

Chapter 8

Potential Energy and Conservation of Energy

First Edition

Hamzeh Khanpour

hamzeh.khanpour@agh.edu.pl

Faculty of Physics and Applied Computer Science

AGH University of Krakow

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Preface

Chapter 7 introduced work as a force-dependent energy transfer and established the work-kinetic-energy theorem. This chapter develops the next step: for an important class of interactions, the work can be represented by a change in a configuration-dependent quantity called potential energy. Once the system has been defined carefully, this representation makes energy conservation a powerful way to connect states without reconstructing every intermediate acceleration.

The word *conservation* needs care. Mechanical energy is not always conserved, and energy does not disappear when mechanical energy decreases. Frictional contact, deformation, electrical heating and other irreversible processes can increase internal energy. Likewise, energy may cross a system boundary through external work or heating. A correct energy balance therefore begins with the same question that underlies all good mechanics: *what is the system?*

This chapter uses three complementary pictures. The force picture asks how forces vary with position. The potential picture replaces a conservative force by an energy landscape. The system picture tracks energy storage and transfer across an explicitly drawn boundary. Switching among these pictures is often more useful than memorizing a long list of energy formulas.

The exposition, examples, exercises and diagrams are original to this manuscript. The references at the end provide complementary treatments of standard definitions, derivations and applications. The exposition distinguishes the stated physical models from the assumptions needed to use them.

The Guideline

Before writing an energy equation, draw or state the system boundary. List the energy stores inside it, identify transfers across it, choose a reference for every potential energy, and state the initial and final states. Only then select a conservation or balance equation.

Pedagogical boundary

The core develops conservative forces, potential energy, mechanical-energy conservation, one-dimensional energy landscapes, system-boundary accounting and irreversible conversion to internal energy. Thermodynamic temperature, heat capacities and microscopic equations of state belong to Chapter 18. Full oscillation dynamics belongs to Chapter 15. Multivariable potential theory is reserved for the advanced pathway and is introduced only as far as needed here.

Conventions and notation

Vectors are bold, as in $\mathbf{F}$ and $\mathbf{r}$. Work done by a conservative force on the chosen moving object is W_c . Potential energy U belongs to an interaction represented inside the chosen system; it is not an intrinsic property of one isolated object. The mechanical energy of the modeled system is $E_{\text{mech}} = K + U$. Internal energy is E_{int} . When thermal-language signs are needed, $Q > 0$ means energy transferred into the system by heating and $W_{\text{by}} > 0$ means work done by the system; equivalently $W_{\text{on}} = -W_{\text{by}}$.

Symbol	Meaning	SI unit
K	kinetic energy in the named inertial frame	J
U	potential energy for an included conservative interaction	J
E_{mech}	mechanical energy, $K + U$ for the model used here	J
E_{int}	internal energy of matter included in the system	J
W_c	work done by a specified conservative force	J
$W_{\text{ext,on}}$	net external work done on the chosen system	J
Q	energy transferred into the system by heating	J
k_s	ideal spring stiffness	N m ⁻¹
U_*, ℓ	energy and length scales in the stipulated double-well model	J, m
E_{diss}	non-negative mechanical-energy conversion in a stated dissipative model	J

Potential-energy reference values are arbitrary constants. Every comparison must use one reference convention consistently. A force obtained from a one-dimensional potential obeys $F_x = -dU/dx$, so adding a constant to U changes neither the force nor the motion.

For numerical teaching examples we use $g = 9.81 \text{ m s}^{-2}$. Values designated as “stipulated” or “synthetic” are instructional model inputs, not measurements. Intermediate digits are retained to avoid rounding drift; they do not imply extra experimental precision.

Note

Core prerequisites are the work–kinetic-energy theorem from Chapter 7, single-variable differentiation and integration, and the force models already developed in mechanics. The advanced section introduces a local Taylor expansion and the gradient/curl language with its mathematical conditions before using it.

Chapter 8

Potential Energy and Conservation of Energy

A maintenance carriage moves between two points on a hillside. One route climbs steeply and then follows a level ledge; another follows a smooth diagonal ramp. If air drag and rolling losses are neglected, does gravity do different work on the two routes because their lengths differ? The answer is no, but the reason matters more than the number. Gravity near Earth's surface is a conservative interaction: its work between two endpoints depends on the endpoint heights, not on the route taken.

That observation is the gateway to potential energy. Instead of carrying the gravitational force integral through every calculation, we can represent the same interaction by a difference in gravitational potential energy. The same strategy works for an ideal spring and many other conservative interactions. It does *not* work for arbitrary dissipative processes, and it never eliminates the need to define the system.

Learning goals

After studying the core, you should be able to:

- decide whether a stated force model is being treated as conservative and derive $\Delta U = -W_c$ with the correct sign;
- explain why potential-energy reference values are arbitrary and show that a reference shift cannot change a force, speed prediction or turning point;
- apply mechanical-energy conservation only when its conditions are satisfied;
- read allowed regions, turning points, force directions and equilibrium stability from a one-dimensional potential-energy curve;
- write consistent energy balances for different system boundaries describing the same physical process;
- distinguish a decrease of mechanical energy from a loss of total energy and represent irreversible conversion by an increase of internal energy;
- validate an energy calculation using units, limiting cases, non-negative kinetic energy and an independent force or work check.

8.1 Conservative forces and potential differences

8.1.1 Path independence and closed paths

A force is called *conservative on a stated domain* when the work it does between two points is independent of the path connecting them. If paths C_1 and C_2 both lead from a to b ,

$$\int_{C_1} \mathbf{F}_c \cdot d\mathbf{r} = \int_{C_2} \mathbf{F}_c \cdot d\mathbf{r}. \quad (8.1)$$

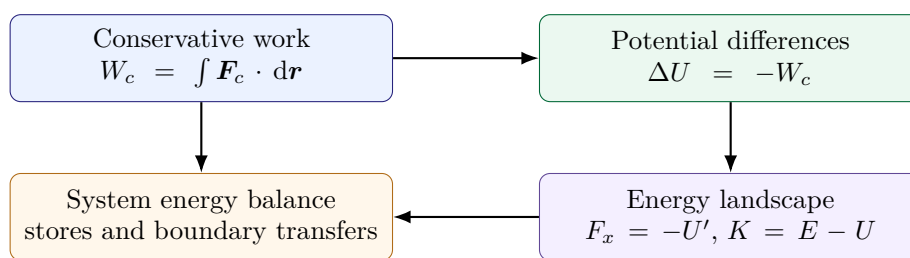


Figure 8.1: Four complementary representations. The conservative-force calculation supplies a potential difference; the potential supplies force and reachability information. Every representation is used within an explicitly chosen system energy balance. This is a conceptual diagram, not a physical energy-flow map.

Reversing C_2 changes the sign of its line integral. Joining C_1 to the reversed C_2 therefore makes a closed path, and Eq. (8.1) is equivalent to

$$\oint_C \mathbf{F}_c \cdot d\mathbf{r} = 0 \quad (8.2)$$

for every admissible closed path in the domain. The word *every* is essential. Finding one particular closed route whose work happens to be zero does not prove that a force is conservative.

Definition 8.1: Potential-energy difference

For a conservative interaction represented inside the chosen system, define the potential-energy difference between configurations a and b by

$$U_b - U_a = - \int_a^b \mathbf{F}_c \cdot d\mathbf{r} = -W_{c,a \rightarrow b}. \quad (8.3)$$

The minus sign means that positive work done by the conservative force is accompanied by a decrease of the associated potential energy.

The definition is endpoint-only because the conservative work is path independent. Reversing the motion gives $W_{c,b \rightarrow a} = -W_{c,a \rightarrow b}$ and hence $U_a - U_b = -(U_b - U_a)$, as required. Potential energy is assigned to the *interaction within the system*. For near-Earth gravity we may casually say “the object’s gravitational potential energy,” but the physical interaction is the object–Earth interaction.

Caution

Do not infer conservativity from dimensional form, from a force being position-dependent, or from one zero-work loop. A path-independence statement must refer to the complete force model and its domain. Time-dependent forcing, frictional contact or a domain with excluded singular regions requires separate analysis.

Example 8.1: compare two paths in a uniform gravitational field

A 2.00 kg carriage moves from $A = (0, 1.0 \text{ m})$ to $B = (3.0 \text{ m}, 4.0 \text{ m})$. Path 1 consists of a vertical segment followed by a horizontal segment; path 2 is a direct ramp. Treat $\mathbf{g} = -g\hat{\mathbf{j}}$ with $g = 9.81 \text{ m s}^{-2}$. Find the gravitational work on each path and the corresponding change in gravitational potential energy.

Model. The carriage–Earth interaction is represented inside the system; g is uniform and air/rolling losses are omitted from this particular work calculation.

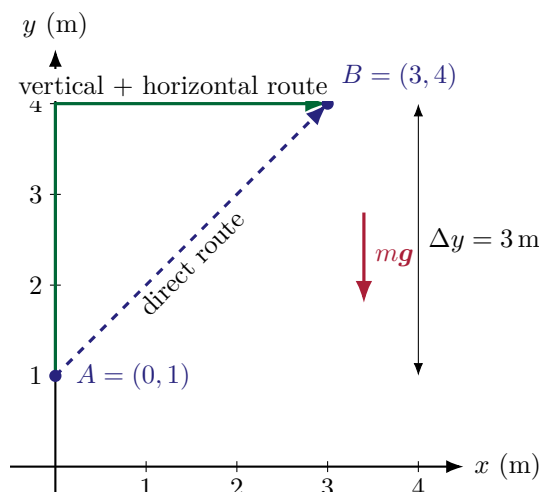


Figure 8.2: Two routes with the exact endpoints of Example 8.1. Gravity points downward and its work is $-mg\Delta y$ for either route. Coordinates and route geometry share one metre scale in both directions; the force-arrow length is schematic.

For any path,

$$W_g = \int_A^B (-mg\hat{j}) \cdot (dx\hat{i} + dy\hat{j}) = -mg \int_{y_A}^{y_B} dy.$$

Thus

$$W_g = -mg(y_B - y_A) = -(2.00)(9.81)(3.00) = -58.86 \text{ J.} \quad (8.4)$$

On path 1 the horizontal segment contributes zero and the vertical segment contributes -58.86 J . On path 2 the same line integral gives the same result. Therefore

$$\Delta U_g = -W_g = +58.86 \text{ J.}$$

Reversing either path changes W_g to $+58.86 \text{ J}$ and ΔU_g to -58.86 J .

Contrast. If a separate idealized resistive force of constant magnitude f always opposes motion, its work is $-f$ times the traveled path length. Two routes of different lengths then give different resistive work. That path dependence belongs to the stated resistive model; it is not a contradiction of the gravitational result.

Guided checks

Diagnostic X08-01. A student walks out along an arbitrary curve and returns exactly along the same curve. The work of a position-dependent force is zero for that out-and-back path. Does this prove that the force is conservative? *No. Reversing the same curve cancels its own line integral for any single-valued position force. Conservativity requires zero work for every admissible closed path, or an equivalent path-independence test.*

8.1.2 From incremental work to a potential function

The endpoint definition becomes useful in calculations when it is connected to the work of a force over a short displacement. Consider motion on a straight guide, with coordinate x increasing to the right. Suppose the conservative force component is a single-valued function $F_x(x)$ and has no explicit dependence on speed, the direction of travel, or time. A small positive displacement dx gives conservative work $dW_c = F_x(x)dx$. The associated potential change is therefore $dU = -F_x(x)dx$.

Dividing by dx and taking the derivative gives

$$F_x(x) = -U'(x), \quad U(x) = U(x_0) - \int_{x_0}^x F_x(\xi) d\xi. \quad (8.5)$$

The symbol ξ is a dummy integration coordinate. It prevents the upper limit x , which specifies the configuration of interest, from being confused with the variable being accumulated inside the integral. The reference value $U(x_0)$ fixes the integration constant; it is not an additional force parameter.

These two operations answer different questions. Integration determines the accumulated potential change over a finite interval. Differentiation determines the force at one position from the local slope of a potential graph. A force is positive where U decreases as x increases, and negative where U increases. Neither the sign of U itself nor the steepness of a distant part of the graph determines the local force. The units also distinguish the operations: a force–position area has units $\text{N m} = \text{J}$, whereas a potential slope has units $\text{J m}^{-1} = \text{N}$.

A continuous position force on a connected one-dimensional interval always admits the construction in Eq. (8.5). This statement is about that complete one-dimensional model. It does not prove that a real mechanism has no friction, hysteresis, time-dependent actuation, or coupling to a second coordinate. For example, the same position may have different forces while loading and unloading a deformable component. One cannot describe both branches by a single $F_x(x)$ and one position-only potential. The branch information is physically relevant even if each individual curve can be integrated.

The word “connected” also matters in using a reference value. If the stated domain comprises disconnected intervals and no permitted path joins them, force integration within one interval does not fix the reference constant in the other. Within the usual connected guide considered here, one reference and the force law determine all potential differences. More general multidimensional domains are treated in the advanced discussion.

The differential relation also explains how a finite potential difference approaches a force measurement. If U is twice differentiable near x , Taylor expansion gives

$$U(x+h) - U(x) = U'(x)h + \frac{1}{2}U''(x)h^2 + \cdots = -F_x(x)h - \frac{1}{2}F'_x(x)h^2 + \cdots.$$

Thus $-[U(x+h) - U(x)]/h$ estimates the force near the interval only when the interval is sufficiently small or the force is nearly constant. A long secant on a potential graph represents an average force; it need not represent the force at either endpoint. The correction term states the source of the discrepancy. A rapidly varying force requires a smaller interval for the same absolute accuracy.

For a centered interval, the estimate $-[U(x+h) - U(x-h)]/(2h)$ cancels the leading asymmetric correction and approximates the force at the midpoint. This is a mathematical advantage, rather than a claim that arbitrarily small intervals always improve experimental results. If the potential values have measurement uncertainty, taking a difference and dividing by a small distance can amplify that uncertainty. Force reconstruction from measured energy values must therefore balance spatial resolution with measurement precision. In the exact analytic models used in this section, differentiation has no such measurement limitation.

There is a parallel distinction between a force being conservative and its work being zero in a particular process. A nonzero conservative force can do zero work along a displacement perpendicular to it. It can also do zero net work along a route containing compensating positive and negative contributions. Conversely, a force can do substantial conservative work along an open path. Conservativity describes the endpoint dependence of the complete work law, not the size of one observed energy transfer.

Example 8.2: A variable force changes sign

An ideal actuator produces the position force

$$F_x(x) = a - bx, \quad a = 12.0 \text{ N}, \quad b = 6.00 \text{ N m}^{-1},$$

on a guide over $0 \leq x \leq 3.00 \text{ m}$. Treat this prescribed force as time independent and conservative on that interval. Given $U(0) = 4.00 \text{ J}$, find the potential, the work from $x = 0.500 \text{ m}$ to $x = 3.00 \text{ m}$, and the location where the force changes sign. Explain why a single average-force estimate can be misleading.

Strategy and model. Construct the potential from the integral of the stated force. No assumption about constant speed or about the net force is needed to calculate this actuator's work. Other guide forces, if present, are not part of this particular potential.

Integration gives

$$U(x) = 4.00 \text{ J} - ax + \frac{b}{2}x^2.$$

Differentiation recovers $-U'(x) = a - bx$, providing a sign and coefficient check. The force vanishes at $x = a/b = 2.00 \text{ m}$; it pushes right on the left of this position and left on its right. The potential has its minimum there, with $U(2.00 \text{ m}) = -8.00 \text{ J}$.

At the requested endpoints,

$$U(0.500 \text{ m}) = -1.25 \text{ J}, \quad U(3.00 \text{ m}) = -5.00 \text{ J}.$$

Consequently $\Delta U = -3.75 \text{ J}$ and $W_c = +3.75 \text{ J}$. The force is negative during the final part of the rightward motion, yet the total work is positive. Splitting at the zero-force position makes the competing contributions explicit:

$$W_{0.500 \rightarrow 2.00} = +6.75 \text{ J}, \quad W_{2.00 \rightarrow 3.00} = -3.00 \text{ J}.$$

Their sum is $+3.75 \text{ J}$, agreeing with the endpoint potential difference.

Check and interpretation. Figure 8.3 shows both signs of the work area and the corresponding potential slopes. Because this force is linear, its exact average on the full interval equals the average of the endpoint forces: $(9.00 - 6.00)/2 = 1.50 \text{ N}$. Multiplying by 2.50 m correctly gives 3.75 J . Using only the initial force would instead give 22.5 J and overlook both the decreasing force and its reversal. Endpoint averaging is exact here because of linearity; it is not a general rule for a curved force graph.

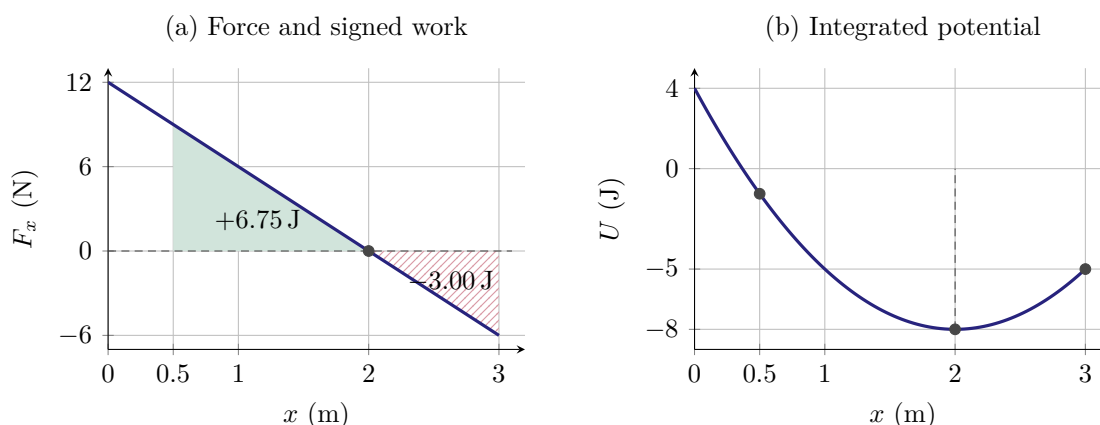


Figure 8.3: Matched force and potential for the affine model in Example 8.2. The shaded areas apply only to the motion from 0.500 to 3.00 m ; their signed sum is 3.75 J . The potential decreases where the force is positive and increases where it is negative. Curve heights have different units in the two panels.

8.1.3 Reading areas and slopes without interchanging them

An integral is a signed accumulation. For motion from a smaller to a larger x , the area above the force axis contributes positive work and the area below it contributes negative work. Reversing the integration limits changes the sign of the whole result. The graph does not acquire new forces when the body returns; the displacement changes sign. A force that opposes velocity instead behaves differently on the return journey, so its work cannot be inferred by reversing a fixed position-force graph.

The potential graph contains the same information in a different representation. Equal potentials at two positions imply zero net conservative work between those positions. They do not imply zero force throughout the interval, equal forces at the endpoints, or an absence of kinetic-energy exchange along the way. The potential can fall and then rise to its original value. Its local slopes, rather than its endpoint heights, explain the intermediate force directions.

When a force is specified by straight graph segments, geometric areas can provide an exact calculation without a long integral. The potential generally consists of curved segments because integrating a line produces a quadratic function. A corner in a continuous force graph makes the potential curvature change abruptly; it does not make the potential itself discontinuous. A jump in a finite force would instead produce a jump in the potential slope, hence a corner in the potential. Such distinctions are useful when a plotted model joins two operating regimes.

Example 8.3: Equal potential values, unequal forces

A conservative guide force is specified by the graph in Fig. 8.4. In SI units it is $F_x = 4.00 \text{ N}$ for $0 \leq x \leq 1.00 \text{ m}$ and $F_x = 8.00 \text{ N} - (4.00 \text{ N m}^{-1})x$ for $1.00 \leq x \leq 3.00 \text{ m}$. Set $U(0) = 0$. Find the work from 0 to 3.00 m, construct $U(x)$, and explain the motion from 1.00 to 3.00 m using both graphs.

Geometric calculation. The first segment contributes the rectangle 4.00 J. From 1.00 to 2.00 m the positive triangular area is 2.00 J. From 2.00 to 3.00 m the negative triangular area is -2.00 J . Thus $W_{0 \rightarrow 3} = 4.00 \text{ J}$ and $U(3.00 \text{ m}) = -4.00 \text{ J}$.

Continuous potential. Let $q = x/(1 \text{ m})$ be the numerical coordinate in metres. Integrating each branch and matching at $q = 1$ gives

$$\frac{U(x)}{1 \text{ J}} = \begin{cases} -4q, & 0 \leq q \leq 1, \\ 2q^2 - 8q + 2, & 1 \leq q \leq 3. \end{cases}$$

Both expressions give -4.00 J at the join. Their first derivatives also agree there: $U' = -4.00 \text{ N}$. The minimum is $U(2.00 \text{ m}) = -6.00 \text{ J}$, where the force is zero.

Between 1.00 and 3.00 m the work is zero because the two triangular areas cancel. Equivalently, $U(1.00 \text{ m}) = U(3.00 \text{ m}) = -4.00 \text{ J}$. Yet $F_x(1.00 \text{ m}) = +4.00 \text{ N}$ and $F_x(3.00 \text{ m}) = -4.00 \text{ N}$. If this is the only force that changes a particle's kinetic energy, the particle gains 2.00 J of kinetic energy on the first half and loses the same amount on the second. Equal final and initial kinetic energies allow substantial intermediate variation.

Check. Differentiating the second branch gives $-U' = 8.00 \text{ N} - (4.00 \text{ N m}^{-1})x$, exactly the stipulated force. The negative potential values follow from the chosen zero and do not indicate a negative kinetic energy.

8.1.4 A decisive counterexample to path independence

In more than one dimension, dependence on position does not by itself produce a potential. Different routes can sample different force components. A single nonzero closed-loop work is enough to disprove conservativity on a domain containing that loop. By contrast, one zero loop cannot establish it. This logical asymmetry makes a simple rectangle particularly useful: its straight sides

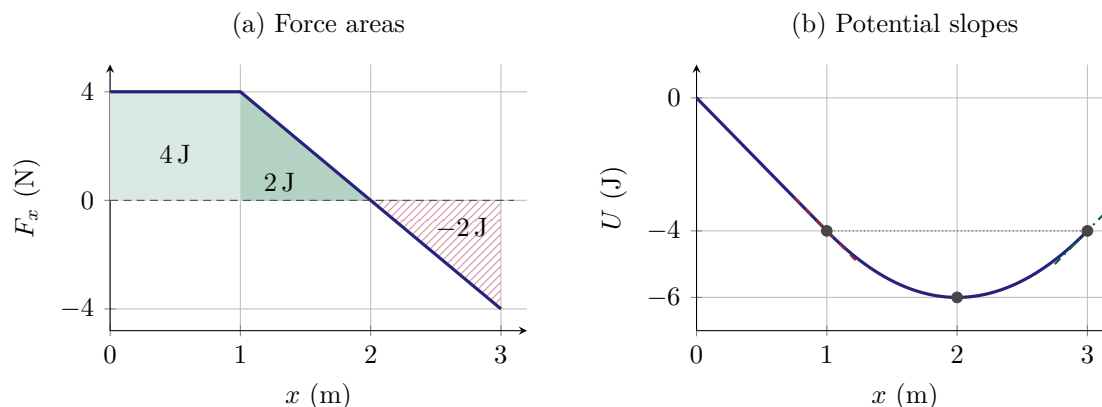


Figure 8.4: The constant-then-linear force and its continuous potential in Example 8.3. The equal potentials at $x = 1.00$ and 3.00 m give zero net work between those positions, while the short dashed tangent segments show opposite slopes and therefore opposite forces. The point at $x = 2.00$ m is a potential minimum.

isolate horizontal and vertical work without requiring advanced vector calculus.

Example 8.4: A position force with path-dependent work

On the whole xy plane, let $\mathbf{F} = \alpha y \hat{\mathbf{i}}$ with $\alpha = 3.00 \text{ N m}^{-1}$. Compare the work from $A = (0, 0)$ to $B = (2.00 \text{ m}, 1.50 \text{ m})$ by the two routes in Fig. 8.5. Route I moves right along the bottom and then up. Route II moves up along the left and then right along the top. Find the work around the loop that follows I outward and II backward, and decide whether a single potential exists.

Resolve the dot product. Here $\mathbf{F} \cdot d\mathbf{r} = \alpha y dx$. Every vertical segment has $dx = 0$, so this force does no work there. On the bottom side $y = 0$ and the force itself vanishes. Route I therefore gives $W_I = 0$.

On the top side, $y = 1.50 \text{ m}$, so the force is the constant 4.50 N to the right. Route II gives

$$W_{II} = (4.50 \text{ N})(2.00 \text{ m}) = 9.00 \text{ J}.$$

Following I and then reversing II produces

$$\oint \mathbf{F} \cdot d\mathbf{r} = W_I - W_{II} = -9.00 \text{ J}.$$

No single endpoint value $U_B - U_A$ can equal both $-W_I = 0$ and $-W_{II} = -9.00 \text{ J}$. The force is thus nonconservative on the plane, despite being time independent and depending only on position.

Check and limitation. For a rectangle of width L and height H in this orientation, the result is $-\alpha LH$, with units of joules. Reversing the loop gives $+\alpha LH$. Retracing route I exactly would instead give zero because the same line integral cancels on reversal. That different zero-work loop does not rescue the failed path-independence test. This prescribed force need not be friction: nonconservative describes its work property, rather than naming a specific dissipation mechanism.

The rectangle result does not contradict energy conservation. A prescribed nonconservative force is a model of an interaction whose work cannot be captured by a configuration-only potential on this plane. If an apparatus generates such a force, a full energy audit must include the energy that the apparatus supplies or receives. Returning the guided particle to its initial coordinates need not return the whole apparatus to its initial state. The impossibility established here is precise: there is no single-valued $U(x, y)$ whose endpoint difference reproduces this field's work for all routes in the

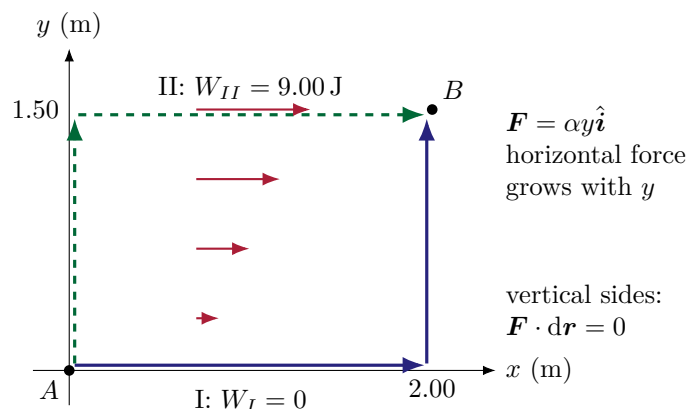


Figure 8.5: Two routes in the prescribed force field of Example 8.4. Solid and dashed lines distinguish routes even in grayscale. The field arrows indicate direction and relative magnitude; they are not motion arrows. The top horizontal segment does positive work on the outward route II, and negative work when traversed backward in the closed loop.

stated domain.

