

Chapter 7

Kinetic Energy and Work

First Edition

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Preface

The previous chapters developed motion, Newton's laws and the forces that change a particle's motion. This chapter introduces a second, complementary way to reason about dynamics: instead of following force components at every instant, we ask how forces transfer energy through displacement. The central quantity is *work*, and the central theorem is that the net work on a fixed-mass particle equals its change in kinetic energy.

Energy methods do not replace Newton's laws. They reorganize the same dynamics around scalar quantities. This can make problems with complicated trajectories easier, because the detailed direction of the velocity need not be tracked when only the speed is required. It can also make errors easier to spot: work has a sign, kinetic energy is non-negative, and a predicted negative final kinetic energy is a warning that the assumed motion cannot continue to the proposed point.

The chapter is deliberately careful about what is doing work and what displacement is used. Work by a selected force is not automatically net work. For an extended or deformable system, the displacement of the force's point of application matters. These distinctions become especially important in engineering devices, deformable structures and later chapters on energy storage and thermodynamics.

The exposition and exercises are written for this course. Standard textbook treatments are compared in the references; the figure credits distinguish independently constructed models from gallery construction methods. Every supplied numerical dataset in this chapter is synthetic unless explicitly stated otherwise.

The Guideline

Before writing an energy equation, name the system, the force whose work is being calculated, and the displacement associated with that force. Then state the initial and final states. A scalar method is not a licence to skip the physical model.

Pedagogical boundary

This chapter develops kinetic energy, work, the work–kinetic-energy theorem, simple spring and gravity work, and power. The systematic language of potential energy, conservative forces and mechanical-energy conservation belongs to Chapter 8. We will therefore derive useful gravity and spring work formulas here without prematurely replacing them by a general potential-energy formalism.

Conventions and notation

Vectors are bold, as in $\mathbf{F}$ and $\mathbf{v}$. A speed $v = |\mathbf{v}|$ is non-negative, while a Cartesian component such as v_x may have either sign. A displacement is $\Delta\mathbf{r} = \mathbf{r}_f - \mathbf{r}_i$. In one dimension, dx carries the sign of the path direction.

Symbol	Meaning	SI unit
m	particle mass, taken constant in the core	kg
$\mathbf{r}, \mathbf{v}$	position and velocity in a named inertial frame	m, m s ⁻¹
K	kinetic energy of the chosen particle or body	J
$W_{\mathbf{F}}$	work done by the specified force $\mathbf{F}$	J
W_{net}	sum of work done by all forces on the chosen particle	J
$\mathcal{P}$	instantaneous power	W = J s ⁻¹
k_s	spring stiffness	N m ⁻¹
x	one-dimensional coordinate; for a spring, signed extension from its relaxed length	m
$\mathbf{R}_{\text{CM}}$	centre-of-mass position of an extended system	m

The notation $W_{\mathbf{F}}$ keeps the source of a work transfer visible when several forces act. Only after summing the work of all forces may we write W_{net} . Power is denoted by $\mathcal{P}$ rather than P so that later chapters can reserve P for other standard uses without ambiguity.

For numerical calculations, use SI units before substitution. We use $g = 9.81 \text{ m s}^{-2}$ when a numerical value is required. Extra digits may be retained during intermediate calculations to avoid rounding drift; they do not imply additional experimental precision.

Note

The core chapter assumes algebra, vectors, elementary trigonometry, Newton's second law and basic single-variable calculus. The advanced section introduces parametrized line integrals and a more careful comparison between work at actual force-application points and centre-of-mass work.

Chapter 7

Kinetic Energy and Work

A robotic carriage accelerates along a guide, an elevator motor lifts a cabin, a spring launcher slows and reverses a moving cart, and a deformable connector is stretched by forces applied at its ends. These systems look different, but each raises the same accounting question: *how much energy is transferred by the forces as the relevant points move?*

Force-based dynamics and energy-based dynamics answer different parts of that question. Newton's second law is a vector law that determines acceleration. Work and kinetic energy give a scalar relation between two states. The scalar relation can be remarkably efficient, but only when its conditions are respected.

Learning goals

After studying the core, you should be able to:

- calculate kinetic energy in a stated reference frame and explain why its value is frame dependent;
- compute work by a constant force as a projection and distinguish the work of one force from net work;
- interpret one-dimensional variable-force work as a signed integral and as signed area on a force–position graph;
- derive and apply the work–kinetic-energy theorem without losing validity at a turning point;
- derive the work of uniform gravity and an ideal spring over a specified displacement;
- calculate average and instantaneous power and use $\mathcal{P} = \mathbf{F} \cdot \mathbf{v}$ with correct signs and limits;
- explain why force-application-point motion matters for extended or deformable systems;
- use numerical quadrature and measurement data to estimate work while checking convergence, calibration and model limits.

Key message

Work is an energy-transfer quantity associated with a specified force and a specified displacement. The net work on a fixed-mass particle changes its kinetic energy. A correct work calculation therefore begins with the physical path and the force that acts along that path, not with a memorized formula.

7.1 Kinetic energy and frame dependence

Kinetic energy quantifies the energy associated with motion in a specified reference frame. For a particle of constant mass m moving at speed v in that frame,

$$K = \frac{1}{2}mv^2. \quad (7.1)$$

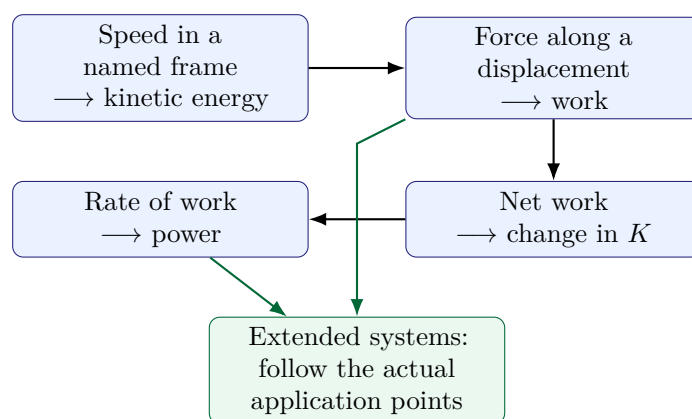


Figure 7.1: Conceptual route through Chapter 7. The core links particle speed, work and kinetic energy; later sections use the same accounting ideas for power, measurements, computation and deformable systems.

The SI unit is the joule,

$$1 \text{ J} = 1 \text{ N m} = 1 \text{ kg m}^2 \text{ s}^{-2}.$$

Because $v^2 \geq 0$, kinetic energy is non-negative in the nonrelativistic model. The equation does not tell us whether the particle is moving toward $+x$ or $-x$; direction information has been compressed into the speed.

Definition 7.1: Kinetic energy

For a fixed-mass particle moving nonrelativistically with velocity $\mathbf{v}$ in a specified inertial frame, the kinetic energy is

$$K = \frac{1}{2} m \mathbf{v} \cdot \mathbf{v} = \frac{1}{2} m v^2.$$

The frame must be identified whenever different observers can have different velocities for the same particle.

7.1.1 Why kinetic energy depends on the observer

Suppose two inertial frames move with constant relative velocity $\mathbf{u}$. In Galilean mechanics their measured velocities are related by

$$\mathbf{v}' = \mathbf{v} - \mathbf{u}. \quad (7.2)$$

The kinetic energies are therefore

$$K = \frac{1}{2} m v^2, \quad K' = \frac{1}{2} m |\mathbf{v} - \mathbf{u}|^2. \quad (7.3)$$

There is no contradiction in obtaining different values. Kinetic energy is not an intrinsic property of a moving object alone; it is a quantity assigned to its motion relative to a chosen frame.

A passenger sitting in a smoothly moving train has zero kinetic energy relative to the carriage but nonzero kinetic energy relative to the station. The mass is the same. The measured velocity is not.

Example 7.1: Energy seen from a moving inspection station

A 1200 kg inspection carriage moves at 8.0 m s^{-1} relative to the laboratory. A second inspection station moves in the same direction at 5.0 m s^{-1} . Find the carriage kinetic energy in the laboratory and in the moving station frame.

Model. Speeds are small compared with the speed of light, so Galilean velocity transformation and the nonrelativistic kinetic-energy expression are appropriate.

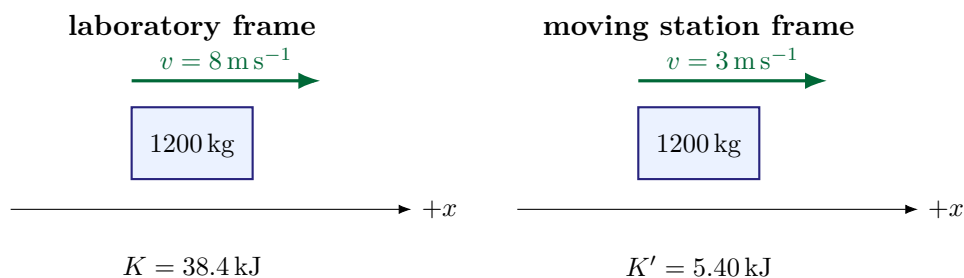


Figure 7.2: The carriage speed is 8.0 m s^{-1} in the laboratory and 3.0 m s^{-1} in a station moving at $u = 5.0 \text{ m s}^{-1}$ in the same direction. The arrows specify direction and are schematic in length. Each kinetic energy uses the speed measured in its labelled inertial frame.

Represent. In the laboratory $v = 8.0 \text{ m s}^{-1}$. In the moving station frame,

$$v' = 8.0 - 5.0 = 3.0 \text{ m s}^{-1}.$$

Solve.

$$K_{\text{lab}} = \frac{1}{2}(1200)(8.0)^2 = 3.84 \times 10^4 \text{ J} = 38.4 \text{ kJ},$$

$$K_{\text{station}} = \frac{1}{2}(1200)(3.0)^2 = 5.40 \times 10^3 \text{ J} = 5.40 \text{ kJ}.$$

Test and interpret. Both results are non-negative. A rider moving with the carriage itself would assign $v'' = 0$ and hence $K'' = 0$. There is no single frame-independent numerical value of a particle's Newtonian kinetic energy.

7.1.2 What does and does not change under a frame shift

The numerical value of K changes when the velocity changes between inertial frames. The mass in the present classical model does not. A statement such as “the object has 20 kJ of kinetic energy” is therefore incomplete if the frame is relevant.

The difference in kinetic energy between two events also depends on the chosen inertial frame in general. Later in this chapter we shall see that the net work computed consistently in that same frame changes by the corresponding amount, so the work–kinetic-energy theorem remains valid within each frame.

Guided checks

Check 1. Can K be negative? *No, not for the fixed-mass nonrelativistic expression $\frac{1}{2}mv^2$.*

Check 2. Can two observers assign different K to the same carriage? *Yes, if they measure different velocities.*

Check 3. Does $K = 0$ prove that no forces act? *No. It only says the particle is instantaneously at rest in the chosen frame. A nonzero net force may give it an acceleration.*

Caution

Kinetic energy depends on speed, not on a signed velocity component. Writing $K = \frac{1}{2}mv_x^2$ is valid only when the motion is known to be one-dimensional along x . In more dimensions use $K = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2)$.

7.1.3 Reading the speed dependence without losing direction

The square in the kinetic-energy formula has several consequences worth separating. At fixed mass, doubling the speed multiplies the kinetic energy by four; increasing the speed by ten per cent multiplies it by $1.10^2 = 1.21$. At fixed speed, doubling the mass doubles the kinetic energy. These comparisons are useful before any detailed calculation: an answer that changes linearly with speed cannot represent $\frac{1}{2}mv^2$ over a finite speed change.

In one dimension the graph of K against the *signed velocity* is a parabola symmetric about $v_x = 0$, as Fig. 7.3 shows. The two points at $v_x = +v_0$ and $v_x = -v_0$ have the same height. This is a genuine loss of directional information, rather than a plotting convention. If an energy calculation gives K_f , then

$$|v_{x,f}| = \sqrt{\frac{2K_f}{m}}.$$

Choosing the positive or negative velocity requires another statement about the trajectory. An object approaching a spring and the same object retreating from it can pass the same position with the same kinetic energy but opposite velocities.

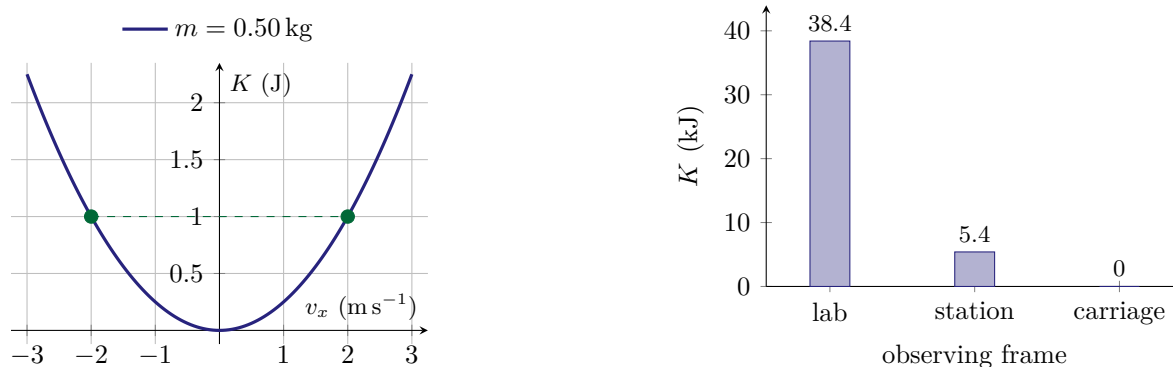


Figure 7.3: Left: kinetic energy discards the sign of a one-dimensional velocity; both marked states of a 0.50 kg particle have $K = 1.00 \text{ J}$. The dotted relation compares states, not successive moments. Right: frame-labelled energies of the earlier 1200 kg inspection carriage with laboratory velocity 8.0 m s^{-1} . The station frame moves at 5.0 m s^{-1} and the carriage frame at 8.0 m s^{-1} , both relative to the lab. One physical carriage has three different frame-specific energies.

An energy description and a vector description therefore answer complementary questions. The velocity tells us where the motion points; kinetic energy tells us how much translational motion is available to change through work in the chosen particle model. A later chapter introduces momentum, another quantity that retains direction. Its conservation rules should not be assumed here merely because kinetic energy happens to be unchanged. For the present chapter, Newton’s vector equation supplies directional information whenever the scalar work equation leaves a choice.

For a small change in speed at fixed mass,

$$dK = mv dv, \quad \frac{dK}{K} = 2 \frac{dv}{v} \quad (v \neq 0).$$

The second relation is a first-order estimate, suitable when the fractional change is small. For a finite change, use the exact ratio $K_f/K_i = (v_f/v_i)^2$. Near rest the fractional expression is undefined, but the original expression $K = \frac{1}{2}mv^2$ remains well behaved. Mathematical convenience does not enlarge a formula’s domain.

Example 7.2: Equal kinetic energies with different masses

Two translating instrument packages move on horizontal low-friction guides in the same laboratory. Package A has mass 0.080 kg; package B has mass 0.320 kg. Each has kinetic energy 4.00 J. Find their speeds. If A moves toward $+x$, does that information determine B's velocity direction? Finally, how much kinetic energy would A have at twice its calculated speed?

Strategy and model. Treat each package as a particle, with negligible rotational kinetic energy. Because the energies are specified in the same frame, invert $K = \frac{1}{2}mv^2$ separately for each mass. Equal energies need not imply equal speeds.

Symbolic reasoning. At a common energy K , $v = \sqrt{2K/m}$, so

$$\frac{v_A}{v_B} = \sqrt{\frac{m_B}{m_A}}.$$

Evaluation. The masses differ by a factor of four; therefore the speeds differ by a factor of two:

$$v_A = \sqrt{\frac{2(4.00)}{0.080}} = 10.0 \text{ m s}^{-1}, \quad v_B = \sqrt{\frac{2(4.00)}{0.320}} = 5.00 \text{ m s}^{-1}.$$

B may move in either direction along its guide. A's stated direction adds no constraint on B. At $v'_A = 20.0 \text{ m s}^{-1}$, A has $K'_A = 4K_A = 16.0 \text{ J}$.

Check and interpretation. Substitution gives $\frac{1}{2}(0.080)(10.0)^2 = \frac{1}{2}(0.320)(5.00)^2 = 4.00 \text{ J}$. The lighter package has the larger speed at fixed energy. The result applies to translational motion in this frame; it does not establish equal collision behaviour or equal momenta.

7.2 Work by a constant force

A force transfers energy through motion of the point on which it acts. For a constant force $\mathbf{F}$ and a displacement $\Delta\mathbf{r}$ of its point of application, the work done by that force is the scalar product

$$W_{\mathbf{F}} = \mathbf{F} \cdot \Delta\mathbf{r} = F \Delta r \cos \theta, \quad (7.4)$$

where θ is the angle between the force and displacement vectors.

The scalar product makes the physical projection explicit. Only the component of $\mathbf{F}$ parallel to the displacement contributes to the work:

$$W_{\mathbf{F}} = F_{\parallel} \Delta r. \quad (7.5)$$

A perpendicular force has zero work over that displacement. A force component opposite the displacement gives negative work.

Definition 7.2: Work by a constant force

For a constant force $\mathbf{F}$ acting at a point whose displacement is $\Delta\mathbf{r}$, the work by that force is

$$W_{\mathbf{F}} = \mathbf{F} \cdot \Delta\mathbf{r}.$$

Positive work means the force component along the displacement points with the motion; negative work means it points against the displacement. Zero work can occur even for a nonzero force.

7.2.1 Selected-force work versus net work

Suppose several forces act on a particle. Each force can do its own work,

$$W_1, W_2, \dots,$$

and the net work is their algebraic sum,

$$W_{\text{net}} = \sum_i W_i. \quad (7.6)$$

The work done by an applied cable is therefore not automatically the work done by the net force. Gravity, friction, a spring or other forces can add positive, negative or zero contributions.

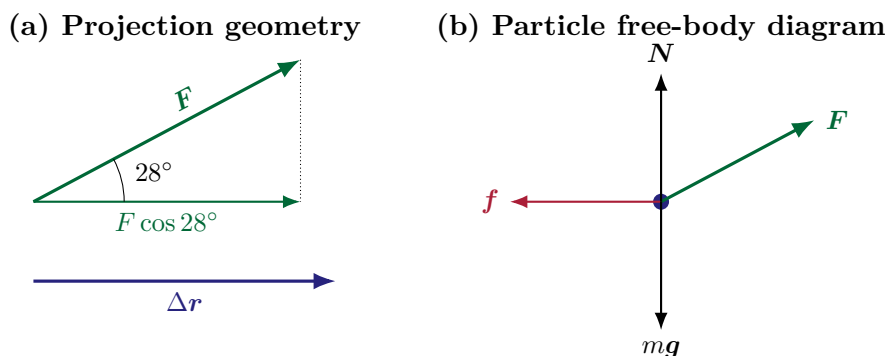


Figure 7.4: An angled cable pulls a cart horizontally. Panel (a) resolves the cable force; the projection is an auxiliary construction. Panel (b) contains each actual force once, with schematic arrow lengths. Gravity and the normal force do zero work in this horizontal particle model. Projection-coordinate construction follows the method studied in Izaak Neutelings, *Work & energy*, TikZ.net; the physics and layout are reconstructed for this chapter.

Example 7.3: An angled cable on a service cart

A service cart is pulled 6.0 m horizontally by a constant cable force of 85 N directed 28° above the horizontal. A constant rolling-resistance force of 22 N opposes the motion. Find the work done by the cable, the work done by resistance and the net work. Assume no vertical displacement.

Model. Treat the cart as a particle for translational motion. The force magnitudes and cable angle are constant over the displacement.

Cable work.

$$W_{\text{cable}} = (85)(6.0) \cos 28^\circ = 4.50 \times 10^2 \text{ J}.$$

Resistance work.

$$W_r = -(22)(6.0) = -132 \text{ J}.$$

Gravity and the normal force are perpendicular to the horizontal displacement, so each does zero work in this particle model.

Net work.

$$W_{\text{net}} = 450 \text{ J} - 132 \text{ J} = 318 \text{ J}.$$

Interpretation. Saying “the cable does 450 J of work” and saying “the net work is 318 J” are different physical statements. The first tracks one interaction; the second tracks the combined effect of all forces on the particle’s kinetic energy.

7.2.2 Zero work does not mean zero force

For circular motion at constant speed, a radial constraint force can be large while remaining perpendicular to the instantaneous displacement. Its instantaneous work is therefore zero. The force still changes the *direction* of the velocity, even though it does not change the speed.

This distinction is central to energy methods: a scalar energy equation may correctly predict the speed while saying nothing by itself about the direction in which the particle turns.

Guided checks

Diagnostic X07-02. A crate moves east while a worker pushes north. Is the worker necessarily doing positive work? *No. If the push is exactly perpendicular to the instantaneous displacement, its work is zero.*

Sign check. If the displacement is reversed while the same constant force is kept fixed, the work changes sign because $\Delta\mathbf{r} \rightarrow -\Delta\mathbf{r}$.

Caution

Do not use the displacement of an arbitrary reference point when computing work. For a particle the issue is simple because every point of the idealized particle has the same position. For an extended or deformable object, the motion of the actual force-application point becomes essential; Section 7.7 develops that distinction.

7.2.3 Using the angle as a quantitative control variable

The projection in $W = Fd\cos\theta$ can be read in two equivalent ways: project the force onto the displacement, or project the displacement onto the force. This equivalence follows from the symmetry of the scalar product. A force arrow is not itself an energy-transfer arrow; the displacement direction must be supplied before a work sign can be assigned.

For fixed F and displacement magnitude d , the work-angle graph in Fig. 7.5 exposes the full sign pattern. The value falls smoothly from $+Fd$ at 0° to zero at 90° , then to $-Fd$ at 180° . Near 90° , even a modest angle error can change the sign of a small computed work. At 60° the parallel force component is half the force magnitude, whereas at 30° it is about $0.866F$. “Partly aligned” is too vague to replace the trigonometric projection.

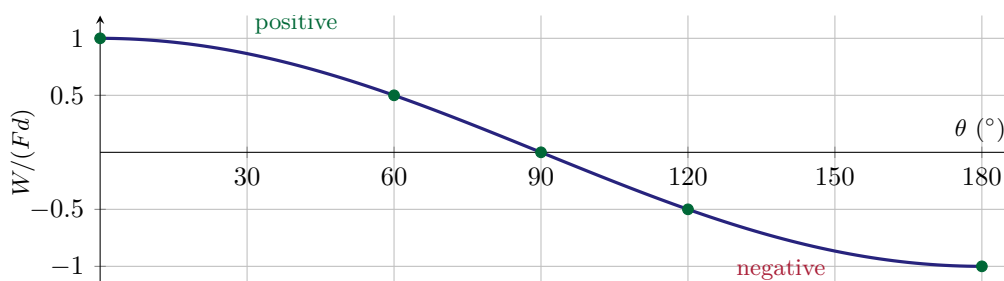


Figure 7.5: Work by one constant force as its angle to a fixed displacement changes. The graph is normalized by Fd , so it applies to any nonzero force and displacement magnitudes. Angles refer to the two vectors, not to a chosen wall, surface, or page edge.

The angle must belong to the actual pair of vectors. If a cart moves along an incline while a cable stays horizontal, the incline angle is the force-displacement angle; if both cable and motion follow the incline, their angle is zero. A helpful check is to write the displacement vector first. For example, with $\Delta\mathbf{r} = (\Delta x)\hat{\mathbf{i}} + (\Delta y)\hat{\mathbf{j}}$ and a constant force $\mathbf{F} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}}$,

$$W = F_x\Delta x + F_y\Delta y.$$

No separate geometric angle is required. This component expression also makes clear that a horizontal force does no work during a purely vertical motion, even when the object travels a substantial distance.

7.2.4 A return trip: displacement, distance, and changing forces

For a force that remains one fixed vector throughout an entire journey, the work is its scalar product with the *total displacement*. Intermediate detours do not change this particular result. A journey

returning to its starting point has zero work by that force. This conclusion concerns that force law; it does not mean that every real interaction does zero work on every return trip.

Kinetic friction and idealized constant rolling resistance normally reverse direction when the motion reverses. Their vector is therefore not constant over a return trip. If a resistance of constant magnitude f always opposes the local motion along a guide, its work is $-f$ times the total distance traveled. Writing $W = -f\Delta x$ for the complete round trip would incorrectly let the losses cancel. The two legs must be treated separately because the physical force vector changes at reversal.

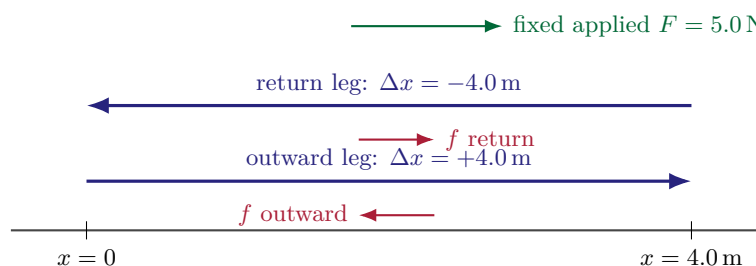


Figure 7.6: A round trip has zero total displacement but nonzero total distance. A fixed rightward applied force has opposite work signs on the two legs. Resistance reverses with the motion and does negative work on both. The separated arrows represent successive legs on the same straight guide.

Example 7.4: One fixed force and resistance on a round trip

A guided 2.0 kg trolley travels from $x = 0$ to $x = 4.0$ m and later returns to $x = 0$. Throughout both legs a device exerts a fixed force $F_x = +5.0$ N. A resistance of magnitude 2.0 N opposes the motion on each leg. Find the work of the device and of resistance, both separately by leg and for the whole trip. May one conclude that the trolley has no change in kinetic energy simply because its total displacement is zero?

Strategy and model. The path is specified; another actuator may be needed to produce the turnaround and return. Compute only the named forces here. Use signed displacement for the fixed vector and path distance for the reversing resistance.

Work on each leg. Outward,

$$W_{F,\text{out}} = (+5.0)(+4.0) = +20 \text{ J}, \quad W_{r,\text{out}} = -(2.0)(4.0) = -8.0 \text{ J}.$$

On the return leg the device still points right, while the resistance points right because the trolley now moves left:

$$W_{F,\text{back}} = (+5.0)(-4.0) = -20 \text{ J}, \quad W_{r,\text{back}} = (+2.0)(-4.0) = -8.0 \text{ J}.$$

Therefore $W_F = 0$ and $W_r = -16 \text{ J}$ over the full 8.0 m distance.

Interpretation and check. Zero displacement makes the work of the *fixed* force vanish. It does not make the resistance work vanish. Nor do the stated forces alone establish the final kinetic energy: any turnaround or driving force must be included in the net work. If an additional actuator arranges equal initial and final speeds, its total work must compensate the 16 J resistance loss. This is an energy requirement, not a claim that the specified device by itself produces the prescribed journey.

Guided checks

Selected work without a friction measurement. A cable pulls a crate 2.50 m horizontally with a known constant force 30.0 N at 40.0° above horizontal. Can its work be calculated before the friction force is known? Yes: its own work depends on its own force and application-point displacement,

$$W_T = (30.0)(2.50) \cos 40.0^\circ = 57.5 \text{ J}.$$

The unknown resistance prevents a numerical *net*-work result, not this selected-force result. If a subsequent measurement establishes a constant opposing resistance of 30.0 N, its work is -75.0 J and the net work is -17.5 J , assuming normal force and gravity do zero work. The cable does positive work while the crate slows. The prescribed forward displacement must be supported by sufficient initial kinetic energy or other stated driving work; positive cable work alone does not establish that it occurs.

7.3 Variable forces and graphical work

Work is a transfer associated with a process, rather than an amount stored inside an object. When a force changes as the particle moves, the constant-force formula cannot simply be applied with an arbitrary average force. In one dimension, divide the path into small displacements Δx_j . Over a sufficiently small interval the force is nearly constant, so the work is approximately

$$\Delta W_j \simeq F_x(x_j^*) \Delta x_j.$$

Adding the pieces and taking the continuum limit gives

$$W_{F_x} = \int_{x_i}^{x_f} F_x(x) dx. \quad (7.7)$$

The integral is signed. If F_x is positive while $dx > 0$, that portion contributes positive work. If the force is negative while the particle continues toward increasing x , that portion contributes negative work.

7.3.1 Signed area on a force–position graph

A plot of F_x versus x gives a geometric interpretation of Eq. (7.7): work is the signed area between the curve and the x axis over the specified interval. Areas above the axis count positively for motion toward increasing x ; areas below count negatively.

Consider the teaching model

$$F_x(x) = F_0 \left(1 - \frac{x}{L}\right), \quad (7.8)$$

with $F_0 > 0$ and $L > 0$. The force crosses zero at $x = L$. For an interval that extends beyond L , positive and negative areas compete.

Example 7.5: A force that changes sign

For the force in Eq. (7.8), let $F_0 = 120 \text{ N}$ and $L = 0.80 \text{ m}$. Find the work as the particle moves from $x_i = 0.20 \text{ m}$ to $x_f = 1.20 \text{ m}$.

