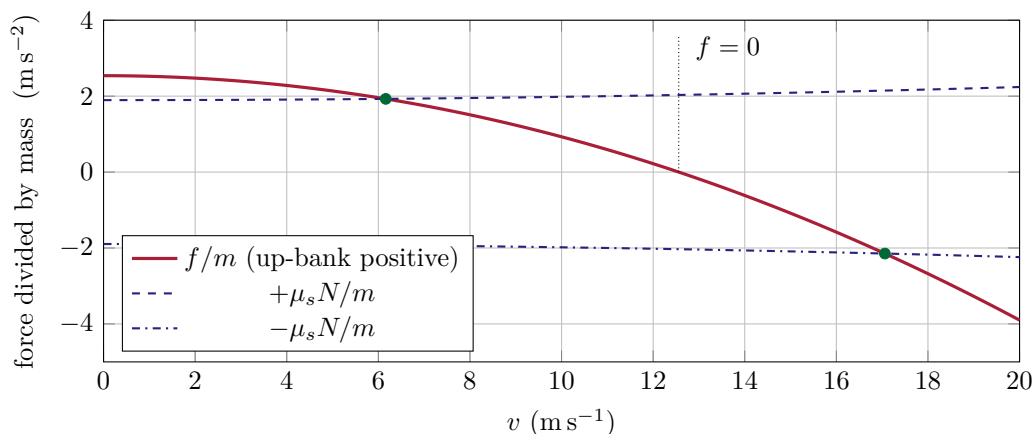


Figure 6.16: Calculated signed friction demand for $R = 60.0$ m, $\beta = 15.0^\circ$ and $\mu_s = 0.200$. Static contact is possible where the red solid curve lies between the two blue limits. The lower-speed endpoint requires uphill friction; the higher-speed endpoint requires downhill friction. Values are model calculations, not road measurements.

At the proposed speed, $a = 10.0^2/60.0 = 1.6667 \text{ m s}^{-2}$. Substituting in the signed equations yields

$$\begin{aligned} N &= 1000(9.81 \cos 15^\circ + 1.6667 \sin 15^\circ) = 9.91 \times 10^3 \text{ N}, \\ f &= 1000(9.81 \sin 15^\circ - 1.6667 \cos 15^\circ) = 929 \text{ N}. \end{aligned}$$

Friction points *up* the bank and $|f| < \mu_s N = 1.98 \times 10^3 \text{ N}$, so the proposal is compatible with static contact.

Check. Substitution into the component equations gives $N \cos \beta + f \sin \beta = 9810 \text{ N}$ and $N \sin \beta - f \cos \beta = 1666.7 \text{ N}$. These recover the required vertical and inward resultants separately. At the low endpoint $f = +\mu_s N$; at the high endpoint $f = -\mu_s N$, confirming the opposite impending-slip directions.

6.4.4 A conical pendulum: the radius is set by the geometry

A conical pendulum is a bob moving in a horizontal circle while its taut string maintains a steady angle θ from the vertical. Suppose the string has length L , the pivot is fixed, air resistance is negligible and the bob is treated as a particle. The circle radius is $r = L \sin \theta$, not L . The vertical component of tension balances weight; its horizontal component supplies the inward resultant. Figure 6.17 shows the instantaneous vertical cross-section and the free-body diagram.

$$T \cos \theta = mg, \quad T \sin \theta = m \frac{v^2}{r}, \quad (6.21)$$

$$T = \frac{mg}{\cos \theta}, \quad v^2 = rg \tan \theta = \frac{gL \sin^2 \theta}{\cos \theta}. \quad (6.22)$$

Writing the period as $\mathcal{T} = 2\pi r/v$ then gives

$$\mathcal{T} = 2\pi \sqrt{\frac{L \cos \theta}{g}}, \quad 0 < \theta < 90^\circ. \quad (6.23)$$

The tension is greater than the weight because only its vertical component balances gravity. At $\theta \rightarrow 0$, the speed and radius both vanish while their ratio has a finite limit. The corresponding period limit describes arbitrarily small conical motion; a bob exactly at rest has no actual circular

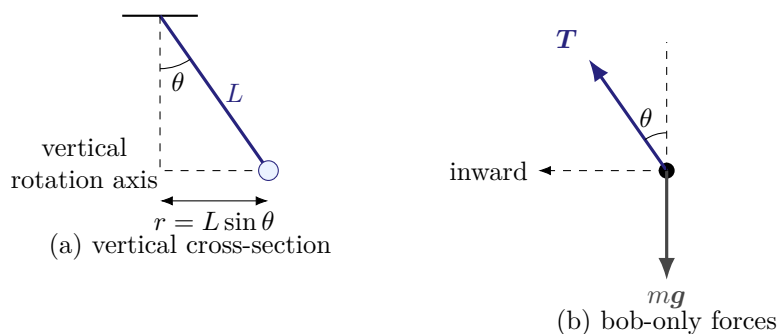


Figure 6.17: A conical pendulum at $\theta = 35^\circ$. The cross-section is drawn with equal scales and uses a string angle measured from the vertical. The circular path is perpendicular to this page; its radius is the horizontal distance to the axis. The body-only force diagram shows tension and weight, with no additional centripetal interaction.

revolution period. The tension and ideal speed diverge as $\theta \rightarrow 90^\circ$, exposing the limits of an indefinitely strong string model.

Example 6.11: Inferring the motion of a conical pendulum

A 0.200 kg bob on a 1.20 m string maintains $\theta = 35.0^\circ$. Determine radius, tension, speed and period under the model above.

The geometry gives $r = 1.20 \sin 35.0^\circ = 0.688$ m. Vertical force balance gives $T = (0.200)(9.81)/\cos 35.0^\circ = 2.40$ N. The radial equation gives

$$v = \sqrt{(0.68829)(9.81) \tan 35.0^\circ} = 2.17 \text{ m s}^{-1}, \quad \mathcal{T} = \frac{2\pi(0.68829)}{2.17437} = 1.99 \text{ s}.$$

The computed inward tension component is 1.37 N, agreeing with mv^2/r . Its vertical component is 1.962 N, agreeing with the weight. Mass affects tension but cancels from speed and period for the prescribed geometry. A period measurement and known L can therefore infer $\cos \theta = g\mathcal{T}^2/(4\pi^2 L)$; the inferred value must lie between zero and one before an angle is assigned.

6.4.5 Inside and outside contact require different radial signs

A surface can push only in its own normal direction. That direction need not be inward. At the top of an *inside* circular guide, the centre is below the body and the guide pushes inward/downward. Weight also points inward, so

$$N + mg = m \frac{v^2}{R}, \quad N = m \left(\frac{v^2}{R} - g \right). \quad (6.24)$$

At the top of an *outside* hill, gravity points inward/downward but the support pushes outward/upward. The equation is instead

$$mg - N = m \frac{v^2}{R}, \quad N = m \left(g - \frac{v^2}{R} \right). \quad (6.25)$$

Both equations require $N \geq 0$. Thus the inside-top condition is $v \geq \sqrt{gR}$ and the outside-top condition is $v \leq \sqrt{gR}$. The same threshold has opposite admissible sides because the contact geometry is different.

Figure 6.18 separates geometry from actual forces. The direction marker is not a force, and the force arrows begin on the isolated-body symbol. A body clamped between two guides, attached by

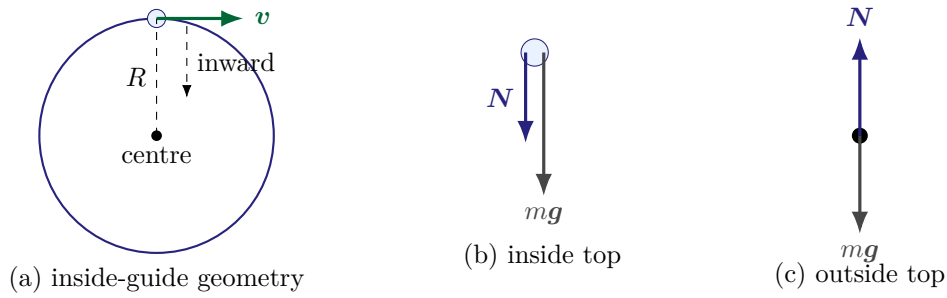


Figure 6.18: Inside-guide and outside-hill contact at the top of a vertical circle. The geometry panel contains velocity and an auxiliary inward marker; the two body-only panels contain the actual forces for different contact arrangements. Force lengths are schematic and a negative calculated N signals failure of the respective assumed contact.

a rigid arm or held by a magnet is a different system; such mechanisms can supply interactions missing from the one-sided model.

Example 6.12: E06-03: contact at the top of an inside guide

A 40.0 kg carriage is intended to follow the inside of a circular guide of radius 3.00 m. Find the minimum local top speed compatible with the stated contact, then test 5.00 and 6.20 m s⁻¹.

Model. Treat the carriage as a particle acted on by weight and a one-sided guide normal, as in Figure 6.18(b). These questions specify *local* speeds; they do not require a model of the approach to the top.

$$v_{\min} = \sqrt{gR} = \sqrt{(9.81)(3.00)} = 5.42 \text{ m s}^{-1}.$$

At the smaller proposed speed,

$$N = 40.0 \left(\frac{5.00^2}{3.00} - 9.81 \right) = -59.1 \text{ N}.$$

The guide cannot provide this tensile/outward normal. The assumed circular path is therefore inadmissible at that instant under the stated forces; the result is not an actual negative contact force. At the larger speed,

$$N = 40.0 \left(\frac{6.20^2}{3.00} - 9.81 \right) = 120 \text{ N} > 0.$$

Contact is locally possible. This does not prove that the full preceding motion reaches that state: the speed history and every earlier contact condition must be checked separately if a complete trajectory is requested.

Check. At $v_{\min}$, $N = 0$ and gravity alone supplies exactly mv^2/R . Increasing the prescribed speed raises the required inward force, so an inward pushing guide supplies a larger positive normal. This is consistent with the sign of Eq. (6.24).

6.4.6 A full vertical circle at a prescribed speed

Let θ now measure position from the bottom of an inside vertical circle, increasing along the rising right side. This angle is distinct from the earlier bank angle β . Position, tangent and inward normal can be written

$$\mathbf{r} = R(\sin \theta, -\cos \theta), \quad \hat{\mathbf{t}} = (\cos \theta, \sin \theta), \quad \hat{\mathbf{n}} = (-\sin \theta, \cos \theta).$$

Weight projects as $-mg \sin \theta$ tangentially and $-mg \cos \theta$ inward. If a tangential actuator applies signed force F_t and the guide applies inward N , then

$$F_t - mg \sin \theta = m\dot{v}, \quad N - mg \cos \theta = m \frac{v^2}{R}. \quad (6.26)$$

At constant *prescribed* speed, $F_t = mg \sin \theta$ and $N = m(v^2/R + g \cos \theta)$. The drive must push along the direction of travel while the carriage rises and oppose travel while it descends. Without that actuator, gravity generally changes the speed; the constant-speed formula would describe an inconsistent model. A tangential drive contributes no normal force in this ideal arrangement.

The radial sign can be checked at three positions: bottom gives $N = mv^2/R + mg$, side gives $N = mv^2/R$, and top gives $N = mv^2/R - mg$. Contact for the *entire* prescribed-speed circle requires its smallest normal, at the top, to be nonnegative. This condition is not needed for an arc that never reaches the top.

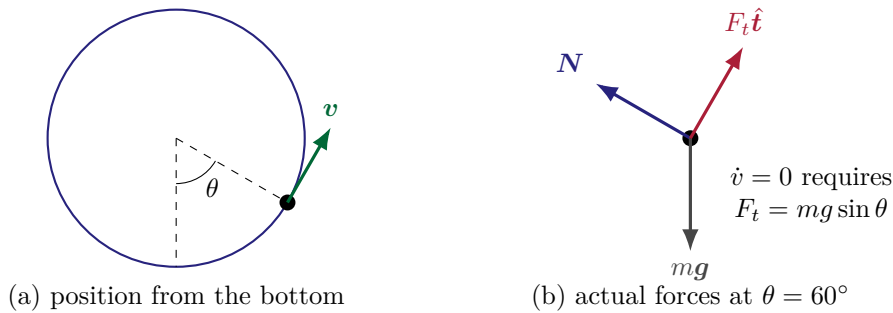


Figure 6.19: A prescribed-speed carriage on an inside vertical circle. The position angle is measured from the bottom and increases along the rising right side. The displayed tangential drive is an actual interaction needed to cancel the tangential gravity component; it is not a consequence of circular kinematics alone. Geometry uses equal scales; force lengths are schematic.

Example 6.13: Loads around a driven vertical circle

A 2.00 kg carriage moves at a prescribed constant 6.00 m s^{-1} in an inside circle of radius 2.00 m. A tangential actuator maintains the speed. Determine the normal load at bottom, side and top, and the drive force at $\theta = 60.0^\circ$.

Here $v^2/R = 18.0 \text{ m s}^{-2}$ and $mg = 19.62 \text{ N}$. Equation (6.26) gives

$$N_{\text{bottom}} = 55.6 \text{ N}, \quad N_{\text{side}} = 36.0 \text{ N}, \quad N_{\text{top}} = 16.4 \text{ N}.$$

The smallest normal remains positive, so one-sided contact is compatible with the whole prescribed motion. On the rising side at 60.0° , the drive is

$$F_t = mg \sin 60.0^\circ = 17.0 \text{ N}$$

along the tangent. It balances the downhill component of gravity, leaving $\dot{v} = 0$. At the corresponding descending position, $\sin \theta < 0$ and the required signed drive is negative: it must brake rather than push along travel.

Interpretation. Normal load changes even though speed is constant. The changing vertical projection of weight changes how much inward support force is required. At the bottom N exceeds the weight; at the top it is smaller. The threshold $\sqrt{gR} = 4.43 \text{ m s}^{-1}$ is below the specified speed, agreeing with the contact test.

6.4.7 Loss of contact followed by free flight

Once N reaches zero, a one-sided surface no longer controls the trajectory unless subsequent motion brings the body back into contact. The body retains its instantaneous position and velocity and evolves under whatever forces remain. It does not acquire a new outward launch force. If gravity is the only remaining force and air resistance is neglected, the continuation is a projectile trajectory, tangent to the old path at release.

To calculate an example without using a work–energy result from the next chapter, consider a particle sliding without friction on the *outside* of a circular hill. Let θ be measured from the top down the right side, so $x = R \sin \theta$, $y = R \cos \theta$. The tangent is $(\cos \theta, -\sin \theta)$ and the inward normal is $(-\sin \theta, -\cos \theta)$. The equations while contact exists are

$$m\dot{v} = mg \sin \theta, \quad mg \cos \theta - N = m \frac{v^2}{R}.$$

The arc length is $s = R\theta$, and $\dot{\theta} = v/R$. Where $v > 0$, the chain rule gives $\dot{v} = (dv/d\theta)(v/R)$. Therefore $v dv = gR \sin \theta d\theta$. Integrating from the top, where speed is v_i , to angle θ gives

$$v^2 = v_i^2 + 2gR(1 - \cos \theta), \quad N = m \left(3g \cos \theta - 2g - \frac{v_i^2}{R} \right). \quad (6.27)$$

The first formula is obtained directly from tangential Newton's law. The second follows from substituting into the normal equation. Initially we need $v_i^2 \leq gR$; otherwise contact already fails at the top. For $0 < v_i < \sqrt{gR}$, the first release angle satisfies

$$\cos \theta_* = \frac{2 + v_i^2/(gR)}{3}. \quad (6.28)$$

Beyond θ_* the circular formula would require negative N and cannot be used as a trajectory model. The physical continuation uses $N = 0$ and free-flight equations instead. An exactly stationary particle placed exactly at the frictionless top has no tangential acceleration there; the familiar small-speed limiting release angle assumes motion is initiated, rather than asserting spontaneous departure from perfectly symmetric rest.

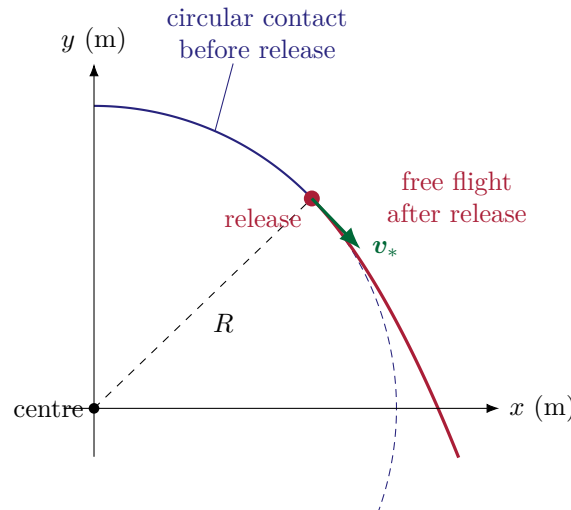


Figure 6.20: Calculated hill-contact and projectile continuation for $R = 5.00$ m and top speed $v_i = 2.00$ m s⁻¹. Release occurs at $\theta_* = 46.1^\circ$; the red continuation uses the release position and tangent velocity as initial conditions. Equal axis scales preserve the geometry. The dashed continuation of the hill locates the surface and is not the body's subsequent path.

Example 6.14: A release state and its projectile continuation

A 0.250 kg particle slides without friction over the outside of the 5.00 m hill in Figure 6.20, with top speed 2.00 m s^{-1} . Find the release angle and speed, then its position 0.300 s after release. The initial normal is $m(g - v_i^2/R) = 2.25 \text{ N} > 0$, so the assumed starting contact is possible. The release condition gives

$$\cos \theta_* = \frac{2 + 4.00/(9.81 \times 5.00)}{3} = 0.693850, \quad \theta_* = 46.1^\circ.$$

At $N = 0$, the radial equation independently gives $v_*^2 = gR \cos \theta_*$, yielding $v_* = 5.83 \text{ m s}^{-1}$. The release state is

$$(x_*, y_*) = (3.60060, 3.46925) \text{ m},$$

$$(v_{x*}, v_{y*}) = (v_* \cos \theta_*, -v_* \sin \theta_*) = (4.04779, -4.20104) \text{ m s}^{-1}.$$

Let τ be elapsed time *since release*. With only gravity acting,

$$x = x_* + v_{x*}\tau, \quad y = y_* + v_{y*}\tau - \frac{1}{2}g\tau^2.$$

At $\tau = 0.300 \text{ s}$, $(x, y) = (4.81, 1.77) \text{ m}$. Its distance from the circle centre is $5.13 \text{ m} > R$, consistent with having separated to the outside. The calculation does not use the negative-contact extension of Eq. (6.27). Differentiating the projectile formula recovers the release velocity at $\tau = 0$ and the remaining acceleration $(0, -g)$ at every later time.