A different question is whether the force model might become conservative when motion is restricted to a particular line. Along a fixed horizontal guide at $y = H$, the force is constant and a one-dimensional potential $U_H(x) = -\alpha Hx + C_H$ can be assigned. That restricted construction works for motions that stay on that guide. It cannot be transferred to a route that changes y , because the arbitrary constants on separate guides do not supply the missing vertical-force consistency. This distinction explains why a successful one-dimensional calibration is not sufficient evidence for a conservative multidimensional model.

8.1.5 Reconstructing a potential from force information

An inverse calculation starts with force information and constructs the potential, instead of starting with a named potential and differentiating it. This is useful for a calibrated elastic element or for a simulated conservative interaction. The reconstruction requires both a reference value and a declared rule between data points. Samples alone do not uniquely determine a continuous force curve. Infinitely many smooth curves pass through the same points and can have different areas between them.

A piecewise-linear interpolation is a transparent benchmark. The work in an interval of width h is exactly the trapezoid $h(F_i + F_{i+1})/2$ for that interpolated model. If the actual force is curved, the same expression is an approximation whose adequacy requires denser sampling or an error estimate. The potential reconstructed from continuous piecewise-linear force is piecewise quadratic and has continuous first derivative. Joining the reconstructed energy values by straight lines would produce a different derivative, and thus a different force model.

The reconstruction should also retain the data domain. In the benchmark below, the force is defined only between the first and last sample positions. The potential can be constructed everywhere within that interval, but the same formula makes no physical prediction beyond it without an additional extrapolation assumption. A plotted curve should not quietly extend a fitted force law into a range in which contact may be lost, a material may yield, or an actuator may saturate. At the introductory level, stopping the graph at the stated interval is often the most accurate way to communicate this limit.

For genuine measurements, repeated loading and unloading would test whether a position-only force law is adequate. A reproducible difference between the two branches would require the force model to include its loading history. A smooth potential reconstructed from one branch could still be a

useful conditional representation, but it would not describe the complete round trip. Conversely, small differences consistent with measurement uncertainty would justify a stated approximation, rather than prove exact reversibility. In the synthetic benchmark all interpolated values are known by construction, so these experimental questions are replaced by an explicit model declaration.

Example 8.5: Synthetic force samples define an explicit benchmark

A time-independent guide force is defined by linear interpolation between the following synthetic values. They are stipulated model data, not experimental measurements.

x (m)	0	0.500	1.00	1.50	2.00
F_x (N)	6.00	4.00	1.00	-3.00	-5.00

Set $U(0) = 2.00$ J. Reconstruct the potential at all sample positions, find the work from 0.500 to 1.50 m, and locate the potential minimum between 1.00 and 1.50 m.

Integrate the stated interpolation. Each interval has width 0.500 m. Its signed work is, in order,

$$2.50 \text{ J}, \quad 1.25 \text{ J}, \quad -0.500 \text{ J}, \quad -2.00 \text{ J}.$$

Subtracting the accumulated work from the reference value gives

x (m)	0	0.500	1.00	1.50	2.00
U (J)	2.00	-0.500	-1.75	-1.25	0.750

The work between 0.500 and 1.50 m is $1.25 - 0.500 = 0.750$ J, consistent with $-[U(1.50) - U(0.500)]$.

The force crosses zero inside the third interval. Write $d = x - 1.00$ m. There,

$$F_x = 1.00 \text{ N} - (8.00 \text{ N m}^{-1})d,$$

so $d = 0.125$ m at the zero. Integrating from $x = 1.00$ m gives

$$U = -1.75 \text{ J} - (1.00 \text{ N})d + (4.00 \text{ N m}^{-1})d^2.$$

The minimum occurs at $x = 1.125$ m, with $U = -1.8125$ J. The minimum lies below both neighboring sample values; simply inspecting the table would miss it.

Check and interpretation. The full signed work is 1.25 J, so $U(2.00) = 2.00 - 1.25 = 0.750$ J. Figure 8.6 uses the same linear-force, quadratic-potential interpolation. A force-sampling table in a multidimensional or hysteretic experiment would not by itself prove conservativity. This example is exact because the one-dimensional conservative benchmark and its interpolation have been explicitly stipulated.

8.2 Reference values and familiar potentials

8.2.1 Why the zero of potential energy is a choice

Equation (8.3) determines differences of potential energy. If $U(x)$ represents a conservative interaction, then

$$\tilde{U}(x) = U(x) + C$$

represents the same interaction for any constant C . The force is unchanged because the derivative of C vanishes. Energy equations remain physically identical provided the *same* reference convention is used for every state in one calculation.

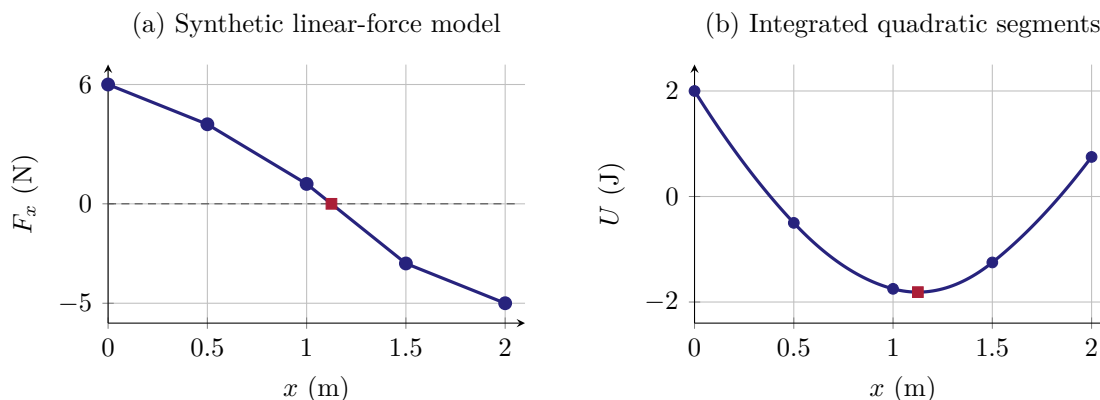


Figure 8.6: A potential reconstructed from the declared synthetic force samples in Example 8.5. Circles identify samples and their accumulated potential values; squares identify the force zero and the potential minimum inside an interval. The energy curve integrates the straight force segments exactly and is not a straight-line interpolation of the energy samples.

For uniform gravity near Earth's surface,

$$U_g(y) = mgy + C_g, \quad (8.6)$$

and for an ideal spring with signed extension x from its relaxed length,

$$U_s(x) = \frac{1}{2}k_s x^2 + C_s. \quad (8.7)$$

The corresponding forces follow from the one-dimensional relation

$$F_x = -\frac{dU}{dx}. \quad (8.8)$$

Thus U_s gives $F_s = -k_s x$, and U_g gives $F_y = -mg$ when y is positive upward.

Theoretical point 8.1: Reference invariance

Adding a constant C to a potential shifts every total mechanical-energy level by the same constant. It changes the numerical label attached to the energy axis, but it cannot change $F = -U'$, $K(x) = E - U(x)$, a turning position, or a speed prediction.

Example 8.6: one spring trajectory, two energy zeros

An ideal spring has $k_s = 200 \text{ N m}^{-1}$. A trajectory has mechanical energy $E_A = 1.80 \text{ J}$ when the relaxed spring is assigned $U_A(0) = 0$. Describe the identical trajectory using instead $U_B(0) = 5.00 \text{ J}$.

With reference A,

$$U_A(x) = 100x^2 \text{ J}, \quad K_A(x) = 1.80 - 100x^2 \text{ J}.$$

Turning points satisfy $K_A = 0$:

$$|x_t| = \sqrt{\frac{2E_A}{k_s}} = 0.134164 \text{ m}. \quad (8.9)$$

With reference B,

$$U_B(x) = 100x^2 + 5.00 \text{ J}, \quad E_B = 6.80 \text{ J}.$$

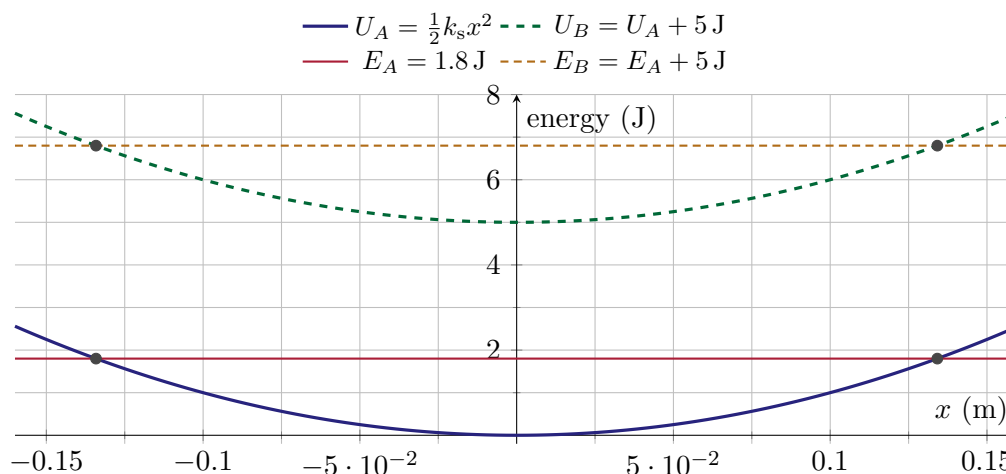


Figure 8.7: The same ideal spring under two potential-energy reference choices, with $k_s = 200 \text{ N m}^{-1}$. Both the potential curve and the total-energy line shift upward by 5 J, so their vertical separation $K = E - U$ and the turning positions remain unchanged.

Then

$$K_B(x) = E_B - U_B(x) = 1.80 - 100x^2 = K_A(x).$$

Moreover,

$$-\frac{dU_A}{dx} = -200x = -\frac{dU_B}{dx}.$$

Both descriptions predict the same force, kinetic energy and turning points. Only the numerical energy origin changed.

Caution

A common error is to shift U but leave the numerical total energy E unchanged. That describes a different physical state. A reference change must shift every energy value containing that potential by the same constant.

Guided checks

Diagnostic X08-02. Two plots differ only by $U_2(x) = U_1(x) - 7 \text{ J}$. A student says the second spring is “weaker” because its graph is lower. *Repair: compare slopes, not heights. The derivatives are identical, so the force law is identical. Shift the total-energy line by the same -7 J and all kinetic energies and turning points remain unchanged.*

8.2.2 Deriving familiar potentials with their signs intact

The familiar spring and gravitational formulas are applications of force integration, rather than separate definitions to memorize. For an ideal spring, let x denote signed displacement of its free end from the relaxed configuration. Hooke’s law is $F_s = -k_s x$, with $k_s > 0$. Integrating from $x = 0$ gives

$$U_s(x) - U_s(0) = - \int_0^x (-k_s \xi) d\xi = \frac{1}{2} k_s x^2.$$

The squared displacement accounts for both stretching and compression. In either case the potential rises as the magnitude of deformation rises. During compression, $x < 0$ and the spring force is positive; during extension, $x > 0$ and the spring force is negative. Those opposite force directions

are encoded by opposite slopes of the same parabola.

A displacement $dx > 0$ from a compressed state can reduce the deformation magnitude, so the spring does positive work even though the spring is still compressed. By contrast, increasing x on the extended side increases the deformation and gives negative spring work. Phrases such as “moving right stores energy” are therefore incomplete: one must specify which side of the relaxed configuration the spring occupies.

The ideal model assumes a linear, reversible force law over the stated deformation range. A tension-only cord can become slack under compression; its force law is then different on that branch. A buckling or permanently deformed component also needs a different model. Calling a real component a spring does not establish that $\frac{1}{2}k_s x^2$ remains valid under every deformation. Within the stated ideal model, however, compression and extension are described by one signed coordinate and one potential.

The work of a controlling agent must also be distinguished from the spring’s work. If the end is moved quasistatically, with negligible change in kinetic energy and no other force doing work, the agent balances the spring and does $W_{\text{agent}} = -W_s = \Delta U_s$. This statement refers to the slow-motion limit. For a rapid push, some external work may increase kinetic energy as well, so equating all agent work to stored spring energy would require additional justification.

A spring constant connects a force scale to a displacement scale. Its units must therefore be force per length, and the energy coefficient $k_s/2$ multiplies a squared length. This is a useful dimensional check when a spring calibration and a displacement measurement use different length units. Converting the displacement alone while leaving the numerical spring constant in incompatible units changes the physical model. A reliable calculation converts the spring constant and all displacements to a common unit system before squaring the displacement.

It is also useful to distinguish the amount of deformation energy relative to the relaxed state from the numerical value under an arbitrary energy reference. Their relation is $U_s(x) - U_s(0) = \frac{1}{2}k_s x^2 \geq 0$. The right-hand side is independent of the additive constant. A shifted potential may have a negative numerical value at a deformed state while its difference from the relaxed state is positive. Only that difference answers how much recoverable energy the ideal spring can release while returning to its relaxed configuration with the other conditions held fixed.

Example 8.7: Compression, extension, and a shifted spring zero

An ideal spring with $k_s = 160 \text{ N m}^{-1}$ is moved slowly from $x_i = -0.0750 \text{ m}$ to $x_f = +0.0450 \text{ m}$. The coordinate is measured from the relaxed length. Find the initial and final spring forces, the spring work, and the controlling agent’s work. Repeat the potential accounting with the reference $\tilde{U}(0.0500 \text{ m}) = 0$.

Model and strategy. Treat the spring as reversible and linear in both compression and extension. The motion is quasistatic and other work and energy stores are negligible. Evaluate endpoint potential differences, then separate the passage through $x = 0$ to interpret the work signs.

The endpoint forces are

$$F_i = -k_s x_i = +12.0 \text{ N}, \quad F_f = -k_s x_f = -7.20 \text{ N}.$$

With $U_s(0) = 0$, the endpoint potentials are $U_i = 0.450 \text{ J}$ and $U_f = 0.162 \text{ J}$. Therefore

$$\Delta U_s = -0.288 \text{ J}, \quad W_s = +0.288 \text{ J}, \quad W_{\text{agent}} = -0.288 \text{ J}.$$

The negative net agent work means that the ideal slow controller receives energy overall. From the compressed state to the relaxed state, the spring releases 0.450 J and the agent does

-0.450 J . From the relaxed state to the final extension, the agent supplies 0.162 J . These contributions sum to -0.288 J .

At $x = 0.0500 \text{ m}$, the original potential is 0.200 J , so the alternative potential is

$$\tilde{U}_s(x) = \frac{1}{2}k_s x^2 - 0.200 \text{ J}.$$

Its endpoint values are $\tilde{U}_i = 0.250 \text{ J}$ and $\tilde{U}_f = -0.0380 \text{ J}$. Their difference is again -0.288 J . The force law and both works are unchanged.

Check. The derivative of the reference offset is zero. The negative final potential in the shifted convention cannot mean that the spring has negative deformation energy relative to its relaxed state: $\tilde{U}_s(x_f) - \tilde{U}_s(0) = 0.162 \text{ J}$ remains positive. Figure 8.8 distinguishes the force signs from the common upward curvature of the spring potential.

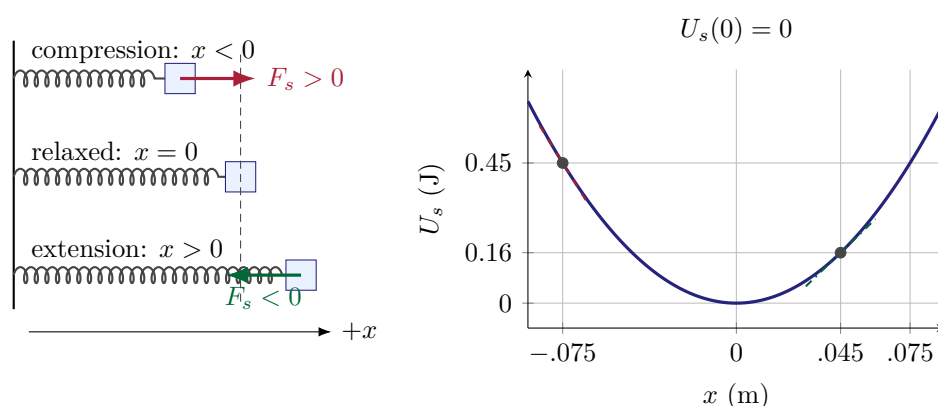


Figure 8.8: Signed deformation of an ideal spring. The left sketches show force directions in compressed, relaxed, and extended states; their lengths are schematic. The right panel is quantitative for the 160 N m^{-1} spring in Example 8.7; tangent segments show negative slope in compression and positive slope in extension. Both directions of deformation raise the potential relative to the relaxed state.

8.2.3 Changing a gravitational datum and changing coordinates

Near Earth's surface, use a fixed vertical axis and the uniform-field approximation $F_y = -mg$. Integrating this force between two heights gives $\Delta U_g = mg(y_b - y_a)$. A convenient explicit form is $U_g = mg(y - y_0)$ when the physical plane $y = y_0$ is assigned zero potential. Heights below that plane then have negative potential. This says where the numerical datum lies; it makes no statement that gravity has reversed or that the interaction is absent.

Changing the potential reference and changing the coordinate origin are related operations that should be distinguished in the algebra. If the coordinate y is retained while the zero-potential plane is moved, the potential function changes by a constant. If a new coordinate $z = y - d$ is introduced, the same physical potential may instead be written $U_g = mg(z + d - y_0)$. Both procedures predict the same force. A calculation becomes inconsistent only if it changes a height convention in one state but leaves the other state in the previous convention.

The field approximation is part of the model. A constant g is appropriate when the vertical extent is small enough that its variation is negligible for the desired accuracy. This chapter uses it for local laboratory and engineering distances. The full radial gravitational potential and orbital motion belong to Chapter 13; using $mg y$ across an orbital-scale height interval would extend the present approximation beyond its justified domain.

Changing the orientation of the vertical coordinate also changes the sign of the force component,

without changing the physical gravitational vector. For example, choose a downward coordinate $s = -y$ with a common zero plane. Then $U_g(s) = -mgs + C$ and the force component is $F_s = -dU_g/ds = +mg$. A positive downward displacement gives positive gravitational work, exactly as the upward coordinate gives $F_y dy = (-mg)(-ds) = mgds$. These are two descriptions of the same scalar work. A formula such as $mg y$ is therefore inseparable from its declared upward coordinate.

The chain rule provides a general check. Rewriting the potential in the new coordinate gives $dU/ds = (dU/dy)(dy/ds)$. The force component must transform with the same orientation factor. Students sometimes keep the upward-coordinate potential $+mgs$ after declaring that s increases downward; differentiating that expression would predict an upward gravitational force and expose the inconsistency. The error is not repaired by a reference constant, because a constant cannot change a slope.

Example 8.8: One lift described from two datum planes

A 3.20 kg instrument is lifted slowly from $y_i = -0.600$ m to $y_f = 1.80$ m, where $y = 0$ is a deck. Use $g = 9.81 \text{ m s}^{-2}$. Find the gravitational potential change and the work of gravity and the lifter. Then describe the same process with zero potential on a platform 1.20 m below the deck and with upward coordinate z measured from that platform.

Deck convention. The instrument–Earth potential is $U_A = mgy$, with $mg = 31.392$ N. Thus

$$U_{A,i} = -18.8352 \text{ J}, \quad U_{A,f} = 56.5056 \text{ J},$$

and

$$\Delta U_g = mg(2.40 \text{ m}) = 75.3408 \text{ J} \simeq 75.3 \text{ J}.$$

Gravity does $W_g = -75.3$ J. In the quasistatic model the kinetic-energy change is negligible and no other force does work, so the lifter does $+75.3$ J.

Lower-platform convention. The new coordinate is $z = y + 1.20$ m, giving $z_i = 0.600$ m and $z_f = 3.00$ m. With $U_B = mgz$, the endpoint potentials are

$$U_{B,i} = 18.8352 \text{ J}, \quad U_{B,f} = 94.1760 \text{ J}.$$

Their difference is still 75.3408 J. At any one physical position, $U_B - U_A = mg(1.20 \text{ m}) = 37.6704$ J. Both endpoint values shifted by this same constant.

Check and meaning. Since $dz = dy$, both coordinate descriptions give the same downward force -31.392 N and the same work integral. The initial potential changed sign only because the initial point lies below the deck and above the lower platform. Figure 8.9 compares the reference planes and the two equal-slope potential lines using the original y axis.

8.2.4 Adding potentials without mixing reference conventions

If a configuration changes several independent conservative interactions, their work contributions add. Consequently their potentials can be added as well: $U_{\text{total}} = U_1 + U_2 + \cdots$, with one constant for each chosen reference. Differentiation distributes over the sum, giving the sum of the associated force components. This procedure requires a shared configuration coordinate or a stated geometric relation between coordinates. A spring extension and a height are not automatically the same variable merely because both have units of length.

For example, take a vertically suspended ideal spring with downward extension s measured from its relaxed length. The spring potential is $\frac{1}{2}k_s s^2$, while the gravitational potential is $-mgs + C$

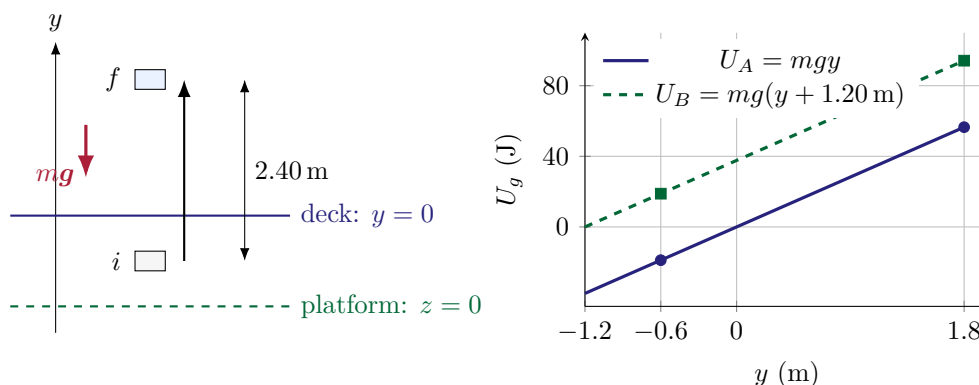


Figure 8.9: Two gravitational reference choices for the same physical lift in Example 8.8. The vertical displacement is identical under both conventions. On the right, both potentials are plotted against the original deck coordinate y ; they are separated by the constant 37.6704 J and have the same slope mg . Circles and squares distinguish corresponding endpoint values.

because increasing s lowers the mass. The combined potential is

$$U(s) = \frac{1}{2}k_s s^2 - mgs + C = \frac{1}{2}k_s \left(s - \frac{mg}{k_s}\right)^2 + C - \frac{(mg)^2}{2k_s}.$$

The downward force is $-U'(s) = mg - k_s s$, so force balance occurs at $s = mg/k_s$. Completing the square reveals why this balance position is displaced from the spring's relaxed configuration. It also shows that changing the total reference constant cannot alter that balance position. The dimensions are consistent: mg/k_s is a length and $(mg)^2/k_s$ is an energy.

This calculation prepares the spring-gravity conservation problems in the next section. The same combined potential can describe static balance, energy exchange during motion, or reachability at a specified energy. Those are different uses of one model. To obtain a motion prediction one must additionally state the kinetic energy and the external transfers; writing a potential alone does not assert conservation of mechanical energy for every possible process.

8.3 Conservation of mechanical energy

Start with the work-kinetic-energy theorem,

$$\Delta K = W_{\text{net}} = W_c + W_{\text{other}}.$$

For included conservative interactions, $W_c = -\Delta U$. Hence

$$\Delta(K + U) = W_{\text{other}}. \quad (8.10)$$

If the chosen system and process have no net contribution from nonconservative work or other mechanical-energy conversion, then

$$\boxed{K_i + U_i = K_f + U_f, \quad E_{\text{mech}} = \text{constant}.} \quad (8.11)$$

This is not a universal identity. It is a conditional consequence of the model and the system boundary.

Method

For a mechanical-energy calculation:

1. choose the system and frame;

2. list the conservative interactions included in U and fix their reference values;
3. state whether any external work, frictional conversion, deformation or other transfer changes $K + U$;
4. write the symbolic balance before substituting numbers;
5. check that every inferred kinetic energy is non-negative and that limiting cases behave sensibly.

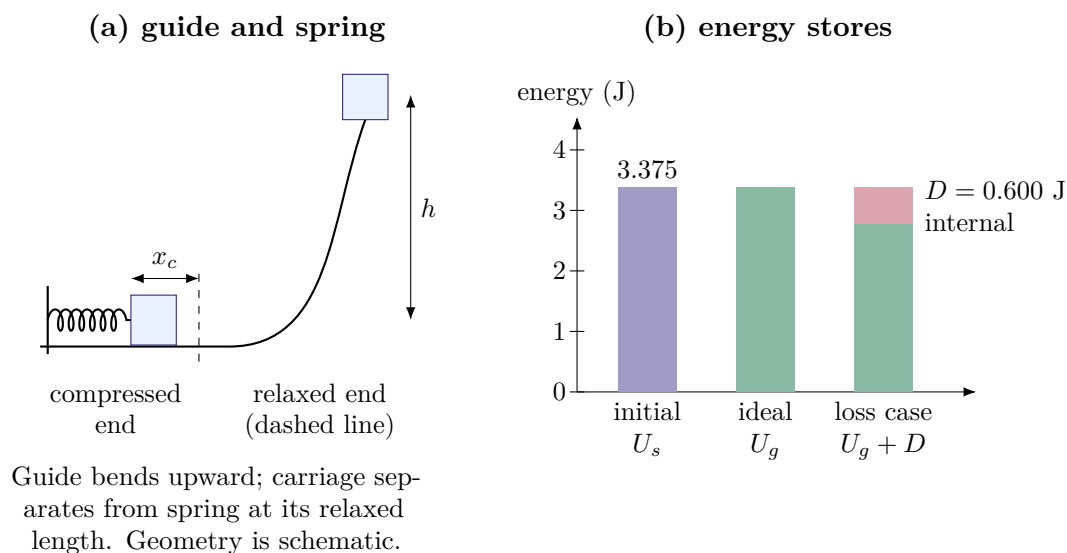


Figure 8.10: A compressed spring pushes a carriage along a stationary guide, after which the carriage rises freely along the guide. The system includes carriage, spring, guide and Earth; h is the rise from the initial carriage level and x_c is compression relative to the relaxed endpoint, not the carriage width. In Example 8.9, calibrated bars show 3.375 J initially, and either 3.375 J gravitational energy or 2.775 J gravitational plus 0.600 J internal energy at the final rest point.

Example 8.9: spring-assisted lift, ideal and with a declared loss

A 0.500 kg carriage is launched upward by an ideal spring with $k_s = 300 \text{ N m}^{-1}$ compressed by 0.150 m. A stationary guide lets the carriage separate from the spring at its relaxed length and subsequently rise. Take $U_g = 0$ at the initial carriage level. The carriage starts and reaches its highest point at rest, and the spring is relaxed after launch. First neglect all losses. Then repeat with a stipulated 0.600 J conversion from mechanical to internal energy.