Integrate the signed force.

$$\begin{aligned} W &= F_0 \int_{x_i}^{x_f} \left(1 - \frac{x}{L}\right) dx \\ &= F_0 \left[(x_f - x_i) - \frac{x_f^2 - x_i^2}{2L} \right]. \end{aligned}$$

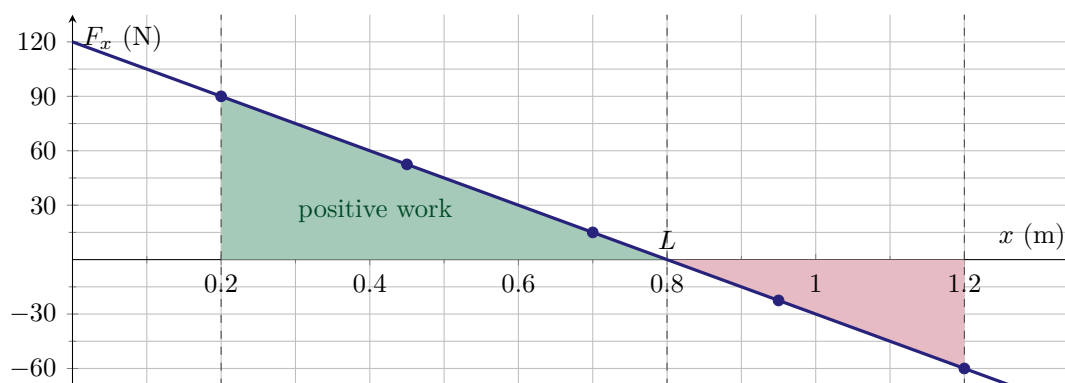


Figure 7.7: Analytical force–position curve for $F_x = F_0(1 - x/L)$ with $F_0 = 120$ N and $L = 0.80$ m. The shaded work from $x_i = 0.20$ m to $x_f = 1.20$ m is the positive green area plus the negative red area.

Substitution gives

$$W = 120 \left[1.00 - \frac{1.20^2 - 0.20^2}{1.60} \right] \text{ J} = 15.0 \text{ J}.$$

Area check. From 0.20 to 0.80 m the area is a positive triangle,

$$W_+ = \frac{1}{2}(0.60 \text{ m})(90 \text{ N}) = 27.0 \text{ J}.$$

From 0.80 to 1.20 m the area is negative,

$$W_- = -\frac{1}{2}(0.40 \text{ m})(60 \text{ N}) = -12.0 \text{ J}.$$

Thus $W = W_+ + W_- = 15.0$ J, confirming the integral.

Reverse-path check. If the particle follows the same one-dimensional path from 1.20 back to 0.20 m while the force law remains the same, the integral limits reverse and the work is -15.0 J.

7.3.2 The integration variable follows the path

Equation (7.7) is one-dimensional. It assumes the path can be described by x and that the force component $F_x(x)$ along that path is known. A force law specified as a function of time is not automatically a force law specified as a function of position. One may need the trajectory $x(t)$ to connect the two descriptions.

Method

For one-dimensional variable-force work:

1. identify the actual initial and final positions;
2. sketch or inspect the sign of F_x over the interval;
3. integrate with the physical limits in their actual order;
4. compare the result with the signed graphical area;
5. check units: force times displacement must reduce to joules.

Guided checks

Diagnostic X07-03. A force–position graph has equal geometric areas above and below the axis. Is the total work necessarily the sum of their absolute areas? *No. Work is the signed area; equal positive and negative areas cancel.*

If the force crosses zero, the particle does not automatically stop there. A zero force at one position only means zero instantaneous acceleration from that force at that position. Whether the particle reaches, passes or turns before that point depends on its kinetic energy and all other forces.

7.3.3 Cumulative work is different from local force

It is often useful to define a running work function along a one-dimensional monotonic leg,

$$W(x) = \int_{x_i}^x F_x(\xi) d\xi.$$

The dummy variable ξ avoids confusing the endpoint x with the variable being summed inside the integral. The fundamental theorem of calculus gives

$$\frac{dW}{dx} = F_x(x).$$

Thus the force graph tells us the *slope* of the cumulative-work graph. A large positive force makes $W(x)$ rise steeply; a small positive force still makes it rise; a negative force makes it fall. Where $F_x = 0$, $W(x)$ can have an extremum. Zero force does not imply zero previously accumulated work.

The distinction is especially useful for a powered slider followed by a braking region. Positive work early in a journey can be removed by negative work later. If the total integral cancels, the running integral may nevertheless have been large at intermediate positions. Once the work–kinetic-energy theorem is established in the next section, the running net work immediately gives the running kinetic energy. An endpoint equation cannot by itself reveal the peak speed or the largest local force.

The original linear-force example also provides a clean comparison of graphical and numerical integration. Sampling $F_x = 120(1 - x/0.80)$ in SI units at a uniform 0.250 m spacing gives

x (m)	0.200	0.450	0.700	0.950	1.200
F_x (N)	90.0	52.5	15.0	−22.5	−60.0

The composite trapezoidal estimate is

$$W_{\text{trap}} = 0.250 \left[\frac{90.0 - 60.0}{2} + 52.5 + 15.0 - 22.5 \right] = 15.0 \text{ J}.$$

It equals the analytical result because every segment of this force graph is straight; each trapezoid gives its exact signed area. A midpoint rule with the same four subintervals uses the forces 71.25, 33.75, −3.75, and −41.25 N, whose sum times 0.250 m is also 15.0 J. The equality is special to this affine force law. It does not establish exactness for arbitrary curved force profiles, nor does an exact quadrature rule eliminate uncertainty in measured force samples. Negative samples must retain their signs in either calculation.

For a force specified as $F_x(x)$, reversing a path changes the integration limits. It does not change the force merely because the particle reverses. By contrast, a speed-dependent resistance such as $F_r = -bv_x$ changes with the motion and cannot be treated as a single-valued function of position unless the particular trajectory supplies $v_x(x)$. The phrase “area under the force curve” is therefore complete only when the curve identifies the force on the actual leg and the direction in which the interval is traversed.

An average force can replace the integral only if it is the appropriate *displacement average*:

$$\overline{F_x}^{(x)} = \frac{1}{x_f - x_i} \int_{x_i}^{x_f} F_x(x) dx, \quad W = \overline{F_x}^{(x)}(x_f - x_i),$$

for distinct endpoints on a monotonic one-dimensional leg. This is not generally the time average. If the particle spends longer in one part of the guide, a time average weights that region more heavily; a displacement average weights equal lengths equally. Neither the largest force nor the arithmetic mean of two arbitrary force readings is automatically the correct average.

Guided checks

Solved graph check. A smooth cumulative-work curve rises to a maximum at $x = a$ and then falls. What can be concluded about the signed force? Before a , the curve's positive slope means $F_x > 0$; after a , its negative slope means $F_x < 0$. At a differentiable maximum, $F_x(a) = 0$. The sign of $W(a)$ itself tells us the work accumulated from the stated starting position, not the instantaneous force there.

7.4 The work–kinetic-energy theorem

The central theorem of this chapter connects the net force description of motion to the scalar work description. Start from Newton's second law for a fixed-mass particle,

$$\mathbf{F}_{\text{net}} = m \frac{d\mathbf{v}}{dt}. \quad (7.9)$$

Take the scalar product with the differential displacement $d\mathbf{r} = \mathbf{v} dt$:

$$\mathbf{F}_{\text{net}} \cdot d\mathbf{r} = m \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} dt \quad (7.10)$$

$$= m \mathbf{v} \cdot d\mathbf{v}. \quad (7.11)$$

Because

$$d(\mathbf{v} \cdot \mathbf{v}) = 2\mathbf{v} \cdot d\mathbf{v}, \quad (7.12)$$

we obtain

$$\mathbf{F}_{\text{net}} \cdot d\mathbf{r} = d\left(\frac{1}{2}mv^2\right) = dK. \quad (7.13)$$

Integrating from an initial state i to a final state f gives

$$W_{\text{net}} = K_f - K_i = \Delta K. \quad (7.14)$$

Theoretical point 7.1: Work–kinetic-energy theorem

For a particle of fixed mass in an inertial frame in Newtonian mechanics, the net work done by all forces as the particle moves from an initial position to a final position equals the change in its kinetic energy:

$$W_{\text{net}} = \Delta K.$$

The theorem is valid whether the path is straight or curved. Every work contribution must refer to the force on the chosen particle and the same physical trajectory.

7.4.1 Why the derivation remains valid at a turning point

A tempting one-dimensional derivation writes $a = v dv/dx$. This relation is useful where the required derivatives exist, but deriving it by dividing through by v can create an artificial problem at a turning point where $v = 0$. Equation (7.11) avoids that problem: it uses $d\mathbf{r} = \mathbf{v}dt$ and the differential identity for v^2 without dividing by the speed.

At an instant when $v = 0$, the kinetic energy may be zero while the net force and acceleration are nonzero. That is exactly what happens when a particle reverses direction under a restoring force.

Caution

Do not conclude that $W_{\text{net}} = 0$ implies every force is zero. It only implies $K_f = K_i$. The particle can change direction while returning to the same speed. Likewise, a nonzero force can do zero work over a segment if it remains perpendicular to the displacement.

7.4.2 Force components that redirect motion

Separate the net force at an instant into a component tangent to the trajectory and a component normal to it:

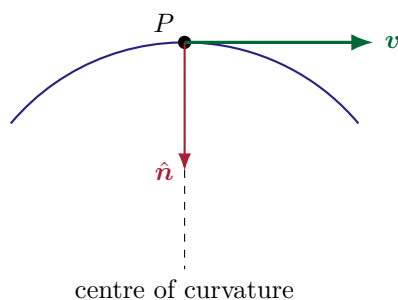
$$\mathbf{F}_{\text{net}} = F_t \hat{\mathbf{t}} + F_n \hat{\mathbf{n}}, \quad \mathbf{v} = v \hat{\mathbf{t}}.$$

The instantaneous rate of work is then

$$\mathbf{F}_{\text{net}} \cdot \mathbf{v} = F_t v.$$

The normal component does not contribute to the instantaneous work. It can nevertheless change the direction of $\mathbf{v}$, which is why knowing K alone does not determine the full trajectory.

(a) Exact local geometry



(b) Force components

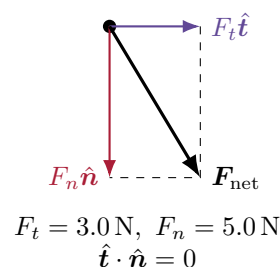


Figure 7.8: At the top of a clockwise circular path, the tangent is horizontal and the inward normal vertical. Panel (b) draws force components on a common force scale: their ratio is 3 : 5, and their vector sum is the diagonal. The normal component redirects motion; only $F_t v$ contributes to the instantaneous change of kinetic energy. Velocity and force are shown in separate panels and use different display scales.

Example 7.6: Redirecting motion versus changing speed

A 0.75 kg guided slider moves at 4.0 m s^{-1} along a curved track. At one instant the net force has tangential component $F_t = 3.0 \text{ N}$ in the direction of motion and normal component $F_n = 5.0 \text{ N}$. Find the instantaneous rate of change of kinetic energy. What role does the normal component play?

Relate. From Eq. (7.13),

$$\frac{dK}{dt} = \mathbf{F}_{\text{net}} \cdot \mathbf{v} = F_t v.$$

Solve.

$$\frac{dK}{dt} = (3.0)(4.0) = 12 \text{ W}.$$

The normal force contribution is $F_n v \cos 90^\circ = 0$.

Interpret. The kinetic energy is increasing at 12 J s^{-1} at that instant. The normal component can still curve the path by changing the direction of the velocity. Energy information and directional dynamics answer complementary questions.

7.4.3 Using the theorem as a reachability test

Suppose a proposed final position leads to

$$K_f = K_i + W_{\text{net}} < 0.$$

The mathematical value is not a physical negative kinetic energy. It signals that the assumed motion cannot reach the proposed final state under the stated model. The particle must turn earlier, encounter a changed force law, or violate one of the assumptions.

This is a powerful diagnostic in spring, gravity and braking problems. Energy methods can reveal an inaccessible region before a detailed time-dependent trajectory is calculated. Check $K(x) \geq 0$ at every intermediate position along the assumed traversal, not merely at the final endpoint.

Guided checks

Diagnostic X07-04. A bead moves around a smooth circular guide at constant speed. The guide force is nonzero. What is the net work by the guide over an arc? *Zero, provided the ideal guide force is everywhere normal to the instantaneous displacement. The velocity direction changes even though K does not.*

7.4.4 A force graph becomes a speed and reachability graph

For one-dimensional particle motion under a known net position-dependent force, the theorem can be used cumulatively:

$$K(x) = K_i + \int_{x_i}^x F_{\text{net}}(\xi) d\xi, \quad v(x) = \sqrt{\frac{2K(x)}{m}} \quad \text{where } K(x) \geq 0. \quad (7.15)$$

The second equation determines speed. The sign of velocity on an individual leg comes from the known direction of motion. This distinction matters when a force changes sign: the slider can continue forward while slowing, and its largest speed occurs where the cumulative kinetic energy is largest, rather than where the force magnitude is largest.

At a position where $K = 0$, a moving particle reaches zero speed. To identify a *turning* point, inspect the force and the continuation. A nonzero force directed back toward the accessible region causes reversal in the usual smooth fixed-mass model. In contrast, an object that arrives at an equilibrium with zero force need not reverse; a singular or nonsmooth model may require further analysis. Reachability is tested along the continuous path from the initial state, not by finding an isolated positive value of K beyond an intervening forbidden interval.

Example 7.7: A variable actuator: acceleration, braking, and a turning point

A 2.00 kg slider enters a horizontal guide at $x = 0$ moving toward $+x$ with speed 2.00 m s^{-1} .

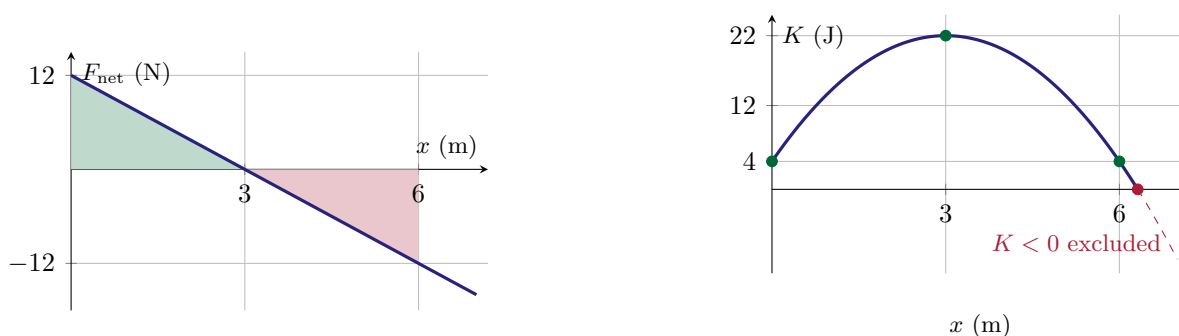


Figure 7.9: The net force $F_{\text{net}} = (12 \text{ N}) - (4 \text{ N m}^{-1})x$ and the corresponding $K(x)$ for $m = 2.00 \text{ kg}$, $x_i = 0$, and $v_i = 2.00 \text{ m s}^{-1}$. Equal positive and negative force areas between 0 and 6 m restore the initial kinetic energy. The maximum K is at the force zero. The dashed negative continuation is a mathematical prediction that the forward trajectory cannot realize.

The net force is

$$F_{\text{net}}(x) = (12.0 \text{ N}) - (4.00 \text{ N m}^{-1})x.$$

Find the speed at $x = 3.00 \text{ m}$ and at $x = 6.00 \text{ m}$, the maximum speed on the forward leg, and the forward turning point. Decide whether the slider reaches $x = 7.00 \text{ m}$ under this model.

Strategy and model. The net force is given directly, so no additional resistance term should be added. Integrate once to obtain $K(x)$; find a kinetic-energy maximum from its slope and a turning point from $K = 0$. Time is not requested.

Symbolic result. With $F = A - Bx$,

$$W(0 \rightarrow x) = Ax - \frac{1}{2}Bx^2, \quad K(x) = \frac{1}{2}mv_i^2 + Ax - \frac{1}{2}Bx^2.$$

Using SI numerical values gives $K(x) = 4.00 + 12.0x - 2.00x^2$ joules when x is entered in metres. At $x = 3.00 \text{ m}$, $K = 22.0 \text{ J}$ and

$$v = \sqrt{\frac{2(22.0)}{2.00}} = 4.69 \text{ m s}^{-1}.$$

At $x = 6.00 \text{ m}$, $K = 4.00 \text{ J}$ and $v = 2.00 \text{ m s}^{-1}$. The total work to 6 m is zero, but the speed was substantially greater in the middle.

Maximum and turning point. Since $dK/dx = A - Bx$, the maximum occurs at $x = A/B = 3.00 \text{ m}$. The forward zero of K is

$$x_{\text{turn}} = \frac{A + \sqrt{A^2 + 2BK_i}}{B} = 3.00 + \sqrt{11.0} = 6.32 \text{ m}.$$

The other quadratic root is negative and is outside the initial forward leg. At the positive root $F_{\text{net}} = -13.3 \text{ N}$, so the slider accelerates back toward smaller x after stopping. At 7.00 m the algebraic expression gives $K = -10.0 \text{ J}$; that point is inaccessible on this forward trajectory.

Checks. The positive and negative triangular work areas from 0 to 6 m are $+18.0 \text{ J}$ and -18.0 J . Differentiating $K(x)$ recovers the force. The units of Bx^2 are joules. The turning point exceeds 6 m, consistently with the nonzero speed there.

7.4.5 Braking distance is governed by an energy budget

An approximately constant opposing net force of magnitude B over a stopping distance d does work $-Bd$. If the final speed is zero,

$$d = \frac{mv_i^2}{2B}. \quad (7.16)$$

This is a mechanical stopping distance after the opposing force begins to act. A reaction interval before braking contributes an additional distance and is outside this particular force model. The distinction is useful in engineering: an energy budget can characterize the braking part without pretending to model the entire sensing and control process.

At a fixed braking-force magnitude, doubling speed quadruples the required stopping distance. At a fixed allowed distance, doubling speed requires four times the opposing work and therefore four times the average opposing force. The mass dependence depends on how the force is produced. If a friction-limited horizontal sliding model has $B = \mu_k mg$ with a constant coefficient and no other horizontal force, then $d = v_i^2/(2\mu_k g)$ and the mass cancels. If B is a fixed actuator limit independent of mass, the stopping distance increases with mass. These are different physical models; the cancellation in one must not be imported into the other.

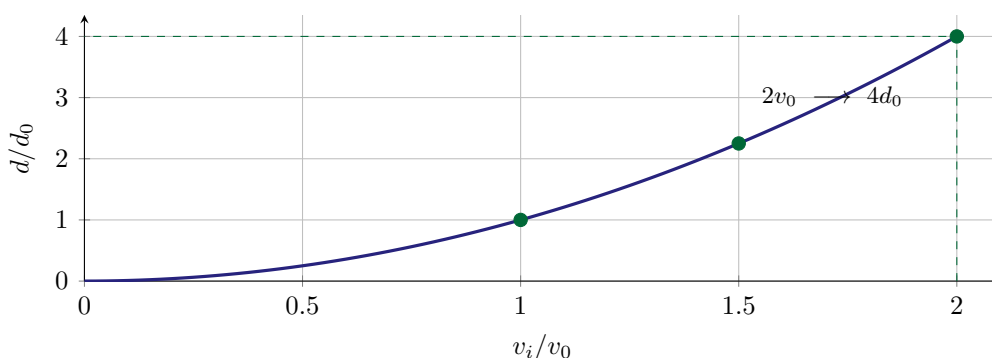


Figure 7.10: For the same mass and constant opposing net force, stopping distance scales with the square of initial speed. Here $d_0 = mv_0^2/(2B)$. This comparison excludes any delay before the braking force acts and any change in force with speed.

Example 7.8: A guided vehicle and a fixed braking-force limit

A 600 kg guided vehicle enters a braking region at 12.0 m s^{-1} . Its approximately constant opposing net force is 1.80 kN. Find the stopping distance. Compare the distance at an entry speed of 18.0 m s^{-1} . If the second vehicle must stop within the first distance, what opposing net force is required?

Strategy and assumptions. Treat the vehicle as a translating particle on a horizontal guide; neglect rotational energy and assume the stated net force is the complete longitudinal resultant after braking starts. The force is opposite the displacement throughout the stop.

Symbolic work balance.

$$-Bd = 0 - \frac{1}{2}mv_i^2, \quad d = \frac{mv_i^2}{2B}.$$

At 12.0 m s^{-1} the energy to remove is

$$K_i = \frac{1}{2}(600)(12.0)^2 = 43.2 \text{ kJ}, \quad d = \frac{43\,200}{1800} = 24.0 \text{ m}.$$

At 18.0 m s^{-1} the entry speed is 1.50 times greater, so

$$K'_i = 97.2 \text{ kJ}, \quad d' = (1.50)^2(24.0) = 54.0 \text{ m}.$$

To remove 97.2 kJ within 24.0 m, the required constant opposing net force is

$$B' = \frac{97\,200}{24.0} = 4050 \text{ N} = 4.05 \text{ kN}.$$

Independent check. Newton's law gives a deceleration magnitude $B/m = 3.00 \text{ m s}^{-2}$ in the first model. The kinematic relation $0 = v_i^2 - 2(B/m)d$ then gives $d = 144/6 = 24.0 \text{ m}$, agreeing with the work calculation. The larger required force is 2.25 times the original, as the energy ratio predicts. A real brake specification would additionally require its thermal and contact-force limits, which are not determined by this particle model.

7.4.6 An inverse use: recovering force from a kinetic-energy profile

The work theorem can be used in reverse. Along a smooth one-dimensional leg with a known $K(x)$,

$$F_{\text{net}}(x) = \frac{dK}{dx}. \quad (7.17)$$

The slope has units $\text{J m}^{-1} = \text{N}$. This equation infers the net force; it does not identify how separate motors, resistances, or springs share that force. A straight kinetic-energy graph corresponds to a constant net force even though the speed changes nonlinearly with position. Conversely, a horizontal kinetic-energy graph on a monotonic one-dimensional leg means zero net longitudinal force there.

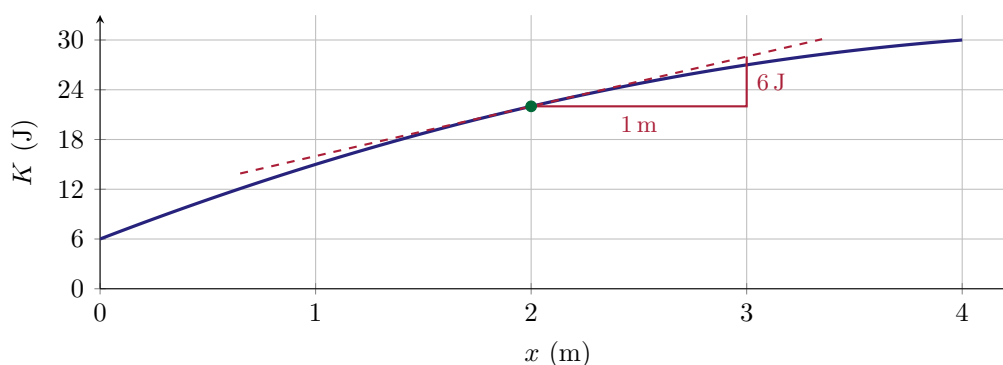


Figure 7.11: The tangent slope of $K(x) = 6 + 10x - x^2$ (SI numerical values) at $x = 2 \text{ m}$ is 6 J m^{-1} , so the net longitudinal force is 6 N there. The dashed tangent is local information, not an alternative kinetic-energy law over the entire guide.

Example 7.9: Inferring an actuator force from a prescribed energy profile

On the outward leg of a guided 3.00 kg slider, a controller produces the theoretical profile

$$K(x) = (6.00 \text{ J}) + (10.0 \text{ N})x - (1.00 \text{ N m}^{-1})x^2, \quad 0 \leq x \leq 4.00 \text{ m}.$$

A separate constant resistance of magnitude 1.50 N opposes the motion. Find the net force and required actuator force at $x = 2.00 \text{ m}$, the speed there, and the net work from 0 to 4.00 m.

Strategy. Differentiate the prescribed kinetic energy to obtain the net force. Then use the signed force sum to infer the actuator contribution. The supplied curve is synthetic controller data, not a measured record.

Force and speed. Differentiation gives

$$F_{\text{net}}(x) = (10.0 \text{ N}) - (2.00 \text{ N m}^{-1})x.$$

At $x = 2.00$ m, $F_{\text{net}} = 6.00$ N. With $F_r = -1.50$ N on this outward leg, $F_{\text{actuator}} = F_{\text{net}} - F_r = 7.50$ N. Also $K(2.00) = 22.0$ J, so

$$v(2.00) = \sqrt{\frac{2(22.0)}{3.00}} = 3.83 \text{ m s}^{-1}.$$

Endpoint work and check. The graph gives $K(0) = 6.00$ J and $K(4.00) = 30.0$ J, hence $W_{\text{net}} = 24.0$ J. Independently,

$$\int_0^4 (10 - 2x) dx = [10x - x^2]_0^4 = 24.0 \text{ J}.$$

The actuator work is 30.0 J and the resistance work is -6.00 J, again totaling 24.0 J. The prescribed interval is accessible because $K(x) > 0$ throughout it. The energy profile alone would not reveal the 1.50 N resistance without the extra information in the statement.

7.4.7 A positive kinetic energy does not guarantee contact

Energy constrains speed; a support force constrains the trajectory. Even when a calculated K is positive everywhere, a unilateral surface may be unable to supply the normal force needed to keep the particle on the assumed path. Unlike a bead threaded onto a wire, an object on a surface can detach when maintaining contact would require a tensile normal reaction from the surface.

Consider a small sliding particle on the inside of a stationary vertical circular track of radius R . Treat it as a particle, neglect friction, and assume the track has no enclosing upper rail. Let ϕ be the angle from the downward radius, increasing as the particle climbs from the bottom. The height above the bottom is $y = R(1 - \cos \phi)$. If it enters with speed corresponding to an earlier frictionless descent from rest through height h , gravity work gives

$$\frac{1}{2}mv^2 = mg[h - R(1 - \cos \phi)], \quad v^2 = 2g[h - R + R \cos \phi].$$

The normal force does no work while contact exists because the fixed-track normal is perpendicular to the velocity. Nevertheless its required magnitude is found from radial Newtonian dynamics. The inward component of gravity is $-mg \cos \phi$, so

$$N - mg \cos \phi = \frac{mv^2}{R}, \quad \frac{N}{mg} = \frac{2h}{R} - 2 + 3 \cos \phi.$$

A surface contact requires $N \geq 0$. At the top, $v_{\text{top}}^2 = 2g(h - 2R)$ gives an energy condition $h \geq 2R$, whereas $N_{\text{top}} \geq 0$ requires the stronger condition $h \geq 5R/2$. This conclusion uses energy and force analysis together.

Example 7.10: Separating an energy test from a contact test

A 0.50 kg particle enters the circular track with $R = 1.00$ m and energy corresponding to $h = 2.20$ m. Does it remain on the surface through the top?

Energy prediction. Were it able to follow the entire circle, the top speed would be

$$v_{\text{top}} = \sqrt{2g(h - 2R)} = \sqrt{3.924} \text{ m s}^{-1} \simeq 1.98 \text{ m s}^{-1}.$$

This positive result is a necessary energy test, but it does not establish contact.

Force test. At the top the required normal force would be

$$N_{\text{top}} = mg(2h/R - 5) = (0.50)(9.81)(-0.60) \text{ N} = -2.943 \text{ N}.$$

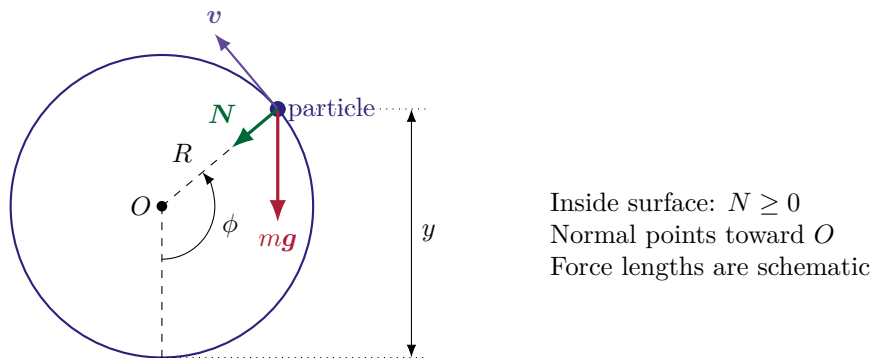


Figure 7.12: Inside contact on a vertical circular track. The angle ϕ is measured from the downward radius; the illustrated particle is climbing counterclockwise. The support pushes inward and gravity is vertically downward. The velocity tangent is drawn separately in purple; it is not an additional force. The height construction determines the gravitational work.