Guided checks

Diagnostic X06-03: what the radial equation establishes. At constant speed on a circle, the acceleration is inward because velocity changes direction. On a flat road the required inward interaction is static friction; on a frictionless bank it is the horizontal component of the normal. At the top of an inside loop a pushing normal cannot point outward. If its equation gives $N < 0$, the assumed contact has failed. In particular $v = 0$ at that inside top requires $N = -mg$ and cannot describe continued circular contact under weight and a one-sided support alone. These answers depend on the specified interaction model; an additional restraining mechanism requires a different diagram.

6.5 Nonuniform curved motion and shared traction

A changing speed and a changing direction are independent features of a velocity vector, but the forces that produce them may draw on the same physical interaction. This matters in a turn with simultaneous acceleration or braking. It also matters for a path whose curvature changes even when speed remains steady. We now extend the circular decomposition to a smooth planar path and use it to formulate one combined contact constraint.

6.5.1 The local radius of curvature

Let s be distance along a smooth path and $\hat{\mathbf{t}}$ the unit tangent in the travel direction. The velocity is $\mathbf{v} = v\hat{\mathbf{t}}$, with $v = ds/dt \geq 0$. Over a short path increment ds , suppose the tangent rotates through $d\psi$. Define the curvature magnitude as $\kappa = |d\psi/ds|$ and the local radius as $R_c = 1/\kappa$. The normal $\hat{\mathbf{n}}$ points toward the local turning side. The derivative of a unit tangent is perpendicular to it, so

$$\frac{d\hat{\mathbf{t}}}{ds} = \frac{1}{R_c}\hat{\mathbf{n}}.$$

Using $ds/dt = v$ and differentiating $v\hat{\mathbf{t}}$ gives

$$\mathbf{a} = \frac{dv}{dt}\hat{\mathbf{t}} + \frac{v^2}{R_c}\hat{\mathbf{n}}, \quad \sum F_t = m\frac{dv}{dt}, \quad \sum F_n = m\frac{v^2}{R_c}. \quad (6.29)$$

The normal direction changes along the path. Its local centre is a geometric construction, not necessarily an object exerting a force. At an inflection point $\kappa = 0$, the normal acceleration vanishes and R_c becomes infinite; the normal direction can switch sides. The vector acceleration itself is still obtained from the trajectory derivatives. A corner with a discontinuous tangent is outside this smooth-path model and cannot be traversed at finite nonzero speed with a bounded ordinary force.

For a graph $y(x)$ traversed with increasing x , elementary differentiation of the tangent angle gives

$$R_c = \frac{[1 + (y')^2]^{3/2}}{|y''|}, \quad y' = \frac{dy}{dx}, \quad y'' = \frac{d^2y}{dx^2}. \quad (6.30)$$

One way to see the factors is to write $\tan\psi = y'$ and $ds/dx = \sqrt{1 + (y')^2}$. Then $d\psi/dx = y''/[1 + (y')^2]$, and dividing by ds/dx produces the curvature above. The absolute value determines magnitude; the sign of y'' determines which side the inward normal points toward.

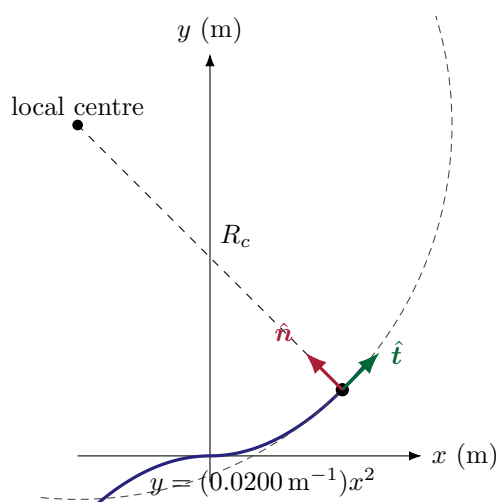


Figure 6.21: Local curvature of an analytic parabolic path at $x = 25.0$ m. The tangent makes 45° with the horizontal, the inward normal is perpendicular to it, and the dashed osculating circle has radius 70.7 m. Equal axis scales and analytic derivatives determine every geometric direction. Only the relevant portion of the circle is shown.

Example 6.15: The force demanded by a noncircular path

A 3.00 kg mobile platform is modelled as a particle following $y = (0.0200 \text{ m}^{-1})x^2$ at $x = 25.0$ m. At that instant its speed is 10.0 m s^{-1} and it is speeding up at 2.00 m s^{-2} . Find the local radius, the resultant force components and the Cartesian acceleration.

Differentiate the path: $y' = 0.0400x = 1.00$ and $y'' = 0.0400 \text{ m}^{-1}$. Thus

$$R_c = \frac{(1 + 1^2)^{3/2}}{0.0400} = 70.7 \text{ m}.$$

The positive second derivative places the inward normal to the left of the tangent in Figure 6.21. At this point,

$$\hat{\mathbf{t}} = \frac{(1, 1)}{\sqrt{2}}, \quad \hat{\mathbf{n}} = \frac{(-1, 1)}{\sqrt{2}}, \quad a_n = \frac{10.0^2}{70.7107} = 1.4142 \text{ m s}^{-2}.$$

The resultants are $\sum F_t = 6.00 \text{ N}$ and $\sum F_n = 4.24 \text{ N}$. Combining components gives

$$(a_x, a_y) = \frac{1}{\sqrt{2}}(2.00 - 1.4142, 2.00 + 1.4142) = (0.414, 2.41) \text{ m s}^{-2}.$$

These are net force requirements, not a list of interactions. To determine the individual forces on an actual device, its support and propulsion model must also be specified.

Check. The acceleration magnitude from Cartesian components equals $\sqrt{a_t^2 + a_n^2} = \sqrt{6} = 2.45 \text{ m s}^{-2}$. The dimension of y'' is inverse length, so Eq. (6.30) produces a length as required.

6.5.2 A simplified isotropic friction budget

Suppose a vehicle on level ground requires tangential acceleration a_t and normal acceleration $a_n = v^2/R_c$. Assume negligible vertical acceleration and other horizontal forces, and approximate the resultant available tyre friction as an isotropic Coulomb bound. Then $N = mg$ and the horizontal force components are $f_t = ma_t$, $f_n = ma_n$. The *single resultant* must satisfy

$$|\mathbf{f}| = \sqrt{f_t^2 + f_n^2} \leq \mu_s N \implies a_t^2 + \left(\frac{v^2}{R_c}\right)^2 \leq (\mu_s g)^2. \quad (6.31)$$

Friction used for braking or speeding up is no longer available for turning in this model. Checking $|f_t| \leq \mu_s N$ and $|f_n| \leq \mu_s N$ separately defines a square in component space, whereas the actual assumed resultant bound is a circle. The square's corners lie outside the circle.

Figure 6.22 links local path geometry with the force-component constraint. An actual circle is used at its bottom point, where the tangent is exactly horizontal and the inward normal exactly vertical in plan view. The force-space panel is dimensionless and uses equal scales; its circle is genuinely circular rather than stretched by unequal axis scales.

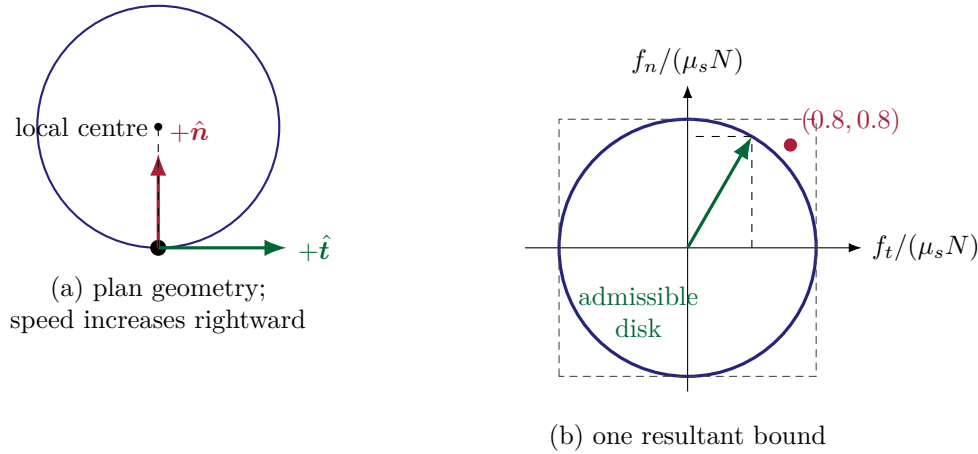


Figure 6.22: Local path geometry and the resultant friction constraint. The tangent and inward normal follow the actual path geometry. In component space the isotropic friction circle bounds the resultant, while separate component tests allow the dashed square and wrongly admit its corners. The marked (0.8, 0.8) demand violates the circular bound even though each component is individually below one.

The isotropic circle is a deliberately simplified contact model. Real tyres can have unequal longitudinal and lateral responses, force dependence on slip and changes in load. Even within this ideal model, added aerodynamic downforce changes N , and vertical road curvature can change the normal load. Such interactions must be included before replacing N by mg . The circle expresses the particular assumptions stated above rather than a universal tyre law.

For prescribed a_t on a constant-radius turn, Eq. (6.31) first requires $|a_t| \leq \mu_s g$. The remaining normal-acceleration capacity is

$$a_{n,\max} = \sqrt{(\mu_s g)^2 - a_t^2}, \quad v_{\max} = \sqrt{R_c a_{n,\max}}. \quad (6.32)$$

If $|a_t| > \mu_s g$, no turning speed, including zero, satisfies the proposed tangential demand under this force model. If $|a_t| = \mu_s g$, no normal capacity remains: $v = 0$ is the only instantaneous speed admitted for finite R_c . For either sign of a_t , the magnitude uses the same fraction of the ideal resultant bound.

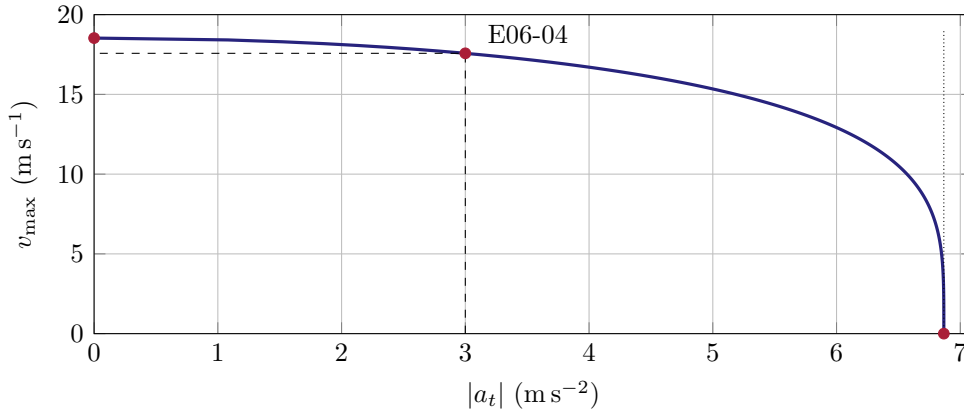


Figure 6.23: Calculated maximum instantaneous turning speed versus simultaneous acceleration or braking demand for $R = 50.0$ m and $\mu_s = 0.700$. The endpoint is $|a_t| = \mu_s g = 6.867 \text{ m s}^{-2}$, where no turning capacity remains. Demands beyond that endpoint are inadmissible in the assumed model and are not plotted.

Example 6.16: E06-04: sharing traction while speeding up

A vehicle is modelled as a particle on a level road with $\mu_s = 0.700$. It follows a radius $R = 50.0$ m while increasing its speed at $a_t = 3.00 \text{ m s}^{-2}$. Find the maximum instantaneous speed compatible with no slip in the isotropic friction-circle model, and compare it with the constant-speed limit.

Strategy. The two acceleration components use one resultant force budget. Since $3.00 < \mu_s g = 6.867 \text{ m s}^{-2}$, the tangential demand alone is admissible. The remaining normal capacity is

$$a_{n,\max} = \sqrt{6.867^2 - 3.00^2} = 6.17703 \text{ m s}^{-2}.$$

Consequently

$$v_{\max} = \sqrt{(50.0)(6.17703)} = 17.6 \text{ m s}^{-1} = 63.3 \text{ km h}^{-1}.$$

At constant speed the tangential demand is zero, and the corresponding limit would be $\sqrt{\mu_s g R} = 18.5 \text{ m s}^{-1}$. The smaller allowable speed with acceleration is the consequence of a shared resultant budget.

Check. At the limiting combined state,

$$\frac{f_t}{\mu_s N} = \frac{3.00}{6.867} = 0.437, \quad \frac{f_n}{\mu_s N} = \frac{6.17703}{6.867} = 0.900.$$

Their squared sum is one, to rounding. Each separately lies below one, yet their resultant has reached capacity. The same speed limit follows for braking with $a_t = -3.00 \text{ m s}^{-2}$. These are calculations within a stipulated teaching model, with no allowance for load transfer, anisotropic tyre response or engineering margins.

Guided checks

Diagnostic X06-04: local direction and one force budget. Constant-speed turning has $a_t = 0$ but generally $a_n \neq 0$, so zero tangential force does not mean zero horizontal resultant. At fixed speed, halving R_c doubles the required normal resultant. The pair $(f_t, f_n) = (0.8, 0.8)\mu_s N$ fails the assumed resultant bound because $\sqrt{0.8^2 + 0.8^2} = 1.13 > 1$. As $R_c \rightarrow \infty$, the budget reduces to $|a_t| \leq \mu_s g$; as $a_t \rightarrow 0$, it reduces to the flat constant-speed turn limit. These limits test the model's equations, while the actual interactions still require a body-only free-body diagram.

6.6 Drag and terminal motion

A contact force can depend on whether two surfaces stick or slide. Resistance from a surrounding fluid introduces a different dependence: the force changes continuously with the body's velocity relative to the fluid. A payload descending through still air, a small particle settling in a liquid, and a vehicle exposed to a moving airstream therefore require a force model as well as a free-body diagram. Newton's second law is unchanged. The challenge is to state a plausible resistance law and to understand which observations would test it.

The detailed resistance can depend on the fluid, the shape and orientation of the body, and the flow pattern. We shall compare two stipulated laws without deriving the fluid mechanics that selects their ranges of validity. The discussion assumes a body of constant mass, a prescribed medium velocity, and resistance parallel and opposite to relative velocity. Lift, changing orientation, and memory effects of the flow are omitted. For vertical falling examples the weight is constant and buoyancy is neglected. This last assumption can be poor for an object immersed in a dense liquid. If a constant upward buoyant force B matters, replace the driving force mg by $mg - B$; do not interpret the omitted force as part of the drag coefficient.

6.6.1 Signed force laws and their dimensions

Let v be a signed velocity component of the body and u the corresponding medium velocity in the same inertial frame. The relative velocity is $v_{\text{rel}} = v - u$. The linear and quadratic laws are

$$F_d = -b(v - u), \quad b > 0, \quad (6.33)$$

$$F_d = -c|v - u|(v - u), \quad c > 0. \quad (6.34)$$

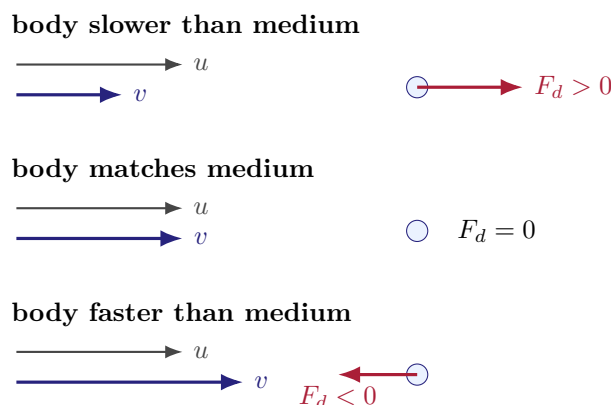
The coefficients cannot be interchanged. Since each right-hand side must have force units,

$$[b] = \frac{\text{N}}{\text{m s}^{-1}} = \text{kg s}^{-1}, \quad [c] = \frac{\text{N}}{\text{m}^2 \text{s}^{-2}} = \text{kg m}^{-1}. \quad (6.35)$$

For positive relative velocity both forces are negative. For negative relative velocity both are positive. Writing $F_d = -cv_{\text{rel}}^2$ without a direction rule is consequently incorrect when the velocity can reverse: it would predict a negative force for either sign of velocity. The absolute value in Eq. (6.34) enforces the physical direction.

The same idealized isotropic laws in more than one dimension are $\mathbf{F}_d = -b\mathbf{v}_{\text{rel}}$ and $\mathbf{F}_d = -c|\mathbf{v}_{\text{rel}}|\mathbf{v}_{\text{rel}}$. The factor $|\mathbf{v}_{\text{rel}}|$ is a scalar speed, while the last factor is the velocity vector. This distinction preserves the direction of the interaction. Neither law says that drag must oppose the laboratory velocity: a moving medium can accelerate a slower body in its own direction.

Figure 6.24 is useful whenever a wind or current is present. Subtract the medium velocity first and choose the force direction second. As a purely algebraic direction check, $F_d v_{\text{rel}} \leq 0$ for either model: it is $-bv_{\text{rel}}^2$ or $-c|v_{\text{rel}}|^3$. The later work-and-power chapter gives this product an energy-transfer interpretation. Here its role is simply to expose a reversed force sign.



Positive direction: right. Arrow lengths are schematic.

Figure 6.24: The medium-relative velocity determines the sign of drag. All three bodies move right in the laboratory, but the drag can point right, vanish, or point left. The arrows compare velocities within a row; they are not measured data.

6.6.2 Terminal velocity is a stable dynamical balance

Take downward as positive for a falling body in a stationary medium. The complete one-dimensional equations are

$$\dot{v} = g - \frac{b}{m}v, \quad \dot{v} = g - \frac{c}{m}|v|v. \quad (6.36)$$

For downward motion, $v \geq 0$, the second expression becomes $g - (c/m)v^2$. A terminal velocity is a constant solution of the dynamical equation, not a condition of zero speed. Setting $\dot{v} = 0$ gives

$$v_{t,L} = \frac{mg}{b}, \quad v_{t,Q} = \sqrt{\frac{mg}{c}}. \quad (6.37)$$

Both are positive in the chosen downward-positive convention. Gravity and drag are still present at terminal motion; their sum vanishes [1, 2, 3].

A force balance also explains stability. At a downward speed below the terminal value, drag is smaller than the weight and the body accelerates downward. Above it, drag exceeds the weight and the acceleration is upward, reducing the downward speed. The acceleration therefore restores the velocity toward the balance value from either side. An upward-moving body, $v < 0$, experiences downward gravity and downward drag, so its acceleration is positive until it turns and begins to descend. There is no negative terminal root of the sign-correct quadratic equation. The apparent root $-\sqrt{mg/c}$ belongs to the incorrectly extended expression $g - (c/m)v^2$.