The initial elastic energy is

$$U_{s,i} = \frac{1}{2}(300)(0.150)^2 = 3.375 \text{ J.} \quad (8.12)$$

In the ideal model, $U_{s,i} = mgh_0$, so

$$h_0 = \frac{3.375}{(0.500)(9.81)} = 0.688 \text{ m.} \quad (8.13)$$

With the declared conversion $E_{\text{diss}} = 0.600 \text{ J}$,

$$mgh = U_{s,i} - E_{\text{diss}},$$

which gives

$$h = 0.566 \text{ m.} \quad (8.14)$$

The result approaches 0.688 m continuously as $E_{\text{diss}} \rightarrow 0$. A proposed loss larger than 3.375 J would make the assumed final rest state at positive height unreachable under this model rather than producing a negative kinetic energy.

Guided checks

Diagnostic X08-03. A solver obtains $K_f = -0.40 \text{ J}$ at a proposed final height. *Interpret the negative value as a failed reachability assumption, not as a physical kinetic energy. The object must turn before that configuration unless another energy transfer has been omitted.*

8.3.1 Spring–gravity motion: the two reference lengths

A vertical spring makes a useful test of whether an energy equation has been constructed from the actual forces. The spring's relaxed length determines its elastic deformation. The equilibrium length of the loaded apparatus is determined by the *combined* spring and gravitational forces. These lengths generally differ. Using displacement from equilibrium directly in $\frac{1}{2}k_s x^2$ as though it were the actual spring extension mixes two different coordinate choices.

Consider a mass attached to a light ideal spring suspended from a fixed support. Choose y positive downward from the position at which the spring has its relaxed length. The spring extension is then y , and the net force on the mass is

$$F_y = mg - k_s y.$$

The mass–spring–Earth system includes both conservative interactions. Taking each potential to vanish at $y = 0$ gives

$$U(y) = \frac{1}{2}k_s y^2 - mgy. \quad (8.15)$$

The negative gravitational term follows from the chosen downward coordinate: lowering the mass decreases gravitational potential energy. Differentiation provides a direct check, $-dU/dy = mg - k_s y$.

The stationary point occurs at $y_{\text{eq}} = mg/k_s$. Completing the square reorganizes the *total* potential without changing the physical system:

$$U(y) = \frac{1}{2}k_s (y - y_{\text{eq}})^2 - \frac{m^2 g^2}{2k_s}. \quad (8.16)$$

With $q = y - y_{\text{eq}}$, the force is $F_q = -k_s q$. If we also shift the energy reference so that the total potential vanishes at equilibrium, the expression becomes $\tilde{U}(q) = \frac{1}{2}k_s q^2$. This familiar quadratic is the spring–gravity potential expressed about equilibrium; it is not a statement that the physical spring stores no elastic energy there. At equilibrium its elastic store remains $\frac{1}{2}k_s y_{\text{eq}}^2$, and gravity supplies the compensating term in the total potential.

Example 8.10: Release from the relaxed length

A 0.800 kg mass is attached to an ideal vertical spring with $k_s = 160 \text{ N m}^{-1}$. It is released from rest when the spring is relaxed. Neglect resistance and take $g = 9.81 \text{ m s}^{-2}$. Find the equilibrium extension, the largest extension, and the speed at equilibrium.

Model and strategy. The spring remains attached throughout. The stationary support does no work, and the mass–spring–Earth mechanical energy is conserved. Use the coordinate and reference of Eq. (8.15). Initially $K = U = 0$, so $E = 0$. At a later position,

$$\frac{1}{2}mv^2 = mgy - \frac{1}{2}k_s y^2.$$

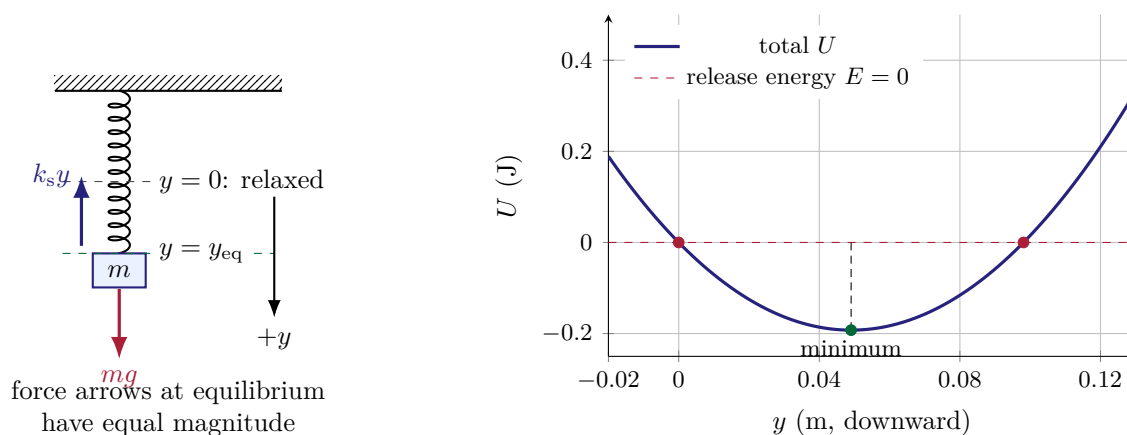


Figure 8.11: Attached vertical spring: the coordinate origin follows the relaxed spring, while the minimum of the combined potential follows the loaded equilibrium. The plot uses $m = 0.800$ kg and $k_s = 160$ N m⁻¹. A release from $y = 0$ has $E = 0$ in this reference and two ordinary turning positions. The schematic and graph use the same downward-positive convention.

The equilibrium extension is $y_{eq} = mg/k_s = 0.04905$ m. At the lower turning point the speed vanishes. Factoring the energy equation gives

$$y \left(mg - \frac{1}{2} k_s y \right) = 0, \quad y_{\max} = \frac{2mg}{k_s} = 0.09810 \text{ m}.$$

The other root, $y = 0$, is the initial upper turning point. At equilibrium,

$$K_{eq} = \frac{m^2 g^2}{2k_s} = 0.192472 \text{ J}, \quad v_{eq} = g \sqrt{\frac{m}{k_s}} = 0.6937 \text{ m s}^{-1}.$$

Interpretation and checks. The largest extension is twice the static extension because the mass passes equilibrium with nonzero kinetic energy. Zero net force at equilibrium means zero instantaneous acceleration, not zero speed. At the lower turning point the net force is $mg - k_s(2mg/k_s) = -mg$, directed upward, so the mass reverses. In the shifted reference the same state has $\tilde{E} = 0.192472$ J and turning displacements $q = \pm 0.04905$ m. Both descriptions predict identical motion.

The attached-spring example also explains why the spring's own energy can rise while the total potential initially falls. For $0 < y < y_{eq}$, the elastic term increases as the spring extends, but the gravitational term decreases more quickly. The net potential slope is negative and gravity still exceeds the spring force. For $y > y_{eq}$, the elastic term rises more quickly than gravity falls, so the net potential rises and the mass decelerates on its downward trip. An energy-bar description and a force description therefore support the same prediction when they use the same coordinate and system.

This model presumes that the spring follows Hooke's law over the entire predicted deformation, that its inertia can be neglected, and that the guide and support are stationary. The energy result itself does not establish these assumptions. In an experiment, a measured force–extension relation would be checked over at least $0 \leq y \leq y_{\max}$. If the relation becomes nonlinear, its elastic potential is obtained by integrating the measured conservative force law rather than retaining the quadratic by habit.

8.3.2 An attached spring and a spring that only pushes

A launcher requires an additional model decision: can its spring pull the moving body after reaching the relaxed length? A mass permanently attached to an ideal spring and a mass merely resting against a compressed spring obey different force laws. An endpoint energy balance is useful only after this physical distinction has been settled.

Let a vertically guided carriage initially compress a spring by $c > 0$, and let z measure upward travel from that initial position. For a spring that only pushes, contact holds while $0 \leq z \leq c$, with upward force $k_s(c - z)$. For $z > c$ the carriage is detached and the spring force is zero. Provided the carriage reaches $z = c$ with positive speed, the potential for the carriage–spring–Earth energy model can be written

$$U_{\text{push}}(z) = mgz + \begin{cases} \frac{1}{2}k_s(c - z)^2, & 0 \leq z \leq c, \\ 0, & z \geq c. \end{cases} \quad (8.17)$$

The spring is left relaxed after detachment. Both U and its first derivative are continuous at $z = c$; the curvature changes because the spring interaction switches off. The potential is a compact way to describe the modeled trajectory, including its contact regime.

An attached spring instead has $U_{\text{attached}}(z) = mgz + \frac{1}{2}k_s(z - c)^2$ throughout the allowed travel. Beyond $z = c$ it pulls downward. Figure 8.12 shows why the two models cannot share the same apex formula. For the push-only case, the maximum height follows from $mgz_{\text{max}} = \frac{1}{2}k_sc^2$. For the attached case, some of the initial elastic store has returned to the spring when the carriage stops above its relaxed position.

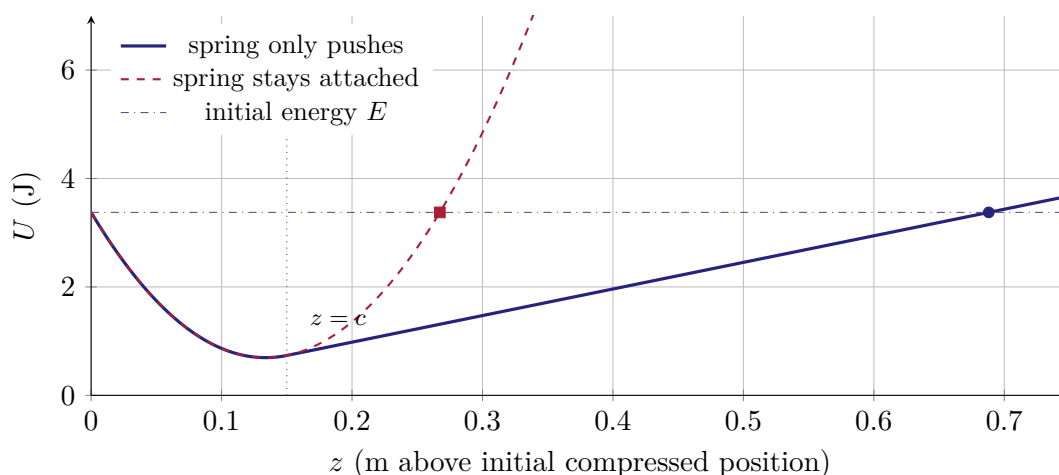


Figure 8.12: Two conservative models with identical initial energy but different later interactions. The solid potential follows a spring that can push but cannot pull; the dashed potential follows an attached spring. With $m = 0.500$ kg, $k_s = 300$ N m⁻¹ and $c = 0.150$ m, their noninitial turning heights are 0.6881 m and 0.2673 m respectively. The diagram is an energy comparison, not a time history.

Example 8.11: Launch height depends on contact

Use the parameters of Fig. 8.12. Determine the speed at the relaxed spring position, the apex of a push-only launch, and the upper turning position if the carriage is instead attached permanently.

Initial energy. The carriage starts from rest at $z = 0$, where $U_g = 0$, and $E = \frac{1}{2}k_sc^2 = 3.375$ J. At $z = c$ both models have zero elastic energy, so

$$K_c = E - mgc = 3.375 - 0.73575 = 2.63925 \text{ J}, \quad v_c = \sqrt{\frac{2K_c}{m}} = 3.249 \text{ m s}^{-1}.$$

Thus detachment is reached with positive upward speed. The spring force falls to zero there,

while gravity continues to decelerate the carriage.

Push-only apex. Above detachment $U_s = 0$. Therefore

$$z_{\max} = \frac{E}{mg} = 0.6881 \text{ m.}$$

The rise *after* detachment is $z_{\max} - c = 0.5381 \text{ m}$. Its gravitational cost is $mg(z_{\max} - c) = K_c$, an independent check using the free-rise phase alone.

Attached upper turning point. Here set the full attached potential equal to the initial energy:

$$mgz + \frac{1}{2}k_s(z - c)^2 = \frac{1}{2}k_sc^2.$$

Besides $z = 0$, the solution is

$$z_{\text{turn}} = 2 \left(c - \frac{mg}{k_s} \right) = 0.2673 \text{ m.}$$

At this point the spring is stretched by 0.1173 m. Its elastic energy is about 2.064 J, leaving about 1.311 J in gravitational potential energy; their sum is 3.375 J. The smaller height is entirely explained by recoverable spring storage, without dissipation.

Regime check. Starting from rest, a push-only launcher initially accelerates upward only if $k_sc > mg$. Reaching detachment with positive speed requires the stronger condition $\frac{1}{2}k_sc^2 > mgc$, or $k_sc > 2mg$. If this stronger condition fails, a positive value of $E/(mg)$ is not enough to justify the assumed detached-flight stage. When $mg < k_sc < 2mg$, the ascending carriage turns before the spring relaxes; when $k_sc \leq mg$, it cannot begin an upward launch from rest. At $k_sc = 2mg$ it reaches the relaxed position with zero speed and has no upward free-rise stage. The energy method must retain the contact model along the path.

8.4 Potential-energy diagrams and stability

In one dimension, a potential-energy graph compresses several pieces of mechanics into one picture. For fixed total mechanical energy E ,

$$K(x) = E - U(x) \geq 0. \quad (8.18)$$

Therefore motion is allowed only where $E \geq U(x)$. A point satisfying $E = U$ is a zero-speed point; it is an ordinary turning point only when the force there drives the particle back into the allowed region. An equilibrium satisfies

$$F_x = -U'(x) = 0. \quad (8.19)$$

A nondegenerate local minimum is stable to sufficiently small one-dimensional disturbances; a local maximum is unstable.

To make these ideas concrete, consider the stipulated analytical landscape

$$U(x) = U_* \left[\left(\frac{x}{\ell} \right)^2 - 1 \right]^2, \quad U_* > 0, \quad \ell > 0. \quad (8.20)$$

With $u = x/\ell$,

$$F_x = -\frac{4U_*}{\ell} u(u^2 - 1). \quad (8.21)$$

The stationary points are $u = -1, 0, +1$. The two minima have $U = 0$ and the central maximum has $U = U_*$. Thus U_* is the barrier height relative to either minimum.

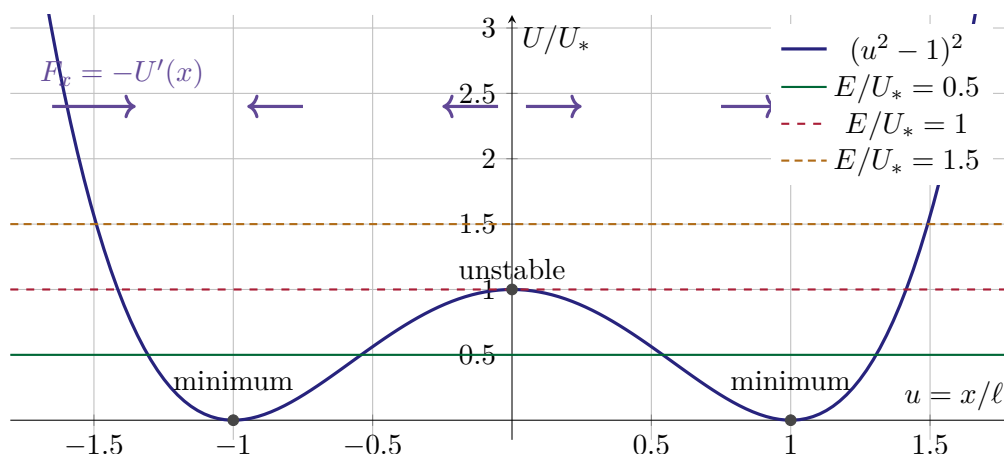


Figure 8.13: Calculated double-well potential from Eq. (8.20). Horizontal lines show energies below, at and above the barrier. Motion is allowed where an energy line lies at or above the potential. The purple arrows indicate the force direction from $F_x = -U'(x)$; x is position, not time.

Example 8.12: carriage in a double-well landscape

Take $U_* = 4.00 \text{ J}$ and $\ell = 0.500 \text{ m}$. Determine allowed intervals and key points for $E/U_* = 0.50$, 1.00 and 1.50 .

Turning coordinates follow from

$$e = (u^2 - 1)^2, \quad e = E/U_*.$$

For $e = 0.50$,

$$u = \pm \sqrt{1 \pm \sqrt{0.50}} = \pm 0.5412, \pm 1.3066.$$

The allowed regions are two disconnected intervals,

$$-1.3066 \leq u \leq -0.5412, \quad 0.5412 \leq u \leq 1.3066.$$

The carriage cannot cross the central barrier because $E < U_*$.

For $e = 1$, roots are $u = 0$ and $u = \pm\sqrt{2}$. The state $u = 0, v = 0$ is the unstable equilibrium at the barrier top; it is not an ordinary finite-time rebound point. For $e = 1.50$, the only outer roots are $u = \pm 1.4916$, and the allowed region is connected across the central barrier.

A reference shift $U \mapsto U + C$ accompanied by $E \mapsto E + C$ changes every vertical numerical label but leaves all roots of $K = E - U$, force directions and stability classifications unchanged.

Guided checks

Diagnostic X08-04. On a potential plot, a student labels every point where $E = U$ as an equilibrium. *Repair:* $E = U$ only states $K = 0$. Equilibrium requires $U' = 0$. A regular turning point usually has $U' \neq 0$, so the force is nonzero and reverses the motion.

8.4.1 Reading the force and the motion from matched plots

The height of a potential curve and its slope answer different questions. Its height relative to E determines the kinetic energy available at a position; its slope determines the force and acceleration there. A high point on a potential graph need not have a large force, and the lowest potential on a trajectory need not be the position of zero speed. The force can be read without knowing the mass, whereas converting kinetic energy into speed requires the mass.

For the double well of Eq. (8.20), introduce dimensionless energy $e = E/U_*$ and dimensionless force $f = F_x \ell/U_*$. Then

$$\frac{U}{U_*} = (u^2 - 1)^2, \quad f(u) = 4u(1 - u^2), \quad u = x/\ell.$$

The scaling is explicit: u and f are dimensionless; the physical force scale is U_*/ℓ , with units of newtons. In the interval $-1 < u < 0$ the force is negative, directing the particle toward the left minimum. In $0 < u < 1$ it is positive, directing the particle toward the right minimum. These directions point away from the central maximum. Outside the minima the force points inward toward the corresponding well.

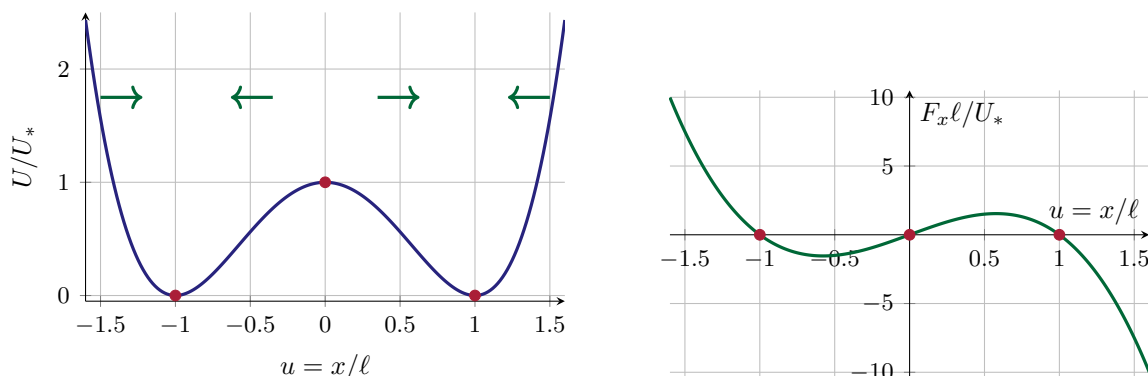


Figure 8.14: Matched potential and force for the same stipulated double well. Every extremum in the left panel corresponds to a force zero in the right panel. The green arrows show force direction, including the directions away from the central maximum. They show acceleration direction and do not specify velocity.

At $u = 0$ the force vanishes even though the potential is higher than at either minimum. Near $u = 0$, a small displacement makes the force point further away from zero, which explains instability without relying on the numerical value assigned to the potential. At $u = \pm 1$, a small displacement produces a force back toward the original point, explaining stability. A particle passing through either minimum can have its *largest* speed because the local kinetic energy $E - U$ is then greatest. It has zero instantaneous acceleration at that passage, so a force zero does not imply a stop.

Example 8.13: A force arrow is not a velocity arrow

A 0.500 kg carriage moves in the stipulated double well with $U_* = 4.00$ J and $\ell = 0.500$ m. At $x = +0.250$ m its mechanical energy is $E = 3.00$ J. Find its possible velocities and its acceleration. If it is moving left, is it speeding up or slowing down at this instant?

Energy and force use different information. Here $u = 0.500$, giving

$$U = 4.00(0.500^2 - 1)^2 = 2.250 \text{ J}, \quad K = 3.00 - 2.250 = 0.750 \text{ J}.$$

Energy determines the speed but leaves its sign undecided:

$$v = \pm \sqrt{2K/m} = \pm 1.732 \text{ m s}^{-1}.$$

The force is uniquely determined by position,

$$F_x = \frac{4U_*}{\ell} u(1 - u^2) = 12.00 \text{ N}, \quad a_x = 24.00 \text{ m s}^{-2}.$$

If $v < 0$, force and velocity oppose one another, so the carriage is slowing down. The instantaneous power $F_x v = -20.78$ W confirms that its kinetic energy is decreasing. With $v > 0$ the same force increases the speed and the power is $+20.78$ W.

Reachability and interpretation. Since $E < U_*$, the carriage remains in the right well. A left-moving carriage does not climb through the barrier: it reaches an inner turning point and returns. Reversing only the velocity at the stated position preserves E and the force, but selects a different direction along the same allowed orbit. Position and total energy are therefore insufficient to specify the full dynamical initial state.

A regular turning point supplies another useful slope check. Suppose a right-moving carriage reaches x_t where $E = U(x_t)$ and $U'(x_t) > 0$. Its acceleration there is $-U'(x_t)/m < 0$, so it begins moving left. Locally the available kinetic energy is approximately $E - U(x) \simeq U'(x_t)(x_t - x)$ on the allowed side. Hence the speed goes to zero like $\sqrt{x_t - x}$, and the corresponding time integral is finite. The speed can vanish without a long wait. This differs from the linear-in-distance speed near the barrier threshold developed below.

It is also possible to have $U'(x_t) < 0$: then the allowed side is to the right and a left-moving carriage turns right. These two cases make the direction of the restoring acceleration explicit. They also explain why a potential plot should show the neighboring allowed region whenever a zero-speed point is described as a turning point. The equal-energy condition alone does not reveal whether a force reverses the trajectory or whether the point is an equilibrium.

8.4.2 Allowed intervals and the initial state

Solving $E \geq U(x)$ identifies all energetically allowed positions, but it does not establish that every one can be reached from a particular initial state. In one-dimensional motion, position changes continuously. If two allowed intervals are separated by a forbidden interval, a trajectory cannot jump across the gap. The well initially occupied selects one connected component of the allowed set.

For $0 < e < 1$, let

$$a = \sqrt{1 - \sqrt{e}}, \quad b = \sqrt{1 + \sqrt{e}}.$$

Then the dimensionless allowed set is $[-b, -a] \cup [a, b]$. Both wells are allowed at the same energy, yet a particle initially in the left well remains there. In a classical deterministic model an initially occupied right well is not selected by a greater probability or by the graph's appearance; it is selected by the actual position and velocity. Noise, external forcing or additional coordinates could change this conclusion, but none is present in the stated conservative one-dimensional model.

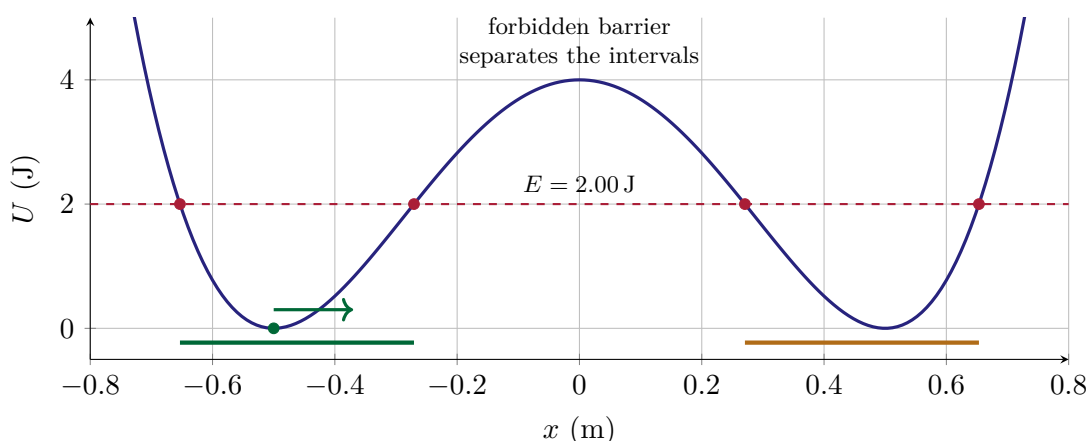


Figure 8.15: For $U_* = 4.00$ J, $\ell = 0.500$ m and $E = 2.00$ J, four zero-speed roots bound two disconnected allowed intervals. The green initial-state mark selects the left interval; the orange interval belongs to possible initial states in the other well. The bars below the curve indicate allowed position sets and are not extra energy curves.

Example 8.14: Two states at the same energy

Use the model of Fig. 8.15 with $m = 0.500$ kg. State A has $x_i = -0.500$ m and $v_i = +\sqrt{8}$ m s⁻¹. State B has $x_i = +0.500$ m and $v_i = -\sqrt{8}$ m s⁻¹. Find the mechanical energy, all roots of $E = U$, and the first turning point reached by each state.

Both initial positions are minima, so $U_i = 0$. Each state has

$$E = \frac{1}{2}(0.500)(\sqrt{8})^2 = 2.00 \text{ J}, \quad e = 0.500.$$

The four roots are $x = \pm \ell a$ and $x = \pm \ell b$, where $a = 0.541196$ and $b = 1.306563$. Thus

$$x = \pm 0.270598 \text{ m}, \quad x = \pm 0.653281 \text{ m}.$$

State A initially moves right from the left minimum, so its first turning point is $x = -0.270598$ m. In this interval $F_x < 0$, consistent with deceleration and subsequent return to the left. State B initially moves left from the right minimum, so its first turning point is $x = +0.270598$ m, where $F_x > 0$.

Complete trajectory information. Each carriage later travels toward its corresponding outer turning point, but neither reaches $x = 0$ because $U(0) = 4.00 \text{ J} > E$. The four algebraic roots therefore do not describe four successive stops of either carriage. Reflection symmetry makes the two trajectories equivalent under $(x, v) \mapsto (-x, -v)$. This is a check on both the roots and the velocity choices.