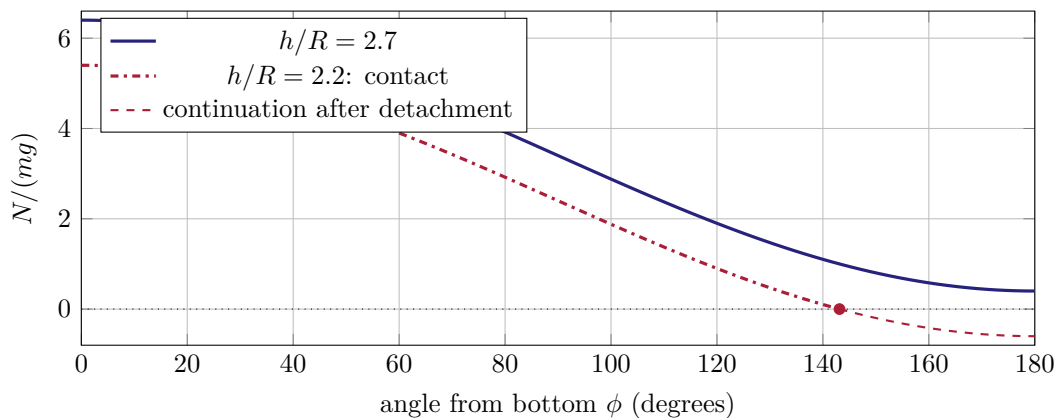


Figure 7.13: The normal force predicted while a particle follows the inside of a frictionless circular track. For $h/R = 2.2$, the dashed negative continuation violates unilateral contact and therefore cannot describe the actual track-following motion. For $h/R = 2.7$, contact remains possible through the top. The curves are analytical, and PGF trigonometric arguments here are in degrees.

The stated surface cannot provide this outward pulling force. Contact fails earlier, at $N = 0$:

$$\cos \phi_{\text{detach}} = \frac{2 - 2h/R}{3} = -0.80, \quad \phi_{\text{detach}} = 143.13^\circ.$$

The detachment height is $R(1 + 0.80) = 1.80 \text{ m}$ and the speed is $\sqrt{2g(2.20 - 1.80)} \simeq 2.80 \text{ m s}^{-1}$. At that event the radial weight component alone supplies mv^2/R , which independently checks the condition. After detachment the particle follows free-flight dynamics until another contact; extending the circular-track energy model through the top would answer the wrong physical question.

7.5 Gravity and springs as work examples

The work–kinetic-energy theorem becomes especially useful when a force law can be integrated once and reused. Two important examples are uniform gravity near Earth’s surface and an ideal linear spring. We derive their work directly here. Chapter 8 will reorganize the same results using potential energy.

7.5.1 Work by uniform gravity

Choose $+y$ upward. Near Earth’s surface, over heights small enough that g can be treated as constant,

$$\mathbf{F}_g = -mg\hat{\mathbf{j}}.$$

For a displacement from y_i to y_f ,

$$W_g = \int_{y_i}^{y_f} (-mg) dy \quad (7.18)$$

$$= mg(y_i - y_f). \quad (7.19)$$

Gravity therefore does positive work when the particle moves downward and negative work when it moves upward. In the uniform-gravity model the result depends only on the endpoint heights, even if the actual path between them is not vertical.

Example 7.11: A descending inspection tool

A 4.0 kg inspection tool is lowered from $y_i = 12.0 \text{ m}$ to $y_f = 7.5 \text{ m}$. What work does gravity do?

$$W_g = (4.0)(9.81)(12.0 - 7.5) = 1.77 \times 10^2 \text{ J}.$$

The positive sign agrees with the fact that the downward gravitational force has a component along the downward displacement.

7.5.2 Work by an ideal spring

Let x be the signed extension of a spring from its relaxed length. For an ideal linear spring,

$$F_s(x) = -k_s x. \quad (7.20)$$

The minus sign means the spring force opposes the displacement from the relaxed state. Its work from x_i to x_f is

$$W_s = \int_{x_i}^{x_f} (-k_s x) dx \quad (7.21)$$

$$= \frac{1}{2} k_s (x_i^2 - x_f^2). \quad (7.22)$$

The expression is valid for extension or compression as long as the ideal linear spring law remains valid. The signs are handled automatically by the squared endpoint values.

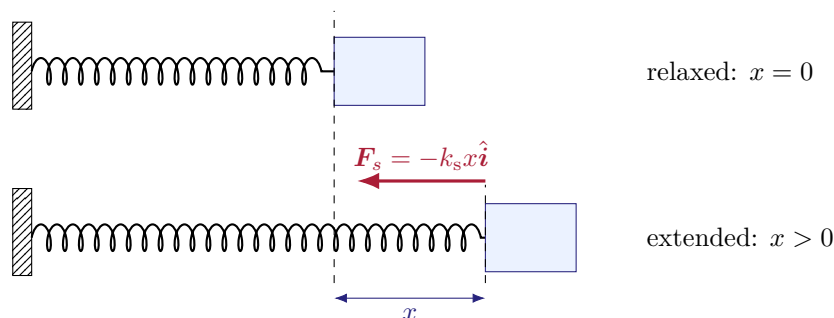


Figure 7.14: Two configurations of an ideal spring with the same fixed left support. The signed extension x is the displacement of the moving spring attachment from its relaxed position; it is positive for extension and negative for compression. Object sizes and spring coils are schematic. Coordinate-based coils and reference guides use the construction method studied in Neutelings, *Springs & masses*, TikZ.net.

Example 7.12: A carriage compressing a spring after initial motion

A 1.50 kg carriage moves on a horizontal frictionless guide with speed 2.50 m s^{-1} when the spring is already compressed by $q_i = 0.040 \text{ m}$. Here $q = -x \geq 0$ is the compression magnitude; the signed extension is $x_i = -0.040 \text{ m}$. It continues in the compression direction until $q_f = 0.120 \text{ m}$. The spring constant is $k_s = 320 \text{ N m}^{-1}$. Find the spring work and the carriage speed at q_f . Also find the maximum compression predicted by the ideal model.

Spring work.

$$W_s = \frac{1}{2} (320) (0.040^2 - 0.120^2) = -2.048 \text{ J}.$$

The initial kinetic energy is

$$K_i = \frac{1}{2} (1.50) (2.50)^2 = 4.6875 \text{ J}.$$

With no other force doing work,

$$K_f = K_i + W_s = 2.6395 \text{ J},$$

so

$$v_f = \sqrt{\frac{2K_f}{m}} = 1.88 \text{ m s}^{-1}.$$

Maximum compression. At the turning point $K = 0$, so

$$0 = K_i + \frac{1}{2} k_s (q_i^2 - q_{\text{turn}}^2).$$

Hence

$$q_{\text{turn}} = \sqrt{q_i^2 + \frac{mv_i^2}{k_s}} = 0.176 \text{ m}.$$

Reachability check. Any proposed compression larger than 0.176 m would produce a negative K in the ideal calculation. The carriage turns before reaching such a point.

7.5.3 Combining several work contributions

If gravity, a spring, a cable and friction all act, the theorem uses the sum

$$K_f - K_i = W_g + W_s + W_{\text{cable}} + W_{\text{friction}} + \dots \quad (7.23)$$

The calculation should preserve the identity of each work term until its sign and model are clear. Combining them too early can hide a sign mistake or an inappropriate displacement.

Guided checks

Diagnostic X07-05. An ideal spring is compressed farther from $q = 0.05$ m to $q = 0.10$ m. Is the spring work positive or negative? *Negative, because $q_f^2 > q_i^2$.*

Here $q = -x$ is the nonnegative compression magnitude. If instead the spring is released from 0.10 m to 0.05 m, Eq. (7.22) gives positive work. The force transfers energy to the moving particle as the deformation magnitude decreases.

7.5.4 Lifting at changing speed: gravity work and cable work

The gravitational work depends on the change of height in the uniform-field model. The cable work generally does not reduce to $+mg\Delta y$ unless the object's initial and final kinetic energies are equal and no other work-producing forces intervene. An upward accelerating lift requires tension greater than mg ; an upward lift slowing to a stop can have tension less than mg . Gravity still does negative work during both upward stages because its force remains downward while the displacement remains upward.

For a load treated as a particle,

$$\Delta K = W_T + W_g, \quad W_T = \Delta K + mg\Delta y.$$

This relation is an accounting result for the stated load and force model. It should be distinguished from the energy supplied to a complete hoist, which may also accelerate rotating parts, overcome machine losses, or deform a cable. Even in the ideal load model, the cable work over one stage can exceed or fall below mg times that stage's height while the complete rest-to-rest lift totals $mg\Delta y$.

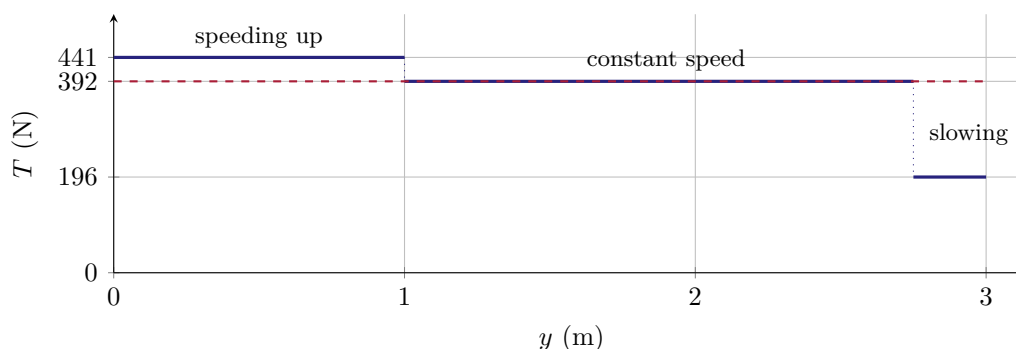


Figure 7.15: An idealized three-stage upward lift of a 40.0 kg load through 3.00 m. Horizontal segments give cable tension in each stage; dotted vertical connectors mark idealized changes of control law and contribute no work. The dashed level is mg . The cable's force-position area exceeds $mg\Delta y$ during acceleration and falls short during upward braking.

Example 7.13: A rest-to-rest lift with three control stages

A 40.0 kg load is raised 3.00 m by an ideal vertical cable. Starting from rest, it accelerates upward at $g/8$ for the first 1.00 m, moves at constant speed for the next 1.75 m, then has a downward acceleration of magnitude $g/2$ over the final 0.250 m. Use $g = 9.81 \text{ m s}^{-2}$. Find the

tension and cable work in each stage, verify that the load stops at the top, and compare the total cable and gravity works.

Strategy and model. The cable remains taut and the load has no other work-producing force. Newton's law determines each constant tension; the work theorem determines the stagewise changes of kinetic energy. The model idealizes instantaneous changes in acceleration between stages, with no jump in speed.

Acceleration stage. With upward positive, $T_1 - mg = m(g/8)$, hence $T_1 = 9mg/8 = 441.45 \text{ N}$. Its work is $W_{T1} = 441.45 \text{ J}$, while gravity does $W_{g1} = -392.40 \text{ J}$. The net work is 49.05 J , giving

$$v_1^2 = \frac{2(49.05)}{40.0} = 2.4525 \text{ m}^2 \text{ s}^{-2}, \quad v_1 = 1.57 \text{ m s}^{-1}.$$

Constant-speed stage. $T_2 = mg = 392.40 \text{ N}$. Over 1.75 m , $W_{T2} = 686.70 \text{ J}$ and $W_{g2} = -686.70 \text{ J}$. Net work is zero, so the speed remains v_1 .

Upward braking stage. $T_3 - mg = -mg/2$, so $T_3 = mg/2 = 196.20 \text{ N}$. Over the upward 0.250 m , $W_{T3} = 49.05 \text{ J}$ and $W_{g3} = -98.10 \text{ J}$. The net work -49.05 J removes exactly the kinetic energy acquired in stage 1, so $v_f = 0$. Tension stays positive; the cable does positive work while the *net* work is negative.

Whole-lift check. $W_T = 441.45 + 686.70 + 49.05 = 1177.20 \text{ J}$, and $W_g = -mg(3.00) = -1177.20 \text{ J}$. Their sum is zero, consistent with equal initial and final speeds. Independently, $v_f^2 = v_1^2 - 2(g/2)(0.250) = 2.4525 - 2.4525 = 0$. Stage details alter the tension history, but not the total ideal cable work for this specified rest-to-rest lift.

7.5.5 Spring compression: a signed coordinate and a positive magnitude

An ideal spring's signed coordinate must be attached to a geometrical convention. For a spring connected to a block on its right, let $x = 0$ place the block at the spring's relaxed length and take rightward as positive. Extension is $x > 0$; compression is $x < 0$. The spring force on the block is $F_s = -k_s x$, so it points left for extension and right for compression. During increasing compression, $dx < 0$ while $F_s > 0$; their product is negative. The spring resists the continuing compression.

One may instead use a nonnegative compression magnitude $q = -x$ on the compressed side. This changes the differential as well as the coordinate: $dx = -dq$. Therefore

$$dW_s = F_s dx = (k_s q)(-dq) = -k_s q dq.$$

The spring's work during increasing compression is still negative. The quantity $+k_s q$ is the rightward Cartesian force component, whereas $-k_s q$ is the force component conjugate to increasing q . Using the first with dq without the minus sign mixes two conventions. Writing compression as a positive magnitude is perfectly legitimate if the work expression is transformed consistently.

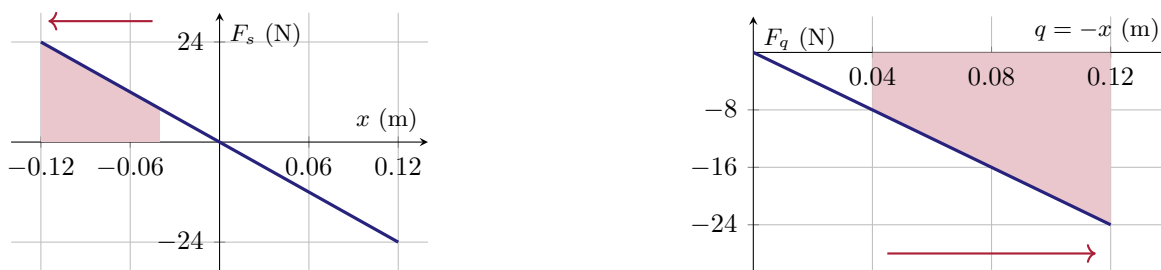


Figure 7.16: Two consistent descriptions of compressing a 200 N m^{-1} spring from $q = 0.040$ to 0.120 m . Left: $F_s = -k_s x$ is positive, but integration runs from $x = -0.040$ to -0.120 m . Right: the force conjugate to q is $F_q = -k_s q$, with increasing limits. Both give $W_s = -1.28 \text{ J}$. Red shading denotes negative work in the specified traversal, even where the Cartesian force lies above the axis.

The endpoint result $W_s = \frac{1}{2}k_s(x_i^2 - x_f^2)$ hides this orientation detail neatly, but the integral reveals why it works. A spring relaxing from either extension or compression reduces x^2 and does positive work. A path crossing the relaxed point can have a positive-work part followed by a negative-work part. The spring does not stop exerting a restoring force merely because the net work over two symmetric endpoint deformations cancels. At $x = 0$, its force is zero, but a moving block can have its largest speed there.

Example 7.14: Spring work with two coordinate conventions

A block connected to a 200 N m^{-1} horizontal ideal spring is compressed from magnitude $q_i = 0.040 \text{ m}$ to $q_f = 0.120 \text{ m}$. Find the work by the spring using both the signed extension $x = -q$ and the compression magnitude q . If a 0.800 kg block has speed 2.00 m s^{-1} at the initial compression and no other force does work, find its speed at the final compression. Finally, find the spring work when it subsequently moves from $x = -0.120 \text{ m}$ to $x = +0.120 \text{ m}$.

Strategy. The work is determined by force times the differential of the same coordinate. After verifying the sign in both conventions, add spring work to the initial kinetic energy. The last journey is split at the relaxed point to expose work cancellation.

Signed-coordinate calculation.

$$W_s = \int_{-0.040}^{-0.120} (-k_s x) dx = \frac{1}{2}k_s [(-0.040)^2 - (-0.120)^2] = -1.28 \text{ J}.$$

Compression-coordinate check.

$$W_s = \int_{0.040}^{0.120} (-k_s q) dq = -\frac{1}{2}k_s (0.120^2 - 0.040^2) = -1.28 \text{ J}.$$

Initially $K_i = \frac{1}{2}(0.800)(2.00)^2 = 1.60 \text{ J}$. Hence $K_f = 0.320 \text{ J}$ and

$$v_f = \sqrt{\frac{2(0.320)}{0.800}} = 0.894 \text{ m s}^{-1}.$$

The block still has kinetic energy at the final compression, so this state is reachable on the initial compression leg.

Crossing the relaxed point. On a later passage from -0.120 to 0 , the spring does $+\frac{1}{2}k_s(0.120)^2 = +1.44 \text{ J}$. From 0 to $+0.120 \text{ m}$, it does -1.44 J . The total is zero, although its work on each part is nonzero. If the later passage starts with 0.320 J , the kinetic energy at $x = 0$ is 1.76 J , corresponding to 2.10 m s^{-1} , then returns to 0.320 J at the symmetric extension.

Physical check. The initial further compression reduces kinetic energy, as a restoring spring should. Equal endpoint deformation magnitudes give equal endpoint kinetic energies in this one-force horizontal model, without implying constant speed between the endpoints. The forces at $x = \pm 0.120 \text{ m}$ are opposite, consistent with their restoring directions.

7.6 Power and energy-transfer rates

Work tells how much energy is transferred over a process. Power tells how rapidly that transfer occurs. The average power associated with a work transfer W during a time interval Δt is

$$\mathcal{P}_{\text{av}} = \frac{W}{\Delta t}. \quad (7.24)$$

The instantaneous power of a force is

$$\mathcal{P} = \frac{dW}{dt} = \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \boxed{\mathbf{F} \cdot \mathbf{v}}. \quad (7.25)$$

The SI unit is the watt: $1 \text{ W} = 1 \text{ J s}^{-1}$.

For one-dimensional motion along the force direction, $\mathcal{P} = Fv$. If force and velocity point oppositely, power is negative: the selected force is removing kinetic energy from the particle at that instant. A perpendicular force has zero instantaneous power.

7.6.1 Average of a product versus product of averages

In general,

$$\frac{1}{\Delta t} \int_{t_i}^{t_f} \mathbf{F}(t) \cdot \mathbf{v}(t) dt \neq \left(\frac{1}{\Delta t} \int \mathbf{F} dt \right) \cdot \left(\frac{1}{\Delta t} \int \mathbf{v} dt \right). \quad (7.26)$$

Thus $\mathcal{P}_{\text{av}}$ is not generally obtained by multiplying average force by average velocity. That shortcut is valid only under additional conditions that make the product behave appropriately.

7.6.2 Averaging a power history

The meaning of an average is fixed by the variable over which the average is taken. For a time interval of duration T , write $\langle q \rangle = T^{-1} \int_0^T q(t) dt$. In one dimension,

$$\langle Fv \rangle = \langle F \rangle \langle v \rangle + \langle (F - \langle F \rangle)(v - \langle v \rangle) \rangle.$$

The final term is the time covariance of force and velocity. When larger force tends to occur at larger velocity, that term can be positive. It can also be negative or zero. This statement concerns the selected force and the velocity of its application point; an average over positions would be a different operation. In several dimensions the same decomposition applies to the scalar product, with the component covariances summed.

Example 7.15: An explicit counterexample to multiplying averages

A 2.00 kg particle moves on a horizontal frictionless guide. Its initial velocity is 1.00 m s^{-1} , and its only horizontal force is $F(t) = (4.00 \text{ N s}^{-1})t$ for $0 \leq t \leq 2.00 \text{ s}$. Find the work and average power; compare the result with the product of the time-averaged force and velocity.

The velocity must follow the prescribed force. Newton's second law gives $a(t) = (2.00 \text{ m s}^{-3})t$, hence

$$v(t) = 1.00 \text{ m s}^{-1} + (1.00 \text{ m s}^{-3})t^2.$$

Using numerical values of t in seconds, the power in watts is $\mathcal{P}(t) = 4t + 4t^3$. Therefore

$$W = \int_0^2 (4t + 4t^3) dt = [2t^2 + t^4]_0^2 = 24.0 \text{ J}, \quad \mathcal{P}_{\text{av}} = 12.0 \text{ W}.$$

The two separate time averages are $\langle F \rangle = 4.00 \text{ N}$ and $\langle v \rangle = (7/3) \text{ m s}^{-1}$. Their product is only $28/3 \text{ W} = 9.33 \text{ W}$. The difference $8/3 \text{ W}$ is the covariance term, rather than a rounding discrepancy.

Independent check. The final speed is 5.00 m s^{-1} . Thus $\Delta K = \frac{1}{2}(2.00)(5.00^2 - 1.00^2) = 24.0 \text{ J}$, agreeing with the power integral. A calculation that multiplied the two averages would predict only 18.7 J of work and fail this check. For constant force, in contrast, $\langle Fv \rangle = F\langle v \rangle$ holds exactly, even if the velocity changes.

Example 7.16: An ideal lifting actuator

An ideal lifting mechanism raises a 650 kg load vertically at constant speed 0.75 m s^{-1} . Find the useful mechanical power delivered to the load. If an explicitly specified overall energy-transfer efficiency is $\eta = 0.82$, what input power is required in this simplified model?

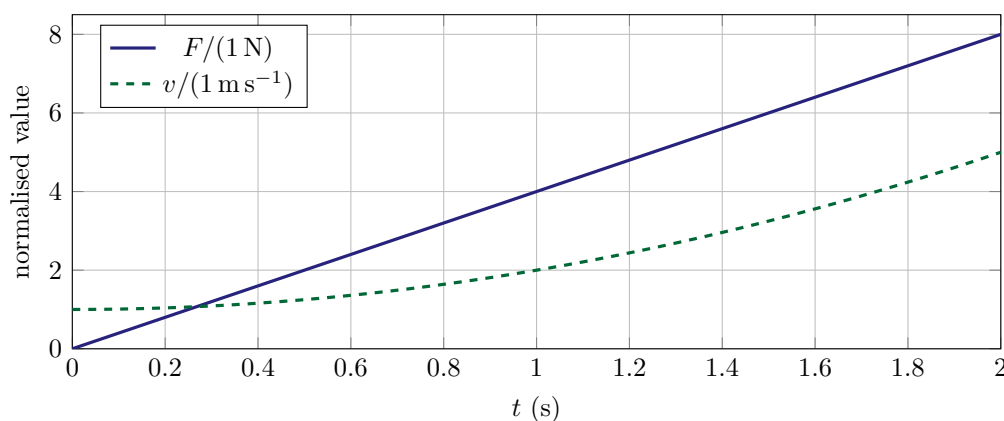


Figure 7.17: Force and velocity histories for the average-product example. Each ordinate is separately normalised to make it dimensionless; equal plotted heights do not mean that force and velocity are interchangeable. Both histories rise, producing a positive covariance and an average power of 12.0 W.

Useful mechanical power. Constant speed implies the upward actuator force balances the weight, so $F = mg$. Therefore

$$\mathcal{P}_{\text{useful}} = mgv = (650)(9.81)(0.75) = 4.78 \text{ kW}.$$

Efficiency. Define efficiency here as

$$\eta = \frac{\mathcal{P}_{\text{useful}}}{\mathcal{P}_{\text{in}}}.$$

Then

$$\mathcal{P}_{\text{in}} = \frac{4.78 \text{ kW}}{0.82} = 5.83 \text{ kW}.$$

Boundary. The value $\eta = 0.82$ is a stipulated teaching parameter, not a claim about a particular commercial actuator. Electrical losses, thermal limits, drive electronics and safety margins require additional device-specific modelling.

7.6.3 Constant force and constant power are different operating laws

If a drive force is constant and parallel to the velocity,

$$\mathcal{P} = Fv, \tag{7.27}$$

so power rises linearly with speed. If instead a drive is idealized as supplying fixed positive power $\mathcal{P}_0$ at nonzero speed,

$$F = \frac{\mathcal{P}_0}{v}. \tag{7.28}$$

The fixed-power formula is singular at $v = 0$. It must not be extrapolated through zero speed. A real actuator has finite force, torque, current, thermal and control limits; an ideal constant-power range begins only after some nonzero speed.

Guided checks

Diagnostic X07-06. A motor delivers constant mechanical power while the driven object speeds up. Must the driving force stay constant? *No. In the one-dimensional idealization $F = \mathcal{P}/v$, so the force decreases as speed rises.*

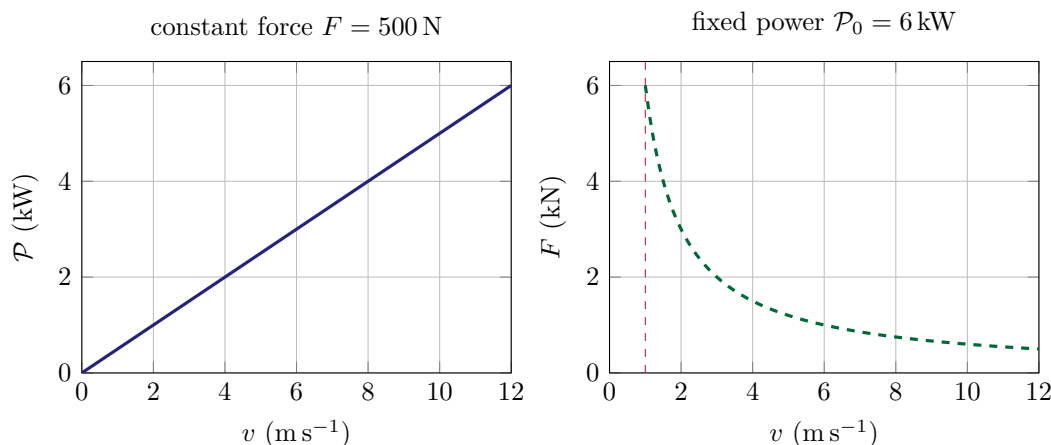


Figure 7.18: analytical operating-law comparison. With constant force, power grows linearly with speed. With fixed positive power, the ideal force varies as $1/v$. The displayed fixed-power branch begins at $v = 1 \text{ m s}^{-1}$ and cannot be extrapolated to $v = 0$.

7.6.4 A work rate needs a named mechanical port

For a simple particle a force and its point of application share the particle velocity. For a transmission, hoist or deformable body, the input and output application points can have different velocities. A pulling force of 100 N moving its contact point at 2.0 m s^{-1} transfers 200 W at that port. A second port delivering 200 N at 1.0 m s^{-1} transfers the same mechanical power. The equality follows only for an ideal transmission without net energy accumulation or loss. The larger output force does not imply a larger energy transfer because its application-point speed is smaller.

For a stated steady input/output model, power bookkeeping can be written $\mathcal{P}_{\text{in}} = \mathcal{P}_{\text{out}} + \mathcal{P}_{\text{loss}}$. During a transient, one must include the rate of energy accumulation in the device, giving $\mathcal{P}_{\text{in}} = \mathcal{P}_{\text{out}} + \mathcal{P}_{\text{loss}} + dE_{\text{device}}/dt$. These balances identify a boundary and a sign convention; they do not supply a missing efficiency. A flywheel can make the output power temporarily exceed the contemporary input by losing stored energy. Conversely, an accelerating motor can absorb input while building its own kinetic energy. Applying a steady efficiency indiscriminately to such a transient can therefore mix different transfers.

The distinction is also relevant to a duty cycle. Suppose an ideal carriage receives 500 J during a 2.0 s movement and then waits for 8.0 s . Its average mechanical power during motion is 250 W , while the average over the full 10.0 s duty is 50 W . Neither number alone determines the peak power within the movement. If a drive uses a stated 20 W of standby electrical power during the wait, that adds 160 J of electrical energy without adding mechanical work to the stationary carriage. The time interval and the receiving system are part of every reported average. These simple examples show why peak, moving average, cycle average and electrical input are different measurable quantities rather than interchangeable specifications.

7.6.5 Acceleration, cruise and stopping in an upward lift

An actuator can require a large force at small speed and a large power at another instant. Its work over a complete motion does not specify either peak. To see the distinction, first write a motion history, then infer the force from Newton's second law. For a load moving vertically with upward-positive coordinate y , the ideal actuator force and its power are

$$F_A = m(g + a_y), \quad \mathcal{P}_A = m(g + a_y)v_y.$$

Gravity supplies $\mathcal{P}_g = -mgv_y$, so the net power is $\mathcal{P}_A + \mathcal{P}_g = ma_yv_y = dK/dt$. These identities also identify the physical source of each transfer. During upward deceleration, the net power is negative;

the actuator power can still be positive if its upward force is smaller than the weight. Regeneration cannot be inferred merely from a decreasing speed.

Example 7.17: A complete lifting duty with three speed phases

An ideal cable raises a 200 kg load from rest to rest. Take $g = 9.81 \text{ m s}^{-2}$ and neglect cable mass and friction. The speed is prescribed as

$$v_y(t) = \begin{cases} (0.50 \text{ m s}^{-2})t, & 0 \leq t \leq 2.00 \text{ s}, \\ 1.00 \text{ m s}^{-1}, & 2.00 \leq t \leq 6.00 \text{ s}, \\ 1.00 \text{ m s}^{-1} - (0.50 \text{ m s}^{-2})(t - 6.00 \text{ s}), & 6.00 \leq t \leq 8.00 \text{ s}. \end{cases}$$

Find the height gain, actuator force and power in each phase, total actuator work and average power. If an overall motoring efficiency of 0.80 is stipulated throughout, estimate the electrical input energy and peak input power.