Example 6.17: Two similar terminal speeds, two different predictions

A body has $m = 0.0800 \text{ kg}$. Compare the linear model $b = 0.160 \text{ kg s}^{-1}$ with the quadratic model $c = 0.0320 \text{ kg m}^{-1}$. Find both terminal speeds and the acceleration at downward speed 2.50 m s^{-1} .

The system is the body. Its only modeled vertical interactions are its weight and fluid drag; the medium is stationary and downward is positive. We first balance forces, rather than assigning a terminal speed from a graph alone:

$$\begin{aligned} mg &= (0.0800)(9.81) = 0.7848 \text{ N}, \\ v_{t,L} &= 0.7848/0.160 = 4.905 \text{ m s}^{-1}, \\ v_{t,Q} &= \sqrt{0.7848/0.0320} = 4.95227 \text{ m s}^{-1}. \end{aligned}$$

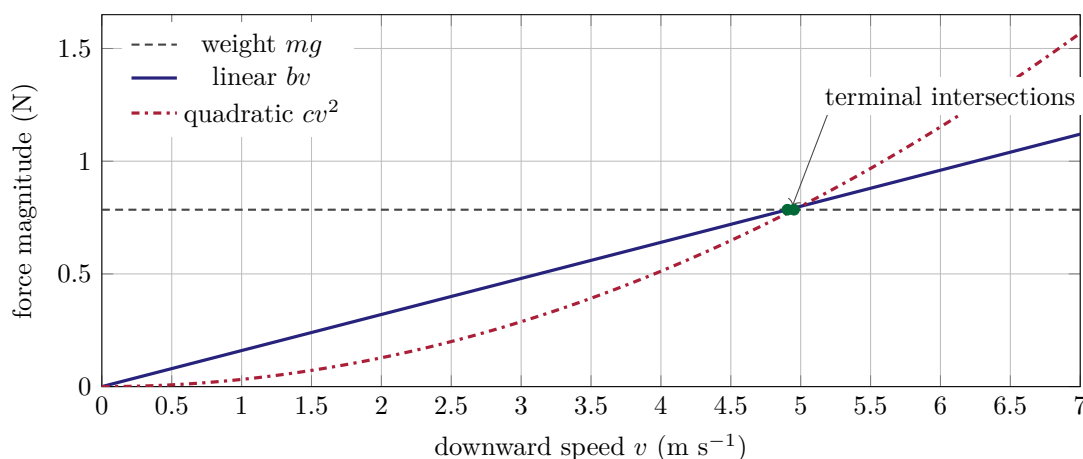


Figure 6.25: The stipulated baseline models have $m = 0.0800$ kg, $b = 0.160$ kg s⁻¹, and $c = 0.0320$ kg m⁻¹. Their terminal intersections nearly coincide, but their force predictions away from the intersections differ. These are calculated force magnitudes; the direction of drag is upward for downward motion.

At the specified speed, $bv = 0.400$ N and $cv^2 = 0.200$ N. Newton's law then gives

$$a_L = 9.81 - \frac{0.400}{0.0800} = 4.81 \text{ m s}^{-2},$$

$$a_Q = 9.81 - \frac{0.200}{0.0800} = 7.31 \text{ m s}^{-2}.$$

The two terminal predictions differ by only 0.0473 m s^{-1} , whereas these transient accelerations differ by 2.50 m s^{-2} . A terminal-speed instrument resolving only 0.10 m s^{-1} would have difficulty separating the stipulated terminal predictions. A resolved transient or force-speed measurement is a more discriminating test.

Both accelerations are positive and smaller than g , consistent with speeds below their terminal values. This comparison concerns two specified hypotheses. It does not determine which hypothesis applies to a particular material or flow.

For a small perturbation $\delta v = v - v_t$ around terminal motion, the linear law gives $\dot{\delta v} = -(b/m)\delta v$ exactly. The quadratic law gives

$$\dot{\delta v} = -\frac{2cv_t}{m}\delta v - \frac{c}{m}(\delta v)^2. \quad (6.38)$$

For $|\delta v| \ll v_t$, the last term is a smaller correction, so the local decay time is $m/(2cv_t) = v_t/(2g)$. The sign of the coefficient of δv confirms stability without relying on a sketch. Figure 6.26 displays the restoring signs and the correct upward-motion branch.

Example 6.18: A wind can push a moving body forward

A prescribed uniform wind moves east at $u = 8.0 \text{ m s}^{-1}$. A test body has linear-drag coefficient $b = 0.50 \text{ kg s}^{-1}$. Determine the horizontal drag if its laboratory velocity is first 5.0 m s^{-1} east, then 10.0 m s^{-1} east, and finally zero.

Choose east positive. For the first state, $v - u = -3.0 \text{ m s}^{-1}$, so $F_d = -0.50(-3.0) = +1.5$ N: drag points east. For the second, $v - u = +2.0 \text{ m s}^{-1}$, so $F_d = -1.0$ N: drag points west. For the stationary state, $v - u = -8.0 \text{ m s}^{-1}$, so $F_d = +4.0$ N east.

The force tends to reduce the relative velocity in all three states. In particular, laboratory rest does not imply zero drag. The wind is treated as a prescribed medium; the reaction on the

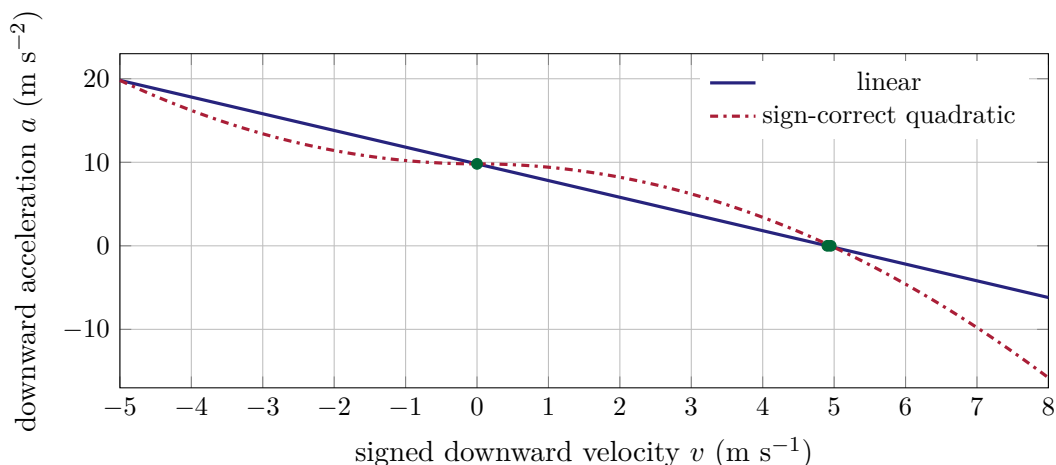


Figure 6.26: Acceleration is a signed quantity. For $v < 0$ the body moves upward and both gravity and drag accelerate it downward. For $v > v_t$ it moves downward while accelerating upward. The baseline parameters are those of Figure 6.25; only the positive terminal equilibria exist.

air and changes to the flow would be needed for a larger body-plus-fluid model. Substituting $v = u$ gives zero drag and checks the physical reference velocity.

6.7 Differential-equation models and their verification

A force that depends on velocity turns Newton's law into an equation for the rate of change of the unknown velocity. Such a differential equation is an initial-value problem: a force law tells us how a state changes, and the initial state selects the particular trajectory. A terminal balance supplies only the constant long-time solution. It cannot supply the time required to approach that solution or the distance traveled during the approach.

6.7.1 Linear drag: separation, arbitrary initial velocity, and position

For the stationary-medium falling model, define the time constant $\tau = m/b$ and terminal velocity $v_t = mg/b$. Notice the physical constraint $v_t = g\tau$: these two parameters are not independent if gravity is the only constant driving force. Then

$$\frac{dv}{dt} = \frac{v_t - v}{\tau}. \quad (6.39)$$

Separation of variables means placing every factor involving v with dv and every factor involving t with dt . If $v_0 \neq v_t$, integrating over the actual change gives

$$\int_{v_0}^{v(t)} \frac{dw}{v_t - w} = \int_0^t \frac{ds}{\tau}, \quad -\ln \left| \frac{v_t - v(t)}{v_t - v_0} \right| = \frac{t}{\tau}. \quad (6.40)$$

The dummy variables w and s prevent confusion between an integration variable and an endpoint. The logarithm involves a ratio of quantities with identical units, so its argument is dimensionless. Its absolute value permits initial speeds above the equilibrium. Continuity of the solution keeps $v_t - v(t)$ on the same side of zero as $v_t - v_0$; restoring that sign gives

$$\boxed{v(t) = v_t + (v_0 - v_t)e^{-t/\tau}}. \quad (6.41)$$

The separated expression divided by zero when $v_0 = v_t$, but that does not invalidate the physics. In that case $v(t) = v_t$ is the constant solution and is already included in the boxed formula.

Differentiating the result gives $a = (v_t - v_0)e^{-t/\tau}/\tau$. At $t = 0$, this is exactly $g - (b/m)v_0$; inserting the solution into Eq. (6.39) verifies it for all times. Initial downward speeds below v_t increase, those above v_t decrease, and either approaches the same stable value without crossing it. An initially negative velocity first increases through zero and then approaches the positive terminal velocity.

Position follows from the definition $dy/dt = v$, with downward y positive. Integrating once more yields

$$y(t) - y_0 = v_t t + \tau(v_0 - v_t)(1 - e^{-t/\tau}). \quad (6.42)$$

A derivative recovers Eq. (6.41), while $t = 0$ gives $y = y_0$. From rest, the short-time expansion $e^{-t/\tau} = 1 - t/\tau + t^2/(2\tau^2) + \dots$ gives $v \simeq gt$ and $y - y_0 \simeq gt^2/2$. Thus the ordinary release-from-rest free-fall relations reappear before drag has grown appreciably. At late times, $y - y_0 \simeq v_t(t - \tau)$ for release from rest: a constant terminal-speed model misses the early distance by a finite amount approaching $v_t\tau$.

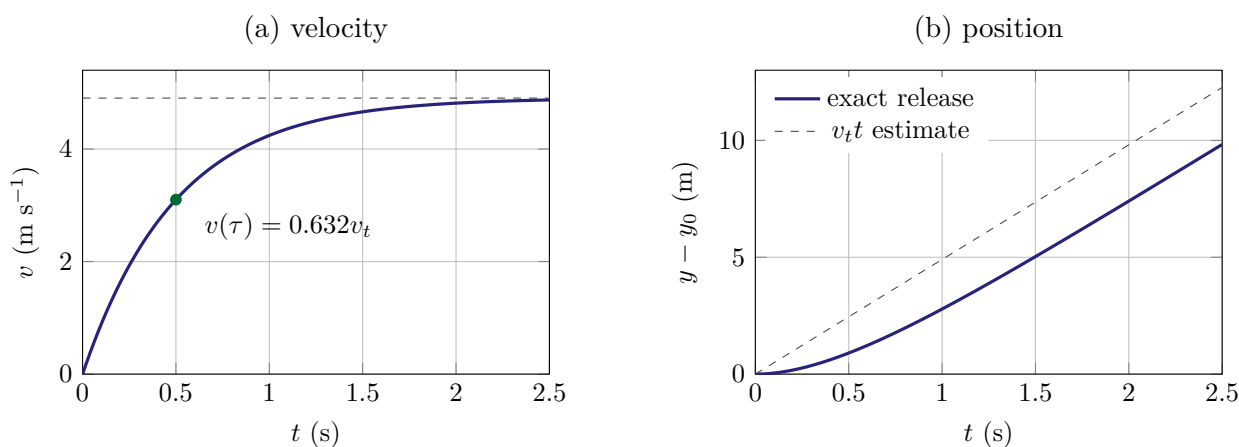


Figure 6.27: A linear-drag release from rest with $m = 0.0800$ kg and $b = 0.160$ kg s⁻¹ has $\tau = 0.500$ s and $v_t = 4.905$ m s⁻¹. The initial position curve has zero slope; replacing the full transient by terminal motion overestimates the fall distance. Both panels are analytical calculations.

Example 6.19: Release from rest: speed, distance, and a 90-percent criterion

For the baseline linear model, find the speed and fall distance after 0.500, 1.50, and 2.50 s. Determine when the speed first reaches 90.0% of terminal speed.

Release means $v_0 = 0$ and $y_0 = 0$. The exact expressions are $v = 4.905(1 - e^{-t/0.500})$ m s⁻¹ and $y = 4.905[t - 0.500(1 - e^{-t/0.500})]$ m, where numerical times are entered in seconds. Evaluating both expressions together gives

t (s)	v (m s ⁻¹)	y (m)
0.500	3.10	0.902
1.50	4.66	5.03
2.50	4.87	9.83

The fraction $p = v/v_t$ satisfies $p = 1 - e^{-t/\tau}$, so

$$t_p = -\tau \ln(1 - p), \quad t_{90} = 0.500 \ln 10 = 1.15 \text{ s}.$$

Exact terminal speed is approached asymptotically; a stated percentage defines a finite operational time. At one, three, and five time constants the fractions are 63.2%, 95.0%, and 99.3%. All calculated positions are smaller than $v_t t$, as Figure 6.27 predicts. Finally $v_t/\tau = 9.81$ m s⁻² checks that the initial acceleration is g .

Example 6.20: An upward launch does not require reversing the drag model

The same body and linear coefficient are used, but the body is launched upward at 2.00 m s^{-1} . Find when it turns and its maximum upward displacement.

Downward remains positive, so $v_0 = -2.00 \text{ m s}^{-1}$. The drag force is initially downward, and $a(0) = 9.81 - 2(-2.00) = 13.81 \text{ m s}^{-2}$. The general solution applies without changing equations. Setting $v(t_*) = 0$ gives

$$e^{-t_*/\tau} = \frac{v_t}{v_t - v_0}, \quad t_* = \tau \ln \left(\frac{v_t - v_0}{v_t} \right) = 0.500 \ln \left(\frac{6.905}{4.905} \right) = 0.171 \text{ s}.$$

Substitution into the position formula gives $y(t_*) - y_0 = -0.161 \text{ m}$; the maximum height gain is therefore 0.161 m . The negative signed displacement is expected because the body moved upward. It is smaller than the no-drag height $v_0^2/(2g) = 0.204 \text{ m}$: during the ascent drag assists gravity in bringing the velocity to zero. After turning, drag changes direction automatically through its signed law.

6.7.2 Quadratic drag: an exact release solution and its distance

For release from rest into a stationary medium, v stays nonnegative. Define $v_t = \sqrt{mg/c}$ and $q = v/v_t$. The ODE becomes

$$\dot{q} = \frac{g}{v_t}(1 - q^2), \quad q(0) = 0. \quad (6.43)$$

Separating and using partial fractions,

$$\int_0^q \frac{dw}{1 - w^2} = \frac{1}{2} \ln \left(\frac{1 + q}{1 - q} \right) = \frac{gt}{v_t}. \quad (6.44)$$

The integration uses $0 \leq q < 1$, which is the physically relevant release branch. Solving the logarithmic relation gives

$$q = \frac{e^{2gt/v_t} - 1}{e^{2gt/v_t} + 1} = \tanh \left(\frac{gt}{v_t} \right), \quad \boxed{v(t) = v_t \tanh \left(\frac{gt}{v_t} \right)}. \quad (6.45)$$

The last equality defines the hyperbolic tangent through exponentials; no prior course in hyperbolic functions is required. Its derivative is $d(\tanh z)/dz = 1 - \tanh^2 z$, obtained by differentiating the exponential fraction. This verifies the ODE, gives $a(0) = g$, and shows that v approaches v_t monotonically.

To obtain distance, define $\cosh z = (e^z + e^{-z})/2$, whose logarithmic derivative is $\tanh z$. Integrating Eq. (6.45) then gives

$$\boxed{y(t) - y_0 = \frac{v_t^2}{g} \ln \cosh \left(\frac{gt}{v_t} \right)}. \quad (6.46)$$

The logarithm is dimensionless and the coefficient $v_t^2/g = m/c$ has length units. At small t , $\ln \cosh z \simeq z^2/2$, recovering $gt^2/2$. At large t , $\ln \cosh z \simeq z - \ln 2$, so the position approaches $v_t t - (v_t^2/g) \ln 2$. These checks examine both the initial free-fall behavior and the eventual constant-speed slope.

Example 6.21: Quadratic release: speed, distance, and time to 90 percent

For $m = 0.0800 \text{ kg}$ and $c = 0.0320 \text{ kg m}^{-1}$, find the speed and fall distance after 1.00 s and the time to reach $0.900v_t$.

The stationary-medium model has $v_t = 4.95227 \text{ m s}^{-1}$. At $t = 1.00 \text{ s}$, the dimensionless

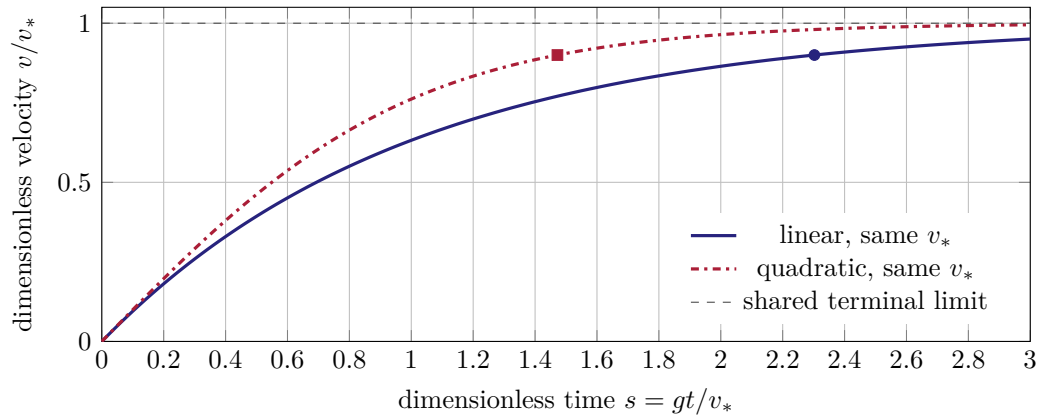


Figure 6.28: Two models calibrated to the same terminal speed v_* still predict distinct transients from rest. For this comparison, $b = mg/v_*$ and $c = mg/v_*^2$; it is a normalized analytical comparison, separate from the slightly different baseline terminal speeds. The 90% times are $s = \ln 10$ and $s = \frac{1}{2} \ln 19$.

argument is $gt/v_t = 1.98091$. Therefore

$$v(1.00) = 4.95227 \tanh(1.98091) = 4.77 \text{ m s}^{-1},$$

$$y(1.00) - y_0 = \frac{4.95227^2}{9.81} \ln \cosh(1.98091) = 3.27 \text{ m}.$$

Solving $p = \tanh(gt_p/v_t)$ with the exponential definition yields

$$t_p = \frac{v_t}{2g} \ln \left(\frac{1+p}{1-p} \right), \quad t_{90} = \frac{4.95227}{2(9.81)} \ln 19 = 0.743 \text{ s}.$$

The speed lies below its terminal value, and the distance lies below the no-drag result $gt^2/2 = 4.905 \text{ m}$. The quadratic model reaches 90% earlier than the baseline linear model because its resistance is weaker at much of the subterminal speed range. Figure 6.28 makes the same distinction even when terminal speeds are exactly equal.