8.4.3 A barrier threshold and finite-time reachability

At $e = 1$, the allowed-position set becomes connected: $-\sqrt{2} \leq u \leq \sqrt{2}$. Nevertheless, a carriage coming from one well does not cross the central maximum in finite time at exactly this energy. The distinction concerns dynamics, rather than another forbidden interval.

For a right-moving trajectory with fixed energy, the speed relation implies

$$dt = \sqrt{\frac{m}{2}} \frac{dx}{\sqrt{E - U(x)}}. \quad (8.22)$$

At the threshold $E = U_*$, the exact expression is

$$E - U = U_*(2u^2 - u^4).$$

Near $u = 0$, the speed is therefore proportional to $|u|$:

$$|v| \simeq 2\sqrt{\frac{U_*}{m}} |u| = \lambda |x|, \quad \lambda = 2\sqrt{\frac{U_*}{m\ell^2}}.$$

The time to approach the maximum contains an integral proportional to $\int dx/|x|$, which diverges logarithmically. The carriage gets progressively closer and slower, without reaching the maximum in finite time. A carriage *initially* at $x = 0$, $v = 0$ stays there in the ideal deterministic model; a small perturbation makes it depart because the equilibrium is unstable.

When $E > U_*$, the barrier speed is instead $v(0) = \pm\sqrt{2(E - U_*)/m}$, nonzero. The integrand in Eq. (8.22) is finite near zero, so crossing takes finite time. The threshold trajectory is often called a *separatrix*: it separates motions confined to one well from motions passing between both wells. Here the term describes this energy boundary; a full phase-space treatment belongs to later dynamics.

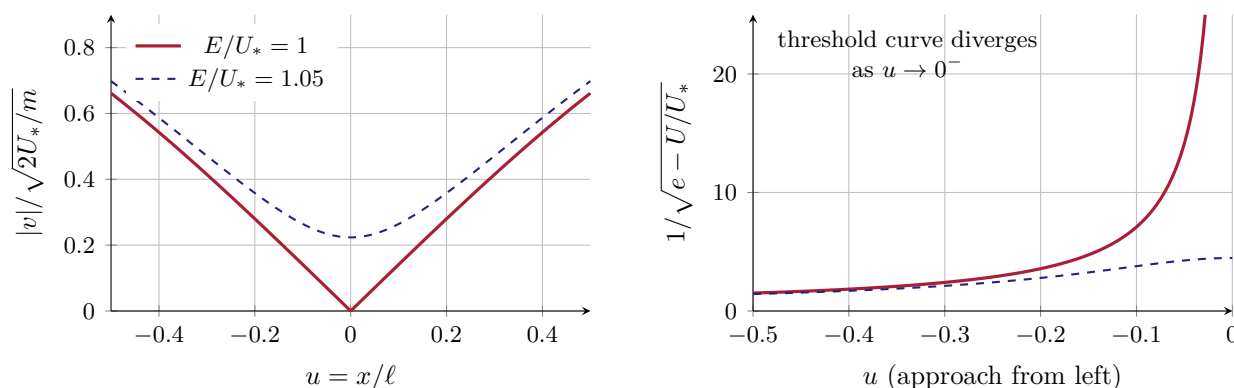


Figure 8.16: At the barrier energy, speed vanishes linearly in distance from the maximum and the time integrand diverges. Slightly above the barrier, the nonzero central speed makes passage possible in finite time. The right panel truncates the divergent red curve at its displayed vertical range; it does not imply a finite value at zero.

Example 8.15: Crossing just above the barrier

For $U_* = 4.00$ J, $\ell = 0.500$ m and $m = 0.500$ kg, compare right-moving passages from $x = -0.250$ m to $x = +0.250$ m at $E = 4.00$ J and $E = 4.20$ J. Find the barrier speed and determine whether a finite crossing time exists.

At $E = 4.00$ J the barrier speed tends to zero. Near $x = 0$ the rate constant is

$$\lambda = 2\sqrt{\frac{4.00}{(0.500)(0.500)^2}} = 11.31 \text{ s}^{-1}.$$

Approach on the left satisfies locally $\dot{x} \simeq -\lambda x$, so its distance from the maximum decreases exponentially. It never changes sign in finite time. There is no finite crossing time to publish for this threshold state.

At $E = 4.20$ J,

$$v(0) = \sqrt{\frac{2(4.20 - 4.00)}{0.500}} = 0.8944 \text{ m s}^{-1}.$$

The exact crossing time is determined by the finite integral

$$\Delta t = \sqrt{\frac{0.500}{2}} \int_{-0.250}^{+0.250} \frac{dx}{\sqrt{4.20 - 4.00[(x/0.500)^2 - 1]^2}} = 0.3343 \text{ s}.$$

This value is a numerical quadrature of the stated energy model; refining the integration agrees at the displayed precision. Over this interval the central speed is the minimum speed, so the time is less than $0.500/0.8944 = 0.5590$ s. The endpoint speed is 2.793 m s^{-1} , the maximum over the interval, giving the lower bound $0.500/2.793 = 0.1790$ s. The computed time lies between these independent bounds. As E approaches 4.00 J from above, the passage takes increasingly long; setting $E = 4.00$ is a qualitative change, not ordinary rounding of the crossing state.

8.4.4 Flat stationary points: inspect the shape

A positive second derivative is a convenient sufficient test for a smooth nondegenerate minimum, and a negative one identifies a nondegenerate maximum. A zero second derivative does not establish neutral equilibrium. Some potentials are flat to quadratic order yet still rise or fall away from their stationary point.

Three exact local models demonstrate the distinction. Write $u = (x - x_0)/\ell$ and choose an energy

scale $U_* > 0$. For $U - U(x_0) = U_*u^4$, both first and second derivatives vanish at $u = 0$, but every nearby displaced configuration has higher potential. Moreover $F_x = -(4U_*/\ell)u^3$ points back toward zero on both sides. The point is stable. For $U - U(x_0) = -U_*u^4$, the force points away and the point is unstable. The model $U - U(x_0) = U_*u^3$ has a horizontal tangent at zero but no extremum; its force $F_x = -(3U_*/\ell)u^2$ points left on both sides. A displacement to the left is driven further away, so the equilibrium is unstable.

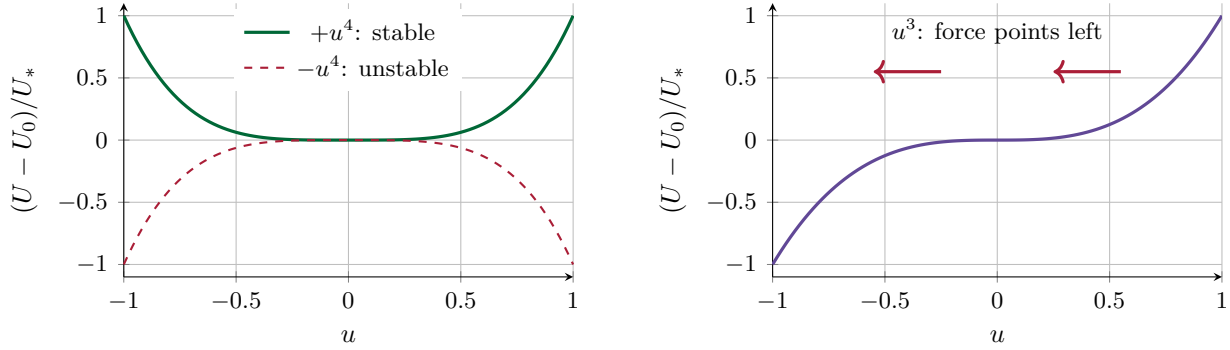


Figure 8.17: The second-derivative test is inconclusive for all three stationary points at $u = 0$. Exact local shape and force directions distinguish a flat minimum, a flat maximum and a stationary inflection. These are local mathematical models; no global confinement is claimed for the negative-quartic or cubic potential.

Stability here means that sufficiently small disturbances remain close to the original equilibrium in the conservative model. It does not mean that a displaced mass settles back to rest: settling requires dissipation. Nor does a flat stable minimum have the usual finite positive quadratic stiffness. The quartic model's restoring force begins at cubic order, so importing the small-amplitude frequency formula from a quadratic well would give no valid finite-frequency description. The advanced section develops the nondegenerate quadratic case separately.

8.4.5 Energy accessibility and a contact constraint

Energy conservation checks the availability of kinetic energy. It does not, by itself, check whether a modeled track can exert the force needed to keep the body on that track. This is especially clear for a particle moving on the *inside* of a vertical circular track, where a smooth surface can push inward but cannot pull the particle outward. An energy-derived speed may be real while the assumed continuous-contact trajectory is impossible.

Consider a particle released from rest at height H above the bottom of a frictionless track leading into a vertical circle of radius R . Ignore the particle's size, rolling and resistance. While it follows the circle, define θ from the downward radius at the bottom, increasing along the direction of travel. Its height is $h = R(1 - \cos \theta)$. Conservation of the particle–Earth mechanical energy predicts

$$v^2 = 2g[H - R(1 - \cos \theta)].$$

The track normal does no work while contact holds because it is perpendicular to the instantaneous velocity. Nonetheless it is dynamically essential. In the inward radial direction, Newton's law gives

$$N - mg \cos \theta = \frac{mv^2}{R}, \quad N = mg \left(\frac{2H}{R} - 2 + 3 \cos \theta \right). \quad (8.23)$$

At the bottom gravity points outward, which explains the minus sign in the radial equation. At the top it points inward; Eq. (8.23) then reduces to $N + mg = mv^2/R$.

The normal force is smallest at the top, where $N_{\text{top}} = mg(2H/R - 5)$. Hence an ideal limiting loop requires

$$H \geq \frac{5}{2}R.$$

The purely energetic condition for the top, $H \geq 2R$, is weaker. At the limiting contact height the particle still has $v_{\text{top}} = \sqrt{gR}$ and zero normal force at the top. Any real perturbation or departure from the ideal geometry requires margin, but that engineering design issue is outside the present stipulated model.

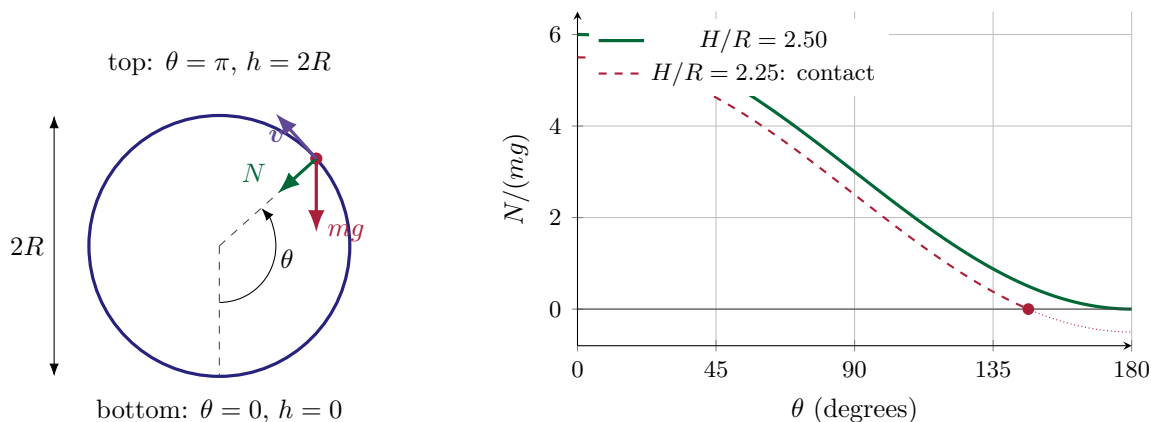


Figure 8.18: Contact must satisfy both energy conservation and an admissible inward normal force. The left schematic shows forces while contact holds. In the right plot, the dotted negative-normal continuation for $H/R = 2.25$ is a rejected mathematical continuation, not a physical contact force. The particle departs the circular constraint where the dashed curve first reaches zero.

The inequality $N \geq 0$ belongs to the physical contact assumption, not to energy conservation. A bead threaded on a rigid wire could be pulled as well as pushed by the constraint; its radial reaction would be a signed quantity and the inside-surface restriction would not apply. Similarly, a cable can sustain tension but not compression, while a rigid linkage may support both. Before rejecting a computed force, identify which constraint is actually present. Before accepting an energy trajectory, identify every constraint inequality that must hold along it.

The normal-force formula is derived only for a particle still following the circle. It cannot be used to describe the subsequent ballistic path by allowing N to become negative. At departure the velocity is tangent to the track, while acceleration changes to gravity alone. The particle's gravitational mechanical energy remains conserved during free flight, but the relation $h = R(1 - \cos \theta)$ no longer describes its position. Thus conservation can remain valid even when a geometrical part of the assumed model fails.

Example 8.16: A real speed with an impossible contact force

A 0.200 kg particle is released from rest at $H = 0.900$ m above the bottom of a smooth inside loop of radius $R = 0.400$ m. An energy-only calculation predicts a positive speed at the top. Test that conclusion and find where continuous contact actually ends.

Energy-only prediction. At the top, $h = 2R = 0.800$ m, so the energy expression gives

$$v_{\text{top}} = \sqrt{2g(H - 2R)} = 1.401 \text{ m s}^{-1}.$$

This is real, but Eq. (8.23) would require

$$N_{\text{top}} = mg(2H/R - 5) = -0.9810 \text{ N}.$$

The track cannot supply this outward pull. Thus the predicted speed belongs to an inadmissible assumed-contact state.

Departure position. Set the normal force to zero at its first zero during ascent:

$$\cos \theta_d = \frac{2 - 2H/R}{3} = -\frac{5}{6}, \quad \theta_d = 146.44^\circ.$$

The height and speed at that point are

$$h_d = R(1 - \cos \theta_d) = 0.73333 \text{ m}, \quad v_d = \sqrt{2g(H - h_d)} = 1.808 \text{ m s}^{-1}.$$

Substitution into mv_d^2/R gives 1.635 N; the inward component of gravity is $-mg \cos \theta_d = 1.635 \text{ N}$, independently confirming the zero-normal condition. After departure the particle follows free-flight motion until another contact occurs, so the circular-path equations must stop there.

Threshold comparison. The minimum ideal release height for continuous contact is $H_{\min} = 2.5R = 1.000 \text{ m}$. At this threshold $v_{\text{top}} = \sqrt{gR} = 1.981 \text{ m s}^{-1}$. The distinction illustrates a general method: first use energy to obtain candidate speeds, then use force constraints to test whether those candidates belong to the declared motion model.

8.5 External work and system boundaries

Energy bookkeeping changes category when the system boundary changes. Consider a sliding block and a rough track. If the system is the block alone, the track is external. The contact force enters the centre-of-mass work–kinetic-energy equation; actual contact energy transfer also depends on deformation and warming. If the system includes both block and track, their contact interaction is internal. The same physical event can then appear as an increase of internal energy rather than an external work term. Both descriptions must predict the same observable motion when used consistently.

For a closed system in the present mechanical-plus-internal bookkeeping, a convenient work-on form is

$$\Delta(K + U + E_{\text{int}}) = W_{\text{ext,on}} + Q. \quad (8.24)$$

The equivalent thermodynamic work-by form is

$$\Delta(K + U + E_{\text{int}}) = Q - W_{\text{by}}, \quad W_{\text{on}} = -W_{\text{by}}. \quad (8.25)$$

The terms used in Eq. (8.24) must exhaust the modeled storage and transfers. An interaction cannot simultaneously be counted as a potential-energy change *and* as external work across the same boundary. Here external work means actual energy transfer at the application points, not an arbitrary force integral along the centre of mass of a deformable system.

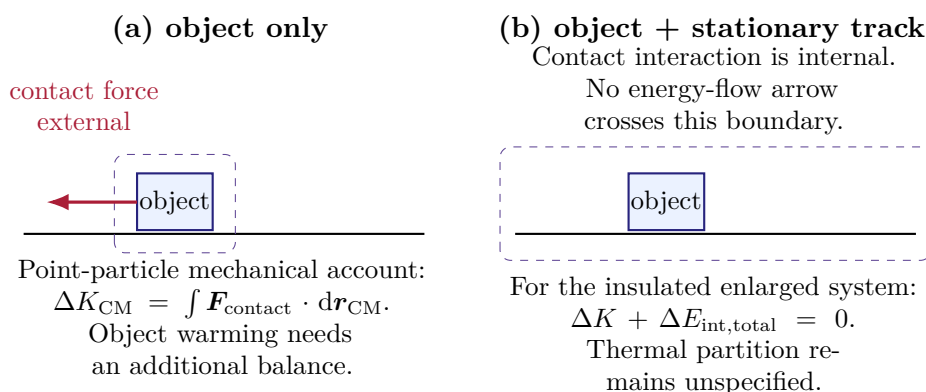


Figure 8.19: The same sliding event has different classifications under two boundaries. The red arrow represents the contact force on the object, not a calibrated energy-flow vector. Its centre-of-mass force integral determines ΔK ; if the object warms, that equation alone is not its complete energy balance. Including both contacting bodies moves the contact interaction inside the boundary, where mechanical energy can become internal energy.

Example 8.17: sliding object alone and object-plus-surface

A 2.00 kg object slows from 4.00 to 2.00 m s⁻¹ while sliding horizontally. Treat the same trajectory in two ways.

The kinetic-energy change is

$$\Delta K = \frac{1}{2}(2.00)(2.00^2 - 4.00^2) = -12.0 \text{ J.} \quad (8.26)$$

Boundary A: object only. In a point-particle translational model the centre-of-mass force integral is

$$\int \mathbf{F}_{\text{contact}} \cdot d\mathbf{r}_{\text{CM}} = \Delta K = -12.0 \text{ J.}$$

Calling this contact work is valid in the ideal rigid, no-warming approximation. For a real warming object the complete account is $\Delta(K + E_{\text{int,obj}}) = W_{\text{contact,on}} + Q$; its actual contact transfer cannot be inferred from ΔK alone. **Boundary B: object plus the contacted part of the track.** If that enlarged system is thermally insulated and no external work is done, then

$$\Delta(K + E_{\text{int}}) = 0,$$

so the decrease of 12.0 J in kinetic energy is accompanied by a 12.0 J increase in the enlarged system's internal energy.

The two accounts describe the same speed change. They do *not* tell us how the 12.0 J internal-energy increase is divided between object and track. Temperature measurements, material thermal properties and a more detailed thermal model would be needed for that partition.

Guided checks

Diagnostic X08-05. “Friction destroyed 12 J of energy.” *Repair: mechanical energy decreased by 12 J in this process, while the enlarged isolated object's-plus-track total energy stayed constant and internal energy increased. The phrase “destroyed energy” confuses a change of energy form with nonconservation.*

8.5.1 What actually crosses a boundary?

An energy-flow arrow answers a different question from a force arrow. A force diagram identifies an interaction on a selected body. An energy-flow diagram identifies a transfer into or out of a selected system over a specified interval. The force can be present while its work is zero, and a system's energy can change through heating without a net force on its centre of mass. Consequently, an arrow marked W should cross the system boundary only if the indicated work is external to that system. An internal conversion should be drawn within the boundary or represented as a change of stores.

Consider a cart raised by a cable. For the cart alone, both the cable and Earth are external. In the particle approximation the mechanical equation is $\Delta K = W_T + W_g$. For cart plus Earth, the gravitational interaction is represented by U_g , so the corresponding equation is $\Delta(K + U_g) = W_T$. The term W_g has become $-\Delta U_g$; it has not become a second contribution that can be added to the new equation. Including the motor and its power source moves further transfers inside the boundary. A decrease in the source's stored energy can then accompany the increase in $K + U_g$ and any warming of the motor. The system need not be isolated merely because its gravitational interaction is internal.

There is an equally useful distinction between an external force that does no work and an external force whose resultant vanishes. On a rigid cart constrained to a stationary frictionless guide, the

normal reaction is perpendicular to each instantaneous displacement of the contact point. Its work is zero even though its magnitude is nonzero and it can change the direction of motion. If the guide moves, the same perpendicular-to-guide statement need not make the contact-point velocity perpendicular to the reaction in the laboratory frame. The boundary conditions of the constraint are part of the energy model.

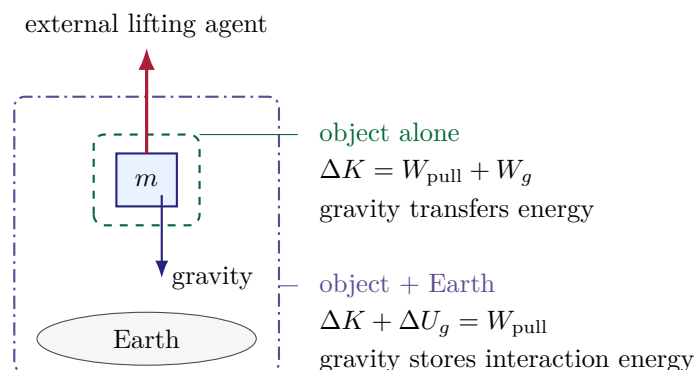


Figure 8.20: Two boundaries for the same ideal lift. Gravity crosses the object-only boundary, but is internal to object plus Earth. Thus $W_g = -\Delta U_g$ moves between equivalent balance equations when the boundary changes. In the larger system, K denotes its modeled kinetic energy; Earth recoil is neglected. Geometry and vector lengths are schematic.

Example 8.18: Loss-free lifting with external work

A cable raises a 5.00 kg cart through 2.00 m. Its initial speed is zero and its final upward speed is 1.00 m s^{-1} . The cart is treated as rigid; drag and heating are negligible. Find the cable work and show two consistent system choices.

Strategy. The work is determined by the endpoint energy changes, so no constant cable tension or constant acceleration is assumed. With cart plus Earth,

$$W_T = \Delta K + \Delta U_g = \frac{1}{2}m(v_f^2 - v_i^2) + mg\Delta y.$$

The terms are 2.50 J and 98.10 J, respectively, giving $W_T = 100.60 \text{ J}$. For the cart alone, $W_g = -98.10 \text{ J}$ and $W_T + W_g = 100.60 - 98.10 = 2.50 \text{ J} = \Delta K$. Both descriptions predict the same motion. The first has positive external cable work although the modeled motion is loss-free. Setting $K + U_g$ constant would therefore be wrong.

Check and limitation. A quasistatic lift to the same height and final rest state needs 98.10 J rather than 100.60 J in this ideal model. If the cart warms or the cable stretches, the relevant energy stores must be included before endpoint data determine the total input. The cable work says nothing by itself about the motor's electrical consumption.

8.5.2 Centre-of-mass accounting and the warming body

For a body of constant mass, Newton's centre-of-mass equation implies

$$\Delta K_{\text{CM}} = \int \mathbf{F}_{\text{ext}} \cdot d\mathbf{r}_{\text{CM}}.$$

The right side is often called *pseudowork* when the force is not applied at the centre of mass or the body deforms. The term distinguishes a mathematical consequence of translation from actual energy transfer at the interacting material points. For a point particle these distinctions collapse, and the ordinary work–energy theorem applies directly. For a finite sliding body they deserve attention.

At a real contact, microscopic regions deform, attach and detach. Forces act at material points whose displacements need not equal the centre-of-mass displacement. Some energy becomes internal

energy in the body and some in the track. Knowing the resultant friction force and the cart's displacement accurately determines its translational kinetic-energy change under the model; it does not generally determine the partition of contact energy or heat between the materials. This is the reason for separating the two equations in the preceding sliding example.

Example 8.19: A speed change does not determine thermal partition

The object in Example 8.17 loses 12.0 J of translational kinetic energy. Suppose the object plus stationary track is insulated, and no external work or other storage change occurs. Does that establish that the object alone gains 12.0 J internally?

Reasoning. The enlarged-system balance establishes

$$\Delta E_{\text{int,obj}} + \Delta E_{\text{int,track}} = 12.0 \text{ J}.$$

It supplies one equation for two unknown store changes. For example, partitions (3.0, 9.0) J and (8.0, 4.0) J both satisfy it. These are illustrative possible allocations, not measured results. Independent thermal information is necessary to select an allocation. Even temperature changes are insufficient unless the relevant thermal capacities and temperature uniformity are known. For the object alone write $-12.0 \text{ J} + \Delta E_{\text{int,obj}} = W_{\text{contact,on}} + Q$. The same missing thermal information now appears in the external transfer terms and the object's internal store. Its centre-of-mass force integral remains -12.0 J regardless of the unknown partition. There is no contradiction between the translational equation and the full balance: they answer different questions.

An energy model may legitimately neglect these details when its purpose is a speed prediction and when the approximation is declared. It becomes misleading when an ideal particle equation is presented as a measured thermal balance for the object. Naming the approximation preserves the usefulness of simple mechanics while keeping the transition to thermodynamics sound.

8.6 Dissipation and total energy

For an enlarged isolated system in which the only relevant change is conversion of mechanical energy into internal energy, define a non-negative dissipated amount E_{diss} by

$$\Delta E_{\text{mech}} = -E_{\text{diss}}, \quad \Delta E_{\text{int}} = +E_{\text{diss}}, \quad E_{\text{diss}} \geq 0. \quad (8.27)$$

This is a *special-case model*, not the definition of energy conservation. If energy also crosses the boundary through heating, electrical power, a moving actuator or mass transfer, those contributions must be added explicitly.

For simple kinetic sliding under the Coulomb model, the mechanical-energy conversion associated with relative sliding distance s_{rel} is often modeled as

$$E_{\text{diss}} = f_k s_{\text{rel}} = \mu_k N s_{\text{rel}}. \quad (8.28)$$

The relative travel belongs to the contacting surfaces. Work by a particular friction force on a particular body is frame- and boundary-dependent; Eq. (8.28) is an internal conversion model for the contact pair, not a universal statement that every friction force always does negative work in every frame.

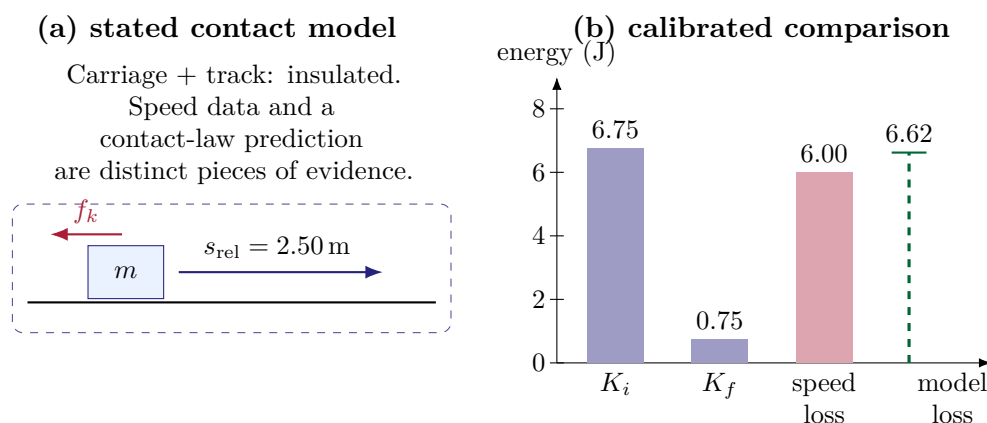


Figure 8.21: Every ordinate uses the same joule scale. For Example 8.20, $K_i = 6.75$ J and $K_f = 0.75$ J imply a 6.00 J mechanical decrease, whereas the stipulated Coulomb model predicts 6.62175 J. The 0.62175 J discrepancy calls for input/model checks; it is not an extra energy sink.