Geometry of the speed history. The displacement is the area under $v_y(t)$. The two triangular parts each contribute 1.00 m and the cruise contributes 4.00 m, giving $\Delta y = 6.00 \text{ m}$. More explicitly, with numerical t in seconds, the height in metres is $0.25t^2$ before 2 s, $t - 1$ during cruise, and $5 + (t - 6) - 0.25(t - 6)^2$ after 6 s. These branches meet continuously at heights 1.00 and 5.00 m.

Force and instantaneous power. The accelerations are $+0.50$, 0 and -0.50 m s^{-2} . Consequently,

$$F_A = \begin{cases} 2062 \text{ N} & \text{acceleration,} \\ 1962 \text{ N} & \text{cruise,} \\ 1862 \text{ N} & \text{deceleration.} \end{cases}$$

Multiply each force by the speed in its own phase. The power rises from zero to 2062 W during acceleration, is 1962 W during cruise, and falls from 1862 W to zero during deceleration. The highest actuator power occurs immediately before the acceleration phase ends. At startup the force is already nonzero while the instantaneous mechanical power is zero.

Integrate rather than extrapolate a cruise estimate. Because force is constant within each phase, the three actuator works are

$$W_{A,1} = 2062(1.00) = 2062 \text{ J}, \quad W_{A,2} = 1962(4.00) = 7848 \text{ J}, \quad W_{A,3} = 1862(1.00) = 1862 \text{ J}.$$

Thus $W_A = 11772 \text{ J}$ and $\mathcal{P}_{A,\text{av}} = 1471.5 \text{ W}$. The first phase increases kinetic energy by 100 J, and the last decreases it by the same amount. The total therefore equals $mg\Delta y = 11772 \text{ J}$, as required by the initial and final zero speeds. Gravity does -11772 J , giving zero total net work over the complete duty.

The stated efficiency applies to a defined transfer. All actuator powers here are nonnegative, so the stipulated motoring model gives $E_{\text{in}} = W_A/0.80 = 14715 \text{ J}$ and $\mathcal{P}_{\text{in,peak}} = 2062/0.80 = 2577.5 \text{ W}$. The average input is 1839.4 W. These values exclude any unprovided standby consumption or acceleration smoothing. The abrupt force jumps are the idealisation of piecewise constant acceleration; a real controller would round those transitions. Smoothing can change a peak while preserving the rest-to-rest work if the final height remains the same.

7.6.6 A finite startup law joined to a constant-power range

A useful bounded actuator model is

$$F_d(v) = \begin{cases} F_{\text{max}}, & 0 \leq v \leq v_*, \\ \mathcal{P}_{\text{max}}/v, & v \geq v_*, \end{cases} \quad v_* = \frac{\mathcal{P}_{\text{max}}}{F_{\text{max}}}.$$

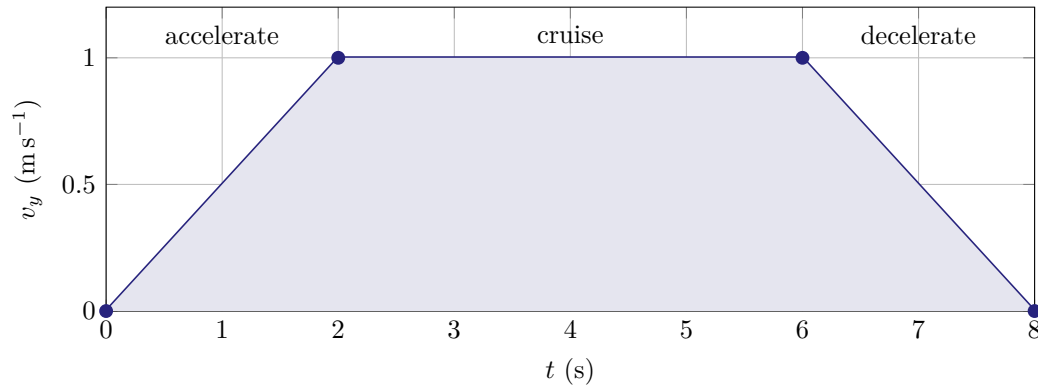


Figure 7.19: Prescribed upward lift speed. The shaded area is 6.00 m of height gain. The two ramp regions have equal area but opposite changes of kinetic energy.

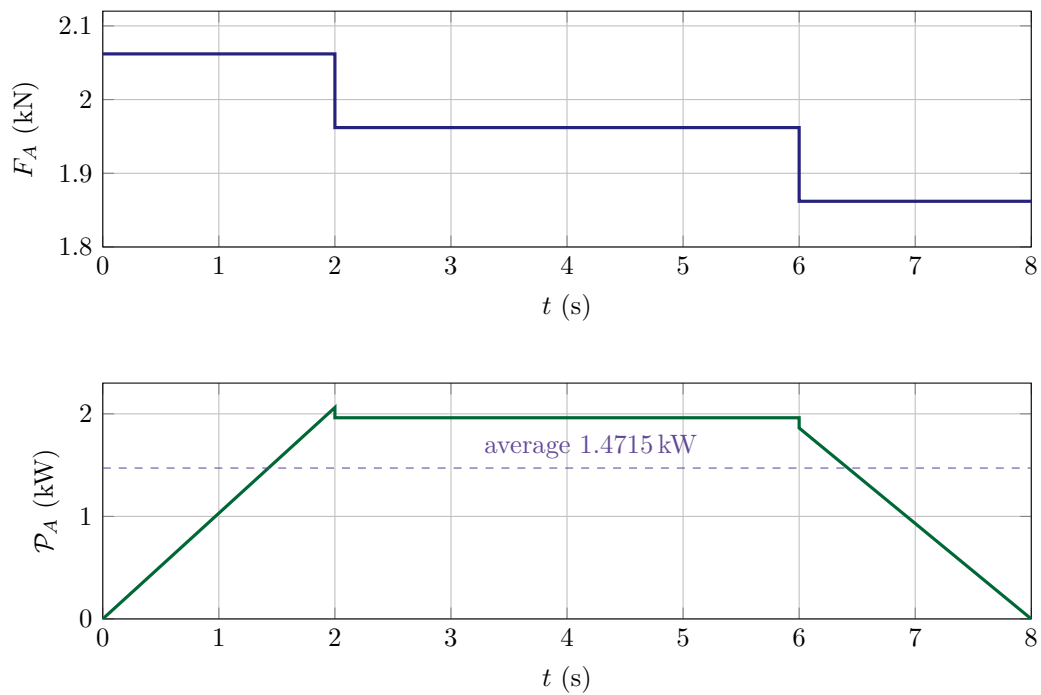


Figure 7.20: Actuator force and mechanical power for the lifting duty. The vertical segments represent idealised changes of acceleration, rather than infinite impulses: the velocity remains continuous. Force, power and average power specify different design requirements.

It is continuous at v_* and has finite force at zero speed. Its power rises as $F_{\max}v$ in the first range, then remains $\mathcal{P}_{\max}$. This model describes forward motoring only. A reverse-motion or braking controller needs its own stated law, and resistive forces must be added before relating the drive power to dK/dt .

Example 7.18: Startup with a force cap and a power cap

A 100 kg carriage starts from rest on a frictionless horizontal guide. Its drive has $F_{\max} = 1000$ N and $\mathcal{P}_{\max} = 5000$ W, following the bounded law above. Find the time and distance to reach 10.0 m s^{-1} , the total work, and compare with an uncapped constant 1000 N drive.

Locate the switch. The crossover speed is $v_* = 5000/1000 = 5.00 \text{ m s}^{-1}$. Before the switch, $a = F_{\max}/m = 10.0 \text{ m s}^{-2}$, so $t_* = 0.500 \text{ s}$ and $x_* = \frac{1}{2}at_*^2 = 1.250 \text{ m}$. The work in this phase is 1250 J.

Solve the constant-power range in its valid domain. For $t \geq t_*$, $mv dv/dt = \mathcal{P}_{\max}$ gives

$$v^2 = v_*^2 + \frac{2\mathcal{P}_{\max}}{m}(t - t_*).$$

The time to increase from 5.00 to 10.0 m s^{-1} is $m(10.0^2 - 5.00^2)/(2\mathcal{P}_{\max}) = 0.750 \text{ s}$. Thus the total time is 1.250 s. To obtain distance, use $dt = (m/\mathcal{P}_{\max})v dv$, hence

$$x - x_* = \frac{m}{3\mathcal{P}_{\max}}(v^3 - v_*^3) = \frac{100}{15000}(1000 - 125) = 5.833 \text{ m}.$$

The full distance is 7.083 m. The constant-power phase supplies $5000(0.750) = 3750 \text{ J}$, and the total 5000 J equals $\frac{1}{2}mv_f^2$.

Comparison and interpretation. An unrestricted constant 1000 N force reaches the same final speed in 1.00 s over 5.00 m. It does the same 5000 J of work but reaches 10000 W at the final speed. The capped drive takes longer and travels farther because its acceleration decreases after the crossover. Its average power is 4000 W, smaller than its 5000 W maximum. A power rating alone therefore does not determine startup performance; the available low-speed force matters.

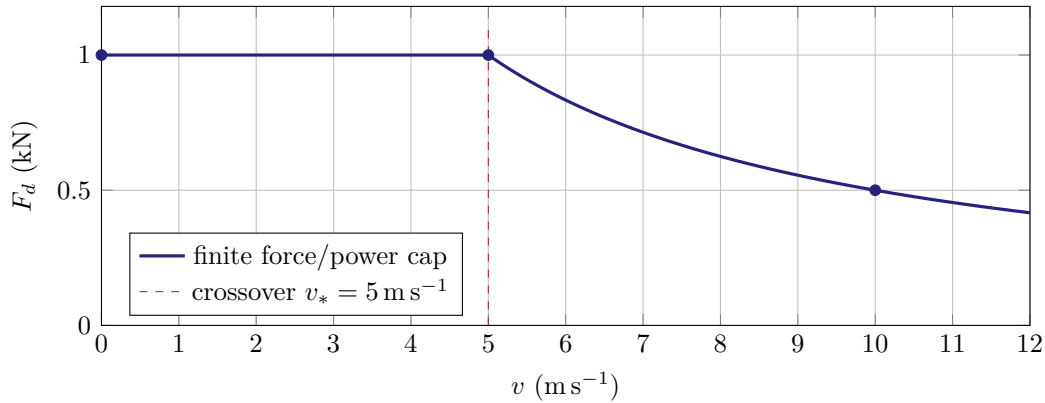


Figure 7.21: A bounded forward drive law. The horizontal branch includes startup at $v = 0$, with finite force and zero mechanical power. The decreasing branch begins at the finite crossover speed. Its construction avoids extrapolating $\mathcal{P}_{\max}/v$ to rest.

7.6.7 Braking power and a stipulated regenerative fraction

Negative mechanical power identifies energy removal from the moving system. It does not identify where the removed energy goes. Frictional braking can convert it into internal energy, while a regenerative actuator can return a specified portion through an electrical port. The efficiency and

the division of braking force must be supplied before a returned electrical energy can be calculated.

Example 7.19: Separate mechanical braking from returned electrical energy

A 1200 kg carriage slows uniformly from 20.0 m s^{-1} to rest in 5.00 s on a horizontal guide. Model it as a translating particle and neglect rotation and aerodynamic drag. Stipulate a directly applied linear regenerative actuator force of -3000 N and a constant sliding-friction brake force making up the remaining braking force. The regenerative conversion returns 70% of its mechanical energy removal to a storage device. Find the braking powers and the returned energy.

Motion and force split. The acceleration is -4.00 m s^{-2} and the net braking force is -4800 N . Thus the friction force is -1800 N and $v(t) = 20.0 - 4.00t$ in SI units. The stopping distance is $\frac{1}{2}(20.0)(5.00) = 50.0 \text{ m}$.

Mechanical power carries the removal sign. The net power on the carriage is

$$\mathcal{P}_{\text{net}} = -4800(20.0 - 4.00t) \text{ W},$$

falling in magnitude from 96.0 kW to zero. Its time integral is -240 kJ , exactly ΔK . The regenerative port removes $3000(50.0) = 150 \text{ kJ}$ and the friction brakes remove $1800(50.0) = 90.0 \text{ kJ}$. Their mechanical powers are $-3000v$ and $-1800v$.

Returned energy uses a different receiving-system sign. Defining positive electrical power as transfer *into* the store gives $\mathcal{P}_{\text{store}} = 0.70(3000)v$, initially 42.0 kW. The returned energy is $0.70(150) = 105 \text{ kJ}$, or 43.75% of the initial kinetic energy. The remaining 45.0 kJ removed through the regenerative port is stipulated conversion loss, and 90.0 kJ is frictional dissipation. These sum with the returned energy to 240 kJ.

A fraction needs a denominator. The conversion efficiency is 70% of regenerative mechanical removal, while the overall recovered fraction is 43.75% of initial kinetic energy. They differ because the friction brake also acts. The average net mechanical braking power is -48.0 kW , and the average returned electrical power is 21.0 kW . Real regenerative operation can change near zero speed or with battery limits; the constant-force split is only the explicitly stipulated teaching model.

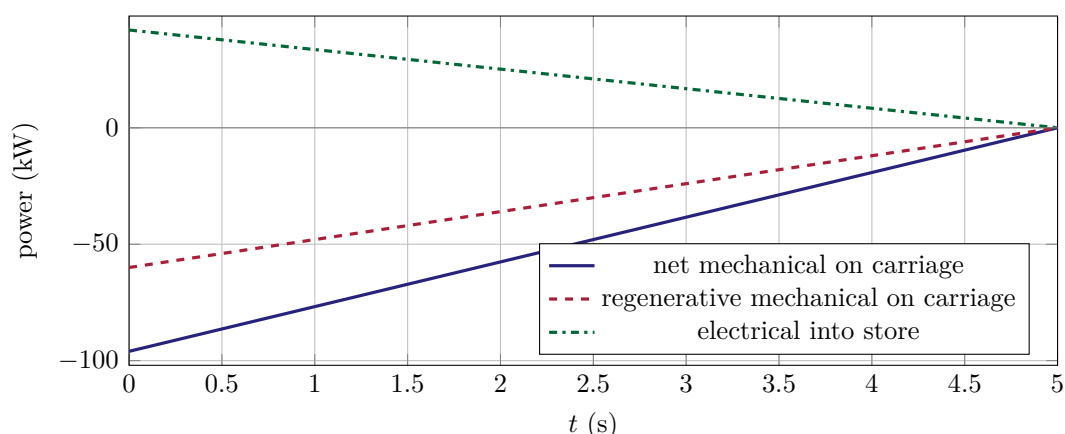


Figure 7.22: Braking and recovery in the stipulated carriage model. Negative powers refer to energy removal from the carriage; positive electrical power refers to energy received by the store. The two sign conventions are explicitly attached to different systems. Each triangular area is the corresponding energy transfer.

7.6.8 Holding and carrying a load: mechanical work and effort

Consider a force gauge supporting a stationary 10.0 kg load. Its upward force is approximately 98.1 N, but the load has zero velocity. The force's mechanical power delivered to that load is

therefore zero. This is a statement about a particular mechanical transfer, not about the operator's metabolic expenditure or a machine's electrical consumption. Muscles can use chemical energy while sustaining a nearly static force; an electromagnet can dissipate electrical energy while holding a body still. Those internal transfers require a different system description from the point-particle work $\mathbf{F} \cdot d\mathbf{r}$.

For an ideal load carried 5.00 m horizontally at constant height, a purely vertical supporting force does zero work on the load because it is perpendicular to the load displacement. Gravity also does zero work on this horizontal segment. If the same load is first raised 0.50 m quasistatically, the support instead supplies $mg\Delta y = 49.1$ J during that lifting segment. One must name the segment and the force before answering whether work is done. Startup and stopping along the horizontal path require horizontal interactions; their individual works must be included if those stages are part of the requested process.

An actual walker and load are extended, deformable systems. The hands can oscillate vertically, the supporting force can have horizontal components, and internal tissues move. The measured load trajectory and contact-force history can then yield nonzero support work, even when the start and finish heights agree. The ideal horizontal carrying result is useful precisely because its assumptions are visible. The correct response to “holding is tiring, so the supporting force must do work on the stationary load” is: effort alone does not specify application-point displacement. Resolve the measured mechanical transfer first, then discuss the additional internal energy budget separately.

7.7 Work in extended or deformable systems

The particle model hides an important geometrical fact: work is tied to the displacement of the point where a force acts. For a force applied at a material point A of an extended body,

$$W_A = \int_A \mathbf{F}_A \cdot d\mathbf{r}_A. \quad (7.29)$$

If another force acts at a different point B , its work must use $d\mathbf{r}_B$. The total external work is the sum of the actual force–application-point integrals.

By contrast, the translational kinetic energy associated with the centre of mass satisfies a different relation under Newtonian conditions:

$$\Delta K_{\text{CM}} = \int \mathbf{F}_{\text{ext,net}} \cdot d\mathbf{R}_{\text{CM}}, \quad K_{\text{CM}} = \frac{1}{2}MV_{\text{CM}}^2. \quad (7.30)$$

Equation (7.30) tracks only centre-of-mass translational kinetic energy. It is not automatically equal to the total external work when the body rotates, deforms or has different application points moving differently.

7.7.1 A simple deformable connector

Consider a connector pulled outward by equal and opposite forces at its ends. The net external force is zero, so the centre of mass can remain at rest. Nevertheless, both forces can do positive work if each end moves in the direction of its applied force. Energy enters the deformable system even though $\Delta K_{\text{CM}} = 0$.

Example 7.20: Equal net force does not determine energy transfer

A deformable connector has two ends A and B . A constant force $\mathbf{F}_A = -400\hat{\mathbf{i}}$ N acts at the left end and $\mathbf{F}_B = +400\hat{\mathbf{i}}$ N acts at the right end. During a slow symmetric extension, A moves 0.030 m left and B moves 0.030 m right. The centre of mass remains fixed. Find the actual external work and compare it with the centre-of-mass work.

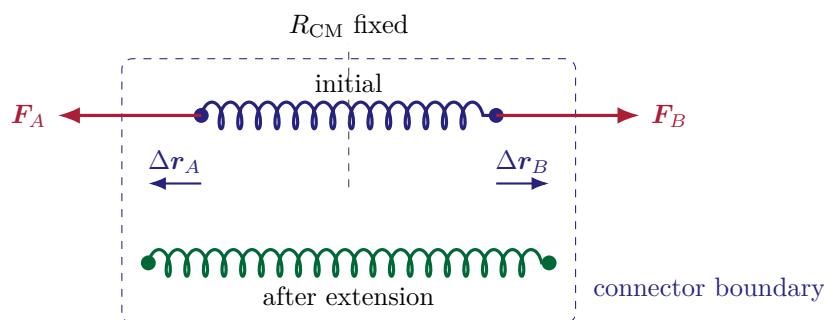


Figure 7.23: Equal and opposite endpoint forces can have zero net force and zero centre-of-mass displacement while both application points move along their respective forces. The dashed boundary selects the connector; endpoint forces transfer energy inward through positive work. The centre-of-mass path is a fixed point, so its pseudowork is zero. Force arrows denote forces, not energy-flow rates.

Application-point work.

$$W_A = (-400)(-0.030) = +12.0 \text{ J}, \quad W_B = (+400)(+0.030) = +12.0 \text{ J}.$$

Therefore

$$W_{\text{ext}} = W_A + W_B = 24.0 \text{ J}.$$

Centre-of-mass work. The net external force is zero and $\Delta \mathbf{R}_{CM} = 0$, so

$$\int \mathbf{F}_{\text{ext,net}} \cdot d\mathbf{R}_{CM} = 0.$$

Interpretation. The missing 24 J has not disappeared. The connector's internal state has changed. A later chapter will introduce systematic internal and stored-energy bookkeeping; here the lesson is purely kinematic: actual external work is determined by the motion of the force-application points, not by net force times centre-of-mass displacement.

7.7.2 When the particle-style shortcut is safe

For a rigid body undergoing pure translation, every material point has the same displacement as the centre of mass. Then

$$\sum_i \int \mathbf{F}_i \cdot d\mathbf{r}_i = \int \left(\sum_i \mathbf{F}_i \right) \cdot d\mathbf{R}_{CM},$$

so the distinction disappears. Rotation or deformation breaks that simple equality because different application points can have different velocities and displacements.

Guided checks

Diagnostic X07-07. Two equal and opposite forces act on a deformable body, so the net force is zero. Must their total work be zero? *No. The answer depends on the separate displacements of the two application points.*

Pedagogical boundary

The present extension does not develop the full energy balance of deformable continua, rotating rigid bodies or thermodynamic systems. Its purpose is to prevent an important conceptual error before Chapter 8: net force and centre-of-mass displacement do not, by themselves, determine

all work transferred to an extended system.

7.8 Engineering, measurement and computational connections

Work and power appear in design specifications because they connect mechanics to energy supply, actuator sizing and measurable force–displacement data. The formulas are simple; the engineering value comes from matching each formula to the correct operating model.

7.8.1 From a force requirement to an energy and power requirement

If a constant horizontal drive force F_d moves a carriage through distance d , its work is $F_d d$. If the same transfer occurs in time Δt , the average mechanical power associated with that selected force is $F_d d / \Delta t$. This does not yet determine electrical input power, motor temperature or battery demand. Those quantities require efficiencies and subsystem losses to be stated explicitly.

For a lifting device operating at constant vertical speed, the simple result $\mathcal{P} = mgv$ is especially useful as a first screening estimate. For an accelerating lift, the actuator force need not equal mg , and the instantaneous power should be calculated from the actual force and velocity.

Remark 7.1: A design estimate is not a component rating

An ideal mechanical-power estimate identifies the scale of energy transfer. A real device rating must also account for duty cycle, efficiency, peak force or torque, control transients, thermal limits and safety factors. None of those should be invented when the problem statement has not supplied them.

7.8.2 Measurement proposal: work from a load–extension curve

A low-force elastic element can be studied by measuring applied force while its extension is changed slowly. A practical procedure is:

1. define the zero-extension reference before loading;
2. calibrate or zero the force sensor with no applied load;
3. record force and extension during loading;
4. return through the same nominal extension values during unloading;
5. plot both force–extension curves;
6. estimate the area under each curve with a stated quadrature rule;
7. repeat to assess reproducibility and drift.

If the loading and unloading curves differ, their enclosed area is evidence that the simple reversible ideal-spring model does not describe the entire cycle. Friction in fixtures, material hysteresis, sensor offsets or rate dependence can contribute. A measured loop should not be silently “corrected” into a straight line.

Table 7.1 is a *synthetic* dataset constructed to demonstrate the calculation. It is not presented as data from an AGH laboratory apparatus.

Table 7.1: Synthetic loading and unloading data for a low-force elastic element. Forces are deliberately close to, but not exactly on, a linear spring law.

Extension x (mm)	Loading force (N)	Unloading force (N)
0	0.00	0.00
5	0.65	0.56
10	1.31	1.15
15	1.96	1.74
20	2.62	2.34
25	3.27	2.94
30	3.92	3.55

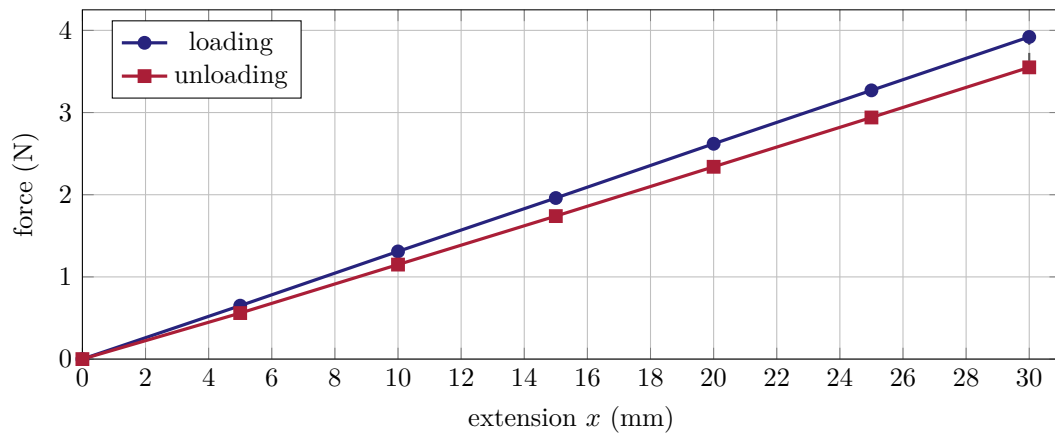


Figure 7.24: Synthetic load–extension curves corresponding to Table 7.1. The loading and unloading branches are joined by the dashed fixed-extension force–port switch at $x = 30$ mm. That vertical segment does zero Fdx work; the closed-loop area is an energy-per-cycle scale for this stipulated quasistatic model.

Example 7.21: Area from synthetic loading and unloading data

Use the trapezoidal rule on Table 7.1. The extension step is $h = 5.0 \text{ mm} = 0.0050 \text{ m}$. For the loading branch,

$$W_{\text{load}} \simeq h \left[\frac{F_0 + F_6}{2} + \sum_{j=1}^5 F_j \right] = 0.05885 \text{ J}.$$

For the unloading-force values integrated from $x = 0$ to $x = 0.030 \text{ m}$,

$$A_{\text{unload}} \simeq 0.052525 \text{ J}.$$

The difference in the two positive geometric areas is

$$A_{\text{loop}} \simeq 6.325 \times 10^{-3} \text{ J} = 6.325 \text{ mJ}.$$

If the actual unloading motion runs from large x back to small x , the displacement element dx is negative, so the external work on that return branch is negative. The geometric loop area therefore measures the mismatch between energy supplied during loading and energy returned through the same force port during unloading, within this quasistatic model.

A least-squares line constrained through the origin for the loading values gives a stiffness scale near 131 N m^{-1} . The ideal linear area $\frac{1}{2}k_s x_{\text{max}}^2$ is about 0.05885 J , close to the loading area because the synthetic loading data were intentionally constructed to be nearly linear.

7.8.3 Closing a loading cycle and interpreting its area

The synthetic branches in Table 7.1 have different endpoint forces at $x = 30 \text{ mm}$: 3.92 N on loading and 3.55 N on unloading. Therefore a closed force–extension cycle requires an explicit transition at fixed extension between these endpoints. Holding $x = 30 \text{ mm}$ while switching the stipulated force branch supplies $\int F dx = 0$ through this mechanical port, because $dx = 0$. The return follows the unloading values in decreasing x , and both branches meet at $(0, 0)$. This construction specifies a genuinely closed cycle rather than treating two disconnected endpoint values as an already closed trajectory.

Using the displayed data without intermediate rounding gives

$$W_{\text{load}} = 0.058850 \text{ J}, \quad W_{\text{return}} = -0.052525 \text{ J}, \quad W_{\text{cycle}} = 0.006325 \text{ J}.$$

Thus the rounded loop value is 6.33 mJ . The sign of the return work follows the direction of motion; 0.052525 J is the magnitude of the positive geometric area. The origin-constrained least-squares stiffness from the loading points is $k_s = 130.769 \text{ N m}^{-1}$, whose ideal triangular area is 0.0588462 J . This numerical agreement reflects the deliberately nearly linear loading data, rather than evidence that the unloading branch obeys the same reversible spring law.

The physical interpretation needs one further boundary condition. If the elastic element and fixture return to the same internal state after the closed cycle, and kinetic-energy changes are negligible, the net positive work must be transferred to other forms such as internal energy. The force–extension plot alone cannot establish whether the element has returned to that state. A drift in temperature, permanent deformation or a controller change can store or exchange energy elsewhere. A fixed-extension switch contributes zero work at the displayed port, but it need not imply zero energy exchange in an unmodelled controller. A careful report calls the computed quantity net work through the measured force–extension port; it calls it dissipation only when the cycle-state and system assumptions support that conclusion.

7.8.4 Calibration offsets and correlated uncertainty

An area calculation can be numerically precise and physically biased. Let x be the indicated displacement and use the simple calibration model

$$F_{\text{raw}} = aF + b + \epsilon_F, \quad F = \frac{F_{\text{raw}} - b}{a}.$$

Here a is a dimensionless gain, b is a common force offset, and ϵ_F represents sample noise. An offset contributes bL to an uncorrected work integral over span L ; unlike independent noise, it does not cancel by taking more closely spaced samples. A force-scale error multiplies the entire area. Displacement-scale error also multiplies the work. Simply translating a consistently measured coordinate origin leaves dx unchanged, but a wrong extension origin can corrupt a spring-law fit or cause incorrect physical integration limits.

Example 7.22: Correcting a synthetic sensor and budgeting work uncertainty

A teaching spring obeys $F = 100x$ in SI units over $0 \leq x \leq 0.0500$ m. Synthetic samples are recorded every 0.0100 m by a sensor with gain $a = 1.020$ and offset $b = 0.200$ N. The raw forces are 0.200, 1.220, 2.240, 3.260, 4.280 and 5.300 N. Compute raw and corrected trapezoidal work. Then estimate a standard uncertainty using independent sample noise $u_F = 0.030$ N, common offset uncertainty $u_b = 0.020$ N, common gain uncertainty $u_a = 0.010$, and displacement scale $s = 1.000$ with $u_s = 0.010$.