6.7.3 A force-speed experiment and parameter identifiability

An inverse problem uses observations to infer a force law or its parameters. A single terminal-speed measurement gives one balance equation. If the measured value is v_* , both $b = mg/v_*$ and $c = mg/v_*^2$ reproduce it exactly. Thus one terminal speed cannot choose between the linear and quadratic hypotheses when each has an unknown coefficient. Several terminal speeds obtained by changing known masses without changing shape, or force measurements over several relative speeds, can add information. Such a procedure must check that the geometry and fluid conditions remain comparable.

A proposed force-speed measurement can hold a test body fixed on a force sensor while a controlled airstream passes it. The sensor reading along the stream estimates drag after subtracting the zero-flow offset. Speed must be measured relative to the body. A realistic study would also inspect sensor calibration, support interference, changing orientation, and the uncertainty in the speed. The following small dataset is explicitly *synthetic*: it was constructed for teaching, not collected in a laboratory. It permits a full fit and residual analysis without implying an experiment was performed.

Example 6.22: Synthetic data distinguish force laws

Assume that relative speeds are known accurately compared with the force uncertainty. Each synthetic force has a stipulated independent standard uncertainty $\sigma_F = 0.010 \text{ N}$:

$v \text{ (m s}^{-1}\text{)}$	0.50	1.00	1.50	2.00	2.50	3.00
$F \text{ (N)}$	0.010	0.030	0.075	0.124	0.206	0.285

Fit $F = bv$ and $F = cv^2$ through the physically stipulated zero-force origin, then compare the residuals.

For a linear prediction $F_i = \alpha x_i$, least squares minimizes $S(\alpha) = \sum_i (F_i - \alpha x_i)^2$. Differentiation gives $dS/d\alpha = -2 \sum_i x_i (F_i - \alpha x_i) = 0$, hence $\alpha = \sum_i x_i F_i / \sum_i x_i^2$. Equal force uncertainties make ordinary and weighted least squares equivalent here. Choose $x_i = v_i$ for the linear hypothesis and $x_i = v_i^2$ for the quadratic hypothesis:

$$\hat{b} = 0.07760 \text{ kg s}^{-1}, \quad \hat{c} = 0.031998 \text{ kg m}^{-1}.$$

The residual is the signed difference $r_i = F_i - F_{\text{model}}(v_i)$. The linear residuals in newtons are approximately $(-0.0288, -0.0476, -0.0414, -0.0312, +0.0120, +0.0522)$; the quadratic residuals are $(+0.00200, -0.00200, +0.00300, -0.00399, +0.00601, -0.00298)$.

A dimensionless residual sum is $\chi^2 = \sum_i (r_i/\sigma_F)^2$. Its values are 86.5 for the linear hypothesis and 0.780 for the quadratic one. Each fit used one parameter for six points, so the residual degrees of freedom are five. The linear residual pattern and magnitude indicate a poor description under the stipulated uncertainty model. The quadratic residuals are small and have no obvious smooth curvature over this range. This supports the quadratic description of these synthetic data; it does not establish a universal drag law. An unusually small residual sum can also reflect conservative uncertainty estimates or an unusually quiet sample.

Figure 6.29 links the fit to its diagnostic residuals. An unrestricted intercept would test a different model, useful if sensor-offset subtraction were uncertain. Here the known zero-force origin is an explicit assumption, rather than an arbitrary aesthetic choice.

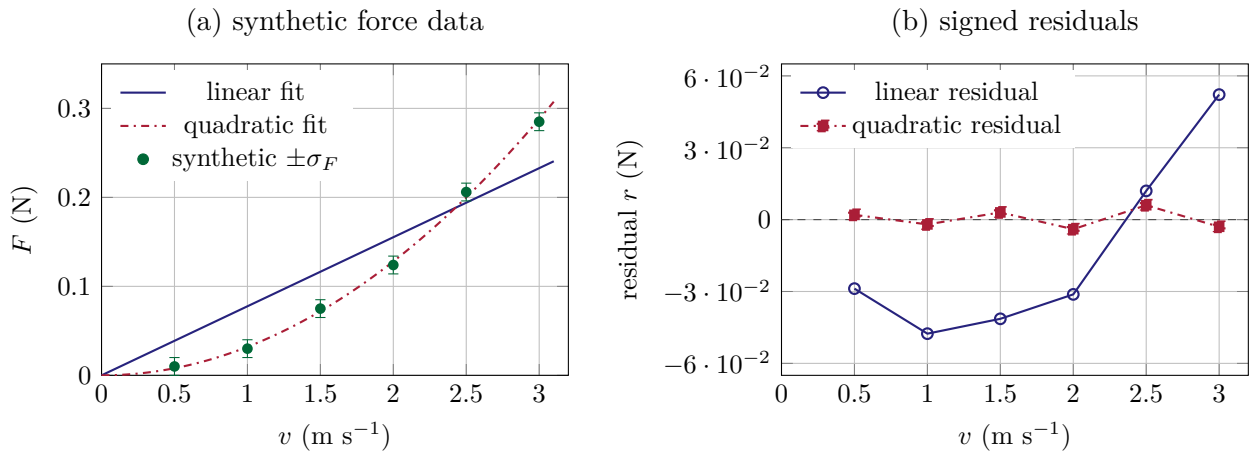


Figure 6.29: A common force axis permits a fair comparison of two fitted laws, while residuals expose structure obscured by the main plot. Force values and error bars are stipulated synthetic teaching data; the error bars represent the assumed standard force uncertainty. Fits and residuals use exactly the tabulated six points.

6.7.4 Numerical integration: accuracy must be demonstrated

An exact solution is not always available, especially when wind or coefficients vary with time. A numerical integrator advances a state using evaluations of its ODE. Let $f(v) = g - (c/m)v|v|$ and $h = \Delta t$. The explicit midpoint method estimates the velocity halfway through a step and uses the

slope there:

$$k_1 = f(v_n), \quad v_{n+1/2} = v_n + \frac{1}{2}hk_1, \quad (6.47)$$

$$v_{n+1} = v_n + hf(v_{n+1/2}). \quad (6.48)$$

The intermediate velocity is an estimate used to evaluate the slope, not a separate measurement. Euler's method would use only $f(v_n)$ for the entire step; midpoint accounts for part of the changing slope and has second-order global error for a sufficiently smooth solution over a fixed time interval.

A fourth-order Runge–Kutta method, abbreviated RK4, samples more slopes:

$$k_1 = f(v_n), \quad k_2 = f(v_n + hk_1/2), \quad (6.49)$$

$$k_3 = f(v_n + hk_2/2), \quad k_4 = f(v_n + hk_3), \quad (6.50)$$

$$v_{n+1} = v_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4). \quad (6.51)$$

The factors hk_i have velocity units, and every k_i has acceleration units. For this one-variable model, the algorithm below gives sufficient information to reproduce either method. Code is optional learning support and is not a compilation dependency.

Listing 6.1: Numerical updates for the sign-correct quadratic model. All inputs and outputs use SI units.

```
from math import sqrt, tanh
m, c, g = 0.0800, 0.0320, 9.81
vt = sqrt(m*g/c)
def f(v):
    return g - (c/m)*abs(v)*v
def midpoint(v, h):
    return v + h*f(v + 0.5*h*f(v))
def rk4(v, h):
    k1 = f(v)
    k2 = f(v + 0.5*h*k1)
    k3 = f(v + 0.5*h*k2)
    k4 = f(v + h*k3)
    return v + h*(k1 + 2*k2 + 2*k3 + k4)/6
# Repeat N = round(2/h) steps from v = 0.
# At every node t compare with vt*tanh(g*t/vt).
```

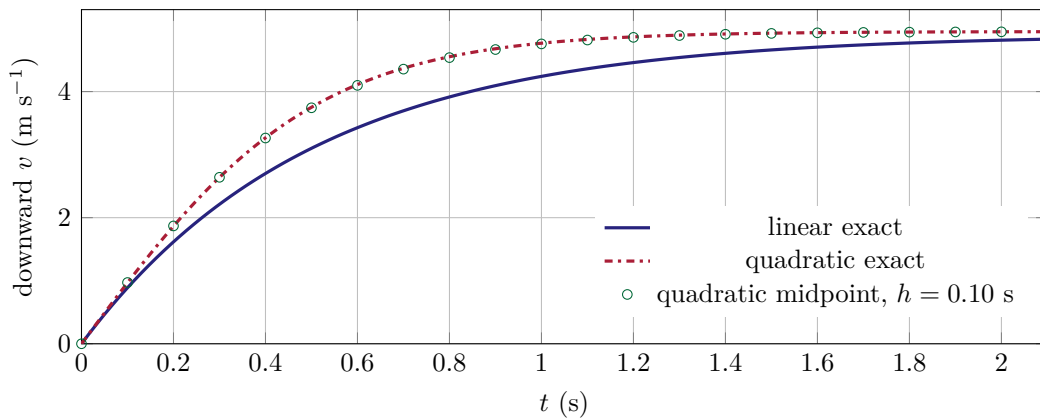


Figure 6.30: Baseline release transients and a computed midpoint solution of the quadratic model. Every marker is a numerical result with the embedded step size and parameters, rather than a measurement. The close visual agreement is useful but must be supplemented by an error calculation on a finer scale.

Example 6.23: Step refinement against an exact benchmark

Integrate the baseline quadratic release on $0 \leq t \leq 2.00$ s using midpoint and RK4. Compare each run with the analytical solution at all its time nodes and interpret the refinement pattern. The physically correct benchmark is Eq. (6.45), not a second numerical run with the same step size. Define $E_h = \max_n |v_n - v(t_n)|$ over the interval. The computed results are

h (s)	midpoint $v(2)$ (m s^{-1})	midpoint E_h (m s^{-1})	RK4 E_h (m s^{-1})
0.200	4.942842	7.0133×10^{-2}	2.6170×10^{-3}
0.100	4.947917	1.3860×10^{-2}	1.3619×10^{-4}
0.050	4.948531	3.1439×10^{-3}	7.7038×10^{-6}
0.025	4.948651	7.4613×10^{-4}	4.5782×10^{-7}
0.0125	4.948679	1.8174×10^{-4}	2.7910×10^{-8}

The exact final speed is $4.948687 \text{ m s}^{-1}$. The maximum error need not occur at the final node: late-time restoring dynamics suppress earlier errors, so checking only $v(2)$ understates the worst error during the transient.

For midpoint the last ratio $E_{0.025}/E_{0.0125} = 4.11$ approaches $2^2 = 4$; for RK4 it is 16.4, approaching $2^4 = 16$. This is consistent with the expected global orders. The coarsest ratios differ because those steps have not fully reached the small-step regime. Figure 6.31 shows the pattern over all five resolutions.

A suitable acceptance criterion for a stated velocity tolerance is to require E_h below that tolerance on the whole interval, with evidence that refinement continues to reduce it. For example, midpoint at $h = 0.025$ s meets a 0.001 m s^{-1} numerical-error requirement here. Such a requirement controls integration error only; uncertainty or error in the physical drag model remains a separate issue.

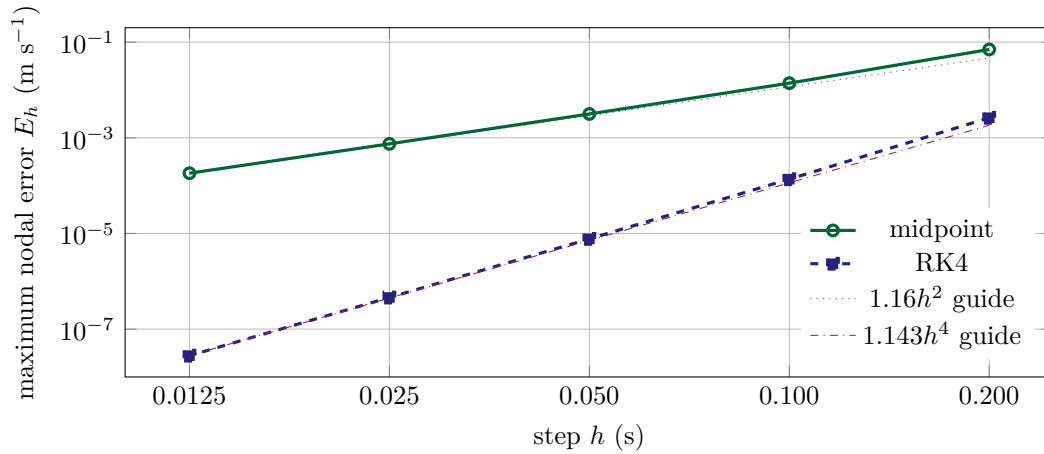


Figure 6.31: Independent benchmark errors on $0 \leq t \leq 2.00$ s reveal second- and fourth-order refinement. The two guide lines show slopes, not exact error laws. For the displayed formulas h is entered as its numerical value in seconds and error is shown in m s^{-1} .

An additional safeguard is qualitative. A release from rest must start with acceleration g and remain between zero and v_t in the exact model. A coarse numerical step can overshoot the equilibrium, oscillate, or even predict an unphysical reversal. The correct response is to investigate resolution and stability; changing the physical force law to rescue a bad integration step would mix two distinct problems. Near terminal motion the local quadratic relaxation equation is $\dot{\delta v} \simeq -\lambda \delta v$, with $\lambda = 2g/v_t$. An explicit method has a finite stable step range for this decay. Convergence as h decreases, rather than a visually smooth coarse curve alone, supplies the needed evidence.

6.8 Advanced Topics for Technical and Engineering Students

This section extends the inertial-frame method to coordinates fixed in an accelerating vehicle or test platform. The prerequisite is vector differentiation with nonrotating axes; the derivation below supplies the required coordinate relation. Rotating frames are excluded. The resulting inertial term is useful for accelerometers and translating inclines, but it must be kept distinct from real interaction forces.

6.8.1 A translating frame and the origin of its inertial term

Let an inertial laboratory frame have origin O , and let a moving origin O' have position $\mathbf{R}(t)$ measured from O . Choose the moving axes parallel to the fixed axes at all times. A body's positions satisfy

$$\mathbf{r}(t) = \mathbf{R}(t) + \mathbf{r}'(t). \quad (6.52)$$

Because the axes do not rotate, differentiating does not introduce derivatives of their basis vectors. Thus

$$\mathbf{v} = \dot{\mathbf{R}} + \mathbf{v}', \quad \mathbf{a} = \mathbf{A} + \mathbf{a}', \quad \mathbf{A} = \ddot{\mathbf{R}}. \quad (6.53)$$

These relations are geometric identities, independent of the body's force law. Applying Newton's law in the inertial frame and rearranging gives

$$\sum \mathbf{F}_{\text{real}} = m(\mathbf{A} + \mathbf{a}') \implies \boxed{\sum \mathbf{F}_{\text{real}} - m\mathbf{A} = m\mathbf{a}'} \quad (6.54)$$

The term $-m\mathbf{A}$ is called an inertial or fictitious force. It accounts for the accelerating origin in the transformed equation. It is not an interaction with another body and has no action–reaction partner. The real interactions, their magnitudes, and their third-law partners remain exactly the same in either description [2].

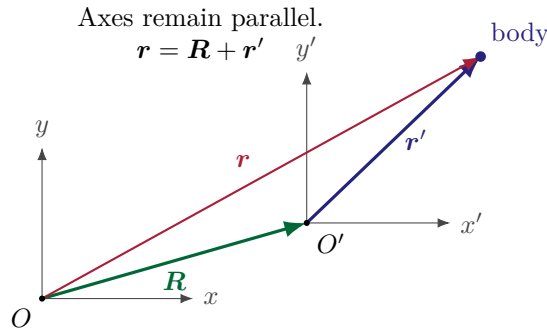


Figure 6.32: Vector geometry for a nonrotating translating frame. The transformation is valid for a prescribed origin motion $\mathbf{R}(t)$, including time-dependent translation. For the steady examples below the platform acceleration is constant. Lengths in this coordinate sketch are schematic.

In particular, a free body with no horizontal real force has $a_x = 0$ in the laboratory but $a'_x = -A_x$ in a platform accelerating horizontally. The relative acceleration comes from the platform changing its motion; it does not imply a previously missing physical interaction. Conversely, a body stationary relative to the platform has $\mathbf{a}' = 0$ but generally $\mathbf{a} = \mathbf{A} \neq 0$. Describing it as “in equilibrium” is meaningful only after specifying the frame and whether one is balancing real forces or the real forces plus the inertial term.

If $\mathbf{A} = 0$, Eq. (6.54) reduces to the ordinary inertial-frame equation. This includes origins moving at constant velocity. When the axes rotate, however, the basis vectors acquire time derivatives. Additional terms depending on angular velocity and angular acceleration appear, including Coriolis, centrifugal, and Euler terms. Adding only $-m\mathbf{A}$ then gives an incomplete equation. These translating examples therefore cannot be transferred unchanged to a rotating space station or to axes turning with a vehicle.

6.8.2 Pendulum accelerometer: geometry, calibration, and limitations

A hanging mass provides a direct visual acceleration indicator. Consider a platform accelerating right at constant $A > 0$, with negligible rotation and no vertical acceleration. After damping has allowed a steady configuration, the bob is stationary relative to the platform. In the laboratory it accelerates right along with the support. The string must therefore pull with a rightward component. Consequently the bob hangs to the *left* of the support, behind the acceleration direction. This geometric conclusion must be established before drawing the diagram.

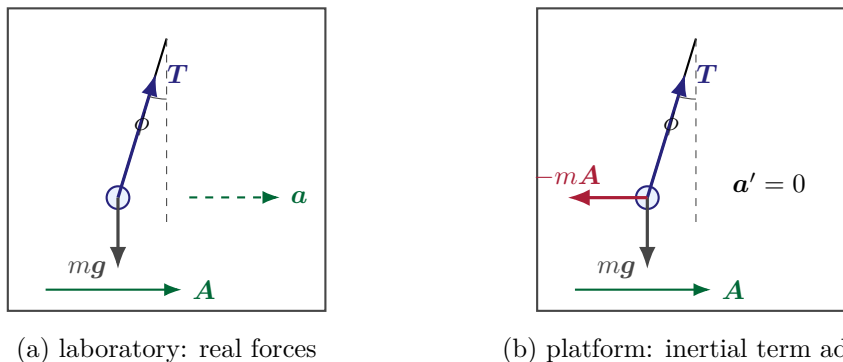


Figure 6.33: Correct string geometry for rightward platform acceleration: the bob hangs left of its support and tension pulls upward and right. Both panels show the same physical configuration, with tilt $\phi = \tan^{-1}(3.00/9.81)$. The laboratory acceleration arrow is dashed to distinguish a kinematic quantity from a force; the platform panel alone includes the inertial term. Force lengths are schematic.