Example 8.20: audit a sliding carriage and track

A 1.50 kg carriage slows from 3.00 to 1.00 ms⁻¹ over a stipulated relative contact travel $s_{\text{rel}} = 2.50$ m on a horizontal track. A simple Coulomb model uses $\mu_k = 0.180$ and $N = mg$. Compare the mechanical-energy decrease inferred from the speeds with the Coulomb-model prediction.

From the speeds,

$$E_{\text{diss}}^{(v)} = K_i - K_f = \frac{1}{2}(1.50)(3.00^2 - 1.00^2) = 6.00 \text{ J.} \quad (8.29)$$

The Coulomb prediction is

$$E_{\text{diss}}^{(C)} = \mu_k m g s_{\text{rel}} = (0.180)(1.50)(9.81)(2.50) = 6.62 \text{ J.} \quad (8.30)$$

The 0.62 J difference is about 10% of the speed-inferred loss. Without uncertainty estimates or additional force/temperature measurements, it would be unjustified to assign the difference to a specific microscopic mechanism. The correct next question is whether the discrepancy exceeds the combined measurement and model uncertainty.

Guided checks

Diagnostic X08-06. A moving conveyor belt can exert a friction force that increases a package's kinetic energy in the laboratory frame. Does that invalidate the statement that sliding contact is dissipative? *No. Work on one selected body and irreversible internal conversion in the two-surface contact are different bookkeeping quantities. Change of system or frame can change the former without reversing the latter.*

8.6.1 Positive friction work and positive dissipation

Let a belt move to the right at fixed speed V while a package moves right at speed $v < V$. The kinetic friction force on the package is $+f$ and the partner force on the belt is $-f$. In an ideal translational description, the instantaneous friction powers are

$$P_{\text{pkg}} = fv, \quad P_{\text{belt}} = -fV.$$

Their sum is $-f(V - v)$, although the power delivered to the package is positive. The positive conversion rate associated with sliding is

$$\dot{E}_{\text{diss}} = f(V - v), \quad v < V. \quad (8.31)$$

For either slip direction, the corresponding Coulomb expression is $f|V - v|$. Thus the sign of friction work on one body is determined by that body's velocity in the chosen frame; the sign of internal conversion is determined by relative sliding. These are complementary statements.

If a motor holds V constant, it supplies the power that friction removes from the belt. The belt's kinetic energy can remain constant while energy passes through it into package motion and internal energy. Consequently, a constant kinetic energy is not evidence of zero energy transfer. The motor is external to a package-plus-belt system unless it has explicitly been included.

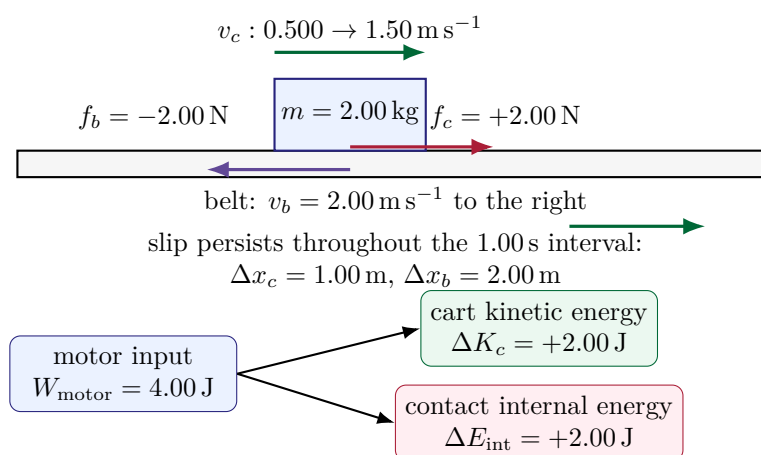


Figure 8.22: Sliding friction can do positive work on a cart moving more slowly than a driven belt. The cart gains 2.00 J , the belt receives -4.00 J of contact work, and 2.00 J becomes internal energy from relative slip. An ideal motor supplies 4.00 J to keep the belt speed constant; vertical motion, belt acceleration and other losses are excluded. Spatial geometry and arrow lengths are schematic; the energy values are quantitative.

Example 8.21: A package accelerated by a moving belt

A stipulated model uses $m = 2.00 \text{ kg}$, $f = 2.00 \text{ N}$ and constant belt speed $V = 2.00 \text{ m s}^{-1}$. Over 1.00 s the package speeds up from 0.500 to 1.500 m s^{-1} . Find the force integrals for package and belt, the sliding conversion, and the required ideal motor input. Neglect belt acceleration and other losses.

Motion check. Newton's law gives $a = f/m = 1.00 \text{ m s}^{-2}$, so the stipulated final speed is consistent. The package travels $s_{\text{pkg}} = v_i t + \frac{1}{2} a t^2 = 1.00 \text{ m}$; the belt travels $s_{\text{belt}} = V t = 2.00 \text{ m}$. Relative travel is therefore 1.00 m , and slip has not reversed because $v_f < V$.

The package force integral is $+f s_{\text{pkg}} = +2.00 \text{ J}$, equal to $\frac{1}{2} m (v_f^2 - v_i^2) = 2.00 \text{ J}$. The partner belt integral is $-f s_{\text{belt}} = -4.00 \text{ J}$. Their sum is -2.00 J , consistent with $E_{\text{diss}} = f s_{\text{rel}} = 2.00 \text{ J}$. Holding the belt speed constant requires ideal motor work $+4.00 \text{ J}$. The package-plus-belt balance is then

$$W_{\text{motor}} = \Delta K_{\text{pkg}} + \Delta E_{\text{int,total}} = 2.00 + 2.00 = 4.00 \text{ J}.$$

The model does not determine which contacting body warms by how much.

Frame check. In the belt frame the package displacement is -1.00 m and its kinetic-energy change is -2.00 J , so its friction force integral is negative. The relative travel and the 2.00 J conversion are unchanged. Work on a body is frame dependent; relative-slip conversion is unchanged by this common Galilean velocity shift.

8.6.2 Storage, recovery and an energy audit over a cycle

Potential energy is recoverable within its conservative idealization: completing a closed configuration cycle gives zero net work for the included conservative force. Sliding conversion is cumulative. If a block traverses a rough segment of length d out and back, and the contact has constant f_k , the contact-pair conversion is $2f_k d$. The final gravitational and elastic potentials may equal their initial values while internal energy has increased. An actuator must supply that increase if the block also returns with its original speed.

This gives a practical test for a proposed storage mechanism. Measure or model the work over the complete charging and discharging cycle, rather than infer recoverability solely from a force-versus-displacement curve on one branch. A hysteresis loop can have nonzero enclosed area because the force depends on loading history. Such a device cannot be described by one single-valued $U(x)$ for both branches without adding internal variables. The area has units of energy and represents the net work under the declared sign convention.

Example 8.22: A rough return cycle

An actuator takes a 1.00 kg block along a horizontal rough guide 0.600 m outward and back. It starts and finishes at rest. Use a constant kinetic friction magnitude $f_k = 1.50$ N whenever it moves and neglect losses during the short reversal interval. Find the net actuator work and compare it with a conservative spring return cycle.

The relative sliding distance is 1.20 m, so $\Delta E_{\text{int, total}} = 1.80$ J for the block-plus-guide. Its net changes in kinetic and potential energy are zero. Thus $W_{\text{actuator}} = 1.80$ J for the enlarged insulated system. Each translating leg requires 0.900 J to offset friction if the intermediate motion is quasistatic; detailed acceleration and braking may redistribute work over the legs but do not alter the endpoint balance when no other stores or transfers change.

For an ideal spring force $F = -k_s x$, the force work is $-\Delta U_s$ and the out-and-back net is zero. A spring and a rough contact can therefore share the same final position without sharing the same cyclic work. This comparison checks the distinction between endpoint storage and path-dependent conversion.

8.7 Engineering, computational and measurement connections

8.7.1 Engineering connection: recoverable and unrecoverable energy

A spring-based launcher, compliant mechanism or energy-storage element can be screened by comparing an ideal recoverable elastic store $\frac{1}{2}k_s x^2$ with a required kinetic or gravitational output. The ideal value is an upper bound within that simplified model. Real devices can also store strain energy in multiple components, produce sound, heat, permanent deformation and vibration, and may be subject to stress, fatigue and safety constraints not represented here.

As a simple scale estimate, a spring with $k_s = 5.0 \text{ kN m}^{-1}$ compressed by 30 mm stores

$$U_s = \frac{1}{2}(5.0 \times 10^3)(0.030)^2 = 2.25 \text{ J}.$$

If a hypothetical process converts 20% of that amount to internal energy before output, at most 1.80 J remains in the modeled mechanical channel. That statement is a teaching energy audit, not a device-efficiency certification.

8.7.2 Computational investigation: energy as a diagnostic

Energy conservation gives a useful test of a numerical trajectory, but it does not prescribe a numerical algorithm. A method can solve the force equation approximately while gradually changing

the energy. That numerical drift is not evidence of physical dissipation unless a dissipative term was actually included. Before interpreting a trajectory, separate the physical model, its numerical approximation, and the diagnostic quantity used to check it.

For the stipulated double well, introduce dimensionless variables

$$u = \frac{x}{\ell}, \quad \theta = \frac{t}{\tau}, \quad \tau = \ell \sqrt{\frac{m}{U_*}}, \quad w = \frac{du}{d\theta}.$$

The scale τ has units of time. Substituting into $m\ddot{x} = -U'(x)$ gives

$$\frac{d^2u}{d\theta^2} = -4u(u^2 - 1), \quad e = \frac{E}{U_*} = \frac{1}{2}w^2 + (u^2 - 1)^2.$$

Here the computational coordinate and velocity are dimensionless. A numerical value of w becomes a physical speed only after multiplying by ℓ/τ . This avoids silently interpreting a drawing scale or a unit-mass benchmark as a universal physical result.

Start at $u_0 = 1.20$, $w_0 = 0$, so $e_0 = 0.1936$. The energy is below the barrier, hence the physical trajectory stays in the right well. Define $f(u) = -4u(u^2 - 1)$. A velocity-Verlet step of size $h = \Delta\theta$ is

$$w_{n+1/2} = w_n + \frac{1}{2}hf(u_n), \quad (8.32)$$

$$u_{n+1} = u_n + hw_{n+1/2}, \quad (8.33)$$

$$w_{n+1} = w_{n+1/2} + \frac{1}{2}hf(u_{n+1}). \quad (8.34)$$

The first half-kick uses the force at the old position, the drift updates the position, and the second half-kick uses the new force. Applying a full old-force velocity change on both ends would be a different method. The diagnostic energy is evaluated from each completed (u_n, w_n) pair, not from the intermediate half-step velocity.

For the same interval $0 \leq \theta \leq 20$, the maximum deviations are

h	$\max e - e_0 $	$\max e - e_0 /e_0$
0.080	0.0029525	0.015251
0.040	0.00073751	0.0038094
0.020	0.00018434	0.00095218
0.010	$4.6083e - 05$	0.00023803

These synthetic results are numerical solutions of the stipulated model, not observations of a physical spring. The maximum energy deviation decreases by about four on halving h in this range. A measured refinement ratio describes this particular error diagnostic and interval; it does not establish that every observable has the same error at every time.

Example 8.23: What does a convergence ratio establish?

Using the table, determine the observed order from $h = 0.040$ and $h = 0.020$, then check whether the trajectory can cross to the left well in this calculation.

The ratio is $R = 0.000737509/0.000184342 = 4.00077$. If this diagnostic behaves as Ch^p in the refinement range, then $p = \log R / \log 2 = 2.00028$. This supports second-order behavior of the energy-error measure over the tested interval. It is a convergence check, not proof of a conservative physical force: the force was specified independently.

At $h = 0.080$ the maximum reported energy is no larger than $e_0 + 0.00295251 = 0.196553$, still far below the barrier at $e = 1$. The exact right-well turning points are $u = \sqrt{1 \pm \sqrt{e_0}}$, namely $\sqrt{0.56} = 0.748331$ and $\sqrt{1.44} = 1.20$. The computed trajectory at $h = 0.010$ has

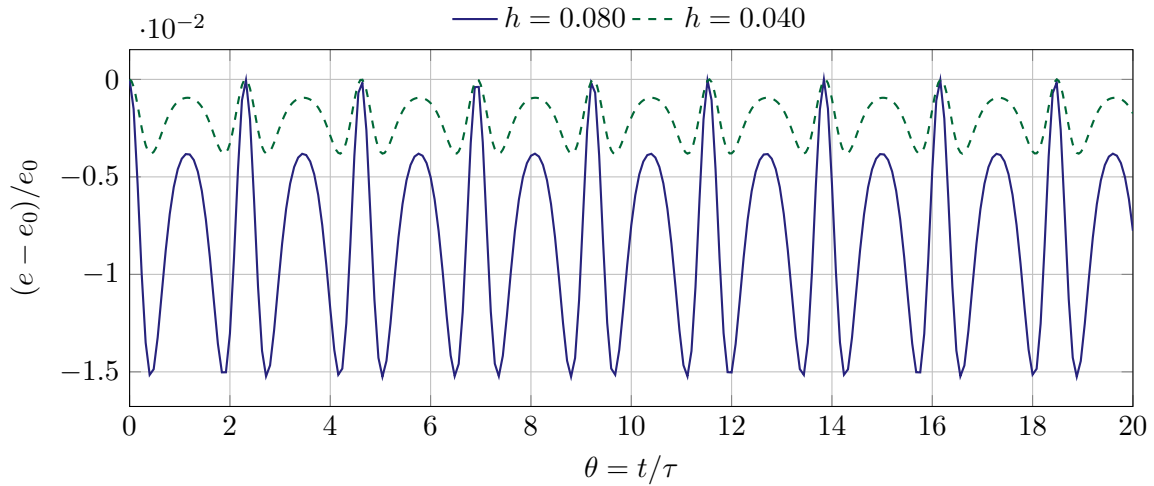


Figure 8.23: Signed energy error in the velocity-Verlet benchmark. Halving the step reduces the bounded error substantially over the same dimensionless interval. These are the embedded computed data, not fitted sinusoidal envelopes. A bounded error plot does not by itself establish exact phase or event timing.

$u_{\min} \simeq 0.74834$ and $u_{\max} = 1.20$. A plotted jump into the left well would be a strong numerical or implementation warning for these initial conditions.

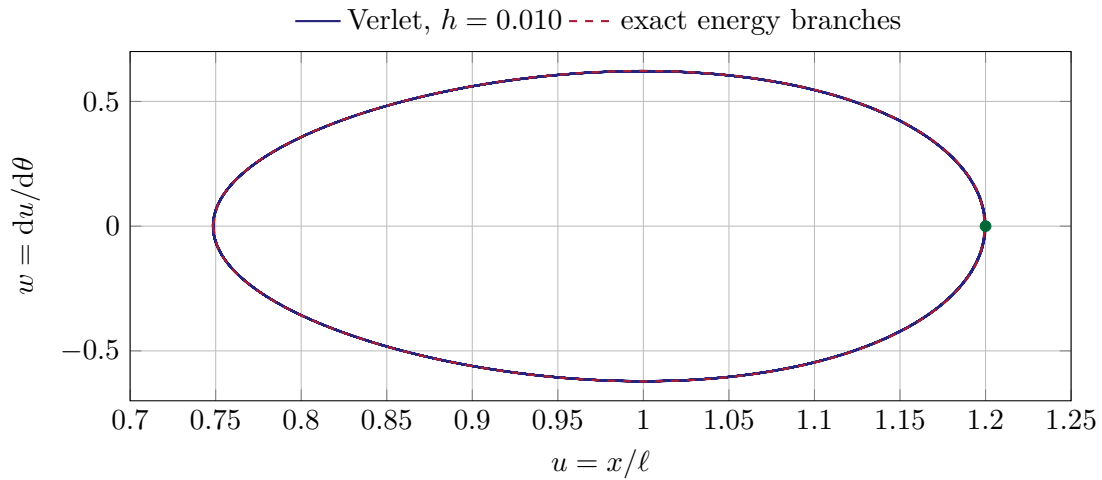


Figure 8.24: Phase-plane comparison for the same right-well trajectory. The two analytical branches follow $w = \pm\sqrt{2[e_0 - (u^2 - 1)^2]}$, while the embedded numerical samples cover $0 \leq \theta \leq 20$. The branches meet at the turning points; they are not two distinct energies. This energy test checks orbit shape but does not independently fix the phase along that orbit.

Listing 8.1: Dimensionless velocity-Verlet update. This optional algorithm is not a compilation dependency.

```
def potential(u): return (u*u - 1.0)**2
def force(u): return -4.0*u*(u*u - 1.0)
def step_verlet(u, w, h):
    whalf = w + 0.5*h*force(u)
    unew = u + h*whalf
    wnew = whalf + 0.5*h*force(unew)
    return unew, wnew
# Completed-state diagnostic: e = 0.5*w*w + potential(u)
# Repeat to theta=20 with h, h/2, h/4; compare over that same interval.
```

An appropriate reference analysis of this computational activity includes the initial energy, turning coordinates, refinement table and a plot against the exact energy branches. Assess conservation error, well confinement and consistency with initial conditions separately. A program that simply rescales the velocity to restore e_0 after every step has imposed the diagnostic rather than independently verified it.

For comparison, explicit forward Euler uses $u_{n+1} = u_n + hw_n$ and $w_{n+1} = w_n + hf(u_n)$. At $h = 0.020$ and $\theta = 20$, this benchmark gives $e = 2.17140$, instead of $e_0 = 0.1936$, and the numerical trajectory has entered the other well. The increase is an algorithmic artifact of this finite-step calculation, not energy supplied by the stated conservative force. For a harmonic benchmark, forward Euler increases the quadratic energy on each step by the exact factor $1 + h^2\omega_0^2$; this separate analytic limit explains why a small-looking position step can still accumulate a large energy error.

If a drag force is deliberately added, mechanical energy should decrease according to its power. Add an accumulated internal store D with $\dot{D} = bv^2$ for stipulated linear drag $-bv$, and test the total mechanical-plus-internal balance instead. A dissipative calculation should neither hold mechanical energy artificially constant nor label all numerical drift as physical heating.

8.7.3 Measurement investigation: is an apparent loss resolved?

A proposed low-energy experiment uses a cart, a spring and a height change. Measure spring compression x , stiffness k_s , initial/final speed, and height difference. Repeat measurements to estimate scatter and calibration uncertainty. Compare

$$E_{\text{unaccounted}} = E_i - E_f$$

with the propagated uncertainty of the measured energy difference. If $|E_{\text{unaccounted}}|$ is not significantly larger than that uncertainty, claiming a resolved loss is not justified. If it is larger, additional measurements such as force-versus-position data, temperature changes, acoustic output or deformation may help identify where energy went. A loss estimate alone does not determine the microscopic mechanism.

8.7.4 A resolved discrepancy and an inverse ambiguity

The uncertainty of an inferred energy difference depends on the actual measured inputs. For a spring-assisted rise ending at rest after the spring has relaxed, define

$$L = \frac{1}{2}k_s x_c^2 - mgh.$$

Within that model a positive L is the mechanical energy not represented by the final gravitational store. It can indicate internal conversion, an omitted energy channel, or a parameter/calibration error. Calling it “frictional heat” requires additional evidence. The uncertainty assessment comes before such a physical interpretation.

For small independent standard uncertainties in k_s, x_c, m, h , the first-order propagation formula is

$$u_L^2 = \left(\frac{1}{2}x_c^2 u_k\right)^2 + (k_s x_c u_x)^2 + (gh u_m)^2 + (mgu_h)^2. \quad (8.35)$$

Each coefficient is the sensitivity of the energy difference to one measured input. The signs of the mass and height derivatives disappear after squaring only because independence has removed covariance terms. A common calibration source may make those terms important. The formula is a local linearization, not a universal exact rule for large uncertainties.

Example 8.24: Is a lower launch height statistically resolved?

Consider proposed measurements, with independent standard uncertainties: $k_s = (300 \pm 6) \text{ N m}^{-1}$, $x_c = (0.150 \pm 0.001) \text{ m}$, $m = (0.500 \pm 0.005) \text{ kg}$, $h = (0.550 \pm 0.005) \text{ m}$. Use $g = 9.81 \text{ m s}^{-2}$ as a stipulated teaching constant and suppose the endpoint speeds are negligible. Find L , its uncertainty, and a defensible interpretation.

The elastic and gravitational terms are 3.37500 and 2.69775 J, so $L = 0.67725 \text{ J}$. The four standard-uncertainty contributions are

$$0.06750, \quad 0.04500, \quad 0.02698, \quad 0.02453 \text{ J},$$

respectively. Their quadrature sum is $u_L = 0.08894 \text{ J}$. A suitable reported result is $L = (0.68 \pm 0.09) \text{ J}$ under the stated standard-uncertainty model. The nominal discrepancy is $7.61u_L$, so measurement scatter represented by this model alone is too small to explain it. This ratio is a comparison with a specified uncertainty budget, not a probability statement valid under arbitrary systematic errors.

Interpretation. At least one part of the ideal endpoint model or the calibrated input description needs investigation. A loss mechanism has not been uniquely identified. If there were no conversion at all, the same final height would instead require

$$k_{\text{inferred}} = \frac{2mgh}{x_c^2} = 239.8 \text{ N m}^{-1}.$$

This is inconsistent with the stipulated $300 \pm 6 \text{ N m}^{-1}$ calibration under that uncertainty model, but it exposes the inverse ambiguity: final height constrains a combination of stiffness, compression and unrepresented energy. Independent stiffness and compression measurements make the audit more informative.

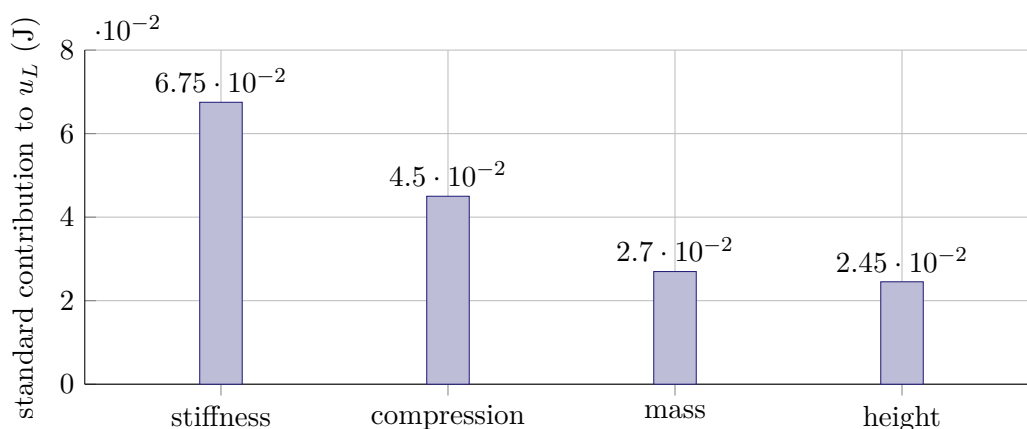


Figure 8.25: Sensitivity-weighted uncertainty contributions for Example 8.24. The bars are standard contributions in joules and are combined in quadrature under the independent-input model; their sum is not u_L . Stiffness calibration is the largest individual contribution in this particular audit.

8.7.5 A proposed investigation with a complete analysis route

A useful investigation records force, displacement and endpoint speeds during both compression and release of a spring mechanism. Record the unloaded encoder zero, sensor sign and force tare before each run. Obtain the spring stiffness from the slope of force against signed extension, rather than from a single ratio F/x that is highly sensitive to an offset near $x = 0$. Integrate the appropriate branch to estimate transferred energy. Repeat the cycle with different speeds to test whether one displacement-only force law describes both branches.

If this is undertaken as an assessed laboratory activity, a defensible reference analysis has four stages. First, state the system, reference and sign conventions. Second, plot raw data with calibration information and fit residuals; identify the data as measured only after an actual acquisition. Third, compare numerical force integrals with endpoint $\Delta K + \Delta U$ and propagated uncertainty. Fourth, interpret any repeatable residual using independently supported mechanisms and state what cannot be resolved. A successful report need not produce zero residual. It should demonstrate a stable result under sampling refinement, a consistent energy balance, and an uncertainty budget that distinguishes scatter from calibration bias.

Example 8.25: An apparent minimum from sensor bias

A synthetic calibration model has true force $F_x = -kx$ with $k = 20.0 \text{ N m}^{-1}$, but a sensor records $F_{\text{rec}} = -kx + b$ with $b = 0.400 \text{ N}$. An analysis chooses $U(0) = 0$ and integrates the recorded force. Find the reconstructed potential and apparent equilibrium. Can an energy-reference shift fix the error?

The integration gives $U_{\text{rec}}(x) = -\int_0^x (-ks + b) ds = \frac{1}{2}kx^2 - bx$. Its derivative vanishes at $x = b/k = 0.0200 \text{ m}$; its curvature remains $k > 0$. At that apparent minimum $U_{\text{rec}} = -b^2/(2k) = -0.00400 \text{ J}$. The true spring has its minimum at $x = 0$, not at the inferred location. Adding a constant changes no derivative and therefore cannot repair the error. Subtracting the calibrated force offset before integration removes the spurious linear term.

The circles in Fig. 8.26 are synthetic noiseless samples. A trapezoidal force integral is exact for this linear model apart from floating-point rounding, so sampling refinement alone cannot reveal the bias: it converges accurately to the wrong potential. The independent tare measurement tests a different assumption from the numerical integration test.

For example, a constant force-sensor offset b produces an energy-integral bias $-b(x_f - x_i)$ in the reconstructed potential. It does not cancel merely because many samples were taken. Repeating a displacement interval twice does not remove the bias either. A force-free reference measurement or an independently calibrated tare is needed. This physical example explains why averaging improves some random errors while leaving a systematic discrepancy intact.

Final height, endpoint speed, force history and temperature change answer different parts of the audit. A richer measurement set is valuable when it resolves an ambiguity; it is not valuable merely because it produces more numbers. The next chapters add further conserved quantities and storage modes, but the same principle remains: compare the information actually measured with the unknowns actually being inferred.

8.8 Chapter summary and a working formula sheet

Summary

For a conservative interaction, work between two configurations is path independent and defines a potential-energy difference by $\Delta U = -W_c$. Potential-energy zeros are conventional; only consistent differences matter. Mechanical energy $K + U$ stays constant when the net effect of boundary transfers and internal conversions on the modeled mechanical stores vanishes throughout the process. A potential curve reveals force through its slope, allowed regions through $E \geq U$, and local stability through its stationary points. Changing the system boundary changes which terms are internal stores and which are boundary transfers. Dissipation converts mechanical energy into internal energy; it does not destroy total energy.