Bias calculation. The linear force law gives exact trapezoidal work for these points. Its true reference value is $W = \frac{1}{2}(100)(0.0500)^2 = 0.1250$ J. The raw result is

$$W_{\text{raw}} = aW + bL = (1.020)(0.1250) + (0.200)(0.0500) = 0.1375 \text{ J}.$$

The uncorrected bias is 0.0125 J, or 10.0% of the reference work. Correcting every force, or equivalently correcting the integrated area, gives $(W_{\text{raw}} - bL)/a = 0.1250$ J. This exact agreement is built into the synthetic model; it is not a claim about a measured apparatus.

Propagate uncertainty with the right correlations. For six equally spaced force values, the trapezoidal weights are $h(1/2, 1, 1, 1, 1, 1/2)$. Independent sample errors therefore give $u_{W,\text{raw}} = hu_F\sqrt{4.5} = 0.0006364$ J. A shared calibration offset instead enters once with the full displacement span. Including the scale factor, write

$$W = s \frac{W_{\text{raw}} - bL}{a}.$$

First-order propagation at the stated central values yields the four standard-uncertainty contributions

$$\frac{u_{W,\text{raw}}}{a} = 0.000624 \text{ J}, \quad \frac{Lu_b}{a} = 0.000980 \text{ J}, \quad \frac{Wu_a}{a} = 0.001225 \text{ J}, \quad Wu_s = 0.001250 \text{ J}.$$

Assuming these four sources are mutually independent, their squares add, giving $u_W = 0.00210$ J or 1.68%. A suitable report is $W = (0.1250 \pm 0.0021) \text{ J}$ with the uncertainty identified as a first-order standard uncertainty, rather than a guaranteed error bound.

Limits of this budget. The coordinate-scale uncertainty is common to all indicated positions; treating it as six independent position errors would describe a different model. Likewise the gain and zero uncertainties are shared, not six independent sensor noises. Drift, fixture compliance, rate dependence and correlated readout noise are excluded because no quantitative models for them were supplied. If present, they require additional measurements or terms in the uncertainty budget. Refining numerical quadrature cannot remove these calibration effects.

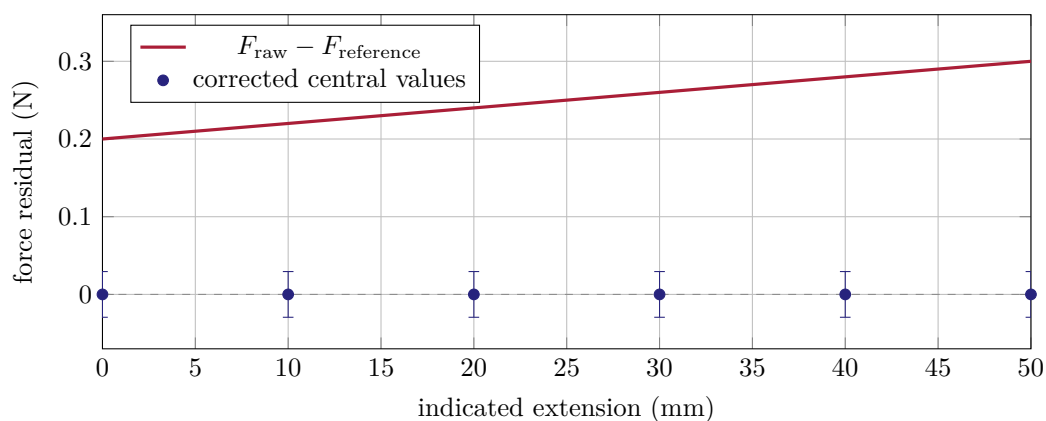


Figure 7.25: Synthetic calibration residuals for the worked example. The raw offset and gain bias make the residual rise from 0.20 to 0.30 N. Corrected central values lie on zero. The displayed bars show only the independent force-noise standard uncertainty u_F/a ; common calibration and displacement-scale uncertainties are accounted for separately in the work budget.

7.8.5 Synchronising force and motion records

A force–position dataset must preserve which force belongs to which part of the motion. If the displacement reverses during a cycle, sorting all samples into increasing position mixes loading and unloading histories and can destroy the signed-work information. Retain the time order and use signed displacement increments, or split the record into separately oriented branches. The same position can legitimately have two different force readings when the material or controller is history dependent. Treating those readings as duplicates and averaging them would change the physical path being integrated.

An alternative measurement computes $W = \int \mathbf{F}(t) \cdot \mathbf{v}(t) dt$. It is equivalent to the force–displacement integral only when the force and application-point velocity describe the same event and share a consistent timing convention. A delay in one sensor can bias the integral even when both channels have correct individual gains. For a simple illustration, take the selected port histories $F(t) = 10 \cos(\omega t)$ N and $v(t) = 0.20 \cos(\omega t)$ m s^{−1}. Their mean product over a complete period is 1.00 W. If the recorded velocity is mistakenly displaced by a quarter-period, it becomes $0.20 \cos(\omega t + \pi/2)$, and the recorded mean product is zero. The correct histories and the shifted histories have identical individual amplitudes and means, but different correlations.

More generally the mean product of these two sinusoidal histories with phase difference ϕ is $(10)(0.20) \cos \phi/2 = \cos \phi$ W. This result follows by expanding the product into a constant term and a double-frequency term whose average over a complete period vanishes. It refers to the selected mechanical port; other forces can be needed to produce the prescribed motion. It illustrates a measurement error without assuming that the displayed force is the net force on a free particle.

A defensible experimental report therefore documents the coordinate direction, force sign, zero reference, sample timing and the rule used to align records. It also states whether the work was integrated in position or time. Agreement between independently processed versions is a valuable consistency check, but it is meaningful only if both refer to the same force port and the same complete trajectory. Sensor calibration, trajectory resolution and channel synchronisation address different sources of uncertainty.

7.8.6 Computational connection: work from tabulated force values

Experimental and simulation forces are often available at discrete positions rather than as a convenient analytic formula. Numerical quadrature approximates the work integral. Two common

composite rules for equally spaced data are the trapezoidal rule,

$$W_T = h \left[\frac{F_0 + F_N}{2} + \sum_{j=1}^{N-1} F_j \right], \quad (7.31)$$

and, for even N , Simpson's rule,

$$W_S = \frac{h}{3} \left[F_0 + F_N + 4 \sum_{j \text{ odd}} F_j + 2 \sum_{j \text{ even}, j \neq 0, N} F_j \right]. \quad (7.32)$$

A single numerical value does not establish accuracy. Refinement is part of the calculation.

Use the analytic teaching law

$$F_x(x) = 90e^{-x/0.50} - 15 \text{ N}, \quad 0 \leq x \leq 1.00 \text{ m}, \quad (7.33)$$

where x is in metres. The exact work is

$$W_{\text{exact}} = \int_0^{1.00} (90e^{-x/0.50} - 15) dx \quad (7.34)$$

$$= 45 (1 - e^{-2}) - 15 \quad (7.35)$$

$$= 23.9099 \text{ J}. \quad (7.36)$$

The following values are synthetic samples generated directly from Eq. (7.33); they are not measured data.

x (m)	F_x (N)
0.000	75.000
0.125	55.092
0.250	39.588
0.375	27.513
0.500	18.109
0.625	10.785
0.750	5.082
0.875	0.640
1.000	-2.820

Table 7.2: Convergence for the analytic force law in Eq. (7.33). Errors are relative to the exact integral 23.909912 J.

Intervals N	W_T (J)	$W_T - W_{\text{exact}}$ (J)	W_S (J)	$W_S - W_{\text{exact}}$ (J)
4	24.717178	$+8.073 \times 10^{-1}$	23.923031	$+1.312 \times 10^{-2}$
8	24.112357	$+2.024 \times 10^{-1}$	23.910750	$+8.382 \times 10^{-4}$
16	23.960563	$+5.065 \times 10^{-2}$	23.909965	$+5.268 \times 10^{-5}$
32	23.922577	$+1.267 \times 10^{-2}$	23.909916	$+3.297 \times 10^{-6}$

The trapezoidal error falls by approximately a factor of four each time N doubles in the asymptotic regime, while Simpson's error falls much faster for this smooth force law. That pattern is evidence about numerical resolution. It is not evidence that the force law itself is physically correct for a particular device.

7.8.7 Choosing a quadrature rule for the measured information

The trapezoidal rule is the exact integral of the piecewise linear interpolant joining adjacent force samples. Its geometric assumption is therefore visible: within an interval, it replaces the unknown force by a straight line. Simpson's rule integrates a quadratic interpolant across each pair of equal intervals. Its higher order is useful when the force is sufficiently smooth and the samples support that model. A higher-order formula does not automatically contain more physical information. With only a few noisy measurements, the curvature inferred by a quadratic can reflect readout noise rather than real force variation.

For unevenly spaced positions, the segmentwise trapezoidal formula remains $\sum_j \frac{1}{2}(F_j + F_{j+1})(x_{j+1} - x_j)$. The equal-spacing Simpson weights displayed earlier must not be reused unchanged. Reversing the order of all samples changes the work sign through the signed position differences; it does not change the positive geometric area of a branch. If positions are repeated during a force change at fixed extension, their interval contributes zero mechanical work even when the force difference is nonzero. These observations connect the numerical rule directly to the oriented path rather than to an unsigned plotting convention.

A useful verification compares at least two representations of the same resolved information. For a synthetic analytic test, evaluate the law independently on refined grids and compare with its integral. For measurements, inspect residuals, repeated runs and new sample locations; refining only the interpolation grid tests the chosen interpolant. In either case, preserve the force source and displacement units before integrating. A table written in newtons and millimetres must have its area multiplied by 10^{-3} to give joules. An omitted millimetre-to-metre conversion creates a factor-of-one-thousand error that numerical convergence will reproduce perfectly.

7.8.8 What convergence establishes, and what it cannot establish

For the smooth exponential force law, the benchmark integral is 23.909912254 J. Recomputing the numerical integrals using unrounded analytic force values gives trapezoidal errors 0.8072656, 0.2024450, 0.05065076 and 0.01266516 J at $N = 4, 8, 16, 32$. The corresponding Simpson errors are 0.01311858, 0.0008381574, 0.00005267694 and 0.000003296901 J. These values validate the displayed convergence table at its stated precision. The trapezoidal-to-Simpson improvement is substantial here because the analytic integrand is smooth and densely sampled without measurement noise.

On logarithmic axes, an error law $|e_N| \simeq CN^{-p}$ has slope $-p$. Doubling the interval count reduces the asymptotic trapezoidal error by about $2^2 = 4$, and the Simpson error by about $2^4 = 16$. For a smooth integrand, estimates $|W_{2N} - W_N|/3$ for the refined trapezoidal result and $|W_{2N} - W_N|/15$ for the refined Simpson result follow by eliminating the leading error coefficient. They are asymptotic estimates requiring the relevant smoothness and error regime, not guaranteed bounds for an arbitrary force table.

The displayed force table rounds values to 0.001 N. If each rounded value differs from the underlying value by at most 0.0005 N, either quadrature rule on this 1.00 m span has a worst-case absolute rounding contribution no larger than 0.0005 J: its nonnegative weights sum to the span. The $N = 32$ Simpson discretisation error is far below that scale. Reporting many output digits from the rounded table would therefore not reproduce the analytic benchmark to those digits. The convergence table uses newly generated full-precision analytic values at each resolution, rather than interpolating the nine rounded values.

Measured tables pose a different limitation. Repeatedly interpolating a fixed coarse dataset does not create missing information about a sharp force peak or a sudden contact change. A smooth curve fitted through sparse points may converge numerically while missing the actual area. New spatial samples, uncertainty estimates or a justified constitutive model are then needed. Noise also changes the usefulness of refinement and extrapolation: the discretisation error can decrease while the measurement uncertainty remains appreciable. A sound computational report states the

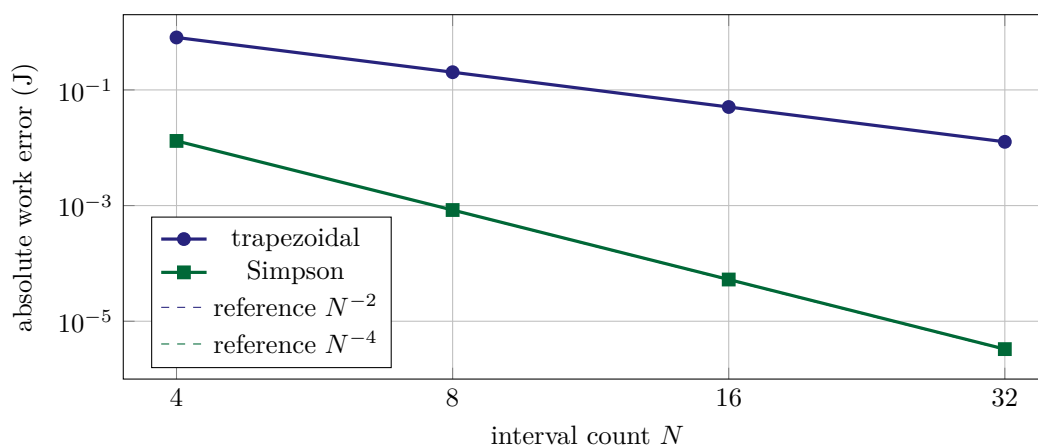


Figure 7.26: Quadrature convergence using full-precision samples of the chapter’s exponential force law. Dashed reference lines are anchored at $N = 4$ and display the expected asymptotic slopes. The near-parallel trends verify numerical behaviour for this analytic test; they do not calibrate a force sensor or validate the force law for a real device.

force model, sample locations, rounding, quadrature rule and the evidence supporting an accuracy estimate.

Example 7.23: Predicting a final speed from numerical work

A 4.0 kg particle enters the interval in Eq. (7.33) with $K_i = 16.0$ J. Using the exact work for verification,

$$K_f = K_i + W = 39.9099 \text{ J},$$

so

$$v_f = \sqrt{\frac{2K_f}{m}} = 4.47 \text{ m s}^{-1}.$$

Using the $N = 8$ Simpson work instead would change the predicted kinetic energy by less than 10^{-3} J in this synthetic test.

If a different force table gave $K_i + W_{\text{quad}} < 0$ before the final sample position, the appropriate conclusion would not be a negative speed. The assumed traversal of the full interval would be physically inconsistent; the turning point or force-domain change would need to be located.

7.8.9 Check accessibility throughout a proposed traversal

To infer speed at a position, it is necessary to check the cumulative work, not only its full-interval value:

$$K(x) = K_i + \int_{x_i}^x F_{\text{net}}(\xi) d\xi \geq 0.$$

A positive formal value at the last position does not guarantee that the particle can reach it. If the cumulative expression first vanishes at an earlier point, the assumed forward path needs to be examined there. At a simple turning point with force directed back toward the starting region, the particle reverses; it cannot cross an interval where the same proposed path would require negative kinetic energy. Energy accounting provides this accessibility test without first solving the complete time history.

Example 7.24: A positive final calculation hides an earlier turning point

A particle begins at $x = 0$ moving right with $K_i = 2.00$ J. Its only horizontal force is stipulated as $F_x = -4.00$ N for $0 \leq x < 1.00$ m and $F_x = +8.00$ N for $1.00 \leq x \leq 2.00$ m. Determine whether it can reach $x = 2.00$ m under this force alone.

First compute the formal cumulative expression. Direct spatial integration gives

$$K_{\text{formal}}(x) = \begin{cases} 2.00 - 4.00x, & 0 \leq x \leq 1.00, \\ -2.00 + 8.00(x - 1.00), & 1.00 \leq x \leq 2.00, \end{cases}$$

where K is in joules when x is in metres. At $x = 2.00$ m the expression is 6.00 J, because the full-interval formal work is $-4.00 + 8.00 = 4.00$ J. An endpoint-only calculation would appear to predict a valid final speed.

Now test the first segment. The kinetic energy reaches zero at $x = 0.500$ m. At this position the force remains -4.00 N and hence accelerates the particle leftward once its forward velocity has vanished. It turns before reaching the force switch at 1.00 m. The negative portion of the formal curve, between 0.500 and 1.250 m, is not physical negative kinetic energy; it marks the failure of the assumed forward traversal. Even the later positive branch is inaccessible from this initial condition under the stipulated force alone.

Interpretation and sampling lesson. The correct answer is that the particle cannot reach 2.00 m. For an elementary discrete calculation, start from the specified initial kinetic energy, add signed work segment by segment, and bracket the first point where the cumulative value reaches zero. In a linearly interpolated force segment, integrate that local linear force and solve the resulting quadratic for the earliest root inside the segment. Then check the local force direction before calling the root a turning point. A zero alone can also mark an equilibrium or a boundary, so the force and the assumed continuation must be inspected. No mass needs to be supplied to locate the turning position from the initial kinetic energy, although mass would be needed for a time or speed calculation. This example also shows why checking only the initial and final energy values is insufficient. A numerical trajectory should monitor cumulative energy at intermediate samples and investigate an approaching zero. If a coarse step spans an inaccessible region, a positive value after the step can hide the failure. Refinement and an appropriate event or root search are required near the first zero; a later positive integral cannot repair an earlier impossible segment.

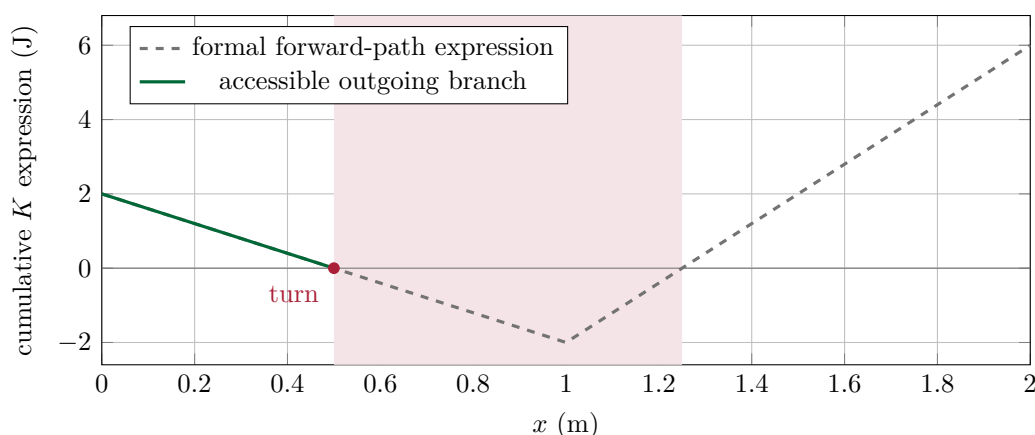


Figure 7.27: An intermediate accessibility failure despite a positive final formal energy. The red band marks positions where the proposed forward-path expression is negative. The actual outgoing motion reaches only the first zero at 0.500 m and turns; the dashed continuation does not represent the realised trajectory.

Listing 7.1: Self-contained reference implementation of two quadrature rules for equally spaced force–position data. The chapter compiles without executing this code.

```

def trapezoid_work(x, force):
    if len(x) != len(force) or len(x) < 2:
        raise ValueError("x and force must have the same length >= 2")
    h = x[1] - x[0]
    if any(abs((x[i+1]-x[i]) - h) > 1e-12 for i in range(len(x)-1)):
        raise ValueError("this simple implementation requires equal spacing")
    return h*(0.5*(force[0] + force[-1]) + sum(force[1:-1]))

def simpson_work(x, force):
    if len(x) != len(force) or len(x) < 2:
        raise ValueError("x and force must have the same length >= 2")
    n = len(x) - 1
    if n < 2 or n % 2:
        raise ValueError("Simpson rule requires a positive even interval count")
    h = x[1] - x[0]
    if any(abs((x[i+1]-x[i]) - h) > 1e-12 for i in range(n)):
        raise ValueError("this simple implementation requires equal spacing")
    return (h/3.0)*(force[0] + force[-1]
                    + 4.0*sum(force[1:-1:2])
                    + 2.0*sum(force[2:-1:2]))

```

Method

A reproducible work-from-data calculation should report: the physical force component being integrated, position units, direction convention, the integration limits, the quadrature rule, grid or sample spacing, a refinement or uncertainty check, and whether the data are measured, simulated or analytically generated.

7.9 Advanced Topics for Technical and Engineering Students

The core chapter was intentionally built around one-dimensional integrals and particle motion. Many engineering systems require a more geometric statement of work because the path curves through two or three dimensions, forces depend on position, or different points of an extended body follow different trajectories. This section develops the required mathematics without yet introducing the general conservative-force and potential-energy framework of Chapter 8.

The additional mathematical prerequisite is a parametrized curve. Let a path be represented by

$$\mathbf{r} = \mathbf{r}(s), \quad s_a \leq s \leq s_b, \quad (7.37)$$

where s is any convenient monotonically increasing path parameter. Then

$$d\mathbf{r} = \frac{d\mathbf{r}}{ds} ds. \quad (7.38)$$

If a position-dependent force $\mathbf{F}(\mathbf{r})$ acts at that moving point, its work along the prescribed path C is the line integral

$$W_C = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{s_a}^{s_b} \mathbf{F}(\mathbf{r}(s)) \cdot \frac{d\mathbf{r}}{ds} ds. \quad (7.39)$$

This is the natural multidimensional extension of $\int F_x dx$. The path is part of the input.

7.9.1 A prescribed line integral along two different paths

Consider the stipulated two-dimensional force field

$$\mathbf{F}(x, y) = ay \hat{\mathbf{i}}, \quad a > 0, \quad (7.40)$$

with a in N/m. We calculate work between the same endpoints $O = (0, 0)$ and $B = (L, H)$ along several prescribed paths. The purpose is to practice the line integral. Chapter 8 develops the broader classification of force fields and path independence.

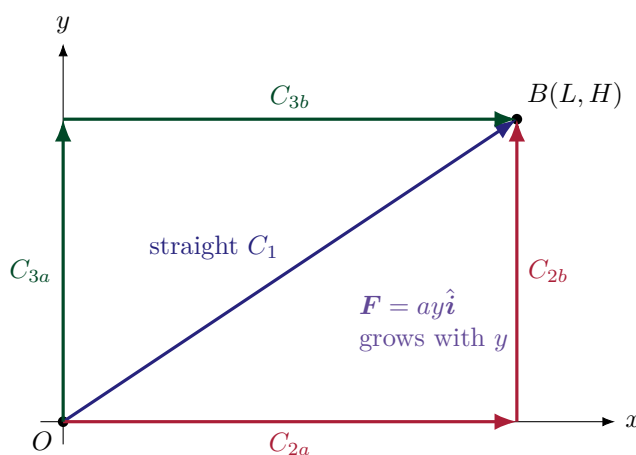


Figure 7.28: Advanced line-integral example. The endpoints are the same, but the force sampled along each prescribed path is different. The paths and arrow lengths are schematic; the axis proportions are not a physical equal-aspect scale. The stated L, H and prescribed paths, rather than drawing lengths, determine the work.

Example 7.25: Line-integral work along prescribed paths

Let $a = 30 \text{ N m}^{-1}$, $L = 0.40 \text{ m}$ and $H = 0.30 \text{ m}$ in Eq. (7.40).

Path C_1 : straight line. Parameterize

$$\mathbf{r}(s) = Ls\hat{\mathbf{i}} + Hs\hat{\mathbf{j}}, \quad 0 \leq s \leq 1.$$

Then $d\mathbf{r} = (L\hat{\mathbf{i}} + H\hat{\mathbf{j}})ds$ and $\mathbf{F} = aHs\hat{\mathbf{i}}$. Hence

$$W_{C_1} = \int_0^1 aHsL ds = \frac{1}{2}aHL = 1.80 \text{ J}.$$

Path C_2 : horizontal then vertical. Along the first segment $y = 0$, so $\mathbf{F} = 0$ and the work is zero. Along the vertical segment $d\mathbf{r} \parallel \hat{\mathbf{j}}$ while $\mathbf{F} \parallel \hat{\mathbf{i}}$, so the work is also zero. Therefore

$$W_{C_2} = 0.$$

Path C_3 : vertical then horizontal. The first vertical segment again gives zero work. Along the top horizontal segment $y = H$ is constant and $\mathbf{F} = aH\hat{\mathbf{i}}$, so

$$W_{C_3} = aHL = 3.60 \text{ J}.$$

Interpretation. A line integral is not specified by endpoints alone unless extra properties of the force are established. Here the numerical work values differ because the prescribed paths sample different force–displacement alignments. The general conditions under which endpoint information becomes sufficient are deferred to Chapter 8.

7.9.2 Local projection form of a line integral

If $d\ell$ is the positive scalar arc-length element and $\hat{\mathbf{t}}$ is the unit tangent to the path, then

$$d\mathbf{r} = \hat{\mathbf{t}} d\ell.$$

Therefore

$$W_C = \int_C (\mathbf{F} \cdot \hat{\mathbf{t}}) d\ell = \int_C F_t d\ell. \quad (7.41)$$

The line integral accumulates only the tangential component of force. A large normal force can control the path while contributing no work locally.

7.9.3 Decomposing actual work into centre-of-mass and relative motion

Let the application point of external force $\mathbf{F}_i$ have position

$$\mathbf{r}_i = \mathbf{R}_{\text{CM}} + \boldsymbol{\rho}_i, \quad (7.42)$$

where $\boldsymbol{\rho}_i$ is measured from the centre of mass. Differentiating gives

$$d\mathbf{r}_i = d\mathbf{R}_{\text{CM}} + d\boldsymbol{\rho}_i.$$

The total differential external work is

$$dW_{\text{ext}} = \sum_i \mathbf{F}_i \cdot d\mathbf{r}_i \quad (7.43)$$

$$= \left(\sum_i \mathbf{F}_i \right) \cdot d\mathbf{R}_{\text{CM}} + \sum_i \mathbf{F}_i \cdot d\boldsymbol{\rho}_i. \quad (7.44)$$

The first term is the centre-of-mass force contribution. The second term records work associated with motion of the application points relative to the centre of mass. It vanishes for pure translation of a rigid body because every $d\mathbf{r}_i = 0$. It need not vanish for deformation or rotation.

The corresponding instantaneous power identity is

$$\mathcal{P}_{\text{ext}} = \sum_i \mathbf{F}_i \cdot \mathbf{v}_i = \mathbf{F}_{\text{ext,net}} \cdot \mathbf{V}_{\text{CM}} + \sum_i \mathbf{F}_i \cdot \dot{\mathbf{r}}_i. \quad (7.45)$$

This identity is kinematic. It does not by itself say how the relative-motion work is partitioned among internal kinetic, elastic, thermal or other forms of energy. Those system-level categories require additional physics.

Example 7.26: A nonlinear deformable connector with fixed centre of mass

A symmetric connector is pulled quasistatically outward at both ends with equal and opposite force magnitude

$$F(\delta) = k_s \delta + b \delta^3,$$

where $\delta = \ell - \ell_0$ is the increase in total end-to-end length, $k_s = 180 \text{ N m}^{-1}$, and $b = 2.0 \times 10^5 \text{ N m}^{-3}$. The centre of mass remains fixed. Find the actual external work required to increase the length by $\delta_f = 0.040 \text{ m}$.

By symmetry each end moves outward by $d\delta/2$. Each of the two equal-magnitude forces is aligned with its endpoint displacement. Thus

$$dW_{\text{ext}} = F \frac{d\delta}{2} + F \frac{d\delta}{2} = F(\delta) d\delta.$$

Therefore

$$\begin{aligned} W_{\text{ext}} &= \int_0^{\delta_f} (k_s \delta + b \delta^3) d\delta \\ &= \frac{1}{2} k_s \delta_f^2 + \frac{1}{4} b \delta_f^4. \end{aligned}$$

Numerically,

$$W_{\text{ext}} = \frac{1}{2} (180) (0.040)^2 + \frac{1}{4} (2.0 \times 10^5) (0.040)^4 = 0.272 \text{ J}.$$

The net external force and centre-of-mass displacement are both zero, so the centre-of-mass work is zero. The nonzero 0.272 J enters through relative endpoint motion.

Validity. The force law is a stipulated teaching model. A real connector can show rate dependence, hysteresis, plasticity or failure, in which case the loading path and constitutive model must be measured rather than assumed.

7.9.4 Numerical line integration along a sampled path

For a discretized trajectory with points $\mathbf{r}_0, \dots, \mathbf{r}_N$ and forces evaluated at segment midpoints, a simple midpoint approximation is

$$W \simeq \sum_{j=0}^{N-1} \mathbf{F} \left(\frac{\mathbf{r}_{j+1} + \mathbf{r}_j}{2} \right) \cdot (\mathbf{r}_{j+1} - \mathbf{r}_j). \quad (7.46)$$

This formula preserves both the vector projection and the actual path. Refining the path discretization should change the result less and less for a sufficiently smooth force and path. A visually smooth trajectory is not, by itself, a convergence test.

Method

For an advanced line-work calculation, record the path parameterization, its orientation, the force law and domain, the application point, the numerical integration rule if used, and at least one independent check: a special case, an analytic result, a reversal-sign test or a convergence study.

Pedagogical boundary

This section intentionally stops before deriving multidimensional conservative-force criteria, potential functions or general mechanical-energy conservation. Those are central topics of Chapter 8. Continuum stress work, control-volume energy equations and thermodynamic work modes are also postponed to later chapters where the necessary system-level framework is available.