Example 6.24: Pendulum accelerometer in two frames

A 0.250 kg bob hangs from a light string in a nonrotating platform accelerating right at 3.00 m s^{-2} . Find the steady tilt from vertical and the string tension. Then infer the acceleration corresponding to a measured steady tilt of $(12.0 \pm 0.5)^\circ$.

The bob is the system. In the laboratory, tension and weight are its only real forces. The bob's acceleration is A right and zero vertically. With ϕ denoting the positive tilt magnitude and the bob behind the support,

$$T \sin \phi = mA, \quad T \cos \phi = mg.$$

Dividing eliminates the unknown tension and mass:

$$\phi = \tan^{-1}(A/g) = 17.0^\circ, \quad T = m\sqrt{g^2 + A^2} = 2.56 \text{ N}.$$

In platform coordinates the bob has $\mathbf{a}' = 0$. Equation (6.54) balances $\mathbf{T} + m\mathbf{g} - m\mathbf{A} = 0$. Its horizontal and vertical components give the same equations and the same tension. Only the acceleration bookkeeping differs.

For the inverse reading, $A = g \tan \phi$ gives $A = 2.085 \text{ m s}^{-2}$. If $\pm 0.5^\circ$ is interpreted as a small standard angular uncertainty and g is treated as fixed, first-order propagation gives

$$\sigma_A = \left| \frac{dA}{d\phi} \right| \sigma_\phi = g \sec^2 \phi \sigma_\phi = 0.0895 \text{ m s}^{-2},$$

where $\sigma_\phi = 0.5\pi/180$ is in radians. The inferred result is $(2.09 \pm 0.09) \text{ m s}^{-2}$ under that uncertainty model. The bob mass affects tension but cancels from the calibration. The direction of tilt identifies the acceleration direction: a steady left tilt implies a rightward acceleration in this setup.

As $A \rightarrow 0$, $\phi \rightarrow 0$ and $T \rightarrow mg$, verifying the inertial limit. The 17.0° tilt is not sufficiently small to replace $\tan \phi$ by ϕ indiscriminately; the exact trigonometric calibration is simple and avoids that approximation.

The steady angle determines horizontal acceleration only when vertical acceleration and rotation are controlled. More generally, a bob fixed relative to a translating platform with acceleration (A_x, A_y) satisfies $T_x = mA_x$ and $T_y = m(g + A_y)$. Hence $\tan \phi = |A_x|/(g + A_y)$ when $g + A_y > 0$. A tilt by itself cannot distinguish an increased horizontal acceleration from a decreased effective downward gravity. Rapidly changing acceleration also produces a transient pendulum motion, so a single instantaneous angle need not equal the steady calibration. This model uses an assumed settled configuration; it does not claim an undamped pendulum automatically settles.

6.8.3 An accelerating incline: static, sliding, and detached regimes

An incline fixed to a translating platform is a useful test of both signs and contact limits. Let the plane rise to the right at angle $0 < \theta < 90^\circ$, and let A be the *signed* horizontal platform acceleration, positive rightward. Choose the tangent $\hat{\mathbf{t}} = (\cos \theta, \sin \theta)$ up the plane and the outward normal $\hat{\mathbf{n}} = (-\sin \theta, \cos \theta)$. The platform-frame effective body force per unit mass is $(-A, -g)$. Projecting, rather than guessing, gives

$$(-A, -g) \cdot \hat{\mathbf{t}} = -(g \sin \theta + A \cos \theta), \quad (-A, -g) \cdot \hat{\mathbf{n}} = A \sin \theta - g \cos \theta. \quad (6.55)$$

Define $D = g \sin \theta + A \cos \theta$ and $Q = g \cos \theta - A \sin \theta$. Both have acceleration units. If the body is in contact and has zero relative normal acceleration, the outward normal force must be

$$N = mQ = m(g \cos \theta - A \sin \theta). \quad (6.56)$$

A simple supporting surface requires $N \geq 0$, or $Q \geq 0$. A negative calculated N is a failed contact assumption, not a surface pulling the block downward.

For rest relative to the plane the required signed static friction is $f_s = mD$ along $+\hat{\mathbf{t}}$. The complete sticking conditions are

$$\boxed{Q \geq 0, \quad |D| \leq \mu_s Q.} \quad (6.57)$$

The friction reverses direction when D changes sign. In particular a frictionless plane can hold a body stationary relative to it only if $D = 0$, giving

$$A = -g \tan \theta, \quad (6.58)$$

so the platform must accelerate toward the lower end of the incline. For that solution $Q = g/\cos \theta > 0$, independently confirming valid supporting contact. Rightward acceleration adds to the down-slope effective component and cannot produce the frictionless balance for this geometry.

If contact remains but sticking fails, friction opposes relative sliding. With v'_t the relative tangential velocity and μ_k the kinetic coefficient,

$$v'_t = -D - \mu_k Q \operatorname{sgn}(v'_t), \quad Q \geq 0. \quad (6.59)$$

This equation applies during established slip. At rest one must test static feasibility first; a sign function at zero is not a substitute for that test. If a sliding body reaches zero relative velocity, the regime must be assessed again. Finally, if $Q < 0$, supporting contact is impossible and $N = 0$ after detachment. The relative outward normal acceleration is then $a'_n = -Q > 0$, while the body's laboratory acceleration is simply gravity if no other real forces act.

Example 6.25: An incline with three distinct outcomes

A 2.00 kg block rests on an incline rising rightward at 30.0° inside a horizontally translating platform. The stipulated coefficients are $\mu_s = 0.400$ and $\mu_k = 0.300$. Find the signed acceleration range permitting rest relative to the incline. Test $A = -6.00, 0, -15.0$, and $+20.0 \text{ m s}^{-2}$.

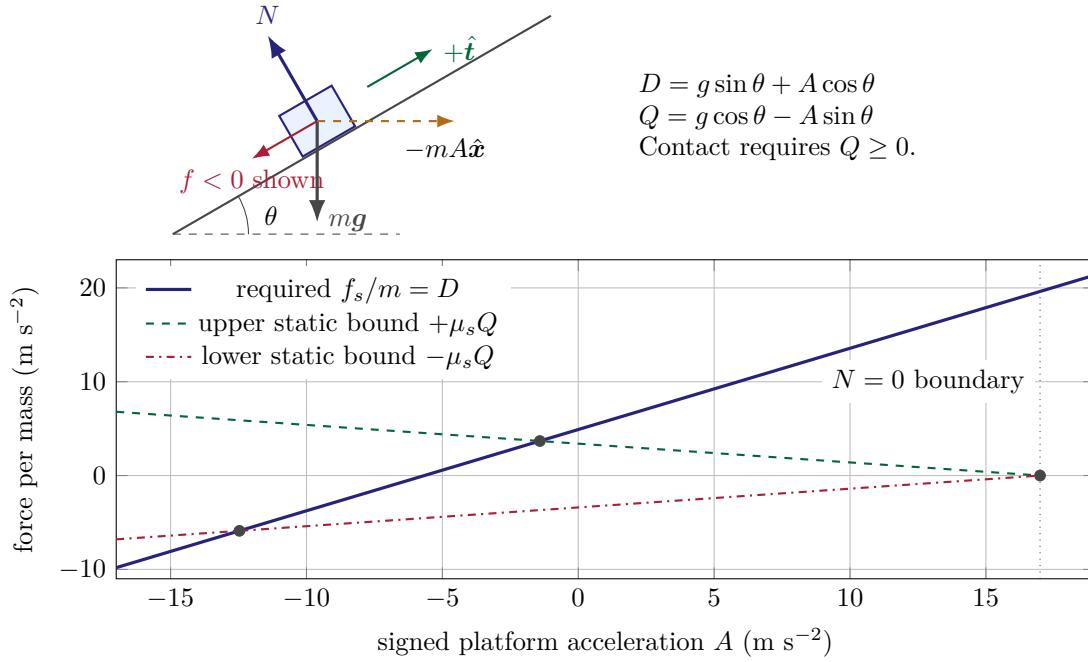


Figure 6.34: A plane rising rightward on a translating platform. The upper panel shows the chosen directions and real forces plus the explicitly identified platform-frame inertial term for a leftward acceleration; its arrow lengths are schematic. The lower panel is quantitative for $\theta = 30.0^\circ$ and $\mu_s = 0.400$. Sticking occurs where the required signed friction lies between the two static bounds and $N \geq 0$. Beyond the contact boundary the supported-contact formulas are inapplicable.

Projection gives $D = 4.905 + 0.866025A$ and $Q = 8.49571 - 0.500A$ in m s^{-2} . The contact inequality is $A \leq 16.9914 \text{ m s}^{-2}$. The static bound $-\mu_s Q \leq D \leq \mu_s Q$ gives, for these particular coefficients,

$$-\frac{g(\mu_s \cos \theta + \sin \theta)}{\cos \theta - \mu_s \sin \theta} \leq A \leq \frac{g(\mu_s \cos \theta - \sin \theta)}{\cos \theta + \mu_s \sin \theta},$$

where both denominators are positive. Numerically,

$$-12.47 \leq A \leq -1.413 \text{ m s}^{-2}.$$

The entire interval satisfies contact positivity. For other angles or coefficients, one must solve the original inequalities and check denominator signs before carrying this interval formula across unchanged.

At $A = -6.00$, $N = mQ = 22.99 \text{ N}$ and the required signed friction is $f_s = mD = -0.5823 \text{ N}$, directed down the plane. Its magnitude is well below $\mu_s N = 9.20 \text{ N}$, so the block sticks. This direction follows from a slight tendency to move uphill in the effective field; it is not determined by the incline alone.

At $A = 0$, static contact is impossible since $\tan 30^\circ > 0.400$. Upon established down-slope slip, $v'_t < 0$ and friction points uphill. Equation (6.59) gives $a'_t = -D + \mu_k Q = -2.36 \text{ m s}^{-2}$; the acceleration is downhill. At $A = -15.0$, contact is still possible but the static demand is below the lower bound. From rest the tendency is uphill; established uphill slip has $a'_t = -D - \mu_k Q = +3.29 \text{ m s}^{-2}$.

At $A = +20.0$, $Q = -1.50429 \text{ m s}^{-2}$, so constrained contact would require $N = -3.01 \text{ N}$. A supporting plane cannot supply it. The block detaches, $N = f = 0$, and its relative outward normal acceleration is $-Q = 1.50 \text{ m s}^{-2}$. In the laboratory its only remaining modeled force is weight. These cases illustrate why friction feasibility and normal-force feasibility must be tested separately.

An acceleration instrument and a constrained block both respond to the same transformed effective field, but their mechanical constraints differ. The string can pull along its length; the plane can push outward and provide a bounded friction force. A formal vector balance must therefore be supplemented by the physical admissibility of each interaction. That is the same modeling discipline used in the core contact and circular-motion problems, now applied in a noninertial coordinate system.

6.9 Exercises and problems

Use $g = 9.81 \text{ m s}^{-2}$. All coefficients below are stipulated parameters of the stated ideal models. Unless indicated otherwise, ignore air resistance, treat each body as a particle, and use an inertial laboratory frame. In calculations, identify the body, resolve the real forces, state the proposed sticking or contact condition, and test it after solving. A negative result for a one-sided normal force means that the assumed contact has failed. The last twelve items include optional mathematical or applied extensions; Problems P06-31–P06-36 require the translating-frame section. Complete solutions follow the statements.

A. Friction and inclined contacts

- P06-01.** A 6.0 kg case rests on a horizontal conveyor with $\mu_s = 0.30$ and $\mu_k = 0.22$. (a) Find the largest belt acceleration for which the case can remain fixed relative to the belt. (b) If the belt accelerates right at 4.0 m s^{-2} , find the case acceleration immediately after sliding is established. Describe the relative slip direction.
- P06-02.** A student writes $f_s = \mu_s N$ for a motionless book on a level desk with no horizontal applied force. Diagnose the error and give the actual friction force. Explain what $\mu_s N$ represents.
- P06-03.** A package slides right in the laboratory while a conveyor beneath it moves right even faster. (a) Draw or describe the force diagram of the package, including friction. (b) Explain why “friction opposes motion” is an incomplete rule. State what happens if the two velocities become equal.
- P06-04.** A 20.0 kg block rests on an 18.0° incline with $\mu_s = 0.40$. (a) Determine whether it can remain at rest with no applied force. (b) Find its actual friction and normal forces if it can. Explain why the limiting-friction formula is unnecessary for the actual friction magnitude.
- P06-05.** A 15.0 kg block lies on a 30.0° incline. A cable parallel to the plane pulls uphill. With $\mu_s = 0.45$, (a) find the complete cable-tension interval for static equilibrium; (b) identify the impending slip and friction directions at each endpoint; (c) identify the tension at which friction vanishes.
- P06-06.** Repeat P06-05 when the cable is directed 20.0° above the plane. (a) Derive the tension interval. (b) Find and compare the normal forces at its two endpoints. (c) Explain why lifting the block partially changes both thresholds, and test the contact condition.
- P06-07.** A worker pushes a crate down and right with force P at 25.0° below horizontal. (a) Derive the no-slip inequality for a level floor. (b) Explain the role of the downward component. (c) State when this inequality gives a finite upper bound on P and what the idealized model predicts if that condition is not met.
- P06-08.** A numerical solver reports the required static-friction magnitude as $1.08\mu_s N$. Why is clipping this answer to $\mu_s N$ while leaving the assumed sticking kinematics unchanged physically incomplete? Describe a consistent next step.

B. Circular and curved motion

- P06-09.** A 0.40 kg puck moves at 6.0 m s^{-1} on a horizontal circle of radius 1.5 m, held by a horizontal string. Find the tension and identify the net centripetal force. Does the string do the job of an additional force named “centripetal force”?
- P06-10.** A car follows a flat circular path of radius 45 m with $\mu_s = 0.60$. (a) Find the maximum constant speed in the no-slip point-particle model. (b) State two real-vehicle effects omitted by the result.
- P06-11.** A frictionless banked turn has $R = 80 \text{ m}$ and bank angle 12.0° . (a) Find the ideal speed. (b) Find the required bank angle if the design speed doubles at the same radius. Explain why the angle itself does not simply quadruple.
- P06-12.** A 25 kg cart moves over the *outside* top of a circular hill of radius 20 m. (a) Derive the normal force as a function of speed. (b) Find the contact-loss speed. (c) Explain why continuing the formula to negative normal force does not describe the detached trajectory.
- P06-13.** A small carriage moves on the *inside* of a vertical circular guide of radius 2.5 m. At its top, its speed is 4.0 m s^{-1} . Assume simple one-sided contact without a retaining rail. (a) Test contact feasibility. (b) Explain precisely which assumption fails and what cannot be inferred about the later trajectory from the data supplied.
- P06-14.** A seat supports a 70 kg passenger at the bottom of a circular arc of radius 18 m at 14 m s^{-1} . Find the normal force and its ratio to the passenger's weight. Explain the passenger's apparent heaviness.
- P06-15.** The local radius of curvature of a path decreases from 60 m to 30 m while the speed remains 12 m s^{-1} . Calculate both normal accelerations and the factor by which the required normal resultant changes. Is either result an individual force without additional information?

C. Shared traction and graphical reasoning

- P06-16.** In the simplified friction-circle model, a vehicle with $\mu_s = 0.75$ negotiates a 60 m radius turn at 15 m s^{-1} . Find the largest simultaneous tangential acceleration magnitude without exceeding the force budget. Explain why the mass cancels.
- P06-17.** At the limit found in P06-16, calculate the normalized components $f_t/(\mu_s N)$ and $f_n/(\mu_s N)$ for positive tangential acceleration. Verify that their squared sum is one. Describe their location in the traction diagram in Figure 6.35.
- P06-18.** A report checks $|f_t| < \mu_s N$ and $|f_n| < \mu_s N$ separately and concludes that no slip can occur. Construct a dimensionless component pair that passes both tests but violates the circular budget. Use Figure 6.35 to explain the erroneous feasible region.
- P06-19.** A vehicle follows a circular path at constant speed. Does $f_t = 0$ imply that the total horizontal tyre force is zero? Explain using tangent–normal components and distinguish constant speed from constant velocity.
- P06-20.** A planner commands $a_t = 4.0 \text{ m s}^{-2}$ and $a_n = 6.0 \text{ m s}^{-2}$ for a surface represented by $\mu_s = 0.70$. (a) Test the budget. (b) Find the largest permitted $|a_t|$ if a_n is retained. (c) Alternatively find a common scale factor that preserves the acceleration-vector direction while placing the command on the boundary.

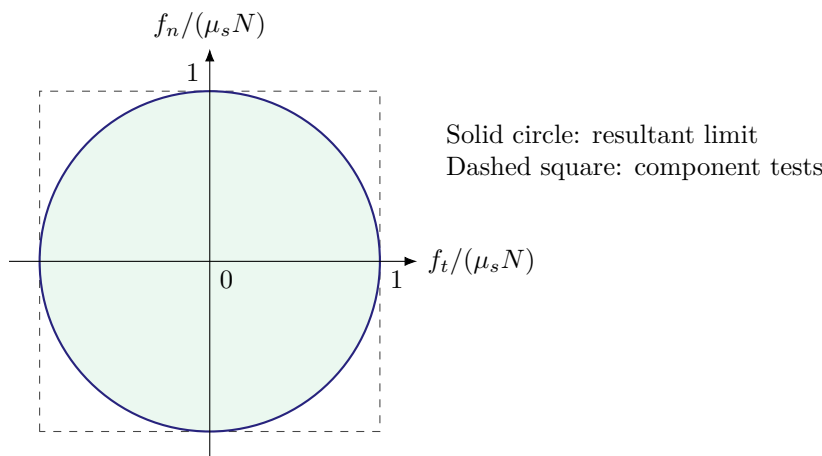


Figure 6.35: Traction-budget geometry for P06-17 and P06-18. The axes are dimensionless force components. The admissible resultant lies in the filled disk; checking each component independently admits the larger dashed square. Students supply the particular force points.