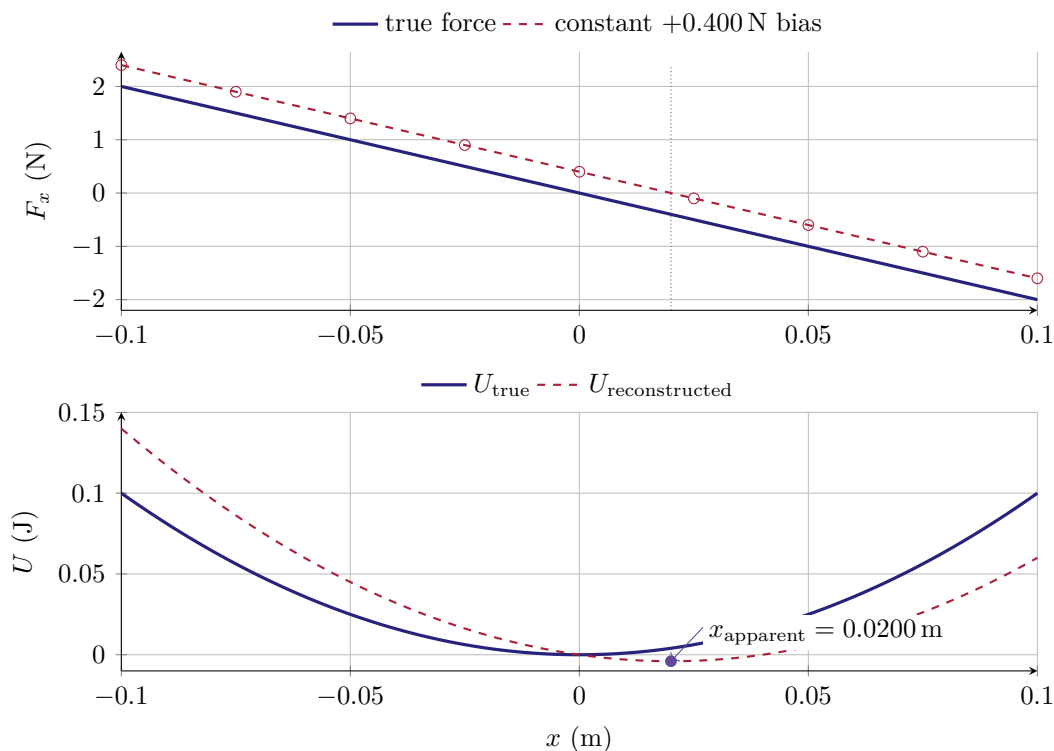


Figure 8.26: A synthetic constant force-sensor bias changes the slope of the reconstructed potential, rather than merely changing its zero. For $k = 20.0 \text{ N m}^{-1}$ and bias $b = 0.400 \text{ N}$, integration gives $U_{\text{reconstructed}} = kx^2/2 - bx$. Its apparent minimum moves from 0 to $b/k = 0.0200 \text{ m}$. Circles are synthetic samples of the stated model, with no random measurement scatter. The offset must be calibrated before interpreting the inferred minimum as a physical equilibrium.

Formula summary

Conservative interaction

$$U_b - U_a = - \int_a^b \mathbf{F}_c \cdot d\mathbf{r}, \quad \oint \mathbf{F}_c \cdot d\mathbf{r} = 0$$

on the stated conservative domain.

Familiar one-dimensional potentials

$$U_g = mgy + C_g, \quad U_s = \frac{1}{2}k_s x^2 + C_s, \quad F_x = -\frac{dU}{dx}.$$

Mechanical-energy conservation, special case

$$K_i + U_i = K_f + U_f.$$

Use only after accounting for nonconservative work, boundary transfer and internal conversion.

General closed-system work-on balance used here

$$\Delta(K + U + E_{\text{int}}) = W_{\text{ext,on}} + Q.$$

Equivalent sign convention: $\Delta E = Q - W_{\text{by}}$.

Dissipative special case for an isolated enlarged system

$$\Delta E_{\text{mech}} = -E_{\text{diss}}, \quad \Delta E_{\text{int}} = +E_{\text{diss}}, \quad E_{\text{diss}} \geq 0.$$

Potential diagram

$$K(x) = E - U(x) \geq 0, \quad F_x = -U'(x).$$

8.9 Advanced Topics for Technical and Engineering Students

The advanced pathway develops two ideas that are useful in vibration analysis, control, stability and multidimensional mechanics: a local quadratic approximation near a stable equilibrium, and the mathematical conditions behind the phrase “conservative force.” Neither result is required for the core exercises.

8.9.1 Small oscillations from local curvature

Let x_0 be a stationary point, $U'(x_0) = 0$, and write $q = x - x_0$. Taylor’s theorem gives

$$U(x_0 + q) = U(x_0) + U'(x_0)q + \frac{1}{2}U''(x_0)q^2 + \mathcal{O}(q^3). \quad (8.36)$$

At the stationary point the linear term vanishes. If

$$k_{\text{eff}} = U''(x_0) > 0, \quad (8.37)$$

then

$$U(x) \simeq U(x_0) + \frac{1}{2}k_{\text{eff}}q^2, \quad F_x \simeq -k_{\text{eff}}q. \quad (8.38)$$

Thus every sufficiently smooth, nondegenerate minimum looks locally like an ideal spring. For a mass m , Newton’s second law then gives the small-oscillation angular frequency $\omega \simeq \sqrt{k_{\text{eff}}/m}$. Chapter 15 develops the time-dependent oscillation theory systematically.

For the double well of Eq. (8.20), expand about the right minimum $x_0 = \ell$. With $\delta = q/\ell$,

$$\frac{U}{U_*} = 4\delta^2 + 4\delta^3 + \delta^4, \quad (8.39)$$

so

$$k_{\text{eff}} = \frac{8U_*}{\ell^2}. \quad (8.40)$$

The cubic term makes the error asymmetric on the two sides of the minimum even though the leading quadratic term is symmetric.

Example 8.26: exact versus local force near a minimum

For the double well, compare the exact and local forces near $x_0 = \ell$. In normalized local coordinate $\delta = q/\ell$,

$$\frac{F_{\text{exact}}\ell}{U_*} = -(8\delta + 12\delta^2 + 4\delta^3), \quad \frac{F_{\text{local}}\ell}{U_*} = -8\delta.$$

For $\delta = 0.050$, the relative force error is 7.08%; for $\delta = 0.100$ it is 13.4%. Solving the 10% criterion separately on the two sides gives approximately

$$-0.0619 < \delta < 0.0723.$$

The asymmetry comes from the cubic term. At $\delta = 0$ the relative force error is undefined because the exact and approximate forces both vanish; an absolute error or curvature comparison is the appropriate check there.

The local model matches U , U' and U'' at the expansion point. It is an approximation with a declared neighborhood, not a new exact potential.

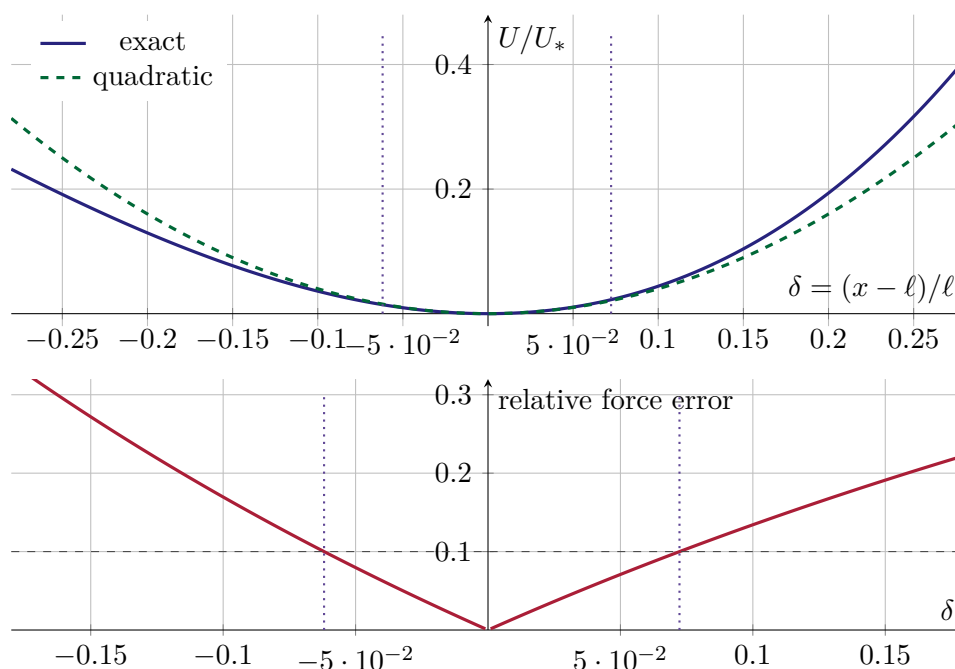


Figure 8.27: Exact and local quadratic descriptions near the right minimum of the stipulated double well. The lower panel uses the force error relative to the exact nonzero force, so the point $\delta = 0$ is omitted because both forces vanish there. The dotted limits mark the approximate interval in which this relative force error is below 10%: $-0.0619 \lesssim \delta \lesssim 0.0723$.

Guided checks

Diagnostic X08-07. A student validates a quadratic approximation only by checking that the curves meet at x_0 . *That is insufficient. Check equal value, zero first derivative, matching second derivative, and an error measure over a finite neighborhood. If $U''(x_0) = 0$, the quadratic test is inconclusive and higher-order terms are required.*

8.9.2 Multidimensional outlook: gradient, curl and domain

For a sufficiently smooth scalar potential $U(x, y, z)$, the conservative force is

$$\mathbf{F} = -\nabla U. \quad (8.41)$$

The gradient points in the direction of steepest increase of U , so the force points locally downhill in the potential. If mixed derivatives are continuous, a necessary local condition is

$$\nabla \times \mathbf{F} = \mathbf{0}.$$

On a simply connected domain this condition is also sufficient for a smooth force field. The domain qualification matters: excluded singular regions can make a curl-free local field fail to admit one single-valued global potential.

As a simple two-dimensional example,

$$\mathbf{F} = \alpha y \hat{\mathbf{i}} + \alpha x \hat{\mathbf{j}}$$

has zero curl and is generated by $U = -\alpha xy + C$. By contrast,

$$\mathbf{F} = \alpha y \hat{\mathbf{i}}$$

has $\partial F_x / \partial y = \alpha$ while $\partial F_y / \partial x = 0$, so it is not conservative on the plane. Around a rectangle of sides a and b its work is $-\alpha ab$ for counterclockwise orientation. Yet an out-and-back path along one line still gives zero work. This is exactly why one chosen zero loop is not a proof of global path independence.

8.9.3 Two-coordinate potentials: the gradient as a calculable force

A one-dimensional potential has a slope. A two-dimensional potential has a slope in every direction, so one derivative cannot describe all the force information. Let x and y denote two independent Cartesian coordinates in metres. The partial derivative $\partial U/\partial x$ means: vary x while holding y fixed. The gradient collects those coordinate slopes,

$$\nabla U = \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y} \right), \quad \mathbf{F} = -\nabla U. \quad (8.42)$$

Each component has units $\text{J m}^{-1} = \text{N}$. The gradient points locally toward the fastest increase in potential energy; the force points toward the fastest decrease. This statement describes the force, not necessarily the velocity. A particle with inertia can move uphill while slowing down, just as a thrown ball rises against gravity.

A contour joins configurations with the same potential energy. Moving an infinitesimal distance along a contour produces no first-order change in U , so the force is perpendicular to that contour wherever the gradient is nonzero. At a stationary point the force vanishes and a direction is not defined by the gradient. Consequently, the absence of an arrow at the origin in the contour diagram has physical meaning. A Cartesian plot is especially useful because its distances are actual coordinate differences; a plot made in angular or generalized coordinates would require a different kinetic-energy and geometric interpretation.

For an infinitesimal displacement, ordinary differentiation along a path gives

$$dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy = -\mathbf{F} \cdot d\mathbf{r}. \quad (8.43)$$

Integrating this exact differential yields $W_{A \rightarrow B} = U(A) - U(B)$. The trajectory may have a complicated shape, but the integral uses its endpoint configurations. It is nevertheless useful to evaluate two different paths once: that calculation checks signs and demonstrates how the separate force components contribute.

A useful engineering model is the coupled quadratic potential

$$U(x, y) = \frac{1}{2}(ax^2 + 2bxy + cy^2), \quad F_x = -ax - by, \quad F_y = -bx - cy. \quad (8.44)$$

Here a, b, c have units N m^{-1} , and the zero is chosen at the origin. The coupling term bxy means that changing one coordinate changes the force associated with the other. Two springs aligned with the coordinate axes would have $b = 0$; an inclined or coupled compliant mechanism can produce a nonzero cross term. Equation (8.44) is a stipulated local model, not a claim that all real mechanisms remain quadratic at large displacement.

Example 8.27: Two paths through a coupled conservative field

A particle is guided quasistatically from $O = (0, 0)$ to $P = (0.20, 0.10)$ m in the field of Eq. (8.44), with $a = 10$, $b = 2$ and $c = 6 \text{ N m}^{-1}$. Find the field's work on the particle along the two axis-segment routes, find the final force, and explain the external agent's energy transfer.

Model and strategy. The supplied field is conservative, stationary and defined throughout the plane. The particle has negligible change in kinetic energy during the slow guided transfer. Calculate field work directly on each segment, then compare with the potential difference. The guide's force need not be parallel to the displacement at every point.

Route 1: horizontal, then vertical. At $y = 0$, $F_x = -10x$, so

$$W_1^{(x)} = \int_0^{0.20} (-10x) dx = -0.200 \text{ J}.$$

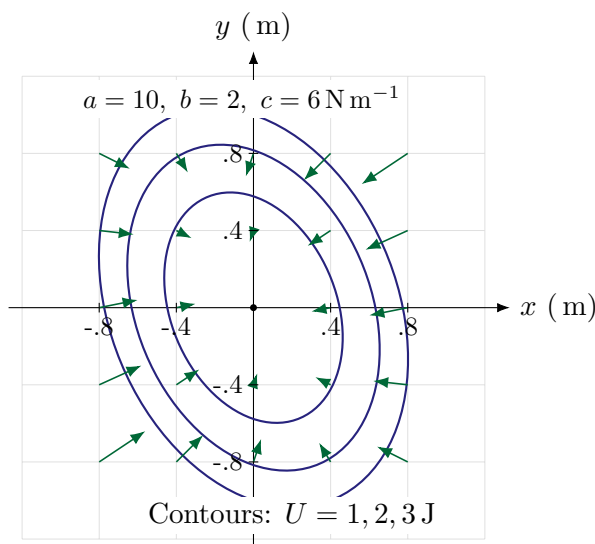


Figure 8.28: An original two-coordinate energy landscape. The nested ellipses are contours of the same potential used to compute every force arrow. Arrows point across contours toward lower U , rather than along the contours; arrow length represents force magnitude with scale 0.025 m N^{-1} . The origin is a stable minimum. The energy contours are representations of configuration energy, not outlines of a physical bowl.

At $x = 0.20 \text{ m}$, $F_y = -0.400 - 6y$ in newtons. Therefore

$$W_1^{(y)} = \int_0^{0.10} (-0.400 - 6y) dy = -0.0400 - 0.0300 = -0.0700 \text{ J}.$$

Their sum is $W_1 = -0.270 \text{ J}$.

Route 2: vertical, then horizontal. At $x = 0$, the vertical work is $-3(0.10)^2 = -0.0300 \text{ J}$. At $y = 0.10 \text{ m}$, the horizontal force is $F_x = -10x - 0.200$, giving

$$W_2^{(x)} = \int_0^{0.20} (-10x - 0.200) dx = -0.200 - 0.0400 = -0.240 \text{ J}.$$

Thus $W_2 = -0.270 \text{ J}$, the same total although the segment contributions differ.

Independent endpoint and force checks. The potential at P is

$$U(P) = \frac{1}{2} [10(0.20)^2 + 4(0.20)(0.10) + 6(0.10)^2] = 0.270 \text{ J},$$

while $U(O) = 0$. Hence $-\Delta U = -0.270 \text{ J}$, confirming both routes. The final force is

$$\mathbf{F}(P) = (-2.20\hat{i} - 1.00\hat{j}) \text{ N}.$$

The quasistatic external agent supplies $+0.270 \text{ J}$ to the particle-plus-interaction energy account. The field does the opposite signed work on the particle. These are two descriptions of the same transfer; they must not be added as two increases of stored energy. Reversing either complete route reverses the field work and restores the original potential configuration.

8.9.4 A minimum in two directions can still hide a saddle

Near a stationary point (x_0, y_0) , put $q_x = x - x_0$ and $q_y = y - y_0$. A multivariable Taylor expansion is the two-coordinate version of the one-dimensional expansion already developed: the constant fixes the reference, the linear terms vanish at equilibrium, and the quadratic terms describe the

leading shape,

$$U \simeq U_0 + \frac{1}{2} (H_{xx}q_x^2 + 2H_{xy}q_xq_y + H_{yy}q_y^2). \quad (8.45)$$

The matrix H , whose entries are second partial derivatives evaluated at the equilibrium, is called the *Hessian*. Equality $H_{xy} = H_{yx}$ follows when the mixed second derivatives are continuous. Its entries have units of stiffness. The local restoring force is $\mathbf{F} \simeq -H\mathbf{q}$.

The Hessian does not describe the absolute energy level: adding a constant leaves every one of its entries unchanged. It records how rapidly the forces change with displacement. A large positive stiffness means a strong restoring response in that direction; a small positive stiffness means a softer response. A negative stiffness means that the force reinforces a displacement instead of opposing it. This directional interpretation allows us to diagnose a mechanism without solving its complete motion.

A stable nondegenerate minimum requires the quadratic energy increase to be positive for *every* nonzero small displacement. Checking just the x and y axes can miss a downhill diagonal. For the symmetric two-coordinate matrix

$$H = \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

completion of the square gives, for $a > 0$,

$$aq_x^2 + 2bq_xq_y + cq_y^2 = a \left(q_x + \frac{b}{a}q_y \right)^2 + \frac{ac - b^2}{a}q_y^2. \quad (8.46)$$

Therefore $a > 0$ and $ac - b^2 > 0$ establish a positive quadratic form. A negative determinant gives one increasing and one decreasing principal direction: a saddle, hence an unstable equilibrium. If the determinant vanishes, the quadratic test is incomplete; higher-order terms must decide what happens along the flat direction.

For the preceding $a = 10$, $c = 6$, $b = 2 \text{ N m}^{-1}$, the determinant is $56 \text{ N}^2 \text{ m}^{-2} > 0$. The two principal stiffnesses, obtained by solving $\det(H - \lambda I) = 0$, are

$$\lambda_{\pm} = \frac{a+c}{2} \pm \sqrt{\left(\frac{a-c}{2}\right)^2 + b^2} = 8 \pm \sqrt{8} \text{ N m}^{-1}. \quad (8.47)$$

Both are positive: approximately 10.83 and 5.17 N m^{-1} . The principal directions are rotated relative to the coordinate axes; this explains the tilted ellipses in Fig. 8.28. These numbers are stiffnesses, not frequencies: mass information and the kinetic-energy model are also needed for a vibration analysis.

Now keep $a = 10$, $c = 6$ but increase the coupling to $b = 8 \text{ N m}^{-1}$. Both coordinate-axis cuts still look like upward parabolas, yet $ac - b^2 = -4 \text{ N}^2 \text{ m}^{-2}$. Along the nearby diagonal $q_x = -0.8q_y$, direct substitution gives

$$U - U_0 = -0.20 \text{ N m}^{-1} q_y^2.$$

The origin is therefore a saddle. Its principal stiffnesses are $8 \pm \sqrt{68} \text{ N m}^{-1}$, approximately 16.25 and -0.246 N m^{-1} . A small perturbation in the negative-stiffness direction experiences a force pushing it away from the origin. A controller can change that behavior, but the passive conservative model by itself is unstable.

8.9.5 Why a curl-free field needs a domain statement

The equality of mixed derivatives tests local compatibility with a potential. A global potential also requires consistent integration around every admissible closed loop. This distinction is visible without a full vector-calculus course. Consider the stipulated planar field

$$\mathbf{F}(x, y) = A \frac{-y\hat{\mathbf{i}} + x\hat{\mathbf{j}}}{x^2 + y^2}, \quad A > 0, \quad (x, y) \neq (0, 0). \quad (8.48)$$

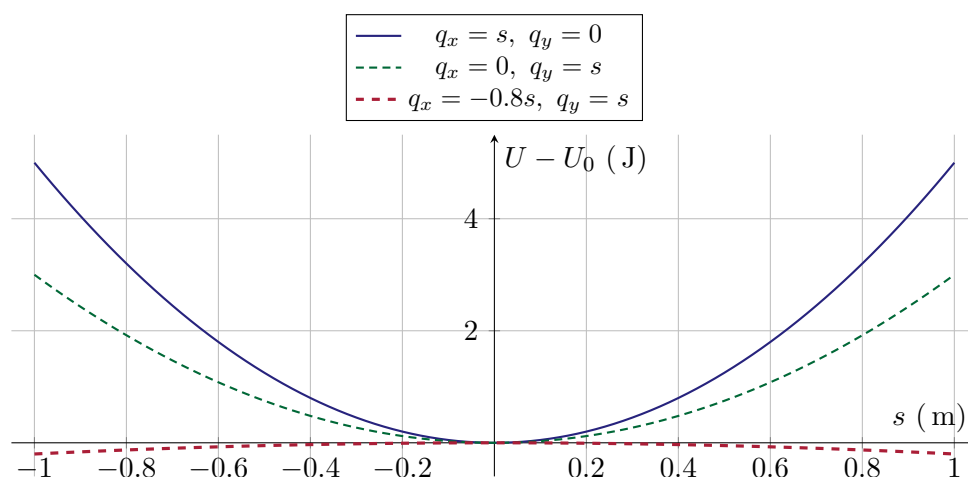


Figure 8.29: Three directional cuts of the same saddle potential with $a = 10$, $b = 8$, $c = 6 \text{ N m}^{-1}$. Looking only along the coordinate axes would falsely suggest a minimum. The diagonal cut decreases below U_0 for every nonzero s , however small. The coordinate s is a path parameter; its diagonal displacement magnitude is $\sqrt{1 + 0.8^2} |s|$, so the diagonal coefficient is not itself a principal stiffness.

A simply connected planar region has no hole that prevents a closed route from contracting while staying inside the region. The full punctured plane fails this condition. Removing an additional ray from the origin opens the angular coordinate into one continuous branch and prevents a route from going all the way around the puncture. On that smaller region a potential can be chosen consistently. Thus a domain restriction changes the global mathematical conclusion even though the force at each retained point has not changed.

The coefficient A has units of joules. The origin is excluded because the force is singular there; no route may pass through it. On a circle, the field points counterclockwise tangent to the circle. Direct differentiation gives

$$\frac{\partial F_y}{\partial x} = A \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial F_x}{\partial y}.$$

The field is curl-free everywhere in its allowed domain. Nevertheless, a loop enclosing the excluded point cannot be continuously shrunk to a point while remaining in that domain. This is the geometric content of the missing simple-connectivity condition.

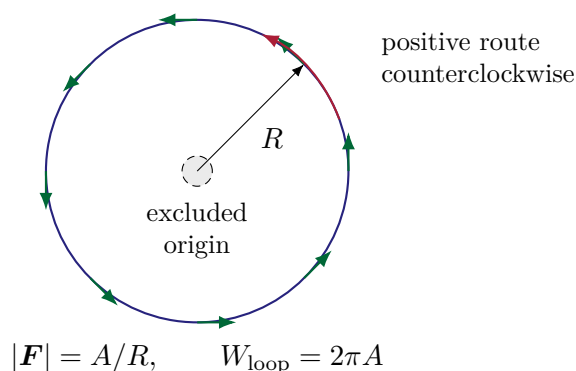


Figure 8.30: A curl-free field on a punctured plane with nonzero circulation around the puncture. The small shaded disk is a drawing aid indicating the excluded singular point; it is not an additional finite-radius singular region. The green arrows show force direction. Reversing the traversal reverses work, while changing the loop radius changes the force magnitude but not its complete-loop integral.

Example 8.28: A loop that rules out a global potential

Take $A = 0.400 \text{ J}$ in Eq. (8.48). Calculate the force magnitude and work for a counterclockwise circle of radius $R = 0.500 \text{ m}$. Compare the upper and lower semicircle routes from $(R, 0)$ to $(-R, 0)$ and decide whether a single-valued potential exists on the entire punctured plane.

Strategy. Parameterize the geometry rather than infer a work value from a local derivative test. On a circle,

$$\mathbf{r} = R(\cos \theta, \sin \theta), \quad d\mathbf{r} = R(-\sin \theta, \cos \theta)d\theta, \quad \mathbf{F} = \frac{A}{R}(-\sin \theta, \cos \theta).$$

Consequently $\mathbf{F} \cdot d\mathbf{r} = Ad\theta$. The complete-loop work and force magnitude are

$$W_{\text{CCW}} = \int_0^{2\pi} Ad\theta = 2\pi A = 2.51 \text{ J}, \quad |\mathbf{F}| = A/R = 0.800 \text{ N}.$$

For the upper route, $\theta : 0 \rightarrow \pi$, so $W_{\text{upper}} = +\pi A = +1.26 \text{ J}$. For the lower route, the continuous angle decreases from 0 to $-\pi$, giving $W_{\text{lower}} = -\pi A = -1.26 \text{ J}$. The two routes have identical endpoints but unequal work. No single-valued global U can produce this field throughout the punctured plane.

Local interpretation and checks. On a region with a chosen angular branch, $U = -A\theta + C$ generates the field. Completing one positive circuit changes that expression by $-2\pi A$, so it cannot assign the same value to the same configuration globally. A larger circular radius would reduce $|\mathbf{F}|$ but increase the circumference in precisely the compensating proportion. The field is a prescribed mathematical force maintained by an external agent: positive loop work represents an energy transfer to the particle, not frictional heating and not energy creation. Local curl-free behavior is insufficient to establish a conservative force on a domain containing this noncontractible loop.

8.9.6 An explicitly changing potential: energy supplied by an actuator

A potential can depend on time as well as position. The force relation at a fixed time remains $F_x = -\partial U/\partial x$, but constant mechanical energy no longer follows. For an ideal spring whose unstretched position is fixed while an external actuator changes its stiffness,

$$U(x, t) = \frac{1}{2}k(t)x^2, \quad F_x = -k(t)x. \quad (8.49)$$

Assume a constant-mass cart, an ideal massless spring, negligible friction and an actuator model with no unlisted energy stores. The spring-plus-cart system excludes the actuator. Differentiate $E = mv^2/2 + U(x, t)$ along the actual motion:

$$\frac{dE}{dt} = mv\dot{v} + \frac{\partial U}{\partial x}\dot{x} + \frac{\partial U}{\partial t} \quad (8.50)$$

$$= v(F_x + \partial U/\partial x) + \frac{\partial U}{\partial t} = \frac{1}{2}\dot{k}(t)x(t)^2. \quad (8.51)$$

If the actuator were included inside an enlarged system, the same $1/2\dot{k}x^2$ term would describe transfer between its internal energy store and the spring-plus-cart mechanical store. It would cease to be an external transfer for that enlarged boundary. The complete energy of an insulated actuator-plus-mechanism system could remain constant while the mechanism's mechanical energy increased. Specifying the boundary therefore resolves the apparent conflict between a conservative instantaneous force law and a changing mechanical energy.