7.9.5 Path shape, physical scales and a change of parameter

A prescribed path has two separate roles: it identifies the positions sampled by a force and orders those positions. The speed at which the path is traversed is a third piece of information. For a force depending only on position, changing traversal speed without changing path or direction leaves the work unchanged. This conclusion need not hold for a force with explicit time dependence or a speed-dependent resistance: then the force history itself changes. Stating the force model is therefore part of specifying a line-integral problem.

Consider the same position-dependent field as above, $\mathbf{F} = ay\hat{\mathbf{i}}$, and the family

$$y = H \left(\frac{x}{L} \right)^p, \quad 0 \leq x \leq L, \quad p > 0. \quad (7.47)$$

The constants L and H are physical lengths; p is dimensionless. Each path connects $(0, 0)$ to (L, H) . The field coefficient has dimensions $[a] = \text{N m}^{-1}$, so ay has force units and aHL has energy units. These assignments are essential: an attractive curve drawn with unitless coordinates does not by itself supply a dimensional force law.

There is also a distinction between plotting coordinates and physical coordinates. If $X = x/L$ and $Y = y/H$, the physical displacement is $d\mathbf{r} = LdX\hat{\mathbf{i}} + HdY\hat{\mathbf{j}}$. Replacing it by $dX\hat{\mathbf{i}} + dY\hat{\mathbf{j}}$ would discard the length scales and give the wrong dimensions. Likewise, an angle measured directly from normalized axes need not equal the physical angle when $L \neq H$. Use the physical derivative $(dx/ds, dy/ds)$ to construct a tangent or evaluate a scalar product, and convert to display coordinates only after calculating the physical quantity. Normalization makes comparisons clearer; it does not eliminate physical geometry.

The word “prescribed” also identifies a modelling choice. The force field alone does not imply that a free particle follows every illustrated curve. A guide, controller or other constraint may be needed. An ideal stationary smooth guide exerts a normal reaction and contributes no work along its allowed tangent. If a controller also imposes speed, its tangential force may do additional work. The integral calculated here is work by the specified field; it becomes net work only after all relevant forces and their work are accounted for. This prevents a path-comparison exercise from silently becoming an unsupported dynamical trajectory calculation.

Figure 7.29 compares these paths on normalized axes. It represents exact mathematical geometry; each prescribed path is traversed from O to B . For $0 < p < 1$, dy/dx diverges at the initial point. The curve and its work remain well defined as improper limits; the picture does not imply a finite-speed actuator can follow that endpoint with arbitrary acceleration. Choosing $p \geq 1$ removes that particular endpoint slope singularity.

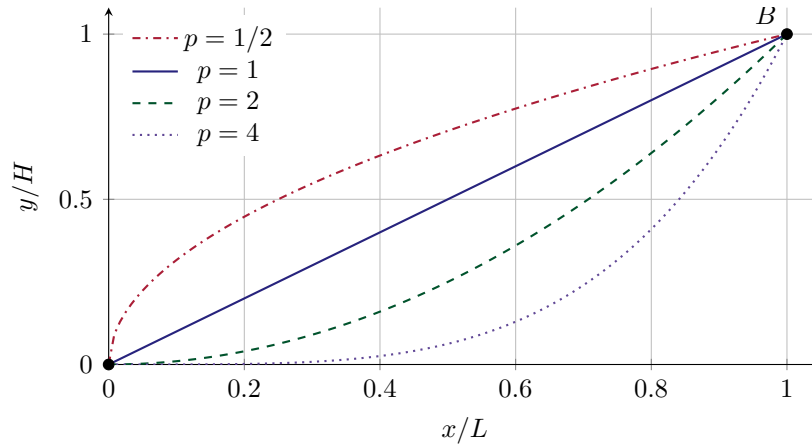


Figure 7.29: Analytical prescribed paths in the field $\mathbf{F} = ay\hat{\mathbf{i}}$. The endpoints coincide, but the height sampled during horizontal motion changes with p . Axes are normalized by the physical lengths L, H ; the four curves are distinguished by line style as well as colour.

Because the force has no vertical component, the work uses only the horizontal displacement:

$$W_p = \int_0^L aH \left(\frac{x}{L}\right)^p dx = aHL \int_0^1 X^p dX \quad (7.48)$$

$$= \frac{aHL}{p+1}. \quad (7.49)$$

The dimensionless integration variable is $X = x/L$. Larger p keeps the path closer to the low-force region for more of the horizontal travel, so the work decreases. The limiting values are $W_p \rightarrow aHL$ as $p \rightarrow 0^+$ and $W_p \rightarrow 0$ as $p \rightarrow \infty$. These limits concern a sequence of paths; the limiting shapes have abrupt corners and are not automatically smooth trajectories of a freely moving particle.

The normalized work in Figure 7.30 makes this geometrical comparison independent of the numerical force scale. It also explains why the diagonal result is halfway between the two right-angle paths: $p = 1$ gives $W = aHL/2$. A figure of the paths alone cannot communicate the energy difference quantitatively; the companion curve supplies that connection.

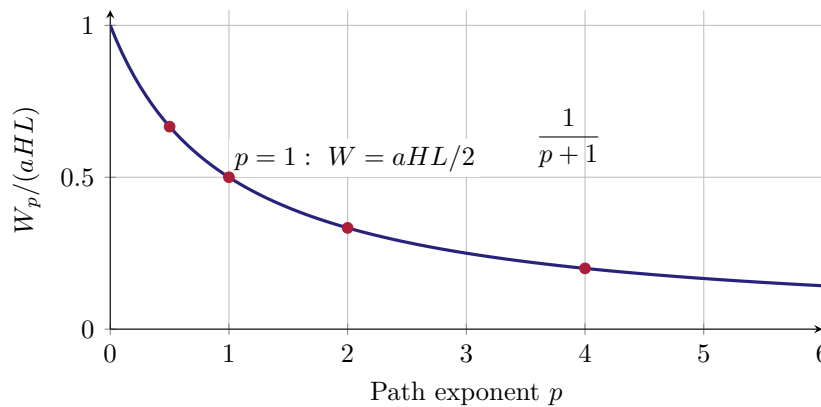


Figure 7.30: Work for the prescribed path family. The curve is an exact dimensionless analytical model, rather than measured data. The endpoint at $p = 0$ represents the limit $p \rightarrow 0^+$; every finite positive p follows Eq. (7.47).

Why a change of parameter cannot change the answer

Suppose a differentiable path $\mathbf{r}(s)$ is re-expressed by $s = g(q)$, with g increasing from s_a to s_b . The chain rule gives

$$\frac{d\mathbf{r}}{dq} = \frac{d\mathbf{r}}{ds} \frac{dg}{dq}.$$

Substitution into the work integral yields

$$\int_{q_a}^{q_b} \mathbf{F}(\mathbf{r}(g(q))) \cdot \frac{d\mathbf{r}}{ds} g'(q) dq = \int_{s_a}^{s_b} \mathbf{F}(\mathbf{r}(s)) \cdot \mathbf{r}'(s) ds.$$

The factor $g'(q)$ is the required displacement conversion. Omitting it amounts to changing the integral, not discovering a physical dependence on notation. Reversing the same path changes its orientation and therefore changes the sign of the work for this position-dependent force. Path shape and traversal direction must be kept distinct.

Example 7.27: A curved path, a nonlinear parameter and reversal

For $a = 30 \text{ N m}^{-1}$, $L = 0.40 \text{ m}$ and $H = 0.30 \text{ m}$, find the work along the parabola $y = H(x/L)^2$. Verify it using $x = Lq^3$, then find the work when the same path is traversed backwards.

Direct integration. Here $p = 2$, so

$$W = \frac{aHL}{3} = \frac{(30)(0.30)(0.40)}{3} = 1.20 \text{ J}.$$

Changed parameter. With $0 \leq q \leq 1$, the path is

$$x = Lq^3, \quad y = Hq^6, \quad dx = 3Lq^2 dq.$$

Consequently

$$W = \int_0^1 (aHq^6)(3Lq^2) dq = 3aHL \left[\frac{q^9}{9} \right]_0^1 = \frac{aHL}{3} = 1.20 \text{ J}.$$

The zero derivative at $q = 0$ does not invalidate the substitution; the mapping is monotonic and the integral is continuous there. It merely gives a different mathematical rate of progress along the path.

Reversal. Keeping the same force field and changing the orientation gives

$$W_{B \rightarrow O} = \int_L^0 aH(x/L)^2 dx = -1.20 \text{ J}.$$

This result assumes the force remains the stipulated field during reversal. A friction force that reverses with slip would require a new force history and would not obey this particular sign argument.

7.9.6 A stationary wall can change centre-of-mass motion without doing work

The decomposition of external work can be tested with a system that has an unambiguous fixed force-application point. Consider two equal masses m connected by an ideal massless spring. Mass A touches a fixed wall; mass B is free to move horizontally on a frictionless guide. Initially the spring is compressed by d , both masses are at rest, and a temporary catch holds the configuration. Release the catch without supplying appreciable work. The spring pushes A against the wall while accelerating B away from it.

Let q be the displacement of B from its initial position. For $0 \leq q \leq d$, the spring compression is $d - q$, and the wall reaction on A is

$$N = k_s(d - q).$$

The contact remains possible because $N \geq 0$ throughout this interval. At $q = d$, the spring is relaxed and the wall reaction vanishes. We stop the analysis there: allowing subsequent stretching would require checking whether contact persists, since a wall cannot pull the mass back toward itself.

Figure 7.31 shows the fixed contact and the moving centre of mass. The wall is an external force source, but its application point has zero displacement. Hence its actual work is zero. This statement is about the ideal fixed-wall model; a compliant wall or a moving support changes the application-point displacement and therefore the work calculation.

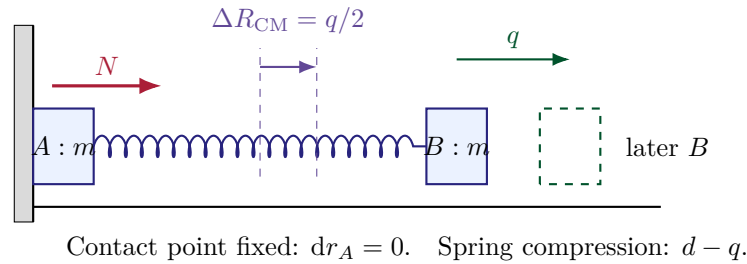


Figure 7.31: A spring-driven pair against an ideal stationary wall. Positions and block sizes are schematic. The wall reaction has positive centre-of-mass pseudowork while its actual work is zero because the contact point is fixed. The dashed block marks the displaced mass, not an additional body.

For equal masses, $dR_{CM} = dq/2$. The centre-of-mass relation therefore gives

$$\Delta K_{CM} = \int_0^d k_s(d - q) \frac{dq}{2} = \frac{1}{4} k_s d^2. \quad (7.50)$$

This positive value is pseudowork; it is not energy supplied by the stationary wall. The spring does actual work on B :

$$W_{s,B} = \int_0^d k_s(d - q) dq = \frac{1}{2} k_s d^2 = \frac{1}{2} m v_B^2.$$

Mass A remains stationary, so the total kinetic-energy increase is twice the centre-of-mass kinetic-energy increase. With $V_{CM} = v_B/2$, direct evaluation gives $K_{CM} = m v_B^2/4$; the remaining $m v_B^2/4$ describes motion relative to the centre of mass. The energy comes from the initially compressed spring, not from the support. General potential-energy bookkeeping is developed in Chapter 8.

The decomposition also shows the cancellation without an energy label. For the wall force, $\rho_A = r_A - R_{CM}$, so $d\rho_A = -dR_{CM}$. Its two terms are

$$N dR_{CM} + N d\rho_A = 0.$$

For $m = 1.00$ kg, $k_s = 200$ N m⁻¹ and $d = 0.100$ m, the spring supplies 1.00 J to particle kinetic energy, $v_B = 1.414$ m s⁻¹, and the centre-of-mass pseudowork is 0.500 J. The initial reaction is 20.0 N, and the centre of mass advances 0.0500 m. These values make the distinction observable without introducing rotational dynamics.

Changing the system boundary provides a second check. For particle B alone, the spring force is external and does the full 1.00 J of work, so the ordinary particle theorem applies directly. For the two particles together, the spring forces act at different application points: their vector sum is zero, but their actual works need not cancel. One point is fixed and the other moves. Including the spring in the system reclassifies the stored elastic energy as an internal resource; the fixed wall still supplies zero actual work. All three descriptions agree when their force identities and energy categories are kept consistent.

An equal and opposite pair of internal forces therefore has two different cancellation properties. Its contribution to net force cancels at an instant, supporting the centre-of-mass equation. Its contribution to actual work cancels only if the two application-point displacements give equal scalar products with opposite signs. Deformation breaks that additional condition. This is the practical reason a net-force diagram alone cannot determine the energy transfer of an extended system, even when every force magnitude is known.

7.9.7 Moving surfaces, frictional work and frame consistency

Sliding friction opposes relative slip at a contact. Its direction is not determined by the sign of a body's velocity in a chosen laboratory frame. A belt moving faster than a block can exert a forward friction force that does positive work on the block, even though sliding still produces dissipation. The useful comparison is between the velocities of the two contacting material points.

For a nonrotating block on a horizontal belt, let the belt move at constant speed $u > 0$ and let $0 < v < u$. The kinetic friction force on the block is $f = \mu_k mg$ in the positive horizontal direction. In the laboratory frame its power is $\mathcal{P}_b = fv > 0$. The equal opposite contact force has power $\mathcal{P}_{\text{belt}} = -fu$. Adding these actual contact powers gives

$$\mathcal{P}_b + \mathcal{P}_{\text{belt}} = -f(u - v) < 0. \quad (7.51)$$

The missing mechanical power becomes thermal or other microscopic energy at the sliding interface. Maintaining the belt speed requires a driver; omitting the driver while treating the belt speed as prescribed would leave the system boundary unexplained.

The combined contact-power result is independent of a constant translational frame change. Subtracting the same frame velocity from both contact velocities changes the two powers by opposite amounts, because the contact forces are equal and opposite. Their sum remains $-f(u - v)$. Thus the positive work received by the block can change sign between frames while the mechanical-energy loss associated with relative sliding retains its value. This statement concerns the contact pair; the driver and support must still be included when constructing a complete balance for the apparatus.

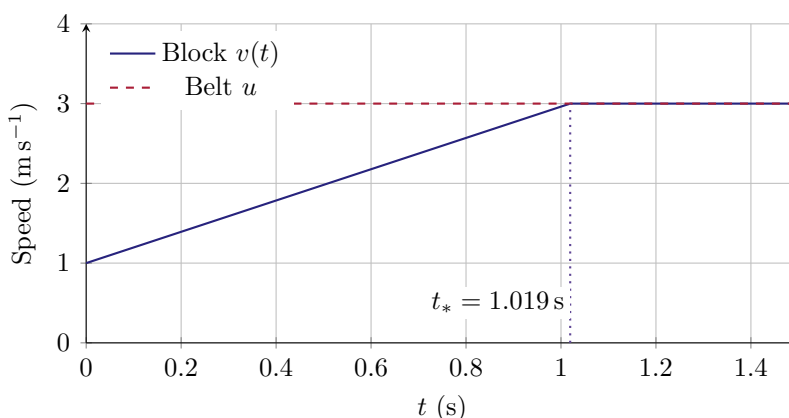


Figure 7.32: A block speeds up on a faster belt in the laboratory frame. The relative slip speed is the vertical difference between the curves. The ideal friction law changes at matching speed: with no other horizontal force, the subsequent contact force can be zero.

Example 7.28: A driven belt: friction work in two frames

A 2.00 kg block starts at $v_0 = 1.00 \text{ m s}^{-1}$ on a belt moving at $u = 3.00 \text{ m s}^{-1}$. Take $\mu_k = 0.200$ and $g = 9.81 \text{ m s}^{-2}$. Find the matching time, block work, work on the belt and interface dissipation. Recalculate block work in the inertial frame moving with the belt.

Motion and matching. Normal balance gives $N = mg$, so $f = 3.924 \text{ N}$ and the block

acceleration is 1.962 m s^{-2} while it slips. Thus

$$t_* = (u - v_0)/(\mu_k g) = 1.01937 \text{ s}, \quad s_b = \frac{1}{2}(v_0 + u)t_* = 2.03874 \text{ m}.$$

The block friction work is $W_b = f s_b = 8.00 \text{ J}$. Independently, $\Delta K_b = \frac{1}{2}(2.00)(3.00^2 - 1.00^2) = 8.00 \text{ J}$, verifying the work–energy relation. The belt contact point travels $s_{mbelt} = ut_* = 3.05810 \text{ m}$, so $W_{belt} = -f s_{mbelt} = -12.00 \text{ J}$. The required driver work is $+12.00 \text{ J}$ in this ideal constant-speed model.

Dissipation. The total relative slip distance is $s_{mbelt} - s_b = 1.01937 \text{ m}$. The interface receives

$$E_{diss} = f(s_{mbelt} - s_b) = 4.00 \text{ J}.$$

Driver work therefore divides into 8.00 J of block kinetic energy and 4.00 J of dissipation. Once speeds match, no horizontal force is required if the belt remains steady and other resistance is absent.

For this constant-speed belt model, eliminating the matching time gives the general result $E_{diss} = \frac{1}{2}m(u - v_0)^2$. Increasing the friction coefficient reduces the matching time and total slip distance, while increasing the force by the compensating factor. The eventual dissipated energy is therefore unchanged for the same initial relative speed. This does not mean friction has no effect: it controls how quickly the transfer occurs. At exactly zero friction there is no matching, so the result for eventual matching cannot be applied by simply setting $\mu_k = 0$ in an expression derived for positive friction.

Belt frame. Here $v'_0 = v_0 - u = -2.00 \text{ m s}^{-1}$ and $v'_f = 0$. Friction is still positive, but the block contact point moves negatively by $s'_b = s_b - ut_* = -1.01937 \text{ m}$. Hence $W'_b = f s'_b = -4.00 \text{ J}$, matching $\Delta K'_b = 0 - \frac{1}{2}(2.00)(2.00^2) = -4.00 \text{ J}$. The belt contact is stationary in this frame and has zero contact work. Relative slip and the 4.00 J dissipation are unchanged; individual work and kinetic-energy changes depend on the frame.

For a constant frame velocity $\mathbf{u}$, the application-point velocity transforms as $\mathbf{v}'_A = \mathbf{v}_A - \mathbf{u}$. Therefore $\mathcal{P}'_A = \mathbf{F}_A \cdot \mathbf{v}'_A = \mathcal{P}_A - \mathbf{F}_A \cdot \mathbf{u}$. Integrating this identity reproduces the difference between the two block-work results. The same application point, force history and time interval must be used in both frames. Mixing laboratory kinetic energy with belt-frame work would fail even though each individual calculation could look plausible. This is why the frame, system boundary and contact-point motion belong at the start of an engineering work calculation.

7.10 Chapter summary and a working formula sheet

Summary

Kinetic energy $K = \frac{1}{2}mv^2$ belongs to motion in a named frame. Work is a scalar energy transfer associated with a specified force and the displacement of its application point. For a fixed-mass particle, net work equals the change in kinetic energy. Variable-force work is a signed integral, and power is the rate at which work is done. Uniform gravity and an ideal spring provide important force-integral examples. For deformable or rotating systems, actual application-point work and centre-of-mass work must not be confused.

Formula summary**Kinetic energy, fixed mass, nonrelativistic**

$$K = \frac{1}{2}mv^2.$$

The speed and therefore K are frame dependent.

Constant-force work

$$W_F = \mathbf{F} \cdot \Delta \mathbf{r} = F \Delta r \cos \theta.$$

The displacement belongs to the force's application point.

One-dimensional variable-force work

$$W = \int_{x_i}^{x_f} F_x(x) dx.$$

The integral is signed and equals signed area on an F_x - x graph.

Work-kinetic-energy theorem

$$W_{\text{net}} = K_f - K_i.$$

The sum includes the work of every force on the chosen fixed-mass particle.

Uniform gravity and ideal spring

$$W_g = mg(y_i - y_f), \quad W_s = \frac{1}{2}k_s(x_i^2 - x_f^2).$$

Power

$$\mathcal{P}_{\text{av}} = \frac{W}{\Delta t}, \quad \mathcal{P} = \mathbf{F} \cdot \mathbf{v}.$$

Extended system: actual application-point work

$$W_{\text{ext}} = \sum_i \int \mathbf{F}_i \cdot d\mathbf{r}_i, \quad \Delta K_{\text{CM}} = \int \mathbf{F}_{\text{ext,net}} \cdot d\mathbf{R}_{\text{CM}}.$$

These two expressions are not generally equal for rotating or deformable systems.

Before accepting an answer, check the frame, system, force identity, path, application point, sign, units and model domain. A numerical answer can be algebraically perfect and physically irrelevant if any of those choices is wrong.

7.11 Exercises and problems

Use $g = 9.81 \text{ m s}^{-2}$, constant masses and a nonrelativistic model unless another assumption is stated. State the system, frame and displacement of the force-application point. A numerical speed prediction is valid only if the particle can reach the requested point along the entire proposed path. Problems 25–31 and 40–43 include advanced extensions; every additional relation needed there is supplied explicitly.

A. Concepts, frames and work signs

1. **Frame dependence.** A 900 kg maintenance vehicle moves east at 14 m s^{-1} in a laboratory frame. An inspection platform moves east at 6.0 m s^{-1} . Find the vehicle's kinetic energy in each inertial frame. Explain why the difference is consistent with mechanics.
2. A passenger is instantaneously at rest relative to a train carriage moving uniformly relative to

the ground. Compare the passenger's kinetic energy in the carriage and ground frames. Can either observer call the other's value incorrect? Identify the information needed for a numerical ground-frame value.

3. A constant force is $\mathbf{F} = (40\hat{\mathbf{i}} - 15\hat{\mathbf{j}})$ N and its application point moves by $\Delta\mathbf{r} = (3\hat{\mathbf{i}} + 4\hat{\mathbf{j}})$ m. Find the work and identify the negative contribution. Explain why multiplying the two vector magnitudes would give the wrong answer.
4. A cart moves 5.0 m horizontally. A 120 N cable pulls at 35° above the motion, and constant 25 N resistance opposes it. Find cable work, resistance work, net work and ΔK . The other forces do no work. Distinguish the selected-force work from the net work.
5. A bead moves at constant speed in a horizontal circle under an ideal inward constraint force. Is that force zero? Is its work over a quarter revolution zero? Explain both answers using the instantaneous vectors, rather than the finite chord between endpoints.
6. A particle moves from $x = a$ to $x = b$ under a prescribed force $F_x(x)$ and then returns along the same path while this position-dependent law remains unchanged. Relate return and outward work. State why the conclusion requires the same force law on the return journey.
7. Check the dimensions of $\mathcal{P} = \mathbf{F} \cdot \mathbf{v}$. Explain why the ideal fixed-power law $F = \mathcal{P}_0/v$ cannot describe starting a drive at rest. State a physically sensible additional constraint that could complete the startup model.
8. Equal and opposite outward external forces act at the ends of a deformable body. Describe a possible motion for which the net force is zero but external work is positive. Identify the displacement information needed to calculate that work, and distinguish it from centre-of-mass displacement.

B. Variable work and the work–kinetic-energy theorem

9. **Signed graphical area.** Figure 7.33a defines a straight force graph through (1 m, 30 N) and (4 m, −30 N). Locate its zero, find the positive and negative work contributions, and obtain the net work from $x = 1$ to 4 m. Give an integral check using the force law inferred from the graph.
10. Use Figure 7.33b to find the work over $0 \leq x \leq 5$ m. Separate rectangles and triangles, keeping their signs. For a 1.0 kg particle with $K_i = 5.0$ J at $x = 0$, find the final speed and verify that the entire path is accessible.
11. A 2.0 kg particle initially moves at 3.0 m s^{-1} . Three forces do +18 J, −5.0 J and zero work during a reachable process. Find the final speed. What can, and cannot, be inferred about the zero-work force?
12. A 0.80 kg cart is released from rest as an ideal 250 N m^{-1} spring relaxes from compression $q_i = 0.150$ m to its relaxed length. The signed spring coordinate is $x = -q$, with $q \geq 0$ a compression magnitude. Neglect all other work. Find spring work and cart speed, explicitly using the signed endpoints.
13. **External cable lift.** A rigid 70 kg equipment load is lifted by an external cable from $y = 0$ to $y = 4.5$ m. Its initial and final speeds are equal. Gravity and the cable are the only forces doing work. Find gravitational work, cable work and net work. Explain why this specified external lift makes application-point work unambiguous.
14. A 0.50 kg cart meets a relaxed 200 N m^{-1} spring at 3.0 m s^{-1} on a frictionless track. Find maximum compression $q \geq 0$, using signed spring coordinate $x = -q$. Check reachability

throughout compression and explain the meaning of a formal prediction $K(q) < 0$ beyond the turning point.

15. Constant forces $\mathbf{F}_1 = (12\hat{i} - 5\hat{j})$ N and $\mathbf{F}_2 = (-2\hat{i} + 9\hat{j})$ N act through $\Delta\mathbf{r} = (4\hat{i} + 3\hat{j})$ m. Find W_1 , W_2 , net work and final kinetic energy for $K_i = 10$ J. Check the result by combining the force vectors first.
16. A 1.20 kg slider starts with $K_i = 6.0$ J at $x = 0$. The only force doing work is $F_x = 30 - 10x$ N, with x in metres, over $0 \leq x \leq 2.0$ m. Find work, final kinetic energy and final speed. Justify reachability from the cumulative work, not just the final value.
17. A 25 kg load moves upward 3.0 m. A constant upward cable force does 1000 J of work; gravity also acts. The initial upward speed is 2.0 m s^{-1} . Find gravitational work and final speed. Use the implied cable force to check that the load does not stop earlier.
18. In Figure 7.34, a 1.0 kg block reaches a relaxed spring at 4.0 m s^{-1} . Its stiffness is 100 N m^{-1} , and constant 5.0 N friction opposes compression. Read the directions from the diagram, write both works in terms of positive compression q and signed spring coordinate $x = -q$, and find maximum compression. Reject any inadmissible root with a physical reason.

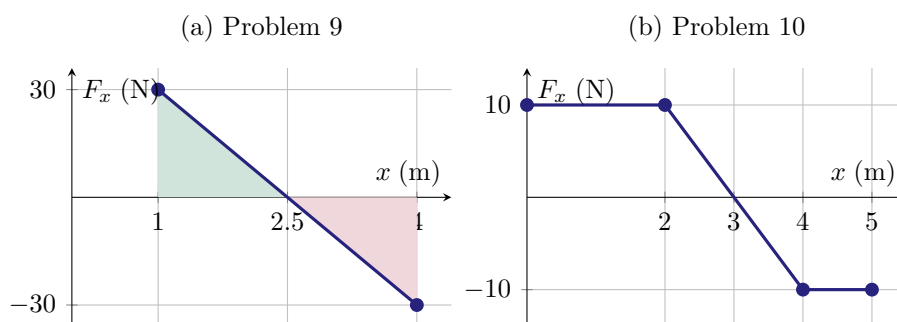


Figure 7.33: Two assessment force graphs. The vertical axis gives signed force, so areas below zero contribute negative work. Lines between the marked endpoints are straight; the domains are exactly those marked.

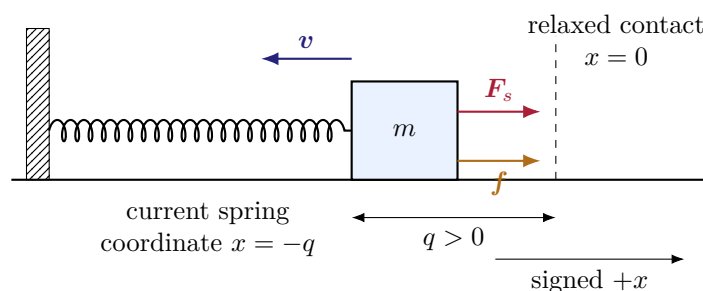


Figure 7.34: Problem 18: the block moves left during compression. Spring force and friction point right. Compression q increases as signed spring coordinate x decreases; consequently both forces do negative work. The spring and the friction arrows label horizontal forces only.

C. Power, engineering, measurement and computation

19. A constant 600 N drive force acts parallel to a vehicle's motion while its speed increases linearly from 2.0 to 8.0 m s^{-1} in 3.0 s. Find initial and final power, displacement, work and average drive power. Explain why the average speed may be used in this case.
20. An ideal drive supplies 12 kW mechanical power in forward motion. Find its force at 3.0 and 8.0 m s^{-1} . Explain the finite-speed domain of this law and distinguish mechanical output power from an unspecified electrical input.

21. A lift of total moving mass 900 kg rises at constant 0.60 m s^{-1} . Find useful mechanical power against gravity. With overall efficiency stipulated as 0.75, estimate input power. State which assumption permits cable tension to be identified with mg .
22. Force samples are $F(0) = 20 \text{ N}$, $F(0.50 \text{ m}) = 15 \text{ N}$ and $F(1.00 \text{ m}) = 4.0 \text{ N}$. Estimate work with trapezoidal and Simpson rules. Explain what their disagreement says about the data, and what further evidence would be needed to declare either estimate accurate.
23. A force sensor has constant uncorrected offset $+0.20 \text{ N}$. It measures work during monotonic extension $\Delta x = +0.040 \text{ m}$. Find the inferred-work bias and its sign. How would the offset contribution change for a monotonic return over the same displacement magnitude?
24. **Computational validation project.** Design a numerical-work validation for a smooth tabulated force law. Include trapezoidal and composite Simpson rules, grids with $n = 4, 8, 16, 32$ equal intervals, an exact or independently resolved benchmark, and a convergence criterion. Provide a worked benchmark using $F(x) = (10 \text{ N}) \exp[x/(1 \text{ m})]$ over $0 \leq x \leq 1 \text{ m}$. Require absolute error below 0.002 J and agreement under the last refinement below 0.005 J . Explain how to handle a proposed trajectory if cumulative $K_i + W$ becomes negative before its endpoint. State what changes when only noisy measured samples are available.