D. Drag, data, and numerical validation

- P06-21.** Determine the SI units of b in $F_d = bv$ and c in $F_d = cv^2$ by dimensional analysis. State the vector direction needed to make these magnitude laws resist motion relative to the medium.
- P06-22.** A 0.060 kg object falls from rest under linear drag with $b = 0.12 \text{ kg s}^{-1}$. (a) Find the terminal speed and time constant. (b) Find its speeds at $t = \tau$ and $t = 3\tau$. (c) Give the corresponding acceleration fractions a/g .
- P06-23.** A 0.10 kg object has quadratic coefficient $c = 0.040 \text{ kg m}^{-1}$. (a) Find its downward terminal speed. (b) At half that speed, find the drag fraction of the weight and the downward acceleration fraction of g .
- P06-24.** The wind moves east at 8.0 m s^{-1} . A balloon moves east at 5.0 m s^{-1} in the laboratory. Determine the linear-drag direction and its signed component in terms of b . Repeat for a balloon velocity of 10.0 m s^{-1} . Take east as positive.
- P06-25.** For downward-positive linear drag in a stationary medium, derive $v(t)$ for arbitrary signed initial velocity v_0 . Verify the ODE, initial condition, and long-time limit directly. Your formula must remain valid while an initially upward-moving body turns around.
- P06-26.** Show that a body falling from rest under linear drag reaches 90% of its terminal speed at $t_{90} = \tau \ln 10$. Evaluate this time for $m = 0.0800 \text{ kg}$ and $b = 0.160 \text{ kg s}^{-1}$.
- P06-27.** Use the explicit-midpoint method for $\dot{v} = g - (c/m)v|v|$, released from rest, with $m = 0.0800 \text{ kg}$ and $c = 0.0320 \text{ kg m}^{-1}$. (a) Give an update algorithm. (b) Compute $v(2.00 \text{ s})$ using $\Delta t = 0.20, 0.10, 0.05 \text{ s}$ and compare with $v_t \tanh(gt/v_t)$. (c) Explain what error ratios test, and why endpoint error alone can mislead.
- P06-28.** Linear and quadratic drag laws are calibrated to have the same terminal speed for one body. Name a transient observable that distinguishes them, derive a relation that demonstrates the distinction, and explain why their identical terminal speed cannot identify the force law.
- P06-29.** An onset-angle measurement is repeated ten times while the contact surface becomes progressively contaminated. Explain why reporting the standard deviation alone does not describe the model error. Propose a diagnostic and a more defensible measurement protocol.
- P06-30.** Refining the time step gives nearly identical terminal speeds but visibly different early-time

curves. Explain why terminal speed is a weak numerical diagnostic. Give two stronger tests and distinguish discretization error from an inadequate force law.

E. Translating frames and advanced challenges

- P06-31.** Derive the translating-frame equation from $\mathbf{r} = \mathbf{R} + \mathbf{r}'$ when the axes do not rotate. Identify where the frame acceleration enters and check the zero-acceleration limit.
- P06-32.** A platform accelerates right at 4.0 m s^{-2} . A block rests on its level floor with $\mu_s = 0.35$. Decide whether it can remain fixed relative to the platform using (a) the laboratory frame and (b) the platform frame. Explain what information is needed to predict the ensuing sliding acceleration.
- P06-33.** A pendulum accelerometer reads a steady angle of 12.0° from vertical. Find the horizontal platform-acceleration magnitude. Does bob mass matter? Describe the direction of platform acceleration relative to the bob's displacement from the point of suspension.
- P06-34.** A frictionless incline rises rightward at angle $0 < \theta < 90^\circ$ inside a platform stipulated to accelerate *right* with magnitude $A \geq 0$. Can a block remain fixed relative to the incline? Derive the required *signed* horizontal acceleration, state its direction, and test the normal force. Resolve the conflict between the stated direction and the equilibrium requirement.
- P06-35.** Explain why the inertial term $-m\mathbf{A}$ has no Newton's-third-law partner. Contrast it with the normal force on the block. Explain why neither term should be added to an inertial-frame force diagram indiscriminately.
- P06-36.** A rotating platform is analyzed by adding only $-m\mathbf{A}$ for acceleration of its centre. Explain why this is incomplete for a body moving relative to rotating axes, without deriving the omitted terms. Identify the assumptions under which the simpler translating-frame equation remains valid.

F. Synthesis, graphical models, and engineering interpretation

- P06-37.** A banked circular turn has $R = 60.0 \text{ m}$, $\theta = 20.0^\circ$, and $\mu_s = 0.25$. The road rises outward. (a) Derive the complete constant-speed interval for sticking contact, allowing friction to point uphill or downhill. (b) Find its numerical endpoints. (c) Explain the friction directions and verify $N > 0$ at both endpoints. Treat the vehicle as a point particle without aerodynamic forces.
- P06-38.** A 10.0 kg block starts at rest on a level floor with $\mu_s = 0.40$ and $\mu_k = 0.30$. A horizontal rightward force P increases smoothly from zero to 60.0 N . (a) Construct the friction-magnitude versus P graph, showing the sticking and established rightward-sliding branches and marking the threshold. (b) Find the friction and acceleration at $P = 20.0 \text{ N}$ and at $P = 50.0 \text{ N}$ on the appropriate branches. (c) Explain the jump between the idealized branches and whether a history that reduces P would necessarily retrace this graph.
- P06-39.** Six *synthetic* onset angles are $20.8^\circ, 21.2^\circ, 20.9^\circ, 21.1^\circ, 20.7^\circ, 21.3^\circ$. Assume independent random angle errors, an unchanged surface, and a common independent calibration standard uncertainty of 0.20° . (a) Find the mean, sample standard deviation, and standard error of the mean. (b) Infer μ_s and propagate both uncertainty contributions to it. (c) Explain why the calibration contribution is not divided by $\sqrt{6}$, and give one physical limitation of interpreting the result as a material constant.
- P06-40.** An 8.0 kg block rests on a 12.0 kg block. The lower block slides right on a level floor with kinetic coefficient 0.20 . Between the blocks, $\mu_s = 0.30$ and $\mu_k = 0.22$. A horizontal force F acts right on the lower block; no other horizontal force acts on the upper block. (a) Find the

largest F allowing both blocks to move together with nonnegative acceleration. (b) Find their common acceleration and interlayer friction at $F = 70.0 \text{ N}$. (c) At $F = 120 \text{ N}$, find both accelerations immediately after interlayer sliding is established, assuming the lower block outruns the upper one. Use Figure 6.36 and check this assumption.

P06-41. A body of mass 0.0800 kg is launched upward at 6.00 m s^{-1} in a stationary medium with linear coefficient $b = 0.160 \text{ kg s}^{-1}$. Take downward as positive and the launch position as $y = 0$. (a) Find the turnaround time. (b) Integrate the velocity to find the maximum height above launch. (c) Explain the drag direction before and after turnaround and compare the height with the vacuum result. Neglect buoyancy.

P06-42. Two drag models have the same downward terminal speed $v_t = 4.905 \text{ m s}^{-1}$ for $m = 0.0800 \text{ kg}$: a linear law and a quadratic law. (a) Find their coefficients. (b) Find the time each takes to reach $0.90v_t$ from rest. (c) At $v = 0.50v_t$, compare their accelerations. (d) Propose a measurement that distinguishes the models while separating timing uncertainty from numerical error. Give a worked reference approach and criteria for judging it.

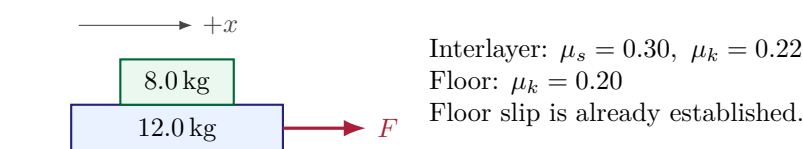


Figure 6.36: Two-contact system for P06-40. The applied force acts only on the lower block. The masses and contact models are given; friction arrows must be inferred from the proposed common motion or relative slip. Geometry is schematic.

6.10 Complete solutions to the exercises and problems

These solutions retain the same stipulated models as their statements. The force diagrams contain forces on the selected body, rather than third-law partners acting on another body. Numerical values retain guard digits during calculation and are rounded for presentation. Symbolic conditions, especially a friction bound or $N \geq 0$, decide whether the proposed regime exists.

P06-01: conveyor threshold and established sliding

(a) For the case, the vertical forces give $N = mg = 58.86 \text{ N}$. If it sticks to the belt, its horizontal acceleration must equal a_b . The required rightward friction is $f_s = ma_b$, with the feasibility condition $|ma_b| \leq \mu_s mg$. Consequently,

$$|a_b| \leq \mu_s g, \quad a_{b,\max} = 2.943 \text{ m s}^{-2} \simeq 2.94 \text{ m s}^{-2}.$$

The maximum available static force is 17.658 N ; at a smaller demand the actual friction is smaller.

(b) At $a_b = 4.0 \text{ m s}^{-2}$, sticking would require 24.0 N , exceeding the bound. Once the belt outruns the case, the case slips left relative to the belt, so kinetic friction on it points right. Thus $f_k = \mu_k mg = 12.9492 \text{ N}$ and

$$a_{\text{case}} = f_k/m = \mu_k g = 2.1582 \text{ m s}^{-2} \simeq 2.16 \text{ m s}^{-2}.$$

Since $a_{\text{case}} - a_b = -1.8418 \text{ m s}^{-2}$, relative motion initially grows in the direction assumed. The block can accelerate right while slipping left relative to its support.

P06-02: static friction is determined by demand

The book has zero acceleration and no other horizontal force. Its horizontal Newton equation therefore gives $f_s = 0$. The quantity $\mu_s N$ is the maximum available magnitude in the dry-friction

model, not an automatically present force. The inequality $|f_s| \leq \mu_s N$ includes zero. A horizontal force applied gradually would cause static friction to adjust to balance that force until the bound is reached. Weight and the normal force act vertically; they do not create a horizontal friction demand merely because contact exists.

P06-03: identify the slipping contact

(a) Taking right as positive, $v_{\text{rel}} = v_{\text{package}} - v_{\text{belt}} < 0$. The package's contacting surface slides left over the belt, so the belt's kinetic-friction force on the package points right. Its other forces are the upward normal force and downward weight, as Figure 6.37 shows. For a level belt without vertical acceleration, $N = mg$.

(b) Kinetic friction opposes *relative sliding at the contact*, not an absolute laboratory velocity. Both laboratory velocities here are positive, while the relative velocity is negative. At equal velocities, the kinetic law's assumed slip direction is no longer established. One must test whether static friction can supply the acceleration required to keep the two surfaces together. Equal instantaneous velocities alone do not guarantee lasting sticking; unequal required accelerations can immediately restart slip.

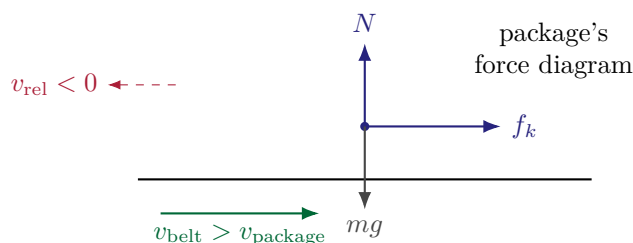


Figure 6.37: Solution of P06-03. Solid arrows are real forces on the package; dashed relative velocity is a kinematic annotation, rather than a force. The belt outruns the package, so friction on the package points right. Arrow lengths are schematic.

P06-04: a bare incline

Choose uphill as positive and outward normal to the plane as the other axis. For rest, $f_s - mg \sin \theta = 0$ and $N - mg \cos \theta = 0$. Therefore the required friction is uphill, with

$$N = 186.597 \text{ N} \simeq 187 \text{ N}, \quad f_s = 60.6291 \text{ N} \simeq 60.6 \text{ N}.$$

Its capacity is $\mu_s N = 74.6389 \text{ N}$, so the required value lies within the admissible interval. Equivalently, $\tan 18^\circ = 0.32492 < 0.40$. Static equilibrium is possible, assuming the block is initially at rest. Substituting $f_s = \mu_s N$ would incorrectly describe impending slip and would predict a nonzero net tangential force for this motionless block. In the zero-angle limit the same equations correctly give $f_s = 0$ and $N = mg$.

P06-05: both ends of the static-tension interval

(a) The parallel cable has no normal component, so $N = mg \cos 30^\circ = 127.4356 \text{ N}$. Taking friction positive uphill, tangential balance gives $f_s = mg \sin 30^\circ - T$. The bound becomes

$$-\mu_s mg \cos 30^\circ \leq mg \sin 30^\circ - T \leq \mu_s mg \cos 30^\circ.$$

Solving both inequalities yields

$$T_- = mg(\sin 30^\circ - \mu_s \cos 30^\circ) = 16.2290 \text{ N}, \quad T_+ = mg(\sin 30^\circ + \mu_s \cos 30^\circ) = 130.921 \text{ N}.$$

Thus $16.2 \text{ N} \leq T \leq 130.9 \text{ N}$ to the displayed rounding; the exact unrounded expressions define the endpoints. Both are nonnegative and $N > 0$ throughout.

(b) At T_- the uphill cable force is too small to balance gravity by itself: impending slip is downhill and friction is uphill at its positive maximum. At T_+ the cable tends to pull the block uphill: impending slip is uphill and friction is downhill. This reversal is why one cannot fix the friction direction before inspecting the demand.

(c) Friction vanishes at $T = mg \sin 30^\circ = 73.575 \text{ N}$. Cable and gravity components then balance without tangential contact force. This tension is inside the static interval, as a consistency check.

P06-06: an angled cable unloads the contact

(a) With $\beta = 20^\circ$ measured above the plane, $N = mg \cos \theta - T \sin \beta$ and $f_s = mg \sin \theta - T \cos \beta$. Setting the two limiting friction signs separately gives

$$T_- = \frac{mg(\sin \theta - \mu_s \cos \theta)}{\cos \beta - \mu_s \sin \beta}, \quad T_+ = \frac{mg(\sin \theta + \mu_s \cos \theta)}{\cos \beta + \mu_s \sin \beta}.$$

Both denominators are positive for these data, so solving the inequalities does not reverse their senses. Evaluation gives $T_- = 20.6532 \text{ N}$ and $T_+ = 119.715 \text{ N}$: the sticking interval is approximately $20.7 \leq T \leq 119.7 \text{ N}$.

(b) The corresponding normals are

$$N(T_-) = 120.372 \text{ N}, \quad N(T_+) = 86.4905 \text{ N}.$$

Both are positive; the upper-tension end is more strongly unloaded. At the lower end, friction is uphill and downhill slip is impending; at the upper end the signs reverse, as in P06-05.

(c) Compared with the parallel cable, the uphill pull per unit tension decreases to $T \cos \beta$, while the available friction also decreases because $T \sin \beta$ lifts away from the plane. Both effects matter: the lower threshold rises and the upper threshold falls. Contact would require $T \leq mg \cos \theta / \sin \beta$, which is less restrictive than the derived interval for these parameters. Merely replacing T by $T \cos \beta$ while retaining $N = mg \cos \theta$ would miss this coupling. At $\beta = 0$, both equations recover the parallel-cable result.

P06-07: a downward push

For a stationary crate, vertical balance gives $N = mg + P \sin 25^\circ$. The floor must supply leftward friction of magnitude $P \cos 25^\circ$. Hence

$$P \cos 25^\circ \leq \mu_s (mg + P \sin 25^\circ), \quad P(\cos 25^\circ - \mu_s \sin 25^\circ) \leq \mu_s mg.$$

The downward component raises the normal load and therefore the friction capacity. If $\cos 25^\circ - \mu_s \sin 25^\circ > 0$, a finite limiting push is

$$P_{\max} = \frac{\mu_s mg}{\cos 25^\circ - \mu_s \sin 25^\circ}.$$

If this coefficient is zero or negative, the ideal inequality is satisfied for every $P \geq 0$: increasing the push raises the modeled capacity at least as rapidly as it raises the tangential demand. This prediction belongs to the ideal constant-coefficient, rigid-contact model; crushing, deformation, or a changed friction coefficient could invalidate its use at large loads. It does not establish unlimited physical load capacity.

P06-08: a failed constraint requires a new regime

The computed demand exceeds the static bound by 8%, so the assumed sticking trajectory cannot satisfy both Newton's law and the contact law. Clipping only the force changes the resultant and therefore the acceleration, contradicting the unchanged kinematics. A consistent calculation releases the no-slip constraint, identifies the resulting relative-slip tendency, applies an appropriate sliding law, and solves again for the motion. If only μ_s is known, the test establishes loss of sticking but does not determine a numerical sliding acceleration. A kinetic coefficient or another contact model is required. At an event where relative velocity returns to zero, a new static-demand test decides whether re-sticking is possible.

P06-09: tension supplies the radial resultant

The vertical normal force and weight balance for the horizontal motion. The only horizontal force on the puck is the inward string tension, so

$$T = m \frac{v^2}{R} = \frac{(0.40)(6.0)^2}{1.5} = 9.6 \text{ N}.$$

The net centripetal force is therefore the tension itself. "Centripetal" specifies the inward component of the real-force resultant required by the circular acceleration; it is not an additional interaction. Adding another inward force of the same magnitude would double-count the force. The result also checks dimensionally because mv^2/R has units kg m s^{-2} .

P06-10: the flat-turn speed bound

The vertical balance is $N = mg$. At constant speed the required horizontal inward force is mv^2/R , supplied by static tyre friction. It is feasible if $mv^2/R \leq \mu_s mg$, giving

$$v_{\max} = \sqrt{\mu_s g R} = \sqrt{(0.60)(9.81)(45)} = 16.2748 \text{ m s}^{-1} \simeq 16.3 \text{ m s}^{-1}.$$

The mass cancels within this model. Real effects omitted include load transfer among wheels and the detailed dependence of tyre force on slip, surface condition, and temperature. Aerodynamic forces, suspension motion, and simultaneous braking also change the available force budget. Any two clearly explained examples meet part (b); merely naming "friction" does not, because friction is already the modeled mechanism. The calculation is a stipulated model limit.

P06-11: a frictionless bank

With inward horizontal and upward vertical axes, $N \sin \theta = mv^2/R$ and $N \cos \theta = mg$. Their ratio removes both N and m :

$$\tan \theta = \frac{v^2}{Rg}, \quad v = \sqrt{(80)(9.81) \tan 12^\circ} = 12.9157 \text{ m s}^{-1} \simeq 12.9 \text{ m s}^{-1}.$$

Doubling the speed at fixed R quadruples $v^2/(Rg)$, so $\tan \theta_{\text{new}} = 4 \tan 12^\circ$ and

$$\theta_{\text{new}} = \tan^{-1}(4 \tan 12^\circ) = 40.3721^\circ \simeq 40.4^\circ.$$

Only the tangent quadruples. The approximation $\tan \theta \simeq \theta$ requires a small radian angle and is not justified at the new angle. The normal remains positive since $N = mg/\cos \theta$ for both designs.