The first two terms cancel because $v = \dot{x}$ and $m\dot{v} = F_x$. The remaining term is the modeled actuator power entering the mechanical store. Increasing k at nonzero extension adds energy; decreasing it

removes energy in the reversible ideal model. Changing k exactly at $x = 0$ gives zero instantaneous transfer by this expression. There is no contradiction with the absence of friction: the system exchanges work with the stiffness-control actuator.

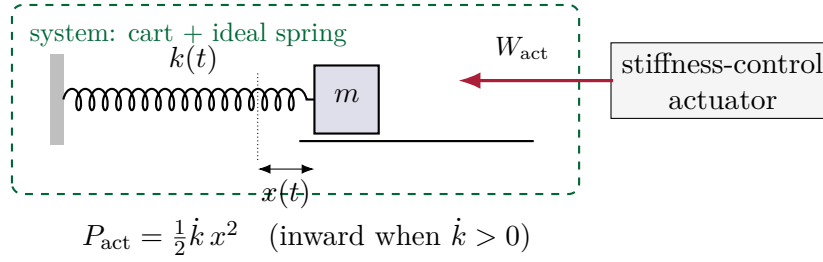


Figure 8.31: The idealized energy boundary for a tunable spring. The actuator is external; its modeled parameter work crosses the dashed boundary, changing the stored mechanical energy. The fixed support does no translational work. The drawing suppresses actuator internals, which would need their own energy accounting in a complete device model. Extension is measured from the unchanged relaxed position.

Example 8.29: Stiffness tuning, release, and a time-dependent energy label

A clamp holds a 0.300 kg cart at spring extension $x_h = 0.120$ m. Over 3.00 s, a reversible actuator raises the stiffness uniformly from 200 to 350 N m⁻¹. Find its work and power, then the cart's maximum speed after release with the final stiffness fixed. Finally repeat the energy ledger using the time-dependent reference $C(t) = (0.400 \text{ J s}^{-1})t$ during the tuning interval.

Held tuning. The clamp prevents motion, so both endpoint kinetic energies vanish. The spring energies are $U_i = 200(0.120)^2/2 = 1.44 \text{ J}$ and $U_f = 350(0.120)^2/2 = 2.52 \text{ J}$. Hence

$$W_{\text{act}} = \frac{1}{2}(k_f - k_i)x_h^2 = 1.08 \text{ J}.$$

Uniform tuning gives $\dot{k} = 50.0 \text{ N m}^{-1} \text{ s}^{-1}$, so $P_{\text{act}} = \dot{k}x_h^2/2 = 0.360 \text{ W}$ and $P_{\text{act}}\Delta t = 1.08 \text{ J}$. The clamp balances a spring force whose magnitude rises from 24.0 to 42.0 N, but its held point does not move: force alone is not work.

Release. After tuning, the stiffness remains fixed, the clamp is removed, and no further transfer is modeled. At $x = 0$, all 2.52 J is kinetic:

$$v_{\text{max}} = \sqrt{\frac{2(2.52)}{0.300}} = 4.10 \text{ m s}^{-1}.$$

This uses a new process interval with its own initial state. Retaining the pre-tuning energy 1.44 J would omit the actuator's work.

Changing reference. If $\tilde{U} = U + C(t)$, the force is unchanged because $\partial C/\partial x = 0$. The reported mechanical-energy change during tuning becomes

$$\Delta\tilde{E} = 1.08 + (0.400)(3.00) = 2.28 \text{ J}.$$

Only 1.08 J crossed the physical boundary. The additional 1.20 J is a change of energy labels, accounted for by

$$\frac{d\tilde{E}}{dt} = P_{\text{act}} + \dot{C}, \quad \Delta(\tilde{E} - C) = W_{\text{act}}.$$

A constant reference shift has $\dot{C} = 0$ and the familiar invariance. A time-dependent shift still leaves trajectories unchanged, but the numerical derivative of the shifted energy requires the extra reference term. It must not be interpreted as another actuator, another energy store or a physical transfer. If the cart moved during tuning, the physical actuator work would instead be $\int \dot{k}x(t)^2 dt/2$; endpoint stiffnesses alone would not determine it without the motion history.

8.9.7 Effective potentials: a preview

In later mechanics, constraints and conserved quantities can sometimes be folded into an *effective potential* whose minima and barriers summarize reduced motion. The method is powerful in rotational and orbital problems, but the effective potential depends on which constants and coordinates have been eliminated. Chapter 13 develops its orbital use; Chapters 10–11 supply the rotational prerequisites. The lesson to carry forward is methodological: always state what has been absorbed into an “effective” energy landscape before interpreting its slopes and turning points.

8.10 Exercises and problems

Use $g = 9.81 \text{ m s}^{-2}$. Numerical inputs are stipulated unless explicitly described as proposed measurements. State the system, reference convention and loss assumptions before selecting an equation. Problems 31–34, 39 and 42 use the optional mathematical enrichment; the other problems use the core treatment. The double-well potential used below is $U(x) = U_*[(x/\ell)^2 - 1]^2$, with $U_*, \ell > 0$.

A. Potential differences and reference choices

1. A 1.50 kg package moves from $y = 2.0 \text{ m}$ to $y = 5.5 \text{ m}$. (a) Find W_g and ΔU_g . (b) Repeat for the reverse trip and check the round-trip work.
2. For $F_x = -ax^3$, where $a = 4.0 \text{ N m}^{-3}$, (a) derive $U(x) - U(0)$ and (b) check the derivative relation and dimensions.
3. An ideal spring has $k_s = 120 \text{ N m}^{-1}$. (a) Compute ΔU_s from $x_i = -0.10 \text{ m}$ to $x_f = +0.20 \text{ m}$. (b) Explain why signed extension does not give the sign of U_s under the conventional relaxed-spring zero.
4. The descriptions $U_1(x) = 3x^2 \text{ J}$ and $U_2(x) = 3x^2 - 11 \text{ J}$ use x in metres. (a) Compare their forces. (b) If a state has $E_1 = 20 \text{ J}$, find the energy label of the same state under U_2 and verify kinetic-energy invariance.
5. For $\mathbf{F} = \alpha y \hat{\mathbf{i}}$, with $\alpha \neq 0$, (a) show that an out-and-back horizontal segment at fixed y has zero work. (b) Evaluate work around a counterclockwise rectangle with corners $(0, 0)$, $(a, 0)$, (a, b) and $(0, b)$, where $a, b > 0$. (c) Explain what this establishes about testing conservativity.
6. A student writes $U_b - U_a = + \int_a^b \mathbf{F}_c \cdot d\mathbf{r}$. Give a physical sign countercheck using a falling object and repair the equation.

B. Mechanical-energy conservation and reachability

7. A 0.40 kg cart starts from rest at height 1.80 m on a frictionless track. Find its speed at 0.50 m. Assume it remains in contact and the connecting track has no intervening energy barrier above its release height.
8. A 0.25 kg block moving at 3.0 m s^{-1} compresses a horizontal ideal spring of stiffness 180 N m^{-1} on a frictionless guide. Find maximum compression.
9. A 0.60 kg mass starts from rest with 4.5 J stored in a spring. That store is fully released before the final configuration. (a) Find its speed after a 0.50 m rise, neglecting losses. (b) Is a 1.0 m rise reachable? Assume no extra barrier on the upward path.
10. Repeat Problem 9 if 0.80 J converts to internal energy before the stated final configuration. (a) Find the speed at 0.50 m. (b) Assess the 1.0 m rise. (c) State the energy condition for reachability and its limitation when an intervening barrier exists.

11. An ideal pendulum bob starts from rest 0.20 m above its lowest point. Neglect air resistance. Find its bottom speed without calculating tension, and explain why tension does not enter the energy balance.
12. A solver shifts the gravitational zero upward by 2.0 m but changes only U_f . Explain the error and demonstrate how a consistent shift cancels from the conservation equation.

C. Energy landscapes

13. For $U(x) = Ax^2$, with $A > 0$, (a) determine the force and (b) classify $x = 0$ physically.
14. For $U(x) = -Ax^2$, with $A > 0$, (a) determine the force and (b) classify $x = 0$ physically.
15. For the stated double well, (a) derive every stationary point and (b) classify each using U'' or slope changes.
16. For $U_* = 9.0$ J, $\ell = 0.30$ m and $E = 4.5$ J, (a) find every dimensional turning position and (b) identify the connected allowed intervals and whether a trajectory can visit both wells.
17. For the same landscape, (a) can a carriage with $E = 8.0$ J cross between wells? (b) State the condition for passage over the barrier in finite time, and explain what happens at $E = 9.0$ J. Distinguish a threshold infimum from an attained minimum crossing energy.
18. An energy line intersects $U(x)$ where $U'(x) > 0$. (a) What is the force sign? (b) If a particle arrives from the left and momentarily has zero speed there, which way does it accelerate?

D. System boundaries and dissipation

19. A 3.0 kg crate slows from 5.0 to 3.0 m s⁻¹ on a horizontal rough surface. (a) Find ΔK . (b) Give the crate's translational work-kinetic account. (c) Give the total-energy account for isolated crate+surface. Explain why the particle account alone does not describe the crate's warming.
20. In Problem 19, can ΔK alone determine the portion of the internal-energy increase in the crate? Explain what extra information would determine that portion.
21. A motor does +120 J of work on a closed system while 35 J leaves by heating. Find $\Delta(K+U+E_{\text{int}})$ using work-on signs.
22. A system receives $Q = +50$ J and does $W_{\text{by}} = +20$ J. Find its total-energy change and check using W_{on} .
23. A 2.0 kg block slides 3.0 m relative to a stationary horizontal surface, with $\mu_k = 0.25$. (a) Compute Coulomb dissipation. (b) If its initial speed is 4.0 m s⁻¹, can it complete that distance without another input? Find its speed there if reachable.
24. Explain why “friction always does negative work” requires a named body and frame. Give a moving-belt counterexample and reconcile positive work on the body with positive thermal conversion in the contact pair.

E. Engineering, computation and measurement

25. A spring store has $k_s = 8.0$ kN m⁻¹ and usable compression 25 mm. (a) Find ideal stored energy. (b) If a stipulated 15% converts to internal energy before launch, find the remaining modeled mechanical output.
26. In the normalized velocity-Verlet benchmark of Section 8.7, the maximum absolute energy deviations for $\Delta t = 0.040$ and 0.020 are 7.38×10^{-4} and 1.84×10^{-4} . (a) Compute their ratio.

- (b) Estimate the apparent error order and explain what this test establishes and what it does not.
27. In a proposed experiment, $m = (0.500 \pm 0.005)$ kg rises through $h = (0.400 \pm 0.003)$ m; uncertainties are independent standard uncertainties and g is exact for this exercise. (a) Propagate the standard uncertainty of mgh . (b) Would an unaccounted 0.010 J be clearly resolved at one standard uncertainty?
28. A force sensor and encoder produce $U(x) - U(0) = -\int_0^x F(s)ds$. Give at least four validation checks before interpreting a reconstructed minimum as stable equilibrium. Explain how your checks distinguish a physical minimum from a calibration or sampling artifact.
29. A measured spring-assisted lift falls short of the ideal predicted height. Give three distinct explanations that final height alone cannot separate, and a further observation that would help distinguish each.
30. A battery drives a motor that accelerates a cart while the motor warms. Construct two energy diagrams: (a) cart alone; (b) battery+motor+cart, isolated for the short process. (c) State which transfer changes category when the boundary expands. Assume the cart's own internal energy and height stay constant. Neglect changes in motor rotational energy and other unlisted stores.

F. Advanced challenges

31. (a) Derive the double well's local effective stiffness directly from U'' . (b) Find angular and ordinary small-oscillation frequencies near $x = +\ell$ for $U_* = 4.0$ J, $\ell = 0.50$ m and $m = 0.80$ kg.
32. (a) Derive the exact relative force-error expression for the local double-well approximation, using $\delta = (x - \ell)/\ell$. (b) Explain why relative error is inappropriate at the minimum and identify any other denominator zeros.
33. For $\mathbf{F} = \alpha y \hat{\mathbf{i}} + \alpha x \hat{\mathbf{j}}$, (a) construct a potential and check both components. (b) Evaluate work from $(0, 0)$ to (a, b) along both piecewise-axis paths.
34. A smooth potential satisfies $U'(x_0) = U''(x_0) = U'''(x_0) = 0$ and $U^{(4)}(x_0) > 0$. (a) Explain why the quadratic test is inconclusive. (b) Find the leading local form and classify the equilibrium.

G. Synthesis: graphs, inverse models and engineering decisions

35. A 0.50 kg carriage starts at $x = 0$ with $v = +2.0 \text{ m s}^{-1}$ on a horizontal guide. Its only tangential force is the position-dependent force in Figure 8.32; the final -2.0 N segment continues for every $x \geq 3.0$ m. Choose $U(0) = 0$. (a) Find work over 0–1, 1–3 and 3–4 m. (b) Construct continuous $U(x)$ on $x \geq 0$. (c) Find speed at $x = 3.0$ m and the first rightward turning position. (d) Locate the potential minimum, classify it, and find the maximum speed before the rightward turn. Read signed areas and slopes before calculating.

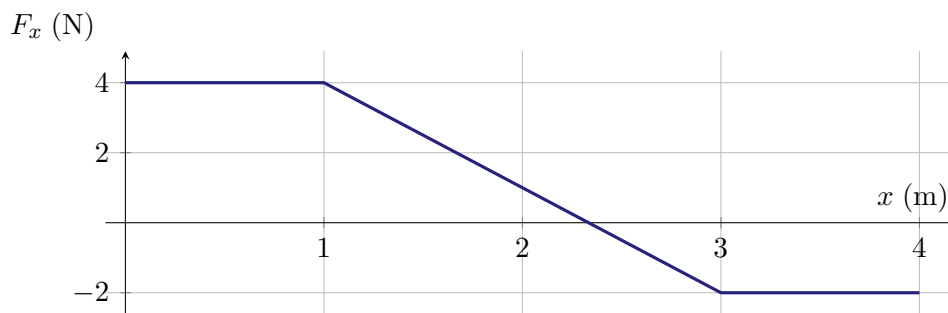


Figure 8.32: Given force profile for Problem 35. The force is constant over the first metre, then varies linearly, and remains -2.0 N beyond $x = 3.0$ m. The diagram supplies the force; it does not supply the potential or stopping point.

- 36.** A 1.20 kg inspection mass falls from rest $h = 0.300$ m above an ideal upright spring of stiffness 480 N m^{-1} . The guided mass compresses the massless spring without dissipation. Let $s \geq 0$ measure downward compression from first contact, as in Figure 8.33. (a) Find maximum compression. (b) Find the compression and speed where speed is greatest. (c) Find the net force at maximum compression, with downward positive, and explain why zero speed does not mean equilibrium.

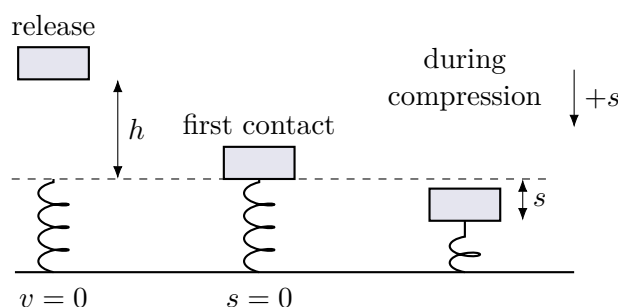


Figure 8.33: Coordinate definition for Problem 36; geometry is schematic. Gravity acts throughout the additional compression distance, so the total descent is $h + s$. The third mass position is an arbitrary intermediate state.

- 37.** A 2.0 kg crate is placed gently, with zero horizontal speed, on a long belt moving at constant laboratory speed $V = 1.50 \text{ m s}^{-1}$. During sliding, $\mu_k = 0.30$; ignore the placement's vertical dynamics. (a) Find sliding duration and laboratory travel distances of crate and belt. (b) Find friction work on the crate and thermal conversion in the crate–belt contact. (c) Find work supplied by the belt motor during that interval and reconcile the full energy budget.
- 38.** A 0.50 kg slider is released from rest 0.440 m above a horizontal rough patch of length $L = 0.800$ m and $\mu_k = 0.20$. Smooth rising tracks flank the patch; they are frictionless, high enough to reverse any surviving motion, and join the patch without impact. The first crossing is from left to right. (a) Find the first right-hand and subsequent left-hand turning heights above the patch. (b) Locate its eventual stop measured from the patch's left edge. (c) Find total internal-energy increase if slider+tracks+Earth are isolated.
- 39.** On the plane excluding the origin, consider $\mathbf{F} = \kappa(-y, x)/(x^2 + y^2)$, where $\kappa = 3.0$ J. (a) Show the field is curl-free for $r > 0$. (b) Find work around a counterclockwise circle of radius 0.80 m and along a radial segment. (c) Decide whether a single-valued potential exists on the whole punctured plane; give a local potential on an angular branch.
- 40.** A quasi-static extension test gives the stipulated data below; the applied load balances the spring's restoring force.

x (m)	0	0.050	0.100	0.150
F_{app} (N)	0	10.0	21.0	33.0

(a) Fit $F_{\text{app}} = kx + \beta x^2$ to the nonzero data and check all points. (b) Derive the stored energy at 0.150 m under that reversible model. (c) Estimate the same area by composite trapezoids, compare results, and identify the sign of the quadrature bias. (d) Can extension data alone establish recoverability or conservativity?

41. A proposed spring lift starts and ends at rest; the spring is fully relaxed at the end. Independent measurements are $k = (120 \pm 3) \text{ N m}^{-1}$, $x = (0.200 \pm 0.002) \text{ m}$, $m = (0.500 \pm 0.005) \text{ kg}$ and $h = (0.400 \pm 0.003) \text{ m}$. All uncertainties are standard uncertainties; take g exact. (a) Find the energy residual $R = \frac{1}{2}kx^2 - mgh$ and its propagated uncertainty. (b) Is R resolved relative to these uncertainties? (c) Find the apparent gravitational-output fraction and its uncertainty. Explain why the residual alone does not identify a heating mechanism.
42. A 0.80 kg carriage crosses the central region of the double well with $U_* = 9.0 \text{ J}$, $\ell = 0.30 \text{ m}$ and $E = 9.18 \text{ J}$, moving rightward from $x = -0.100$ to $+0.100 \text{ m}$. (a) Find its speed at the barrier top. (b) Estimate the transit time with composite Simpson quadrature using $n = 8$ equal intervals; use

$$t = \sqrt{m/2} \int_{-0.100}^{0.100} \frac{dx}{\sqrt{E - U(x)}}, \quad \int_a^b f dx \simeq \frac{h_x}{3} \left[f_0 + f_n + 4 \sum_{\text{odd}} f_j + 2 \sum_{\substack{\text{even} \\ 0 < j < n}} f_j \right],$$

where $h_x = (b - a)/n$ and f_j is the integrand including $\sqrt{m/2}$. (c) Explain why time diverges as $E \downarrow 9.0 \text{ J}$ and why equality does not give finite-time crossing. Figure 8.34 shows the central energy geometry.

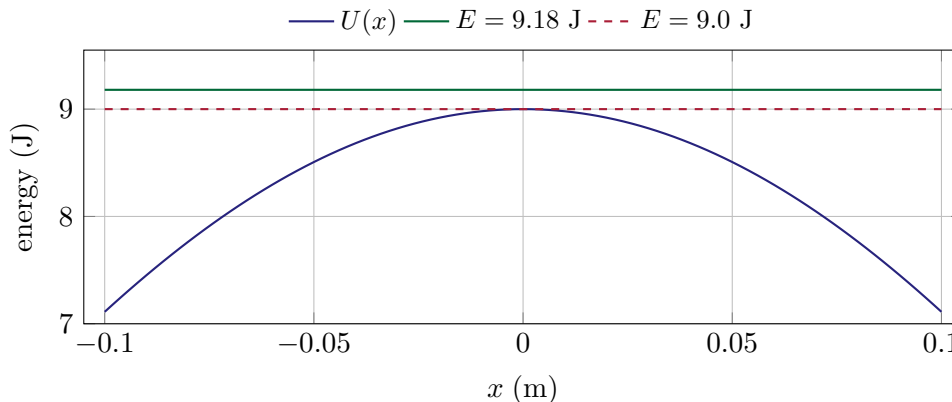


Figure 8.34: Barrier geometry for Problem 42. Compare the strictly positive kinetic-energy gap at 9.18 J with the tangent threshold line at 9.0 J. The narrow central gap calls for a quadrature-refinement check.

8.11 Complete solutions

Each solution answers every stated part. Numerical rounding occurs after the energy equation has been evaluated. In open-ended tasks the exemplar is one defensible response; the stated criteria explain how to judge alternatives.

Problem 1: gravity's signs

(a) Choose package+Earth, with uniform g and any fixed height zero. The signed height change is $\Delta y = 5.5 - 2.0 = +3.5 \text{ m}$. Thus

$$\Delta U_g = mg\Delta y = (1.50)(9.81)(3.5) = +51.5025 \text{ J}, \quad W_g = -51.5025 \text{ J}.$$

To three significant figures these are +51.5 and −51.5 J. The force opposes the upward displacement, so negative work is the expected sign. **(b)** For the return, $\Delta y = -3.5$ m, so $W_g = +51.5$ J and $\Delta U_g = -51.5$ J. The two works sum to zero, as a conservative interaction requires. Here $\text{kg m}^2 \text{s}^{-2} = \text{J}$.

Problem 2: integrating a cubic force

(a) The force is a single-valued position function on the real line; take $U(0)$ as its unspecified reference. Then

$$U(x) - U(0) = - \int_0^x (-as^3) ds = \frac{a}{4} x^4.$$

For the given a , this is x^4 J when the numerical coordinate is in metres. **(b)** Differentiating, $-U'(x) = -ax^3$, exactly the original force. The coefficient has units N m^{-3} , so ax^4 is $\text{N m} = \text{J}$. The potential rises on either side of the origin although the restoring force changes sign; this also checks the physical meaning of the even fourth power.

Problem 3: compression and extension

(a) For spring+block, use $U_s = 0$ at relaxed length. The endpoint energies are $U_i = \frac{1}{2}(120)(0.10)^2 = 0.600$ J and $U_f = \frac{1}{2}(120)(0.20)^2 = 2.400$ J. Therefore $\Delta U_s = +1.800$ J. Spring work on the moving end is -1.800 J. **(b)** Both compression and extension deform the spring, so their energies are nonnegative under this zero and depend on x^2 . The sign of x controls force direction through $F = -k_s x$; it does not control stored-energy sign. A larger absolute deformation means more energy.

Problem 4: the same state under a shifted zero

(a) Both derivatives are $U' = 6x$, giving $F_x = -6x$ N when x is in metres; the coefficient 3 in the original potential carries units J m^{-2} . A constant cannot change its slope. **(b)** The potential shift is $C = -11$ J. The energy of the same physical state must shift by the same amount: $E_2 = 20 - 11 = 9$ J. At any x ,

$$K_2 = E_2 - U_2 = (E_1 - 11) - (U_1 - 11) = E_1 - U_1 = K_1.$$

Changing U without changing E would describe a different velocity state, not a harmless reference change.

Problem 5: one zero loop is insufficient

(a) At fixed $y = y_0$, outward work through distance a is $\alpha y_0 a$ and return work is $-\alpha y_0 a$. Their sum is zero. **(b)** Traverse the rectangle in the stated counterclockwise order. The lower horizontal segment has $y = 0$ and gives zero; both vertical segments give zero because $F_y = 0$. Along the upper segment, $y = b$ and x runs from a to 0, so

$$W_{\text{loop}} = \int_a^0 \alpha b dx = -\alpha ab \neq 0.$$

Units require α in N m^{-1} . **(c)** Any single nonzero closed-loop integral disproves conservativity. Zero work on an out-and-back path merely reflects cancellation on that particular path; the conservative criterion quantifies over *all* loops.

Problem 6: repairing the potential sign

For a falling object, displacement is downward, parallel to gravity, so $W_g > 0$. Its separation from Earth decreases, and $\Delta U_g < 0$. The student's plus sign predicts the opposite. The correct relation is $U_b - U_a = - \int_a^b \mathbf{F}_c \cdot d\mathbf{r} = -W_c$. This test uses the same Earth-object interaction on both sides; it does not identify the work of an unrelated applied force with $-\Delta U_g$.

Problem 7: speed from a height difference

Choose cart+Earth. With no friction, a fixed ideal track's normal force does no work. The stated contact and barrier assumptions allow the cart to reach the target. Conservation gives

$$\frac{1}{2}mv_f^2 = mg(1.80 - 0.50), \quad v_f = \sqrt{2(9.81)(1.30)} = 5.05035 \text{ m s}^{-1}.$$

Thus the speed is 5.05 m s^{-1} . As a numerical budget check, $K_f = 5.1012 \text{ J}$ equals the lost gravitational energy. Mass cancels because both energies scale with m ; path shape still affects transit time and the normal force needed to maintain contact.

Problem 8: horizontal spring stopping

Use block+spring on a fixed horizontal frictionless guide. Height is constant, and at maximum compression the block's speed is momentarily zero. Hence

$$\frac{1}{2}mv_i^2 = \frac{1}{2}k_s s_{\max}^2, \quad s_{\max} = v_i \sqrt{m/k_s} = 3.0 \sqrt{0.25/180} = 0.111803 \text{ m}.$$

The physical compression magnitude is 0.112 m. Squaring introduces two signed displacements; geometry chooses the compression branch. Substitution gives $\frac{1}{2}k_s s_{\max}^2 = 1.125 \text{ J}$, exactly the initial kinetic energy.

Problem 9: spring store used against gravity

Use mass+spring+Earth, with initial gravity zero and all spring energy released by the target. **(a)** The gravitational increase is $mgh = (0.60)(9.81)(0.50) = 2.943 \text{ J}$, leaving

$$K_f = 4.500 - 2.943 = 1.557 \text{ J}, \quad v_f = \sqrt{2(1.557)/0.60} = 2.27816 \text{ m s}^{-1}.$$

The answer is 2.28 m s^{-1} . **(b)** At 1.0 m, gravity would require 5.886 J, exceeding 4.500 J. A negative predicted K means the state is inaccessible. The maximum rise in this model is $4.500/(0.60 \times 9.81) = 0.764526 \text{ m}$. This conclusion uses complete release and the stipulated absence of other barriers.

Problem 10: declared internal conversion

The enlarged isolated system still conserves total energy; its mechanical store after conversion is $4.50 - 0.80 = 3.70 \text{ J}$. **(a)** At 0.50 m, $K_f = 3.700 - 2.943 = 0.757 \text{ J}$ and $v_f = \sqrt{2(0.757)/0.60} = 1.58850 \text{ m s}^{-1}$. **(b)** A 1.0 m rise remains impossible, since $5.886 > 3.700 \text{ J}$; the greatest rise is $3.700/5.886 = 0.628610 \text{ m}$. **(c)** At an endpoint, $K_f = E_i - U_f - E_{\text{diss}} \geq 0$ is necessary. Every intermediate configuration must also have nonnegative kinetic energy after its cumulative losses. An endpoint inequality alone does not prove that an intervening hill can be crossed; barrier equality may additionally require infinite approach time.