D. Advanced line-work and deformable-system challenges

25. For $\mathbf{F} = ay\hat{\mathbf{i}}$, with a in N/m , derive work along the straight path from $(0, 0)$ to (L, H) using $\mathbf{r}(s) = (Ls, Hs)$, $0 \leq s \leq 1$. Compare with the path that first moves vertically to $(0, H)$, then horizontally to (L, H) . Identify the source of the difference.
26. A constant force $\mathbf{F} = F_0\hat{\mathbf{i}}$ acts along the upper semicircle $\mathbf{r}(\theta) = R(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}})$ traversed from $\theta = \pi$ to zero. Evaluate line work directly and check against $\mathbf{F} \cdot (\mathbf{r}_f - \mathbf{r}_i)$. Explain why using arc length alone fails.
27. For $\mathbf{F} = -ky\hat{\mathbf{i}} + kx\hat{\mathbf{j}}$, where $k > 0$ has units N/m , parameterize one counterclockwise circle of radius R centred at the origin and calculate work. Repeat the sign argument for clockwise traversal. Does returning to the initial position force the work to vanish?
28. Equal and opposite outward forces of 300 N act on a deformable component. The left application point moves 0.020 m left; the right moves 0.050 m right. Find total external work, showing each signed force and displacement. Explain why multiplying net force by a representative displacement fails.
29. Two external forces are $F_1 = +100 \text{ N}$ and $F_2 = -40 \text{ N}$. The centre of mass moves $+0.10 \text{ m}$, and the application points move relative to it by $\Delta\rho_1 = +0.020 \text{ m}$ and $\Delta\rho_2 = -0.030 \text{ m}$. Calculate actual external work, the centre-of-mass force contribution and the relative-motion contribution. Verify their sum explicitly.
30. A symmetric connector has force magnitude $F(\delta) = 150\delta + 1.0 \times 10^5\delta^3 \text{ N}$, with δ in metres and the coefficient units chosen accordingly. Here δ means the *total increase in end-to-end separation*, so each end moves outward by $\delta/2$. The centre of mass is fixed while δ grows from zero to 0.030 m . Find actual external work and centre-of-mass force work. Explain why an extra factor of two must not be inserted.
31. For $\mathbf{F} = [(2.0 \text{ N m}^{-1})y]\hat{\mathbf{i}} + [(1.0 \text{ N m}^{-1})x]\hat{\mathbf{j}}$, apply midpoint line-work quadrature to $(0, 0) \rightarrow (1, 0) \rightarrow (1, 1)$, coordinates in metres. Give both segment contributions and the reversed-path result. Determine whether the midpoint estimate is exact for these straight segments.
32. A 2.0 kg cart has $K_i = 10 \text{ J}$. A drive does $+25 \text{ J}$, friction -8.0 J and a spring -4.0 J in a reachable process. Find final speed. If the guide is redesigned to curve the path while these work transfers

remain identical and its ideal force remains normal to motion, does the speed change? What is needed to find final velocity direction?

E. Synthesis, inverse models and evidence from graphs

33. **A misleading endpoint.** The step graph in Figure 7.35a defines the net position-dependent force on a 2.0 kg particle launched toward increasing x with $K_i = 6.0 \text{ J}$ at $x = 0$. Calculate the formal cumulative $K(x)$ on both graph segments and the formal value at $x = 4 \text{ m}$. Is that endpoint reachable? Locate the first turning point and distinguish algebraic continuation from the realized motion.
34. **Infer force from kinetic energy.** Figure 7.35b gives a synthetic prescribed $K(x)$ for a 2.0 kg particle moving toward increasing x . Lines join $(0, 4)$, $(2, 12)$ and $(5, 6)$, with x in metres and K in joules. Infer the net force in each interval, calculate net work over the full interval, and find speeds at $x = 2$ and 5 m . State how the corner in the graph should be interpreted in this idealized model.
35. **Speed-dependent braking.** A 1200 kg vehicle brakes from 20 m s^{-1} on level ground. During forward motion its net resistive force has magnitude $B + Cv^2$, with $B = 1200 \text{ N}$ and $C = 2.0 \text{ kg m}^{-1}$. Derive stopping distance using $dK = F_x dx$. Compare with estimates that hold resistance fixed at its initial value and at B alone. Explain the ordering without solving the time-dependent equation.
36. **Signed power record.** Figure 7.36 defines the net power of a 10 kg particle. Initially $K = 100 \text{ J}$. Read the signed areas to find net work and average power over 0–8 s, and final speed. Locate the time and value of maximum kinetic energy. Check that the negative-power interval does not make the prescribed motion impossible.
37. A 60 kg rigid load moves 5.0 m up a 20° incline at constant speed. Constant resistance is 30 N down the incline; a cable makes 15° above the incline. Find gravitational work, resistance work and cable tension. Verify that the implied normal reaction remains positive. Assume a straight, fixed incline and no acceleration normal to it.
38. **Force-limited startup followed by fixed power.** A 400 kg vehicle starts from rest on a level frictionless track. Its drive supplies $F_{\max} = 2000 \text{ N}$ until $F_{\max}v = \mathcal{P}_{\max}$, then supplies constant $\mathcal{P}_{\max} = 10 \text{ kW}$. Derive transition speed and time. Find speed and distance after 5.0 s, treating both regimes separately and checking continuity.
39. **Work from calibrated data.** The table gives equally spaced reported force samples. Each has independent random standard uncertainty 0.20 N; positions are exact for this model. A common calibration correction of -0.50 N applies to all readings, with independent standard uncertainty 0.10 N on that common offset. Find trapezoidal reported and corrected work, random uncertainty of the integrated work, calibration uncertainty, and their combined standard uncertainty. State why the calibration uncertainty is not reduced by treating it as five independent errors.

$x \text{ (m)}$	0	0.10	0.20	0.30	0.40
$F_{\text{reported}} \text{ (N)}$	5.0	7.0	9.0	11.0	13.0

40. **Fixed end, moving centre of mass.** In Figure 7.37, two 1.0 kg blocks are connected by an ideal 200 N m^{-1} spring. Block A is rigidly anchored to the wall while block B is released from rest. Spring extension decreases from $\delta_i = 0.100 \text{ m}$ to zero. Gravity and vertical support do no work. Find actual wall work and spring work on each block, the speed of B, and the kinetic-energy changes of the centre of mass and relative motion. Verify the centre-of-mass pseudowork using wall force and centre-of-mass displacement. Use $K_{\text{relative}} = K_A + K_B - K_{\text{CM}}$, with the ideal spring interaction included among the two-block subsystem's internal forces.

41. A 2.0 kg particle changes its velocity from $+5.0$ to $+9.0 \text{ m s}^{-1}$ in a laboratory inertial frame. A second frame moves at $U = +3.0 \text{ m s}^{-1}$. Find ΔK in both frames and Δp in the laboratory, where $p = mv$. Derive and verify $\Delta K' = \Delta K - U\Delta p$ for constant mass. Explain why different net work values are consistent.
42. A prescribed force is $\mathbf{F}(t) = (4.0 \text{ N})\hat{\mathbf{i}} + [(3.0 \text{ N s}^{-1})t]\hat{\mathbf{j}}$. Its application point follows $\mathbf{r}(t) = [(2.0 \text{ m s}^{-1})t]\hat{\mathbf{i}} + [(0.50 \text{ m s}^{-2})t^2]\hat{\mathbf{j}}$ from $t = 0$ to 2.0 s . Find instantaneous power, endpoint power and work. Compare with the incorrect shortcut $\mathbf{F}(2 \text{ s}) \cdot \Delta \mathbf{r}$. The prescribed force need not be the net force maintaining this path.
43. **Three paths, one pair of endpoints.** In Figure 7.38, a force field $\mathbf{F} = -ay\hat{\mathbf{i}} + 2ax\hat{\mathbf{j}}$ has $a = 2.0 \text{ N m}^{-1}$. Calculate work along path A (horizontal then vertical), path B (straight diagonal), and path C ($y = x^2/(4 \text{ m})$). All coordinates are in metres. Rank the works and give the work around the closed loop formed by A outward and B in reverse.
44. **Reversal with friction.** A 1.0 kg particle initially moves at 3.0 m s^{-1} . Constant kinetic resistance of magnitude 2.0 N always opposes its velocity. It travels 1.0 m outward, undergoes an ideal reversal that changes direction but does no work, then travels the same distance back. Find work and speed after each leg. Explain why Problem 6's cancellation rule does not apply. The reversal can be realized as a smooth short guide with normal constraint forces; resistance during that idealized turning segment is neglected.
45. A forward-moving 500 kg vehicle receives constant drive power 6000 W and experiences resistance cv^2 , with $c = 10 \text{ kg m}^{-1}$. At $v = 10 \text{ m s}^{-1}$, find drive force, net power and acceleration. Derive the terminal speed and explain its stability from the sign of dK/dt . Would doubling mass change that terminal speed under the same force and power laws?
46. **Identify a spring and friction from stopping data.** An unknown ideal spring and constant friction stop a block during positive compression q . A first trial has incident kinetic energy 2.0 J and stops at $q = 0.100 \text{ m}$; a second has 6.0 J and stops at $q = 0.200 \text{ m}$. Infer stiffness and friction magnitude. Predict stopping compression for $K_i = 8.0 \text{ J}$. Explain why two identical compression values would not provide independent equations for these two parameters.

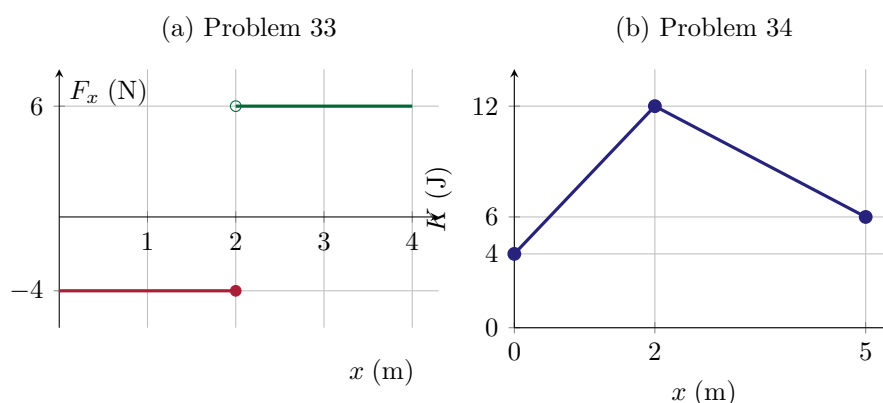


Figure 7.35: Assessment graphs for cumulative-work reachability and inverse force inference. The value of a step force at the isolated switching point has no effect on its work integral. The kinetic-energy graph is a separate, entirely positive synthetic record.

7.12 Complete solutions

These solutions cover every numbered problem and every requested subpart. Rounded final values retain the precision appropriate to the statements; calculations below use unrounded intermediate



Figure 7.36: Problem 36: a signed net-power history. Every connecting segment is straight; the area under the graph is net work. Kinetic energy reaches its maximum when the final descending segment crosses zero power.

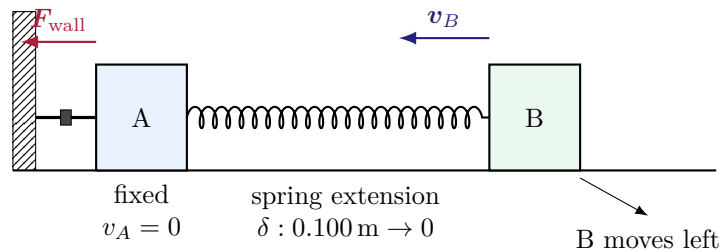


Figure 7.37: Problem 40: block A is rigidly attached to the wall by the short fixed anchor shown; the wall force acts at this stationary block, whereas the spring accelerates B and changes the two-block centre-of-mass motion. Actual wall work is zero; its centre-of-mass pseudowork need not be zero. The figure represents the initially extended spring.

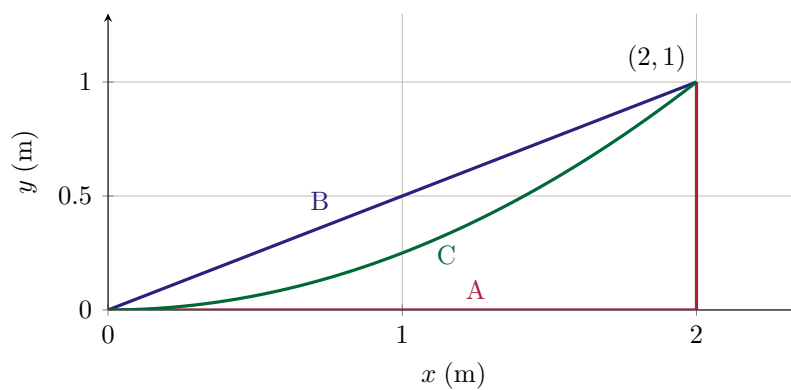


Figure 7.38: Problem 43: all three paths run from $(0, 0)$ to $(2\text{ m}, 1\text{ m})$. A uses two axis-parallel segments, B is straight, and C is parabolic. Each calculation must use the field values along its own path.

values. They distinguish work by one force, net work, and the centre-of-mass contribution. A positive algebraic endpoint kinetic energy alone does not establish accessibility.

Problem 1: kinetic energy in two inertial frames

Choose east positive and use the Galilean velocity transformation, valid in the nonrelativistic model. The platform observer measures $v' = 14 - 6 = 8.0 \text{ m s}^{-1}$, not 20 m s^{-1} . Thus

$$K_{\text{lab}} = \frac{1}{2}(900)(14)^2 = 88\,200 \text{ J} = 88.2 \text{ kJ}, \quad K_{\text{platform}} = \frac{1}{2}(900)(8)^2 = 28\,800 \text{ J} = 28.8 \text{ kJ}.$$

Both are positive and both have units $\text{kg m}^2 \text{ s}^{-2} = \text{J}$. Kinetic energy depends on the velocity relative to the selected frame; it is not required to be the same in two inertial frames. Each observer must apply the work–kinetic-energy theorem with force-application displacements measured in that observer’s own frame. No disagreement about a physical outcome follows from these different numerical assignments.

Problem 2: a passenger at rest in a carriage

In the instantaneous carriage frame the passenger has $v' = 0$, so $K' = 0$. In the ground frame the passenger shares the train’s speed V at that instant and has $K = \frac{1}{2}mV^2$. A numerical value requires the passenger’s mass and the train’s ground-frame speed, neither of which is specified. The uniformly moving carriage defines an inertial frame under the idealization; its energy assignment is as legitimate as the ground-frame assignment. If the train were accelerating, the inertial-frame theorem would need modification in carriage coordinates, but that situation is excluded here. Being at rest is a statement relative to a frame, so there is no observer-independent claim that this passenger has zero kinetic energy.

Problem 3: component-by-component work

For a constant force the work equals the dot product with application-point displacement:

$$W = F_x \Delta x + F_y \Delta y = (40)(3) + (-15)(4) = 120 - 60 = 60 \text{ J}.$$

The negative y contribution comes from a downward force component acting through upward displacement. The x contribution is positive and larger, yielding positive net selected-force work. Products of newtons and metres give joules. Multiplying magnitudes would replace $\mathbf{F} \cdot \Delta \mathbf{r}$ by $F \Delta r$, implicitly assuming that the vectors are parallel. They are not: their directions differ. The dot product is the correct operation because it retains the force component tangent to the displacement and the physical sign of energy transfer.

Problem 4: cable work and net work

The application point moves horizontally, so only the horizontal cable component contributes:

$$W_c = (120)(5.0) \cos 35^\circ = 491.49 \text{ J}.$$

Resistance is opposite the displacement and therefore does $W_r = -(25)(5.0) = -125 \text{ J}$. The other forces contribute zero by the statement, giving

$$W_{\text{net}} = 491.49 - 125 = 366.49 \text{ J}, \quad \Delta K = W_{\text{net}} \simeq 366 \text{ J}.$$

The positive sign means an increase in translational kinetic energy for the particle model. It does not say that all supplied cable energy becomes that increase: 125 J is removed by resistance. A check is that the horizontal net force $120 \cos 35^\circ - 25 = 73.299 \text{ N}$ times 5.0 m gives the same net work. No final speed can be found without mass and initial kinetic energy.

Problem 5: a nonzero normal force with zero work

An inward force is required for the bead's centripetal acceleration. For radius R and nonzero speed v , its magnitude is mv^2/R , so the force is not zero. At every point the instantaneous displacement is tangent to the circle, whereas the constraint force is radial; hence $\mathbf{F} \cdot d\mathbf{r} = 0$ locally. Integrating that zero quantity over a quarter revolution gives zero work. Constant speed is consistent with zero change of kinetic energy while velocity direction changes. The chord between endpoints is not a substitute for the tangent displacement in a variable-direction-force integral. A zero-work result constrains energy transfer, not acceleration direction or the force magnitude.

Problem 6: return work for an unchanged position law

With exactly the same position-dependent force on both traversals,

$$W_{a \rightarrow b} = \int_a^b F_x(x) dx, \quad W_{b \rightarrow a} = \int_b^a F_x(x) dx = -W_{a \rightarrow b}.$$

Reversing the integration limits accounts for reversed displacement; no separate sign must be inserted into the force law. The total work on this out-and-back path is zero even if each leg has substantial nonzero work. This conclusion is restricted to the stated unchanged law. A force such as kinetic friction reverses direction when velocity reverses, so its return integrand is a different function. A time-controlled force could also change between visits to the same position. Those changes invalidate this cancellation argument, as Problem 44 demonstrates.

Problem 7: dimensions and the startup domain

Power has dimensions

$$[\mathbf{F} \cdot \mathbf{v}] = \text{N m s}^{-1} = \text{J s}^{-1} = \text{W}.$$

For forward collinear force and velocity, $\mathcal{P}_0 = Fv$ implies $F = \mathcal{P}_0/v$ only for $v > 0$. At rest a finite force produces zero instantaneous mechanical power because its application point has zero velocity. Maintaining positive fixed power at exactly $v = 0$ would instead require an infinite force, outside the finite-force model. One sensible completion is a force cap, $F = \min(F_{\max}, \mathcal{P}_0/v)$ for $v > 0$, with $F = F_{\max}$ at startup. Then the drive initially operates at bounded force and transitions to fixed power at $v = \mathcal{P}_0/F_{\max}$. Problem 38 works out this completion quantitatively.

Problem 8: zero net force does not imply zero external work

Take leftward force $-F$ at the left end and rightward force $+F$ at the right end. If those ends move outward by $\Delta x_L = -d_L$ and $\Delta x_R = +d_R$, where both d values are positive, then $W_{\text{ext}} = (-F)(-d_L) + Fd_R = F(d_L + d_R) > 0$. The forces sum to zero, but their application points undergo different displacements. The body can stretch while its centre of mass remains fixed, with work transferred into deformation or internal kinetic energy. Therefore one needs both application-point motions, or an equivalent deformation coordinate and its force law. Multiplying net force by centre-of-mass displacement would give zero and describes only the translational centre-of-mass contribution, not this actual external work.

Problem 9: cancellation of signed areas

The slope read from the graph is $(-30 - 30)/(4 - 1) = -20 \text{ N m}^{-1}$, giving $F_x = 50 - 20x \text{ N}$ when x is entered in metres. Its zero is $x = 2.5 \text{ m}$. The positive triangle has base 1.5 m and height 30 N, so $W_+ = \frac{1}{2}(1.5)(30) = 22.5 \text{ J}$. The negative triangle contributes $W_- = -22.5 \text{ J}$. Hence $W = 0$. Independently,

$$\int_1^4 (50 - 20x) dx = [50x - 10x^2]_1^4 = 40 - 40 = 0 \text{ J}.$$

Zero net work reflects cancellation, not an absence of force or work throughout the interval. Adding the absolute triangle areas would incorrectly give 45 J and erase the distinction between transfer into and out of the particle's kinetic energy.

Problem 10: piecewise graphical work and accessibility

The first rectangle contributes $(10)(2) = 20$ J. Between 2 and 3 m the positive triangle contributes +5 J; between 3 and 4 m the negative triangle contributes -5 J. The final rectangle contributes $(-10)(1) = -10$ J. Therefore $W = 20 + 5 - 5 - 10 = 10$ J, and

$$K_f = 5 + 10 = 15 \text{ J}, \quad v_f = \sqrt{2(15)/1.0} = 5.48 \text{ m s}^{-1}.$$

For reachability, K rises from 5 to 30 J by $x = 3$ m and then falls to 25 J at $x = 4$ m and 15 J at $x = 5$ m. Its minimum over the whole interval is the positive initial 5 J. Thus no earlier zero blocks the endpoint. The theorem yields speed magnitude; forward direction follows from this continuously positive-energy trajectory and its stated initial direction.

Problem 11: sum the work before finding speed

The initial energy is $K_i = \frac{1}{2}(2.0)(3.0)^2 = 9.0$ J. Sum all forces' work, including signs:

$$W_{\text{net}} = 18 - 5 + 0 = 13 \text{ J}, \quad K_f = 9 + 13 = 22 \text{ J}, \quad v_f = \sqrt{2(22)/2.0} = \sqrt{22} = 4.69 \text{ m s}^{-1}.$$

The larger final speed agrees with positive net work. A zero-work force could be a nonzero normal force perpendicular to motion at every instant, or could have positive and negative contributions that cancel over the process. Therefore zero total work does not establish zero force or zero acceleration. The statement supplies process reachability; the three total work values alone do not reveal every intermediate energy or the final velocity direction.

Problem 12: relaxing a compressed spring

Use the signed extension coordinate rather than assigning the compression magnitude the sign of an extension. The endpoints are $x_i = -0.150$ m and $x_f = 0$. Since $F_s = -k_s x$,

$$W_s = \int_{x_i}^0 -k_s x \, dx = \frac{1}{2}k_s(x_i^2 - x_f^2) = \frac{1}{2}(250)(0.150)^2 = 2.8125 \text{ J}.$$

Only the spring does work and $K_i = 0$, so

$$v_f = \sqrt{\frac{2W_s}{0.80}} = 2.65 \text{ m s}^{-1}.$$

As compression relaxes, the spring force and the cart's displacement point in the same direction; positive work is therefore expected. Throughout $-0.150 \leq x \leq 0$, $K(x) = \frac{1}{2}k_s(x_i^2 - x^2) \geq 0$, so the relaxed endpoint is accessible after release. Squared lengths remove the sign from the final formula, but the signed integral fixes its physical interpretation.

Problem 13: externally lifting a rigid load

Uniform gravity points downward through an upward displacement:

$$W_g = -mg\Delta y = -(70)(9.81)(4.5) = -3090.15 \text{ J}.$$

Equal endpoint speeds give $\Delta K = 0$, and the two working forces therefore satisfy $W_c + W_g = 0$. Thus $W_c = +3090.15 \text{ J} \simeq 3.09 \text{ kJ}$ and $W_{\text{net}} = 0$. The cable supplies the work that gravity removes from the load's translational kinetic-energy account. Because the load is rigid and externally translated, the cable application point shares the specified vertical displacement, making its actual work well defined. Equal endpoint speeds do not require constant tension throughout the lift; they constrain its integrated work. A self-propelled climber would require an internal-energy and contact analysis instead, which is why that physically different system is not assumed here.

Problem 14: a spring turning point

The initial kinetic energy is $K_i = \frac{1}{2}(0.50)(3.0)^2 = 2.25 \text{ J}$. From $x = 0$ to $x = -q$, spring work is $W_s = -\frac{1}{2}k_s q^2$. At maximum compression the speed is zero, hence

$$0 = 2.25 - \frac{1}{2}(200)q^2, \quad q_{\max} = \sqrt{2(2.25)/200} = 0.150 \text{ m}.$$

For $0 \leq q \leq 0.150 \text{ m}$ the cumulative kinetic energy $K(q) = 2.25 - 100q^2$ is nonnegative, reaching zero first at the stated value. Spring force opposes compression and then reverses the cart's motion, so the zero is a genuine turning point. A formal $K < 0$ beyond it means the assumed continued compression is inaccessible; it is not a negative physical kinetic energy and must not be converted into an imaginary speed. The positive compression magnitude corresponds to signed coordinate $x = -0.150 \text{ m}$.

Problem 15: independent force contributions

Compute each force's dot product:

$$W_1 = (12)(4) + (-5)(3) = 33 \text{ J}, \quad W_2 = (-2)(4) + (9)(3) = 19 \text{ J}.$$

Therefore $W_{\text{net}} = 52 \text{ J}$ and $K_f = 10 + 52 = 62 \text{ J}$. As an independent algebraic check, the combined force is $(10\hat{i} + 4\hat{j}) \text{ N}$, whose work is $10(4) + 4(3) = 52 \text{ J}$. This agreement follows from the distributivity of the dot product; adding forces first or adding their works first is equivalent when both constant forces act through the same particle displacement. A mass is not supplied, so the problem determines final kinetic energy rather than a numerical final speed. Positive net work means kinetic energy increases by 52 J .

Problem 16: variable force with a reachability check

Integrate the force over its specified domain:

$$W = \int_0^2 (30 - 10x) dx = [30x - 5x^2]_0^2 = 40 \text{ J}.$$

Then $K_f = 6 + 40 = 46 \text{ J}$ and

$$v_f = \sqrt{2(46)/1.20} = 8.76 \text{ m s}^{-1}.$$

The entire path is accessible because $K(x) = 6 + 30x - 5x^2$ starts at 6 J and has derivative $30 - 10x > 0$ throughout $0 \leq x \leq 2 \text{ m}$. It therefore rises monotonically without an intervening zero. A graphical check is the positive trapezoid under the force graph: average force $(30 + 10)/2 = 20 \text{ N}$ over 2 m gives 40 J . Both checks agree with acceleration during the full stated interval.

Problem 17: an upward load with nonzero net work

Gravity does $W_g = -(25)(9.81)(3.0) = -735.75 \text{ J}$. Initially $K_i = \frac{1}{2}(25)(2.0)^2 = 50 \text{ J}$. With cable work 1000 J ,

$$K_f = 50 + 1000 - 735.75 = 314.25 \text{ J}, \quad v_f = \sqrt{2(314.25)/25} = 5.01 \text{ m s}^{-1}.$$

The constant cable force inferred from its work is $T = 1000/3.0 = 333.33 \text{ N}$. It exceeds weight 245.25 N , so net upward force is 88.083 N and $K(y) = 50 + 88.083y$ stays positive over the ascent. This verifies reachability instead of relying only on the positive endpoint value. The larger upward speed is physically consistent with a net positive upward force and positive net work.

Problem 18: compression while friction also acts

The diagram shows motion left and both resistive forces right. Use $q \geq 0$ for compression and $x = -q$ for signed spring coordinate. Spring work is $W_s = -\frac{1}{2}k_s q^2 = -50q^2$ J; friction work is $W_f = -fq = -5q$ J. With $K_i = \frac{1}{2}(1.0)(4.0)^2 = 8.0$ J, the stopping condition is

$$50q^2 + 5q - 8 = 0, \quad q = \frac{-5 \pm \sqrt{25 + 1600}}{100}.$$

The admissible root is $q = 0.353113 \text{ m} \simeq 0.353 \text{ m}$; the other is -0.453113 m and violates the compression domain. The cumulative $K(q) = 8 - 50q^2 - 5q$ decreases strictly for $q \geq 0$, so this first zero is reached without an earlier forbidden interval. At the root the spring removes 6.23444 J and friction removes 1.76556 J, totaling the incident 8.0 J. The signed spring coordinate is $x = -0.353113 \text{ m}$.

Problem 19: power from a known velocity history

For collinear force and velocity, $\mathcal{P}(t) = Fv(t)$. Thus $\mathcal{P}_i = 600(2.0) = 1200$ W and $\mathcal{P}_f = 600(8.0) = 4800$ W. The stated linear speed history has average speed $(2 + 8)/2 = 5.0 \text{ m s}^{-1}$, so displacement is $5.0(3.0) = 15.0 \text{ m}$. Drive work is $W_d = 600(15.0) = 9000$ J and average drive power is $9000/3.0 = 3000$ W. Integrating $\mathcal{P}(t) = 1200 + 1200t$ W over 0–3 s confirms 9000 J. Taking the endpoint-average speed works because the time dependence is linear; it is not a general substitute for $\int v(t)dt$. These are drive contributions; unspecified additional forces could change net work.