P06-12: contact on an outside hill

At the crest, inward is downward. Gravity points inward and the road's normal force points outward. For circular contact,

$$mg - N = m \frac{v^2}{R}, \quad N = m \left(g - \frac{v^2}{R} \right).$$

For $m = 25 \text{ kg}$ and $R = 20 \text{ m}$, this is $N = 25(9.81 - v^2/20) \text{ N}$ when v is inserted in m s^{-1} . The contact boundary is $N = 0$, giving

$$v_{\text{loss}} = \sqrt{gR} = \sqrt{196.2} = 14.0071 \text{ m s}^{-1} \simeq 14.0 \text{ m s}^{-1}.$$

For a larger instantaneous crest speed the road would have to pull the cart downward to enforce this curvature. One-sided contact cannot do so. After detachment, $N = 0$ and the cart is in free flight; its path curvature must be determined from its actual position and velocity. Continuing the circular-path formula to $N < 0$ extrapolates an inadmissible constraint rather than describing that flight.

P06-13: insufficient speed at an inside-loop top

At the top of an inside guide, both gravity and an admissible normal point toward the centre, downward. Therefore

$$mg + N = m \frac{v^2}{R}, \quad \frac{N}{m} = \frac{4.0^2}{2.5} - 9.81 = -3.41 \text{ m s}^{-2}.$$

This requires an outward pulling normal, which the stipulated one-sided guide cannot supply. The minimum speed for admissible contact there is $\sqrt{gR} = 4.95227 \text{ m s}^{-1}$, greater than 4.0 m s^{-1} . The circular-contact assumption has failed. A negative algebraic normal is not a permissible force for this model. The earlier detachment point and release history cannot be inferred from the given top speed. After the contact assumption fails, a speed alone also leaves the full velocity direction unspecified. If a complete instantaneous position and velocity are supplied, the subsequent free-flight trajectory is determined directly by those initial conditions and gravity. A retaining rail would introduce a different contact model and could change the conclusion.

P06-14: apparent weight at the bottom

At the bottom, inward is upward. The passenger's normal force is inward and gravity is outward, so

$$N - mg = m \frac{v^2}{R}, \quad N = 70 \left(9.81 + \frac{14^2}{18} \right) = 1448.922 \text{ N} \simeq 1.45 \text{ kN}.$$

The weight is $mg = 686.7 \text{ N}$, and $N/(mg) = 2.10998 \simeq 2.11$. The passenger feels heavier because the seat must both oppose gravity and supply an upward resultant for the changing velocity direction. Gravity itself has not increased. At $v = 0$ the same local force equation gives $N = mg$, consistent with no curvature acceleration at that instant.

P06-15: curvature and the normal resultant

The two values are $a_{n,1} = 12^2/60 = 2.40 \text{ m s}^{-2}$ and $a_{n,2} = 12^2/30 = 4.80 \text{ m s}^{-2}$. At fixed mass the required net normal component ma_n doubles. Neither value by itself determines a normal contact force, tension, or friction. Those depend on which interactions have normal components. The word "normal" here refers to the direction perpendicular to the path tangent; it does not automatically mean the surface contact force N . Constant speed means $a_t = 0$ but leaves the direction-changing normal acceleration intact.

P06-16: turning and changing speed share one budget

Under the flat-road model, $N = mg$, $f_t = ma_t$, and $f_n = mv^2/R$. The resultant must obey $\sqrt{f_t^2 + f_n^2} \leq \mu_s mg$. Dividing by m gives

$$a_t^2 + \left(\frac{v^2}{R} \right)^2 \leq (\mu_s g)^2.$$

Here $a_n = 15^2/60 = 3.75 \text{ m s}^{-2}$ and the available total acceleration is $0.75g = 7.3575 \text{ m s}^{-2}$. Thus

$$|a_t|_{\max} = \sqrt{7.3575^2 - 3.75^2} = 6.33011 \text{ m s}^{-2} \simeq 6.33 \text{ m s}^{-2}.$$

Both the force demand and the capacity scale with m , which is why it cancels. The symmetric magnitude bound includes idealized speeding up and braking; it does not assert that a real vehicle has identical performance in both directions.

P06-17: normalized traction components

For positive limiting tangential acceleration,

$$\frac{f_t}{\mu_s N} = \frac{6.33011}{7.3575} = 0.860361, \quad \frac{f_n}{\mu_s N} = \frac{3.75}{7.3575} = 0.509684.$$

Their squared sum equals one before rounding because the tangential bound was obtained by the Pythagorean resultant equation. Rounded to three decimals the coordinates are $(0.860, 0.510)$, on the first-quadrant circular boundary in Figure 6.35. Reversing the tangential component puts the same normal demand on the upper-left portion of the boundary; it does not change the available magnitude.

P06-18: the square is too large

Choose $(f_t/(\mu_s N), f_n/(\mu_s N)) = (0.8, 0.8)$. Both magnitudes are less than one, so each separate component test passes. The squared normalized resultant is $0.8^2 + 0.8^2 = 1.28$, giving resultant magnitude $\sqrt{1.28} = 1.13137 > 1$. This point lies inside the dashed square but outside the solid circle. The independent tests therefore admit forces that the circular model cannot supply. The issue is vector magnitude, not the signs of the components or insufficient numerical precision. Other pairs within the square and outside the disk are equally valid counterexamples.

P06-19: constant speed still needs an inward force

At constant speed $a_t = dv/dt = 0$, but the unit tangent rotates. The normal acceleration is $a_n = v^2/R$, so $f_n = mv^2/R$ remains nonzero when $v > 0$ and R is finite. The total horizontal force is therefore inward despite vanishing tangential force. Constant velocity requires both unchanging speed and unchanging direction; circular motion satisfies only the first. A third-law force from the tyres on the road acts on the road and does not cancel the road's inward force in the vehicle's own equation.

P06-20: repair an inadmissible command

(a) The commanded magnitude is $\sqrt{4.0^2 + 6.0^2} = \sqrt{52} = 7.21110 \text{ m s}^{-2}$, while the available model magnitude is $0.70g = 6.867 \text{ m s}^{-2}$. The command exceeds the capacity by about 5.01% relative to the capacity, so the proposed no-slip motion is infeasible.

(b) Retaining $a_n = 6.0 \text{ m s}^{-2}$ leaves

$$|a_t| \leq \sqrt{6.867^2 - 6.0^2} = 3.34001 \text{ m s}^{-2}.$$

Reducing $|a_t|$ to this boundary, or below it, respects the stated budget. In a planned circular turn, lowering speed or increasing radius also lowers a_n .

(c) Scaling both components by λ retains their direction. The boundary value is $\lambda = 6.867/\sqrt{52} = 0.952282$, giving $(a_t, a_n) = (3.80913, 5.71369) \text{ m s}^{-2}$. Their resultant is exactly 6.867 m s^{-2} before rounding. A practical planner would need justified margins and a validated vehicle model in addition to this ideal constraint.

P06-21: coefficient dimensions and force direction

Because $[F] = \text{kg m s}^{-2}$ and $[v] = \text{m s}^{-1}$,

$$[b] = [F]/[v] = \text{kg s}^{-1}, \quad [c] = [F]/[v]^2 = \text{kg m}^{-1}.$$

For isotropic resistance, the force opposes the medium-relative velocity $\mathbf{v}_{\text{rel}}$. Thus $\mathbf{F}_d = -b\mathbf{v}_{\text{rel}}$ or $\mathbf{F}_d = -c|\mathbf{v}_{\text{rel}}|\mathbf{v}_{\text{rel}}$, with nonnegative coefficients. The scalar expression cv^2 specifies a magnitude; using $-cv^2$ as a signed one-dimensional force regardless of velocity would give the wrong direction for negative v . The dissipative relative power is $-b|\mathbf{v}_{\text{rel}}|^2$ or $-c|\mathbf{v}_{\text{rel}}|^3$, respectively.

P06-22: linear-drag scales and accelerations

For downward positive, $m\dot{v} = mg - bv$. Terminal balance gives $v_t = mg/b = 4.905 \text{ m s}^{-1}$ and the relaxation time is $\tau = m/b = 0.500 \text{ s}$. Starting from rest,

$$v(t) = v_t(1 - e^{-t/\tau}), \quad a(t) = ge^{-t/\tau}.$$

Consequently $v(\tau) = 3.10055 \simeq 3.10 \text{ m s}^{-1}$ and $v(3\tau) = 4.66079 \simeq 4.66 \text{ m s}^{-1}$. The corresponding acceleration fractions are $e^{-1} = 0.367879$ and $e^{-3} = 0.0497871$. After 3τ the speed is close to terminal but the acceleration is still nonzero. At release $a = g$, as required when drag vanishes; at long times $v \rightarrow v_t$ and $a \rightarrow 0$.

P06-23: quadratic drag at half terminal speed

Terminal balance is $cv_t^2 = mg$, giving

$$v_t = \sqrt{\frac{(0.10)(9.81)}{0.040}} = 4.95227 \text{ m s}^{-1} \simeq 4.95 \text{ m s}^{-1}.$$

At $v = v_t/2$, the drag magnitude is $c(v_t/2)^2 = mg/4$, one quarter of the weight. The resultant downward force is $3mg/4$, so $a = 0.75g = 7.3575 \text{ m s}^{-2}$. The drag fraction is quadratic in the speed fraction, unlike linear drag, where half terminal speed gives half-weight drag and half- g acceleration. This contrast is a direct transient discriminator.

P06-24: drag in a moving medium

With east positive, $v_{\text{rel}} = v - u$ and $F_{d,x} = -b(v - u)$. At $v = 5.0 \text{ m s}^{-1}$, $v_{\text{rel}} = -3.0 \text{ m s}^{-1}$ and $F_{d,x} = +b(3.0 \text{ m s}^{-1})$: drag points east. At $v = 10.0 \text{ m s}^{-1}$, the relative velocity is $+2.0 \text{ m s}^{-1}$ and $F_{d,x} = -b(2.0 \text{ m s}^{-1})$: drag points west. A numerical force magnitude cannot be reported without a value of b . The first case shows why a resistive force can point along the laboratory velocity: it reduces relative motion against the medium.

P06-25: solve and verify the signed linear ODE

The force law $m\dot{v} = mg - bv$ remains valid for either sign of v . Define $v_t = mg/b$ and $\tau = m/b$. Then $\dot{v} = -(v - v_t)/\tau$. Setting $w = v - v_t$ gives the homogeneous equation $\dot{w} = -w/\tau$, whose solution with $w(0) = v_0 - v_t$ is

$$v(t) = v_t + (v_0 - v_t)e^{-t/\tau}.$$

This derivation also includes the special case $v_0 = v_t$, for which the speed remains terminal; dividing by $v - v_t$ in separation should not silently discard that constant solution. Differentiation gives

$$\dot{v} = -\frac{v_0 - v_t}{\tau}e^{-t/\tau} = \frac{v_t - v}{\tau} = g - \frac{b}{m}v,$$

which verifies the ODE independently of a numerical substitution. At $t = 0$, the expression returns v_0 , and as $t \rightarrow \infty$ it tends to v_t . For initially upward motion, $v_0 < 0$, the signed drag $-bv$ points downward until the turning point, vanishes at $v = 0$, and points upward during the subsequent fall. No piecewise change of the linear force law is needed.

P06-26: time to ninety percent

From rest, $0.90 = 1 - e^{-t_{90}/\tau}$, so $e^{-t_{90}/\tau} = 0.10$ and

$$t_{90} = \tau \ln 10.$$

Here $\tau = 0.0800/0.160 = 0.500$ s, giving $t_{90} = 1.15129$ s $\simeq 1.15$ s. The logarithm acts on a dimensionless fraction; the time units come from τ . At this time the drag equals 90% of the weight and the acceleration is $0.10g$, so reaching 90% speed does not mean force balance is exact. Exact terminal speed is approached asymptotically in this ideal release-from-rest solution.

P06-27: a transient numerical benchmark

(a) Let $f(v) = g - (c/m)v|v|$. At each step compute $k_1 = f(v_n)$, $v_{n+1/2} = v_n + \Delta t k_1/2$, and $v_{n+1} = v_n + \Delta t f(v_{n+1/2})$. Start with $v_0 = 0$ and perform exactly $2.00/\Delta t$ steps. This is an explicit midpoint rule: it estimates the force at a predicted midpoint velocity rather than at the beginning of the full step. All three requested step sizes divide the final time exactly.

(b) With $c/m = 0.400$ m⁻¹, the terminal speed is $v_t = 4.95227221$ m s⁻¹. The exact benchmark at 2.00 s is 4.94868723 m s⁻¹. Iteration gives:

Δt (s)	$v(2)$ (m s ⁻¹)	endpoint error (m s ⁻¹)	max nodal error on $[0, 2]$ (m s ⁻¹)
0.20	4.94284250	0.00584473	0.0701326
0.10	4.94791721	0.00077003	0.0138603
0.05	4.94853125	0.00015598	0.00314389

The errors in the table are absolute differences from the analytic curve evaluated at the corresponding time nodes. The coarsest numerical result underestimates the endpoint by about 0.118%, but its larger error earlier in the transient is more revealing.

(c) A second-order method has error $C(\Delta t)^2$ in its asymptotic refinement regime; halving the step should then reduce a consistent error norm by about four. Here the maximum-error ratios are 5.06 and 4.41, approaching that behavior. Endpoint-error ratios are 7.59 and 4.94, farther from four because the endpoint lies near a strongly attracting terminal state, where earlier errors have decayed. Using maximum nodal error over the same interval reduces this endpoint bias. Agreement with an analytic curve and step refinement validate the discretization of the stipulated ODE; neither demonstrates that the quadratic law describes a real experiment.

P06-28: identical equilibrium, different transients

Let the shared terminal speed be V . Terminal balance requires $b = mg/V$ for the linear law and $c = mg/V^2$ for the quadratic law. At an intermediate speed $v = \alpha V$, downward acceleration is $g(1 - \alpha)$ in the linear model and $g(1 - \alpha^2)$ in the quadratic model. At $\alpha = 1/2$ these are $g/2$ and $3g/4$, respectively. Measuring acceleration at a known speed fraction, or the elapsed time to reach that fraction after release, therefore distinguishes the models. Terminal speed probes only the single force-balance point $F_d(V) = mg$; different functions can share this value while differing elsewhere. Initial acceleration is also insufficient by itself because both models start at g when released from rest.

P06-29: distinguish scatter from a changing surface

Progressive contamination violates the assumption that all trials estimate one unchanged onset threshold. The standard deviation mixes random measurement scatter with a possible systematic drift, while the mean may correspond to no single contact condition. Plot onset angle against trial number and inspect the ordering, not merely its histogram. A steadily increasing or decreasing

sequence signals a history-dependent surface change even if the scatter about that trend is small. A better protocol controls or renews the surface, records load and environmental conditions, repeats measurements in randomized blocks where possible, and reports calibration uncertainty separately. If a changing-surface study is intended, the drift itself becomes a modeled observable. A low standard error of the mean cannot repair an invalid common-parameter assumption.

P06-30: test the dynamics, not just its attracting limit

Many discretizations share the same fixed point: if $f(v_t) = 0$, the update may preserve v_t exactly even while its transient accuracy is poor. Thus agreement in terminal speed is a weak test. Stronger checks include a comparison with an available exact solution over the entire time interval, and a step-refinement study of the maximum nodal error or a selected early-time observable. One can also check $v(0) = 0$, $\dot{v}(0) = g$, and the sign of acceleration below and above terminal speed. Discretization error should decrease in a predictable way under refinement. A discrepancy between a converged solution and actual measured motion instead suggests a deficient force law, parameter error, or measurement issue. Numerical convergence answers whether the chosen equations were solved accurately; physical validation answers whether the equations are appropriate.

P06-31: transform acceleration with fixed axis directions

The nonrotating assumption means that laboratory and moving-frame vector components use axes with the same fixed directions. Differentiating $\mathbf{r} = \mathbf{R} + \mathbf{r}'$ once gives $\mathbf{v} = \dot{\mathbf{R}} + \mathbf{v}'$ and twice gives $\mathbf{a} = \ddot{\mathbf{R}} + \mathbf{a}' = \mathbf{A} + \mathbf{a}'$. Newton's law in the inertial frame is $\sum \mathbf{F}_{\text{real}} = m(\mathbf{A} + \mathbf{a}')$. Rearranging,

$$\sum \mathbf{F}_{\text{real}} - m\mathbf{A} = m\mathbf{a}'.$$

The frame acceleration enters at the second derivative of its origin position; the inertial term is a reorganization of this kinematic relation. If $\mathbf{A} = 0$, ordinary Newton balance is recovered even if the origins have a nonzero constant relative velocity. Converting any moving-frame result back using $\mathbf{a} = \mathbf{A} + \mathbf{a}'$ checks that the physical acceleration and real-force resultant agree.

P06-32: the same sticking test in two frames

(a) In the laboratory, sticking demands $a_x = A = 4.0 \text{ m s}^{-2}$. With $N = mg$, required friction is mA right, but capacity is $0.35mg$. The maximum supported acceleration is $0.35g = 3.4335 \text{ m s}^{-2} < 4.0$, so sticking is impossible.

(b) In the platform frame, hypothetical rest requires $a'_x = 0$ and horizontal balance $f_s - mA = 0$. The same capacity inequality follows, so the conclusion is identical. Once sliding begins the block lags left relative to the platform, and kinetic friction on the block points right. Its laboratory acceleration would be $\mu_k g$ in the stated horizontal sliding model, and its platform-frame acceleration would be $\mu_k g - A$. A numerical value cannot be supplied because no kinetic coefficient is given. The failure of static equilibrium does not license using μ_s automatically as a kinetic coefficient.

P06-33: pendulum direction and acceleration

At a steady relative angle, the bob shares the platform's horizontal acceleration and has no vertical acceleration. Real forces give $T \sin \phi = mA$ and $T \cos \phi = mg$ when ϕ denotes the positive tilt magnitude. Dividing yields

$$A = g \tan 12^\circ = 2.08518 \text{ m s}^{-2} \simeq 2.09 \text{ m s}^{-2}.$$

The mass cancels because both required force components scale with it. The bob is displaced behind the suspension point relative to the acceleration direction: rightward platform acceleration requires rightward tension, so the bob lies left of its support. The tilt magnitude alone gives the acceleration magnitude, while the observed side identifies its direction. The result assumes a steady state without platform rotation and without another significant force on the bob.

P06-34: the required direction contradicts the premise

Let laboratory x point right and write the signed platform acceleration A_x , rather than assuming its sign. The uphill unit tangent is $\hat{\mathbf{t}} = (\cos \theta, \sin \theta)$ and the outward normal is $\hat{\mathbf{n}} = (-\sin \theta, \cos \theta)$. In the platform frame, relative rest requires

$$N\hat{\mathbf{n}} + (0, -mg) - mA_x(1, 0) = \mathbf{0}.$$

The uphill projection gives $-mg \sin \theta - mA_x \cos \theta = 0$, hence

$$A_x = -g \tan \theta.$$

For $0 < \theta < 90^\circ$, this is leftward, toward the lower end of the incline, with magnitude $g \tan \theta$. The stipulated rightward acceleration with $A \geq 0$ therefore admits no such equilibrium: both gravity and the rightward-frame inertial term project downhill. The normal projection gives $N = mg \cos \theta - mA_x \sin \theta = mg / \cos \theta > 0$ for the required leftward acceleration, so the corrected-direction equilibrium is physically admissible. The inertial-frame check is $N(-\sin \theta) = -mg \tan \theta = mA_x$ and $N \cos \theta - mg = 0$. At $\theta = 0$ the required acceleration tends to zero, as expected for frictionless relative rest on a horizontal platform.

P06-35: interaction forces have partners; a frame term does not

The normal force is an interaction exerted by the support on the block. Its equal and opposite third-law partner is the force the block exerts on the support; that partner belongs in the support's diagram. The term $-m\mathbf{A}$ arises from differentiating a moving coordinate origin and has no other body as its source, so no physical interaction partner exists. In an inertial-frame diagram it is absent: real forces already determine $m\mathbf{a}$. In a translating-frame equation it is included to obtain $m\mathbf{a}'$. The normal is included in either frame if physical contact exists. Mixing an inertial acceleration with the extra frame term would double-count the coordinate correction and give the wrong dynamics.

P06-36: rotating axes change the derivative rule

A vector fixed in space can have changing components in rotating axes because the basis directions themselves change. Differentiating position twice therefore produces additional acceleration terms depending on angular velocity, angular acceleration, relative position, and relative velocity. These give centrifugal, Euler, and Coriolis terms in a rotating-frame force balance. In particular, a body moving relative to the axes requires the velocity-dependent Coriolis term; merely correcting acceleration of the centre cannot account for it. The simpler equation derived in P06-31 applies when the axes do not rotate, so their directions remain fixed, and the origin translates with a known $\mathbf{A}(t)$. Uniform acceleration is useful for steady examples but is not required for the instantaneous transformation. A frame whose centre has zero acceleration can still be noninertial if its axes rotate.

P06-37: the complete speed window on a rough bank

(a) Let inward horizontal be positive. The road rises outward, so its normal has inward and upward components $N \sin \theta$ and $N \cos \theta$. Define f as positive *downhill*, giving inward component $f \cos \theta$ and downward component $f \sin \theta$. This local signed f is the negative of an uphill-positive friction component; changing that sign convention does not change the admissible speed interval. Newton's equations are

$$N \sin \theta + f \cos \theta = mv^2/R, \quad N \cos \theta - f \sin \theta = mg.$$

Solving this two-by-two rotation of components, with $q = v^2/(Rg)$, gives

$$N = mg(\cos \theta + q \sin \theta), \quad f = mg(q \cos \theta - \sin \theta).$$

For these data $N > 0$ for every $q \geq 0$. The static constraint $|f| \leq \mu_s N$ yields

$$\frac{\sin \theta - \mu_s \cos \theta}{\cos \theta + \mu_s \sin \theta} \leq q \leq \frac{\sin \theta + \mu_s \cos \theta}{\cos \theta - \mu_s \sin \theta}.$$

Both denominators and the lower numerator are positive here. In a different parameter regime one must also intersect this interval with $q \geq 0$ and inspect any denominator sign before using the same displayed inequality.

(b) The endpoints are $q_- = 0.104464722$ and $q_+ = 0.675429272$, giving

$$v_- = 7.84142 \text{ m s}^{-1}, \quad v_+ = 19.9388 \text{ m s}^{-1}.$$

Thus the constant-speed sticking window is approximately $7.84 \leq v \leq 19.9 \text{ m s}^{-1}$.

(c) Below the frictionless design speed, $q < \tan \theta$, so $f < 0$: friction acts uphill to prevent downward slip and reduces the net inward component. At the low-speed endpoint it is at that uphill limit. At the high-speed endpoint $f > 0$, so friction is downhill and adds inward force. The endpoint normal loads are $N_-/m = 9.56889 \text{ m s}^{-2}$ and $N_+/m = 11.4846 \text{ m s}^{-2}$, both positive. Substitution into the component equations confirms $|f| = \mu_s N$ with the correct signs. The mass independence follows from the common factor m in both demand and capacity.

P06-38: the response graph is conditional on motion history

(a) The vertical force balance gives $N = mg = 98.1 \text{ N}$. While the block sticks, the leftward friction magnitude equals P up to $P_c = \mu_s N = 39.24 \text{ N}$. Once sustained rightward slip is established, the ideal kinetic magnitude is $\mu_k N = 29.43 \text{ N}$. Figure 6.38 shows these two branches for the specified increasing-force experiment. It is a response graph rather than a universal single-valued force law for every history.

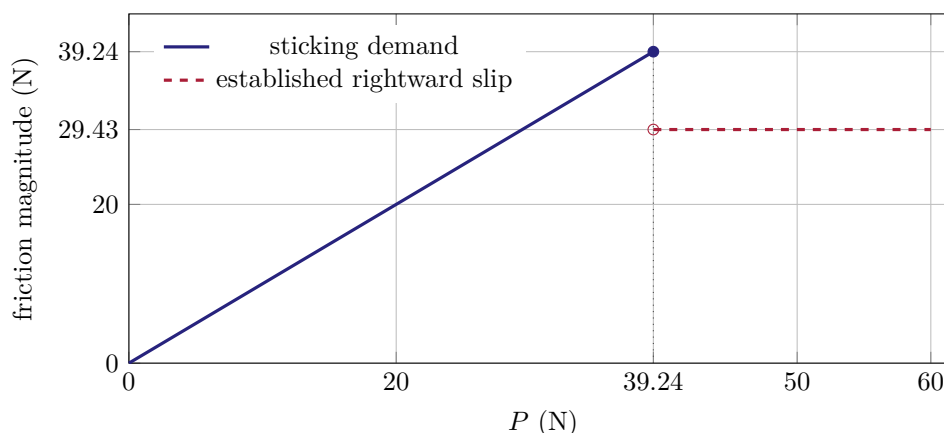


Figure 6.38: Solution response graph for P06-38. The solid static branch supplies the demanded force; the dashed branch supplies a smaller constant kinetic magnitude after rightward sliding is established. The open marker distinguishes that sliding branch from the initially stationary threshold state.

(b) At $P = 20.0 \text{ N}$, static friction is 20.0 N left and $a = 0$. At $P = 50.0 \text{ N}$, the sliding branch gives $f_k = 29.43 \text{ N}$ left and $a = (50.0 - 29.43)/10.0 = 2.057 \text{ m s}^{-2}$ right. At the end $P = 60.0 \text{ N}$, the acceleration is 3.057 m s^{-2} . These accelerations apply to the established branch rather than a hypothetical constrained rest solution.

(c) The ideal transition drops the friction magnitude by $39.24 - 29.43 = 9.81 \text{ N}$, reflecting the stipulated difference between static and kinetic coefficients. Real contacts may transition more smoothly. During decreasing force, a block still moving right remains on the kinetic branch even when $P < P_c$; it need not stop immediately. Re-sticking requires the relative velocity to reach zero and the static demand to be feasible. A force-only graph therefore cannot determine the return history without initial velocity and the time dependence of P .

P06-39: random uncertainty and common calibration uncertainty

(a) The mean is $\bar{\theta} = 21.0^\circ$. The sum of squared deviations is 0.28 degree^2 . With $n = 6$ and the sample divisor $n - 1$,

$$s_\theta = \sqrt{0.28/5} = 0.236643^\circ, \quad u_{\bar{\theta}, \text{random}} = s_\theta/\sqrt{6} = 0.0966092^\circ.$$

The sample standard deviation describes trial scatter, while the standard error estimates uncertainty of the mean under the stated independence assumption.

(b) At the limiting bare-incline angle, $\mu_s = \tan \bar{\theta} = 0.383864$. Combine the independent random and calibration standard uncertainties before propagation:

$$u_{\bar{\theta}} = \sqrt{(0.0966092)^2 + (0.20)^2}^\circ = 0.222111^\circ = 0.00387657 \text{ rad}.$$

Because $d(\tan \theta)/d\theta = \sec^2 \theta$ with θ in radians,

$$u_\mu = \sec^2(21^\circ)u_{\bar{\theta}} = 0.00444779.$$

A suitable report is $\mu_s = 0.3839 \pm 0.0044$ with the uncertainty identified as a combined standard uncertainty from these two specified sources. This first-order propagation is justified by the small angular uncertainty; it is not an interval guaranteed to contain all possible model effects.

(c) A common calibration error shifts every trial in the same direction, so repeating the measurement does not average it away. Its contribution must not be divided by $\sqrt{6}$. Further limitations include surface contamination, differing normal load, vibration near onset, or a coefficient that depends on the materials' condition. Any one explained limitation meets the question. These synthetic numbers illustrate an analysis; they are not a claim that this experiment was performed or that μ_s is a universal material constant.

P06-40: two contacts require two balances

(a) The upper block's normal load is $N_{12} = m_1g$, while the floor supports both masses: $N_f = (m_1 + m_2)g = 196.2 \text{ N}$. The floor's established leftward drag is $f_f = 0.20N_f = 39.24 \text{ N}$. If the blocks move together, adding their horizontal equations cancels the interlayer forces and gives

$$a = \frac{F - f_f}{m_1 + m_2} = \frac{F - 39.24}{20.0}.$$

The upper block's only horizontal force is rightward interlayer friction $f_{12} = m_1a$. Sticking requires $|a| \leq 0.30g = 2.943 \text{ m s}^{-2}$. With the requested $a \geq 0$, the interval is $39.24 \leq F \leq 98.10 \text{ N}$. The maximum force is therefore 98.1 N . Taking floor load as m_2g would omit the transmitted weight of the upper block and understate its drag.

(b) At $F = 70.0 \text{ N}$, $a = (70.0 - 39.24)/20.0 = 1.538 \text{ m s}^{-2}$. The upper-block friction is $f_{12} = 8.0a = 12.304 \text{ N}$ right. This is below the interlayer capacity $0.30(8.0)(9.81) = 23.544 \text{ N}$, so sticking is consistent. The lower block receives the opposite interlayer force, leftward; its equation is $70.0 - 39.24 - 12.304 = 12.0a$, confirming the common acceleration.

(c) At $F = 120 \text{ N}$, hypothetical common acceleration is 4.038 m s^{-2} , exceeding the interlayer bound. In the stated slipping regime, the upper block lags behind the lower one, so it receives rightward kinetic friction

$$f_{12,k} = 0.22(8.0)(9.81) = 17.2656 \text{ N}.$$

Therefore

$$a_1 = 17.2656/8.0 = 2.1582 \text{ m s}^{-2}, \quad a_2 = \frac{120 - 39.24 - 17.2656}{12.0} = 5.2912 \text{ m s}^{-2}.$$

The difference $a_2 - a_1 = 3.1330 \text{ m s}^{-2} > 0$ confirms the lower block initially outruns the upper one. Interlayer forces cancel in the whole-system balance: $m_1a_1 + m_2a_2 = 120 - 39.24 = 80.76 \text{ N}$, a separate force-accounting check. After slipping starts there is no single common acceleration, although both blocks can accelerate right.

P06-41: launch, reversal, and fall with one signed law

(a) Here $\tau = m/b = 0.500 \text{ s}$, $v_t = g\tau = 4.905 \text{ m s}^{-1}$, and $v_0 = -6.00 \text{ m s}^{-1}$. The signed linear solution is $v = v_t - (6.00 + v_t)e^{-t/\tau}$. At turnaround $v = 0$, so

$$t_* = \tau \ln \left(\frac{6.00 + v_t}{v_t} \right) = 0.399483 \text{ s} \simeq 0.399 \text{ s}.$$

The logarithm argument is greater than one, giving a positive turning time.

(b) Integrating from $y(0) = 0$ gives

$$y(t) = v_t t + (v_0 - v_t)\tau(1 - e^{-t/\tau}).$$

At $t = t_*$, use $e^{-t_*/\tau} = v_t/(6.00 + v_t)$ to obtain $y(t_*) = -1.040535 \text{ m}$. Since downward is positive, maximum height above launch is $h = -y(t_*) = 1.04054 \text{ m} \simeq 1.04 \text{ m}$. Differentiating $y(t)$ returns the stated velocity and its initial position, checking the integral rather than merely its units.

(c) Before reversal $v < 0$, so $-bv > 0$: drag points downward with gravity and makes the rise shorter than in vacuum. After reversal $v > 0$, drag is upward and reduces the falling acceleration. At the turning point drag vanishes and the acceleration is g downward. The vacuum height is $v_0^2/(2g) = 1.83486 \text{ m}$, greater than the drag height, as physically expected. Both the signed ODE and its solution remain continuous through the reversal; changing the force to always point upward would incorrectly model the launch phase.

P06-42: an identifiable comparison and a measurement exemplar

(a) Balancing each law at the same $V = v_t = 4.905 \text{ m s}^{-1}$ gives

$$b = \frac{mg}{V} = 0.1600 \text{ kg s}^{-1}, \quad c = \frac{mg}{V^2} = 0.0326198 \text{ kg m}^{-1}.$$

These models share the stipulated terminal speed exactly; this quadratic coefficient differs from the nearby illustrative 0.0320 kg m^{-1} model.

(b) From rest, the linear solution reaches fraction 0.90 at $t_{90,L} = (V/g) \ln 10 = 1.15129 \text{ s}$. For the quadratic model, $v/V = \tanh(gt/V)$, so

$$t_{90,Q} = \frac{V}{g} \operatorname{artanh}(0.90) = \frac{V}{2g} \ln \frac{1 + 0.90}{1 - 0.90} = 0.736110 \text{ s}.$$

The quadratic model reaches this fraction about 0.415183 s earlier. A force proportional to v^2 is smaller than the calibrated linear resistance at speeds between zero and V , explaining the quicker approach at fixed terminal speed.

(c) At $v = V/2$, $a_L = g/2 = 4.905 \text{ m s}^{-2}$ and $a_Q = 3g/4 = 7.3575 \text{ m s}^{-2}$. The two laws have the same initial acceleration g and the same zero terminal acceleration; the intermediate behavior supplies information the endpoints lack.

(d) One defensible reference approach is to release the same object repeatedly from rest in a stationary medium, record calibrated position versus time, estimate velocity with a stated filtering method, and interpolate the first crossing of $0.90V$. Record release-time uncertainty and speed-scale uncertainty; use a fitting method or uncertainty propagation to quantify their effect on the crossing time. Compare the measured interval with both theoretical predictions. For instance, if the combined timing uncertainty were 0.03 s and other model assumptions were controlled, the predicted 0.415 s separation would be readily resolvable. This is a hypothetical uncertainty requirement, not fabricated measured performance. If integration is used, refine the numerical step until its contribution is small compared with the stated measurement uncertainty, or use these analytic curves. A strong response states the observable, initial conditions, calibration and uncertainty

assumptions, an independent numerical check, and a criterion for distinguishing predictions. A weak response measures only terminal speed, omits release timing, or treats solver convergence as experimental validation. If neither model describes the resolved transient, investigate the force law or neglected effects instead of declaring the closer curve correct by default.

6.11 Summary and formula guide

Contact dynamics combines Newton's law with an admissible range for each interaction. A normal contact can push; a string can pull; static friction can take any required value inside its bound. The direction of friction follows relative sliding or its impending tendency. After a regime fails, the old kinematic constraint must be replaced rather than kept with an artificially clipped force.

Curved motion does not introduce a new interaction called centripetal force. The net inward component of the real forces supplies mv^2/R_c , while the tangential component controls speed change. In an ideal isotropic traction model these demands share one vector budget. Drag depends on velocity relative to the medium. Terminal motion is a stable force balance, and transient data can discriminate models that agree at that balance.

Formula summary

Dry friction and admissibility

$$N \geq 0, \quad |f_{s,\text{req}}| \leq \mu_s N, \quad \mathbf{f}_k = -\mu_k N \hat{\mathbf{v}}_{\text{rel}} \quad (v_{\text{rel}} \neq 0).$$

Static equality holds only at impending slip. The normal equation must include every normal-directed interaction.

Incline pulled at β above the surface

$$N = mg \cos \theta - T \sin \beta, \quad f_{s,\text{req}} = mg \sin \theta - T \cos \beta.$$

Test $T \geq 0$, $N \geq 0$, and both sides of the friction inequality. With no cable, rest requires $\tan \theta \leq \mu_s$.

Curved motion

$$\sum F_t = m\dot{v}, \quad \sum F_n = m \frac{v^2}{R_c}.$$

At the top of an inside loop, $N = m(v^2/R - g)$; at the top of an outside hill, $N = m(g - v^2/R)$. Both require $N \geq 0$.

Level-surface isotropic traction model

$$a_t^2 + (v^2/R_c)^2 \leq (\mu_s g)^2.$$

This is a stipulated particle/contact model, not a measured tyre envelope.

Formula summary

One-dimensional resistance in a medium

$$F_d = -bv_{\text{rel}}, \quad [b] = \text{kg s}^{-1}, \quad F_d = -c|v_{\text{rel}}|v_{\text{rel}}, \quad [c] = \text{kg m}^{-1}.$$

For downward fall in a stationary medium, neglecting buoyancy:

$$v_{t,L} = mg/b, \quad \tau_L = m/b, \quad v_L(t) = v_{t,L} + (v_0 - v_{t,L}) \exp(-t/\tau_L).$$

For release from rest with quadratic drag:

$$v_{t,Q} = \sqrt{mg/c}, \quad \tau_Q = v_{t,Q}/g, \quad v_Q(t) = v_{t,Q} \tanh(t/\tau_Q).$$

Use the sign-correct drag law for arbitrary motion; the displayed tanh branch assumes $v_0 = 0$ and downward-positive fall.

Translating, nonrotating accelerated frame

$$\mathbf{a} = \mathbf{A} + \mathbf{a}', \quad \sum \mathbf{F}_{\text{real}} - m\mathbf{A} = m\mathbf{a}'.$$

The inertial term is a coordinate correction, not a new interaction. Rotating axes require additional terms.

6.12 Bridge to work and kinetic energy

Resolving forces instant by instant is indispensable when constraints and interaction limits matter. Sometimes the desired result is a comparison between two states rather than a full time history. The next chapter derives work, kinetic energy, and power from Newton's law and develops another way to organize that information. The contact tests learned here remain necessary: an energy calculation does not make an inadmissible normal force or an over-limit static force physical.

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