Problem 20: fixed mechanical power at finite speed

The ideal parallel-drive model gives $F = \mathcal{P}/v$ for $v > 0$. With $\mathcal{P} = 12\,000$ W,

$$F(3.0) = 12\,000/3.0 = 4000 \text{ N}, \quad F(8.0) = 12\,000/8.0 = 1500 \text{ N}.$$

Units check as $\text{W}/(\text{m s}^{-1}) = \text{N}$. The force decreases at higher speed because a given work rate can be delivered through faster application-point motion. The model does not cover zero-speed startup, where it would demand unbounded force. A real force or torque cap sets another regime. The supplied number is mechanical output power; electrical input would be larger if efficiency were below unity, but no efficiency is supplied here. Inventing an input-power value would therefore exceed the given information.

Problem 21: lift power and stipulated efficiency

Constant upward speed means zero acceleration. With no additional resistive force, cable tension is $mg = 900(9.81) = 8829$ N. The useful mechanical power is

$$\mathcal{P}_{\text{useful}} = mgv = 8829(0.60) = 5297.4 \text{ W} \simeq 5.30 \text{ kW}.$$

Efficiency is defined here by $\eta = \mathcal{P}_{\text{useful}}/\mathcal{P}_{\text{in}}$, hence

$$\mathcal{P}_{\text{in}} = 5297.4/0.75 = 7063.2 \text{ W} \simeq 7.06 \text{ kW}.$$

The remaining 1765.8 W is the implied non-useful input-power share under the stipulated overall efficiency. The weight-power result assumes constant speed and only gravity opposing useful ascent. During acceleration the cable would supply additional kinetic-energy power; unspecified cable or guide losses would also require an altered force balance. The present calculation is an operating power scale, not a complete motor rating.

Problem 22: sparse-data quadrature

The spacing is $h = 0.50$ m, with two intervals. The trapezoidal and Simpson estimates are respectively

$$W_T = h \left[\frac{F_0 + F_2}{2} + F_1 \right] = 0.50[12 + 15] = 13.5 \text{ J},$$

$$W_S = \frac{h}{3}(F_0 + 4F_1 + F_2) = \frac{0.50}{3}(20 + 60 + 4) = 14.0 \text{ J}.$$

Their 0.5 J difference indicates that interpolation curvature matters at the current sampling density. It does not identify which number is exact: infinitely many smooth force laws can pass through the three readings with different integrals. Additional samples, a defensible force model or an independent benchmark are needed to estimate error. Averaging the two without such evidence has no general convergence justification. If the force were known to be exactly quadratic, Simpson would be exact, but that information is not provided.

Problem 23: a common offset in force integration

Write $F_{\text{measured}} = F_{\text{true}} + b$ with constant $b = +0.20$ N. The inferred-minus-true work is

$$\Delta W_{\text{bias}} = \int b \, dx = b(x_f - x_i) = (0.20)(0.040) = +0.0080 \text{ J} = +8.0 \text{ mJ}.$$

For increasing x the positive offset overestimates work. On the monotonic return, $\Delta x = -0.040$ m, so the same offset gives -8.0 mJ. It is a signed displacement effect, not automatically a positive error proportional to path length. A constant offset can cancel on an exactly retraced closed integration, but that cancellation does not prove that the underlying work measurement is otherwise correct. Calibration and random sampling uncertainties are distinct issues treated in Problem 39.

Problem 24: a worked computational benchmark and assessment rubric

First define the physical domain and an exact reference independent of the numerical rules. For $\ell = 1$ m and $F_0 = 10$ N,

$$W_* = \int_0^\ell F_0 e^{x/\ell} \, dx = F_0 \ell (e - 1) = 17.1828182846 \text{ J}.$$

For each even interval count n , sample $x_j = j\ell/n$, set $h = \ell/n$, and compute

$$T_n = h \left[\frac{F_0^{(s)} + F_n^{(s)}}{2} + \sum_{j=1}^{n-1} F_j^{(s)} \right],$$

$$S_n = \frac{h}{3} \left[F_0^{(s)} + F_n^{(s)} + 4 \sum_{\substack{j=1 \\ j \text{ odd}}}^{n-1} F_j^{(s)} + 2 \sum_{\substack{j=2 \\ j \text{ even}}}^{n-2} F_j^{(s)} \right],$$

where $F_j^{(s)}$ denotes a sample, avoiding confusion with the amplitude F_0 . The resulting validation table is

n	T_n (J)	S_n (J)	$ T_n - W_* $ (J)	$ S_n - W_* $ (J)
4	17.27221905	17.18318842	0.08940076	0.00037013
8	17.20518592	17.18284155	0.02236764	0.00002326
16	17.18841129	17.18281974	0.00559300	0.00000146
32	17.18421660	17.18281838	0.00139832	0.00000009

Trapezoidal error decreases by approximately four under each doubling, and Simpson error by approximately sixteen. This is the expected second- and fourth-order behaviour for this smooth function. For the last trapezoidal refinement, $|T_{32} - T_{16}| = 0.00419468 \text{ J} < 0.005 \text{ J}$ and its benchmark error is $0.00139832 \text{ J} < 0.002 \text{ J}$: both criteria pass. Simpson's last difference is $1.36 \times 10^{-6} \text{ J}$, and its benchmark error is below 10^{-7} J ; it also passes. Passing only a refinement test can conceal a systematic model error, which is why a benchmark is useful.

For a dynamics check, construct the cumulative integral at every sample and inspect $K(x) = K_i + W(0 \rightarrow x)$ throughout the path. If it first reaches zero while the force opposes the stated motion, refine or interpolate around that first zero, bracket the turning point and solve $K = 0$. Do not report $\sqrt{2K/m}$ where $K < 0$, and do not continue to a later positive endpoint through a forbidden interval. Distinguish a true physical zero from a quadrature or force-model error by refining the calculation and checking the governing force law.

A full-credit report states sample locations, units and force model; implements both rules with correct endpoint weights; reports all four grids and an independent reference; checks observed error reduction and both numerical tolerances; and explains the whole-path kinetic-energy test. A report missing a benchmark must explicitly identify that limitation. With noisy measured data, inventing intermediate samples by interpolation is not equivalent to collecting new independent data. The error budget must include sampling uncertainty and calibration bias, and high-order quadrature need not outperform a lower-order method in that setting. The benchmark demonstrates algorithmic correctness for a smooth known law; it does not by itself certify an experimental sensor.

Problem 25: straight and two-segment line work

Along the straight path, $\mathbf{r}(s) = (Ls, Hs)$ and $d\mathbf{r} = (L, H)ds$. The force is $(aHs, 0)$, so

$$W_B = \int_0^1 aHsL ds = \frac{1}{2}aHL.$$

For the vertical first leg, $dx = 0$ and the force has no vertical component, giving zero work. The horizontal leg lies at $y = H$, where $F_x = aH$ is constant, so its work is aHL . Thus the two-segment work is twice the straight-path work for positive a, L, H . Units are $(\text{N/m})(\text{m})(\text{m}) = \text{J}$. The differing force values sampled during horizontal motion produce the difference: on the diagonal y rises gradually, while the second path performs all horizontal displacement at the maximum $y = H$. Endpoint agreement alone does not ensure identical work in this field.

Problem 26: constant force along a semicircle

Differentiate the parameterization:

$$\frac{d\mathbf{r}}{d\theta} = R(-\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}}).$$

Its dot product with the constant force is $-F_0R\sin\theta$. Retaining the actual decreasing parameter limits,

$$W = \int_{\pi}^0 -F_0R\sin\theta d\theta = [F_0R\cos\theta]_{\pi}^0 = 2F_0R.$$

The endpoint displacement is $(2R)\hat{\mathbf{i}}$, so the constant-force shortcut gives the same $2F_0R$. The positive sign is consistent with net motion from the left endpoint to the right endpoint under a rightward force. Using F_0 times the semicircle length πR would treat the force as tangent everywhere; it is tangent only at selected points and its local tangential component varies. The endpoint shortcut succeeds because the force vector is spatially constant, not because the path is straight.

Problem 27: nonzero work on a closed circle

Take $\mathbf{r}(\theta) = R(\cos \theta, \sin \theta)$ for $0 \leq \theta \leq 2\pi$. The field along it is $kR(-\sin \theta, \cos \theta)$, while $d\mathbf{r} = R(-\sin \theta, \cos \theta)d\theta$. Therefore

$$W_{\text{CCW}} = \int_0^{2\pi} kR^2(\sin^2 \theta + \cos^2 \theta) d\theta = 2\pi kR^2 > 0.$$

The force is tangent and aligned with counterclockwise displacement at every point. Clockwise motion reverses displacement through the same field and yields $W_{\text{CW}} = -2\pi kR^2$. Work has units $\text{N/m} \times \text{m}^2 = \text{J}$. Returning to the initial position does not force work to vanish for a general position-dependent vector field. The calculation supplies a direct example: the complete loop transfers energy even though its net endpoint displacement is zero.

Problem 28: work at two deforming ends

Choose right positive. At the left end $F_L = -300 \text{ N}$ and $\Delta x_L = -0.020 \text{ m}$, giving $W_L = +6.0 \text{ J}$. At the right end $F_R = +300 \text{ N}$ and $\Delta x_R = +0.050 \text{ m}$, giving $W_R = +15.0 \text{ J}$. Consequently $W_{\text{ext}} = W_L + W_R = 21.0 \text{ J}$. Both end motions are along their respective forces, so both signs must be positive. Although $F_L + F_R = 0$, there is no single common application-point displacement to factor out of the work sum. The body deforms, and the actual end displacements differ. Net-force information suffices for centre-of-mass acceleration but does not determine this external work or its partition among internal energy forms.

Problem 29: centre-of-mass and relative-motion contributions

Recover actual application-point displacements first:

$$\Delta x_1 = 0.10 + 0.020 = 0.120 \text{ m}, \quad \Delta x_2 = 0.10 - 0.030 = 0.070 \text{ m}.$$

Actual external work is $100(0.120) - 40(0.070) = 12.0 - 2.8 = 9.2 \text{ J}$. The centre-of-mass force contribution is $(100 - 40)(0.10) = 6.0 \text{ J}$. Relative work is $100(0.020) + (-40)(-0.030) = 2.0 + 1.2 = 3.2 \text{ J}$. Their sum is $6.0 + 3.2 = 9.2 \text{ J}$, verifying the decomposition directly. The negative force does negative actual work because its actual point moves right, but it contributes positive relative work because that point moves left relative to the translating centre. Distinguishing actual and relative displacements prevents this apparently conflicting sign assignment from becoming an error.

Problem 30: the deformation coordinate fixes the factor

Each end moves by magnitude $d\delta/2$ along its outward force. Thus the total incremental actual work is

$$dW = F(\delta) \frac{d\delta}{2} + F(\delta) \frac{d\delta}{2} = F(\delta) d\delta.$$

The coordinate δ already represents total separation increase; it is not the displacement of either individual end. Integrating gives

$$W_{\text{ext}} = \int_0^{.030} (150\delta + 10^5 \delta^3) d\delta = \left[75\delta^2 + 25000\delta^4 \right]_0^{.030} = 0.0675 + 0.02025 = 0.08775 \text{ J}.$$

The net force is zero and the centre of mass is fixed, so the centre-of-mass force work is zero. The positive actual work comes from the relative end motions. Inserting another factor of two would double-count them; using δ for each end's displacement would describe a different deformation model. The explicit coordinate definition is therefore essential to the result.

Problem 31: midpoint work on a polyline

The first segment has midpoint $(0.5, 0)$ m, force $(0, 0.5)$ N and displacement $(1, 0)$ m. Its contribution is zero. The second has midpoint $(1, 0.5)$ m, force $(1, 1)$ N and displacement $(0, 1)$ m, giving 1.0 J. Thus the midpoint sum is 1.0 J. Reversal retains both midpoints and force samples while negating both segment displacement vectors; it yields -1.0 J. Here the midpoint result is exact, not merely approximate: the force field is affine in position, and along each straight segment its tangential component is at most linear in the segment parameter. Midpoint quadrature integrates such a linear function exactly. This exactness need not hold for nonlinear fields or curved segments.

Problem 32: changed guide, identical work transfers

The three works sum to $25 - 8 - 4 = 13$ J. Hence $K_f = 10 + 13 = 23$ J and

$$v_f = \sqrt{2(23)/2.0} = \sqrt{23} = 4.80 \text{ m s}^{-1}.$$

For the redesigned reachable path the ideal guide remains normal to the local motion, so it still does zero work. With the other three integrated transfers stipulated unchanged, the same work theorem gives the same final speed. This result does not determine the velocity vector: one needs the tangent direction of the new guide at the final point and the sense in which it is traversed. Constraint dynamics would additionally be needed to check whether the guide can supply the required normal force. Equal work transfers supply a scalar energy result, while direction and guide feasibility require geometric and dynamical information.

Problem 33: a positive endpoint behind a forbidden interval

The graph gives $F_x = -4$ N up to $x = 2$ m, then $+6$ N up to 4 m. Formal cumulative integration yields, with x expressed in metres,

$$K(x) = \begin{cases} 6 - 4x & \text{J,} & 0 \leq x \leq 2, \\ -2 + 6(x - 2) & \text{J,} & 2 \leq x \leq 4. \end{cases}$$

The algebraic endpoint is $K(4) = 10$ J and would correspond to $\sqrt{10} \text{ m s}^{-1}$ for $m = 2$ kg. But that speed is not realized by this launch. The first zero is $6 - 4x = 0$, or $x_t = 1.50$ m. The force there is still leftward, so the initially rightward particle stops and turns before reaching the force switch. Formal K is negative from 1.50 to 2.333 m, an inaccessible interval. The work integral is mathematically defined there, but its algebraic continuation is not the realized trajectory. An additional external intervention would be needed to cross it. This example shows exactly why a whole-path test must precede reporting an endpoint speed.

Problem 34: force is the slope of kinetic energy

For a particle moving along x , $dK = F_{\text{net},x}dx$. Thus the first graph slope is $(12 - 4)/(2 - 0) = 4.0$ N and the second is $(6 - 12)/(5 - 2) = -2.0$ N. Total net work is the endpoint difference, $6 - 4 = 2.0$ J; independently, $4(2) - 2(3) = 2.0$ J. At $x = 2$ m,

$$v_2 = \sqrt{2(12)/2.0} = 3.46 \text{ m s}^{-1}, \quad v_5 = \sqrt{2(6)/2.0} = 2.45 \text{ m s}^{-1}.$$

All graph energies are positive, so there is no turning point over the displayed domain. The corner idealizes a sudden change in finite net force, not an impulse: kinetic energy and speed remain continuous while their position derivatives change. Real measured corners may be rounded, and slope inference would then require uncertainty-aware fitting rather than exact differentiation of noisy data.

Problem 35: integrate braking as speed changes

During forward motion $F_x = -(B + Cv^2)$ and $dK = mv dv$. Therefore

$$dx = -\frac{mv dv}{B + Cv^2}, \quad d = \int_0^{v_i} \frac{mv}{B + Cv^2} dv = \frac{m}{2C} \ln \left(\frac{B + Cv_i^2}{B} \right).$$

Substitution gives $d = 300 \ln(2000/1200) = 153.25$ m. The logarithm argument is dimensionless; m/C has dimensions of length. Initial kinetic energy is $\frac{1}{2}(1200)(20)^2 = 240\,000$ J. Holding resistance at its maximum initial value 2000 N would give 120 m; holding it at its minimum $B = 1200$ N would give 200 m. The actual resistance decreases continuously between these bounds as speed falls, so $120 < 153.25 < 200$ m is expected. The derivation uses the changing force rather than assuming an unsupported constant deceleration. At the stopping point the forward-motion model ends; it is not extended into reversed velocity without revisiting the braking law.

Problem 36: integrate a signed power history

The first triangular area is $\frac{1}{2}(2)(400) = 400$ J, the plateau area is $(4)(400) = 1600$ J, and the last trapezoid contributes $(2)(400 - 200)/2 = 200$ J. Thus total work is 2200 J, average net power $2200/8 = 275$ W, and final $K = 100 + 2200 = 2300$ J. The final speed is $\sqrt{2(2300)/10} = 21.45$ m s⁻¹. On the last segment, writing $\tau = t - 6$ in seconds, $\mathcal{P} = 400 - 300\tau$ W. It crosses zero at $\tau = 4/3$ s, or $t = 7.333$ s. The positive triangle after $t = 6$ contributes $\frac{1}{2}(4/3)(400) = 266.667$ J. Therefore

$$K_{\max} = 100 + 400 + 1600 + 266.667 = 2366.667 \text{ J}.$$

The negative-power tail removes only 66.667 J, leaving positive 2300 J. Throughout the history the minimum energy is its initial 100 J, so the record never demands negative kinetic energy. Negative power means decreasing energy, not negative energy itself.

Problem 37: work along an incline and a contact check

The vertical rise is $d \sin 20^\circ$, giving $W_g = -mgd \sin 20^\circ = -1006.565$ J. Resistance work is $W_r = -30(5.0) = -150$ J. Constant speed gives zero net work; the cable therefore does 1156.565 J. Since its angle to the actual displacement is 15° ,

$$Td \cos 15^\circ = 1156.565 \text{ J}, \quad T = 239.473 \text{ N}.$$

For the normal direction, the cable has an outward component $T \sin 15^\circ$ and gravity an inward component $mg \cos 20^\circ$. Zero normal acceleration requires

$$N = mg \cos 20^\circ - T \sin 15^\circ = 491.123 \text{ N} > 0.$$

Thus the assumed fixed contact with the incline is consistent. The cable's transverse component contributes no work along the incline but affects the normal reaction. Using Td without the cosine would miss that distinction. The calculations require the fixed straight path and constant-speed assumptions stated in the problem.

Problem 38: a continuous startup model

Transition occurs when the force-limited mechanical power reaches the cap: $v_c = \mathcal{P}_{\max}/F_{\max} = 10000/2000 = 5.0$ m s⁻¹. In the first regime $a = F_{\max}/m = 5.0$ m s⁻², so $t_c = v_c/a = 1.0$ s and $x_c = \frac{1}{2}at_c^2 = 2.50$ m. Thereafter $dK/dt = \mathcal{P}_{\max}$, giving

$$v^2 = v_c^2 + \frac{2\mathcal{P}_{\max}}{m}(t - t_c).$$

At $t = 5$ s, $v^2 = 25 + 50(4) = 225$ and $v = 15.0$ m s⁻¹. To integrate distance, use $dt = (m/\mathcal{P}_{\max})v dv$:

$$x - x_c = \frac{m}{3\mathcal{P}_{\max}}(v^3 - v_c^3) = \frac{400}{30000}(3375 - 125) = 43.333 \text{ m.}$$

Total distance is 45.833 m. Both force laws give 2000 N at transition; speed, power and acceleration are continuous there. Initially finite force gives zero power at zero speed, avoiding the fixed-power singularity. A work check is $2000(2.50) + 10000(4) = 45000$ J = $\frac{1}{2}(400)(15)^2$.

Problem 39: separate random and common calibration uncertainties

Trapezoidal weights are $\mathbf{w} = (0.05, 0.10, 0.10, 0.10, 0.05)$ m. The reported work is $0.05(5) + 0.10(7+9+11) + 0.05(13) = 3.60$ J. Correcting every force by -0.50 N subtracts $0.50 \sum w_j = 0.50(0.40) = 0.20$ J, leaving 3.40 J. For independent random sample errors,

$$u_{W,\text{random}} = 0.20\sqrt{0.05^2 + 3(0.10)^2 + 0.05^2} = 0.03742 \text{ J.}$$

The common offset contributes $u_{W,\text{cal}} = 0.10(0.40) = 0.0400$ J. As the two sources are stipulated independent, the combined standard uncertainty is

$$u_W = \sqrt{0.03742^2 + 0.0400^2} = 0.05477 \text{ J.}$$

Report approximately (3.40 ± 0.05) J as a standard-uncertainty result under this model. A common offset shifts every reading together, so its contributions are correlated; treating them as five independent errors would spuriously reduce its effect. This budget covers the stated force uncertainties, not an additional unknown interpolation or position error.

Problem 40: fixed anchor, changing centre-of-mass motion

Block A and its fixed anchor remain at rest, so actual wall work and spring work on A are both zero. Spring work on B is

$$W_{s,B} = \frac{1}{2}k_s(\delta_i^2 - 0) = \frac{1}{2}(200)(0.100)^2 = 1.0 \text{ J.}$$

Hence $K_B = 1.0$ J and $v_B = \sqrt{2(1.0)/1.0} = \sqrt{2}$ m s⁻¹ leftward. Because the masses are equal and $v_A = 0$, the centre-of-mass speed is $v_B/2$. Thus $K_{\text{CM}} = \frac{1}{2}(2.0)(v_B/2)^2 = 0.50$ J and $K_{\text{relative}} = 1.0 - 0.50 = 0.50$ J. With right positive, $F_{\text{wall}} = -k_s\delta$ and $dR_{\text{CM}} = dx_B/2 = d\delta/2$. The centre-of-mass pseudowork is therefore

$$\int_{\delta_i}^0 F_{\text{wall}} dR_{\text{CM}} = \int_{\delta_i}^0 -\frac{k_s\delta}{2} d\delta = \frac{1}{4}k_s\delta_i^2 = 0.50 \text{ J.}$$

It equals ΔK_{CM} although actual wall work is zero. The internal spring interaction has zero net force on the two-block subsystem yet does positive net work on its constituent blocks because their application points move differently. That internal work supplies the total 1.0 J increase in block kinetic energy; centre-of-mass pseudowork accounts for only its translational part. No wall-energy transfer or momentum-conservation assumption is needed.

Problem 41: frame transformation of a kinetic-energy change

Laboratory change is $\Delta K = \frac{1}{2}(2)(9^2 - 5^2) = 56$ J. In the moving frame the velocities are 2 and 6 m s⁻¹, giving $\Delta K' = \frac{1}{2}(2)(6^2 - 2^2) = 32$ J. Laboratory momentum change is $\Delta p = m(v_f - v_i) = 2(4) = 8.0$ kg m s⁻¹. Expand the transformed kinetic energy:

$$K' = \frac{1}{2}m(v - U)^2 = K - Up + \frac{1}{2}mU^2.$$

For constant m and U the last term cancels between endpoints, leaving $\Delta K' = \Delta K - U\Delta p = 56 - 3(8) = 32$ J. The product $U\Delta p$ has units of energy. Net force is the same under a Galilean inertial transformation, but the application-point displacement over a given time differs between frames, so net work changes consistently with kinetic-energy change.

Problem 42: line work from a time parameter

Differentiate the specified path: $\mathbf{v}(t) = (2.0 \text{ m s}^{-1})\hat{\mathbf{i}} + [(1.0 \text{ m s}^{-2})t]\hat{\mathbf{j}}$. The force power is therefore

$$\mathcal{P}(t) = \mathbf{F}(t) \cdot \mathbf{v}(t) = 8.0 \text{ W} + (3.0 \text{ W s}^{-2})t^2.$$

At 2.0 s it is 20.0 W, and integration gives $W = \int_0^2 (8 + 3t^2) dt = 16 + 8 = 24.0 \text{ J}$. Endpoint displacement is (4, 2) m and endpoint force is (4, 6) N, so the shortcut would give $4(4) + 6(2) = 28 \text{ J}$. It overestimates the changing vertical-force contribution by applying its final value throughout. This is selected-force work on a prescribed path; no claim is made that this force alone generates that motion. Only net force work can be equated with the corresponding particle kinetic-energy change.

Problem 43: compare three path integrals

Along A's horizontal segment $y = 0$, so $F_x = 0$ and work is zero. Along its vertical segment $x = 2 \text{ m}$, the upward force is $2ax = 8 \text{ N}$ through 1 m, giving $W_A = 8.0 \text{ J}$. For B set $(x, y) = (2s, s) \text{ m}$, $0 \leq s \leq 1$. Then $\mathbf{F} = (-2s, 8s) \text{ N}$ and $d\mathbf{r} = (2, 1) \text{ m ds}$, yielding $W_B = \int_0^1 4s ds = 2.0 \text{ J}$. For C use $y = x^2/(4 \text{ m})$ and $dy/dx = x/(2 \text{ m})$:

$$W_C = \int_0^{2 \text{ m}} \left[-ay + 2ax \frac{dy}{dx} \right] dx = \int_0^{2 \text{ m}} \frac{3ax^2}{4 \text{ m}} dx = 4.0 \text{ J}.$$

Thus $W_A > W_C > W_B$. A outward followed by B reversed has work $8 - 2 = 6.0 \text{ J}$. The reversal changes B's work sign because the same field is traversed oppositely. The unequal works and nonzero loop work are direct properties of the field, not numerical-integration errors or inconsistent endpoints.

Problem 44: friction remains negative on a return leg

Initial kinetic energy is $\frac{1}{2}(1)(3)^2 = 4.5 \text{ J}$. Outward resistance does $W_1 = -2(1) = -2.0 \text{ J}$, leaving $K_1 = 2.5 \text{ J}$ and $v_1 = \sqrt{5} = 2.24 \text{ m s}^{-1}$. The ideal reversal does zero work and preserves this speed. On the return the resistive force reverses to oppose the new velocity, so it again does $W_2 = -2.0 \text{ J}$. The final $K_2 = 0.50 \text{ J}$ gives speed $v_2 = 1.0 \text{ m s}^{-1}$, now in the return direction. Both legs are accessible because energy decreases linearly yet remains positive even at the final point. Total work is -4.0 J , not zero. Problem 6 assumed an unchanged position-only force; here velocity reversal changes the force at each position. The force and displacement both reverse on return, keeping their dot product negative.

Problem 45: terminal speed from a power balance

At $v = 10 \text{ m s}^{-1}$ the drive force is $\mathcal{P}/v = 600 \text{ N}$, while resistance is $cv^2 = 1000 \text{ N}$. Net force is -400 N and acceleration is $-400/500 = -0.80 \text{ m s}^{-2}$. Net power is $\mathcal{P} - cv^3 = 6000 - 10000 = -4000 \text{ W}$, consistent with deceleration. Terminal speed satisfies zero net power at positive speed:

$$v_t = (\mathcal{P}/c)^{1/3} = 600^{1/3} = 8.43 \text{ m s}^{-1}.$$

Below v_t net power is positive and speed increases; above v_t it is negative and speed decreases. This establishes stability in the forward finite-speed operating regime. Since v_t contains no mass, doubling mass would not change it under these same stipulated force and power laws. The larger mass would reduce acceleration at a given speed and alter the approach time. Startup still requires a force cap if the vehicle begins at rest.

Problem 46: infer two resisting parameters

For each trial, stopping requires the incident energy to equal the magnitude of spring and friction work: $E_i = \frac{1}{2}k_s q_i^2 + f q_i$. The two measured trials give

$$0.005k_s + 0.100f = 2.0, \quad 0.020k_s + 0.200f = 6.0,$$

with SI units understood. Subtract twice the first equation from the second to get $0.010k_s = 2.0$, hence $k_s = 200 \text{ N m}^{-1}$. Back-substitution gives $f = 10.0 \text{ N}$. Both positive parameter values are physically compatible with opposing compression. For $E = 8.0 \text{ J}$ solve

$$100q^2 + 10q - 8 = 0, \quad q = \frac{-10 + \sqrt{3300}}{200} = 0.237228 \text{ m}.$$

The negative root -0.337228 m is outside the compression domain. The energy decreases monotonically as $8 - 100q^2 - 10q$, so the accepted zero is the first reachable stopping point. If the compression values were identical, the coefficient rows would be identical and could not independently identify both parameters. Different energies with exactly identical compression would instead signal inconsistent data or an incomplete model. With finite measurement uncertainty, closely spaced compressions can also make parameter inference poorly conditioned.

7.13 Bridge to Chapter 8

This chapter treated work as a transfer calculated from force and displacement, and it used the work–kinetic-energy theorem to connect that transfer to a particle’s speed. The gravity and spring examples showed something suggestive: their work could be written using endpoint configurations rather than every detail of the intervening motion.

Chapter 8 develops that observation into a systematic framework. It introduces potential energy for appropriate interactions, identifies when a force can be represented by a configuration-dependent energy function, and builds mechanical-energy conservation under clearly stated conditions. The distinction made here between selected-force work, net work and actual application-point work remains essential. Energy conservation is not a replacement for careful system definition; it is a more structured accounting method built on it.

The extended-system warning also carries forward. For a deformable body, energy can enter through relative motion of force-application points even when the centre of mass does not accelerate. Later chapters will enlarge the bookkeeping to include internal, thermal and other forms of energy, but the geometric starting point remains the same: identify where forces act and how those points move.

Take-home message

Force answers how motion changes locally. Work answers how a specified force transfers energy through displacement. The work–kinetic-energy theorem links the total transfer to a change in speed. Knowing which of these questions you are asking is often more important than the algebra used to answer it.

References and further reading

- [1] D. Halliday, R. Resnick and J. Walker, *Fundamentals of Physics*, 12th edition, Volume 1, John Wiley & Sons, 2022. Chapter 7, “Kinetic Energy and Work,” with Chapter 8, Section 8.4 consulted for system-work distinctions.
- [2] R. A. Serway and J. W. Jewett, Jr., *Physics for Scientists and Engineers with Modern Physics*, 9th edition, Cengage Learning, 2014. Chapter 7, especially Sections 7.2–7.5; Section 8.5 for power; Section 9.8 for deformable-system energy accounting.
- [3] S. J. Ling, J. Sanny and W. Moebs, *University Physics, Volume 1*, OpenStax, Rice University, 2018. Chapter 7, Sections 7.1–7.4, “Work and Kinetic Energy.”