Problem 11: a pendulum without tension algebra

Use bob+Earth with the lowest point as $U_g = 0$. A fixed support and inextensible string constrain the bob to a circular arc. Tension is radial, whereas displacement is tangent, so their scalar product and tension work vanish. Consequently

$$\frac{1}{2}mv_b^2 = mg(0.20), \quad v_b = \sqrt{2(9.81)(0.20)} = 1.98091 \text{ m s}^{-1}.$$

The speed is 1.98 m s^{-1} . This holds for the stated ideal pendulum, including large angles; a small-angle approximation was never used. An extensible string or moving support would require a new work account.

Problem 12: shifting both endpoints

Raising the zero by $H = 2.0$ m changes $U(y) = mgy$ to $\tilde{U}(y) = mg(y - H) = U(y) - mgH$. Therefore $\tilde{U}_i = U_i - mgH$ and $\tilde{U}_f = U_f - mgH$. In the conservation equation

$$K_i + U_i - mgH = K_f + U_f - mgH,$$

the identical constants cancel. Changing only U_f inserts an artificial energy difference mgH and gives a spurious kinetic-energy prediction. The force, $-d\tilde{U}/dy = -mg$, is a second independent invariance check.

Problem 13: a stable quadratic well

(a) Differentiation gives $F_x = -2Ax$; A has units J m^{-2} and $2A$ acts as a stiffness. (b) At $x = 0$, $F = 0$ and $U'' = 2A > 0$. A small positive displacement produces negative force, and a negative displacement produces positive force. Both restore the particle toward zero, so this is a stable equilibrium. A disturbed conservative particle oscillates; it need not settle back at rest because no dissipation is included.

Problem 14: an unstable quadratic hill

(a) Here $F_x = +2Ax$. (b) Although $F(0) = 0$, $U''(0) = -2A < 0$. A small displacement is reinforced by the force, so $x = 0$ is unstable. Exact placement with zero velocity remains stationary in the ideal deterministic model; that special solution does not imply stability against perturbations.

Problem 15: the three stationary points

(a) Write $u = x/\ell$. Then

$$U'(x) = \frac{4U_*}{\ell}u(u^2 - 1), \quad U''(x) = \frac{4U_*}{\ell^2}(3u^2 - 1).$$

Stationarity gives $u = 0, \pm 1$, namely $x = 0, \pm\ell$. (b) At $x = 0$, curvature is $-4U_*/\ell^2 < 0$ and $U = U_*$, so the central point is an unstable maximum. At $x = \pm\ell$, curvature is $8U_*/\ell^2 > 0$ and $U = 0$, so both are stable minima. Units of curvature are $\text{J m}^{-2} = \text{N m}^{-1}$. These conclusions concern equilibrium with zero initial velocity, not arbitrary moving particles passing through a force zero.

Problem 16: four roots, two accessible components

(a) At a turning point, $K = 0$ and $(u^2 - 1)^2 = E/U_* = 1/2$. Thus

$$u^2 = 1 \pm \sqrt{1/2}, \quad x = \pm\ell\sqrt{1 \pm \sqrt{1/2}}.$$

The inner magnitudes are 0.162359 m and the outer magnitudes 0.391969 m, or $x = \pm 0.1624, \pm 0.3920$ m. (b) Nonnegative K requires $0.292893 \leq u^2 \leq 1.707107$. Hence the allowed intervals are $[-0.391969, -0.162359]$ m and $[0.162359, 0.391969]$ m. A trajectory started in one cannot jump the forbidden middle interval where $U > E$. All four algebraic roots belong to the energy level, but any one conservative trajectory visits only the two roots of its component.

Problem 17: threshold versus actual crossing

(a) At the central barrier, $U(0) = 9.0$ J. An 8.0 J trajectory would need $K(0) = -1.0$ J and therefore cannot cross. (b) Finite-time crossing from one side requires $E > 9.0$ J; then $v(0) = \sqrt{2(E - 9.0)/m} > 0$. At exact equality,

$$E - U(x) = \frac{2U_*}{\ell^2}x^2 + O(x^4), \quad |v| \simeq \frac{2}{\ell}\sqrt{U_*/m}|x|.$$

The approach time contains $\int dx/|x|$, which diverges. A carriage approaches the top asymptotically; if placed exactly there at rest it stays. The infimum of crossing energies is 9.0 J, but no smallest member of the strict interval $(9.0\text{J}, \infty)$ attains finite-time crossing. This is why merely solving $U = E$ is insufficient at a force-zero root.

Problem 18: direction at an ordinary turn

(a) Since $U' > 0$, the force is $F_x = -U' < 0$. (b) Arriving from the left means positive velocity before the root. At the root, velocity momentarily vanishes while acceleration $a_x = F_x/m$ remains negative. The subsequent motion is leftward. The nonzero slope distinguishes this ordinary turn from the zero-force barrier equilibrium of Problem 17.

Problem 19: the same slowdown with two boundaries

(a) The translational kinetic change is

$$\Delta K = \frac{1}{2}(3.0)(3.0^2 - 5.0^2) = -24.0 \text{ J}.$$

(b) For the crate's centre-of-mass motion, net translational force integrated over displacement gives $\Delta K = -24.0 \text{ J}$. With a stationary floor and the elementary particle sliding model, this is $W_{\text{friction}} = -24.0 \text{ J}$; weight and normal force do zero translational work. That account deliberately tracks translation only. A real finite crate may warm, so it cannot by itself serve as the full thermodynamic account of the crate's energy or of contact heat partition. (c) For isolated crate+surface, the contact interaction is internal, and height is unchanged. Therefore

$$\Delta K + \Delta E_{\text{int}} = 0, \quad \Delta E_{\text{int}} = +24.0 \text{ J}.$$

The mechanical decrease becomes combined internal energy. Do not add a second external friction-work term to this enlarged-system equation: it would count the same conversion twice.

Problem 20: why heat partition is underdetermined

No. Problem 19 determines only the combined 24.0 J internal increase, not its partition. For example, crate gains of 6 J or 18 J would both fit the total if the surface gained 18 J or 6 J. Determine the crate's share with calibrated temperature changes, heat capacity and allowance for gradients or subsequent heat transfer, or with a validated contact thermal model using the material properties and contact history. The ratio of body masses alone does not fix the partition.

Problem 21: heating leaving a closed system

"Closed" excludes mass transfer, not energy transfer. With work-on signs, $W_{\text{on}} = +120 \text{ J}$ and $Q = -35 \text{ J}$ because heating is outward. Thus $\Delta(K + U + E_{\text{int}}) = W_{\text{on}} + Q = 120 - 35 = +85 \text{ J}$. The positive net gain checks the direction of the larger transfer. No division among kinetic, potential and internal stores is determined without further information.

Problem 22: work-by and work-on consistency

For the same boundary and process, $W_{\text{on}} = -W_{\text{by}} = -20 \text{ J}$. The work-by form gives $\Delta E = Q - W_{\text{by}} = 50 - 20 = +30 \text{ J}$; the work-on form gives $\Delta E = Q + W_{\text{on}} = 50 + (-20) = +30 \text{ J}$. Both state that incoming heat exceeds outgoing work by 30 J. Interchanging these signs while retaining the same numerical work label would be an inconsistent convention.

Problem 23: checking a proposed sliding distance

(a) A horizontal stationary surface gives $N = mg$ and $f_k = \mu_k mg = (0.25)(2.0)(9.81) = 4.905 \text{ N}$. With constant friction and relative slide $s = 3.0 \text{ m}$, $E_{\text{diss}} = f_k s = 14.715 \text{ J}$. (b) The initial kinetic

energy is $K_i = \frac{1}{2}(2.0)(4.0)^2 = 16.000$ J. Since $K_f = 16.000 - 14.715 = 1.285$ J is positive, the distance is reachable and

$$v_f = \sqrt{2K_f/m} = 1.13358 \text{ m s}^{-1}.$$

As an independent check, stopping distance is $v_i^2/(2\mu_k g) = 3.26198$ m, exceeding 3.0 m. Once stopped, kinetic friction's formula is not an instruction to continue into negative K .

Problem 24: positive friction work can still dissipate

In the laboratory, a package initially slower than a forward-moving belt experiences forward friction. Its forward displacement makes work on the package positive, increasing its kinetic energy. The belt's contacting material moves farther than the package during slip, so the motor must supply more energy than the package receives as mechanical energy. The difference becomes internal energy of the contact pair: $\dot{E}_{\text{diss}} = f_k |V_{\text{belt}} - v_{\text{package}}| \geq 0$. In another inertial frame the body's displacement and friction work can change sign. Relative slip and its irreversible conversion remain positive. Work on one body and combined contact dissipation are distinct quantities.

Problem 25: an engineering store with declared loss

(a) Convert units first: $k = 8000 \text{ N m}^{-1}$ and $x = 0.025$ m. Then $U_s = \frac{1}{2}(8000)(0.025)^2 = 2.500$ J. (b) The stipulated conversion is $0.15(2.500) = 0.375$ J, leaving $0.85(2.500) = 2.125$ J. The two outputs sum to the initial store. This result describes the hypothetical audit; it neither measures a real device's efficiency nor includes extra output channels such as vibrations.

Problem 26: interpreting step refinement

(a) The error ratio is $r = (7.38 \times 10^{-4})/(1.84 \times 10^{-4}) = 4.01087$. (b) If this error measure scales as $C\Delta t^p$ over the observed range, halving the step gives $r = 2^p$, hence $p = \ln r / \ln 2 = 2.00391$, consistent with second-order convergence. This compares the same initial state, time interval and error definition. It supports numerical refinement of the stated model; it does not establish physical realism, exact energy conservation, long-time stability for every step, or a proof of global order from two rounded values. A third refinement and a trajectory benchmark strengthen the inference.

Problem 27: uncertainty of a gravitational increment

(a) With $E = mgh$, first-order propagation for independent inputs is

$$\sigma_E^2 = (gh\sigma_m)^2 + (gm\sigma_h)^2, \quad E = (0.500)(9.81)(0.400) = 1.962 \text{ J}.$$

The separate contributions are 0.019620 and 0.014715 J, giving $\sigma_E = 0.024525$ J. Equivalently the relative uncertainty is $\sqrt{0.010^2 + 0.0075^2} = 0.0125$. Report approximately $E = (1.962 \pm 0.025)$ J. (b) A 0.010 J residual is only 0.408 times this uncertainty, so it is not clearly resolved. A residual involving another measured store would normally require that store's uncertainty as well; neglecting it would make the resolution test overly optimistic.

Problem 28: a worked validation checklist

A defensible exemplar has the following connected checks.

- (i) Calibrate force sign with a known restoring spring and encoder direction. Integrating $F = -kx$ should give $U - U(0) = kx^2/2$, in joules, rather than its negative.
- (ii) Measure unloaded force zero. A constant force bias b creates a spurious potential tilt $-bx$ and can move or invent a minimum.

- (iii) Differentiate the reconstructed curve back to $-U'$ and compare with raw force within uncertainty. Avoid smoothing so strongly that a force zero or sign change is introduced artificially.
- (iv) Repeat with finer position spacing and alternative defensible quadrature. A genuine feature must persist within numerical and measurement resolution.
- (v) Propagate calibration and repeatability uncertainties; check that the force reverses restoringly on both sides over a resolvable neighbourhood. One noisy zero sample is insufficient evidence of stable equilibrium.
- (vi) Compare loading and unloading and vary speed. Hysteresis or velocity dependence means a position-only curve may be a path-specific work integral rather than a universal conservative potential.

Four or more well-explained checks earn full credit if they address sign/units, numerical reliability, uncertainty and physical-model validity. Equivalent benchmark, repeatability and independent-force checks are acceptable. The conclusion may be “unresolved” rather than a forced stability label.

Problem 29: final height does not identify a mechanism

The ideal rest-to-rest relation is $mgh = \frac{1}{2}kx^2$. A shortfall supplies one scalar constraint, so several different changes can produce it. A worked exemplar is: (i) the initial store is overestimated because stiffness or compression calibration is wrong; measure the loading force–displacement area independently. (ii) Sliding or deformation converts mechanical energy into internal energy; inspect force along the path and temperature or permanent deformation. (iii) At the reported height, energy remains in rotation, vibration or residual translation; measure those motions instead of assigning all unaccounted energy to heat. A biased height datum is another defensible explanation, checked with an independent length measurement. A complete response names three physically distinct possibilities, links each to the energy equation, and proposes an observation that can test it. Repeating three names for friction without different predictions does not meet that criterion.

Problem 30: constructing two boundary diagrams

- (a) For cart alone, motor work crosses the boundary: $\Delta K_{\text{cart}} = W_{\text{motor,on}} > 0$, with height and cart internal energy fixed by the assumptions. (b) For the isolated enlarged system, no net energy crosses the boundary. Battery chemical energy decreases while cart kinetic energy and motor internal energy increase:

$$\Delta E_{\text{chem}} + \Delta K_{\text{cart}} + \Delta E_{\text{int,motor}} = 0.$$

- (c) The motor–cart interaction changes from an external transfer to an internal redistribution. Warming is also internal to the enlarged system; it is not an external loss unless energy actually leaves the boundary. Figure 8.35 is an acceptable diagram. An equivalent drawing earns full credit if it names both boundaries, directs transfers correctly, and gives consistent equations without counting motor work twice.

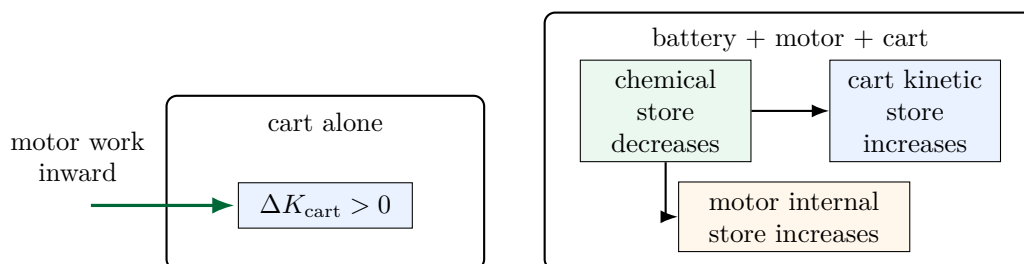


Figure 8.35: Solution 30: expanding the boundary changes external motor work into internal conversion. The right-hand outer boundary has no transfer arrow because the enlarged system is stipulated isolated.

Problem 31: local stiffness and frequency

(a) From Problem 15, $U''(x) = 4U_*(3x^2/\ell^2 - 1)/\ell^2$. At $x = +\ell$, $k_{\text{eff}} = U''(\ell) = 8U_*/\ell^2$. The local displacement $q = x - \ell$ obeys $m\ddot{q} \simeq -k_{\text{eff}}q$ after higher powers are neglected. (b) Substitution gives $k_{\text{eff}} = 8(4.0)/(0.50)^2 = 128 \text{ N m}^{-1}$. Therefore

$$\omega = \sqrt{128/0.80} = 12.6491 \text{ s}^{-1}, \quad f = \omega/(2\pi) = 2.01317 \text{ Hz}.$$

These are 12.65 s^{-1} and 2.01 Hz . Units check: $(\text{N/m})/\text{kg} = \text{s}^{-2}$. The frequency applies to sufficiently small amplitude about one minimum; large motion that samples the anharmonic well or barrier has a different period.

Problem 32: relative error and force zeros

(a) Substitute $x = \ell(1 + \delta)$ into the exact force:

$$F_{\text{exact}} = -\frac{U_*}{\ell}(8\delta + 12\delta^2 + 4\delta^3), \quad F_{\text{local}} = -\frac{8U_*}{\ell}\delta.$$

Where $F_{\text{exact}} \neq 0$, the relative error is

$$\varepsilon_F = \frac{|F_{\text{local}} - F_{\text{exact}}|}{|F_{\text{exact}}|} = \frac{|12\delta^2 + 4\delta^3|}{|8\delta + 12\delta^2 + 4\delta^3|}.$$

(b) The denominator factors as $4\delta(\delta + 1)(\delta + 2)$, so it vanishes at $\delta = 0, -1, -2$, the three stationary positions. At the expansion minimum, both forces vanish; their relative error is undefined even though the local approximation is correct there. Use absolute error or curvature. The relative error tends to zero as $\delta \rightarrow 0$ through nonzero values; a limiting value must not be confused with the undefined ratio at the point itself. The other zeros lie outside the intended neighbourhood of this local approximation.

Problem 33: checking both paths

(a) Integrate $-\partial U/\partial x = \alpha y$ to obtain $U = -\alpha xy + f(y)$. The second component requires $-\partial U/\partial y = \alpha x - f'(y) = \alpha x$, hence $f' = 0$. Thus $U(x, y) = -\alpha xy + C$ and both components are verified. (b) Along $(0, 0) \rightarrow (a, 0) \rightarrow (a, b)$, the first work is zero and the second is $\int_0^b \alpha a \, dy = \alpha ab$. Along $(0, 0) \rightarrow (0, b) \rightarrow (a, b)$, the first is zero and the second is $\int_0^a \alpha b \, dx = \alpha ab$. Both agree with $U(0, 0) - U(a, b) = \alpha ab$. The smooth plane-wide potential, rather than agreement of just two samples, establishes global conservativity.

Problem 34: a flat stable minimum

(a) The usual test needs a nonzero quadratic coefficient $U''(x_0)$. Here it vanishes, so neither positive nor negative curvature classifies the point. (b) Taylor expansion with $q = x - x_0$ gives

$$U(x) - U(x_0) = \frac{U^{(4)}(x_0)}{24}q^4 + o(q^4).$$

The leading positive even power raises energy on both sides, giving a strict local minimum and stable equilibrium. Its force is restoring, $F \simeq -U^{(4)}(x_0)q^3/6$. Because the linear restoring coefficient is zero, the harmonic-frequency formula does not describe finite-amplitude motion here. The vanishing cubic term is essential: an earlier nonzero odd term would change the conclusion.

Problem 35: signed area to a continuous potential

Model. The fixed guide does no tangential work; the given position-only force has a potential on the stated line. Initial energy is $E = K_0 = \frac{1}{2}(0.50)(2.0)^2 = 1.00$ J. **(a)** Signed areas give $W_{0 \rightarrow 1} = 4.00$ J, $W_{1 \rightarrow 3} = \frac{1}{2}(4 - 2)(2) = 2.00$ J and $W_{3 \rightarrow 4} = (-2)(1) = -2.00$ J. The negative part of the sloping segment must be subtracted, not added by magnitude. **(b)** With $U(0) = 0$ and $U = -W_{0 \rightarrow x}$,

$$U(x) = \begin{cases} -4x, & 0 \leq x \leq 1, \\ \frac{3}{2}x^2 - 7x + \frac{3}{2}, & 1 \leq x \leq 3, \\ 2x - 12, & x \geq 3, \end{cases}$$

where numerical x is in metres and U is in joules. The joining values are $U(1) = -4$ J and $U(3) = -6$ J on both branches. Taking $-U'$ gives 4, $7 - 3x$, and -2 N, recovering the graph. **(c)** At 3.0 m, $K = 1 - (-6) = 7.00$ J and $v = \sqrt{2(7)/0.50} = 5.29150$ m s⁻¹. Beyond that point, $K = 1 - (2x - 12) = 13 - 2x$ J, so the first rightward turn is $x_t = 6.50$ m. Force there is -2 N, so it reverses the carriage. **(d)** In the middle segment, $U' = 3x - 7 = 0$ at $x = 7/3 = 2.33333$ m. With $U'' = 3$ J m⁻² > 0, it is stable. The minimum is $U_{\min} = -20/3$ J, giving $K_{\max} = 23/3$ J and $v_{\max} = \sqrt{2(23/3)/0.50} = 5.53775$ m s⁻¹. The fastest point is where force vanishes and potential is least, not where the force magnitude is largest.

Problem 36: gravity continues during compression

Choose mass+spring+Earth, with gravitational zero at first contact. Release energy is mgh ; while compressed, gravity is $-mgs$ and spring energy is $\frac{1}{2}ks^2$. Thus

$$K(s) = mg(h + s) - \frac{1}{2}ks^2.$$

(a) At maximum compression, set $K = 0$ and select the positive root:

$$s_{\max} = \frac{mg + \sqrt{m^2g^2 + 2kmgh}}{k} = 0.148285 \text{ m}.$$

The other root is negative and outside the compression geometry. The spring then stores 5.27721 J, equal to $mg(h + s_{\max})$. **(b)** Maximum K occurs at $dK/ds = mg - ks = 0$, so $s_v = mg/k = 0.024525$ m. Here $K_{\max} = mgh + (mg)^2/(2k) = 3.675954$ J and $v_{\max} = \sqrt{2K_{\max}/m} = 2.47519$ m s⁻¹. **(c)** At the deepest point the downward-positive force is $F = mg - ks_{\max} = 11.772 - 71.17668 = -59.4047$ N, upward. The mass is reversing, not in equilibrium. As a useful check, maximum speed occurs at force balance, whereas maximum compression has zero speed and a substantial restoring acceleration.

Problem 37: motor work and relative slip

(a) During slip, friction is forward with magnitude $f = \mu_k mg = 5.886$ N. Therefore $a = f/m = 2.943$ m s⁻² and

$$t_s = V/a = 0.509684 \text{ s}, \quad d_c = \frac{1}{2}at_s^2 = 0.382263 \text{ m}, \quad d_b = Vt_s = 0.764526 \text{ m}.$$

Crate speed reaches V at the end; subsequent ideal common motion has no slip. **(b)** Friction work on the crate is $W_c = fd_c = 2.250$ J, equal to its gain $\frac{1}{2}mV^2$. Relative slip is $d_b - d_c = 0.382263$ m, so contact conversion is $E_{\text{diss}} = f(d_b - d_c) = 2.250$ J. **(c)** Maintaining belt speed requires motor work $W_{\text{motor}} = fd_b = 4.500$ J against the crate's friction on the belt. Thus $4.500 = 2.250 + 2.250$ J: motor supply becomes crate kinetic energy plus combined internal energy. Belt kinetic energy stays constant. Using fd_c as the *entire motor work* would miss half the budget; using absolute crate displacement as a universal heating distance would also fail in a general moving-surface example.

Problem 38: count complete crossings before stopping

Every complete rough-patch crossing converts $D = \mu_k mgL = (0.20)(0.50)(9.81)(0.800) = 0.7848$ J. Initial energy is $E_0 = mg(0.440) = 2.1582$ J. **(a)** After the first crossing, $E_1 = 1.3734$ J, which produces right-hand height $h_1 = E_1/(mg) = 0.280$ m. On the smooth return this same energy reaches the patch; after the second crossing, $E_2 = 0.5886$ J, giving left-hand height $h_2 = 0.120$ m. **(b)** The third crossing starts at the left edge, but $E_2 < D$. With $f_k = \mu_k mg = 0.9810$ N, stopping distance is $s = E_2/f_k = 0.600$ m from that edge, inside the 0.800 m patch. **(c)** Final mechanical energy is zero relative to patch height, so combined internal increase is 2.1582 J. Independently, $2D + f_k s = 1.5696 + 0.5886 = 2.1582$ J. The position requires tracking crossing direction; total loss alone would not identify the stopping side.

Problem 39: curl-free locally, nonconservative globally

(a) Let $r^2 = x^2 + y^2$. Direct differentiation gives

$$\partial_x F_y = \kappa \frac{y^2 - x^2}{r^4}, \quad \partial_y F_x = \kappa \frac{y^2 - x^2}{r^4},$$

so the out-of-plane curl is zero for $r > 0$. **(b)** In polar coordinates, $\mathbf{F} = (\kappa/r)\hat{\mathbf{e}}_\theta$. For a counterclockwise circle, $d\mathbf{r} = r\hat{\mathbf{e}}_\theta d\theta$, and $W = \int_0^{2\pi} \kappa d\theta = 2\pi\kappa = 18.8496$ J, independent of radius. A radial segment has displacement perpendicular to force, so its work is zero. **(c)** Nonzero closed-loop work excludes a single-valued global potential on the punctured plane. On a branch that assigns a continuous angle, $U = -\kappa\theta + C$ gives $F_\theta = -(1/r)\partial_\theta U = \kappa/r$ and $F_r = 0$. After one full revolution this expression changes by $-2\pi\kappa$; it cannot be made a single-valued function everywhere without a branch cut. The missing hypothesis in the curl criterion is simple connectedness.

Problem 40: fitting a nonlinear energy store

(a) At 0.100 and 0.150 m, $0.100k + 0.0100\beta = 21.0$ and $0.150k + 0.0225\beta = 33.0$. Multiply the first equation by 1.5 and subtract it from the second: $0.0075\beta = 1.5$. Thus $\beta = 200 \text{ N m}^{-2}$ and $k = 190 \text{ N m}^{-1}$. The remaining point predicts $190(0.050) + 200(0.050)^2 = 10.0$ N; zero extension also gives zero force. **(b)** The spring force is negative of the applied load. Under the stipulated reversible model,

$$U(x) - U(0) = \int_0^x F_{\text{app}}(s) ds = \frac{1}{2}kx^2 + \frac{1}{3}\beta x^3.$$

At 0.150 m this is $2.1375 + 0.2250 = 2.3625$ J. **(c)** Three trapezoids give $U_T = 0.050[\frac{1}{2}(0 + 33) + 10 + 21] = 2.3750$ J. The excess is 0.0125 J, or 0.5291% of the exact model area. Since $F''_{\text{app}} = 2\beta > 0$, straight chords lie above the convex force curve and trapezoids overestimate its area. **(d)** Loading alone does not establish recoverability. An unloading curve, speed dependence and repeatability are needed to exclude hysteresis or dissipation. The fitted integral is a model of the loading work; calling it a conservative store requires the additional reversible assumption.

Problem 41: a resolved residual with an unresolved cause

(a) The initial spring energy is $S = \frac{1}{2}kx^2 = 2.400$ J, and gravitational output is $G = mgh = 1.962$ J. Therefore $R = S - G = 0.438$ J. For independent inputs, first-order propagation gives

$$\sigma_S^2 = (\frac{1}{2}x^2\sigma_k)^2 + (kx\sigma_x)^2, \quad \sigma_G^2 = (gh\sigma_m)^2 + (gm\sigma_h)^2.$$

Their standard uncertainties are $\sigma_S = 0.0768375$ J and $\sigma_G = 0.0245250$ J; hence $\sigma_R = \sqrt{\sigma_S^2 + \sigma_G^2} = 0.0806565$ J. Report $R = (0.438 \pm 0.081)$ J. **(b)** The ratio $R/\sigma_R = 5.43$ exceeds one standard uncertainty substantially. It is resolved within this measurement-error model, provided the calibration

and physical assumptions are valid; an omitted systematic bias is not covered by the arithmetic. (c) The apparent fraction is $\eta = G/S = 0.8175$, with

$$\sigma_\eta = \eta \sqrt{(\sigma_G/G)^2 + (\sigma_S/S)^2} = 0.02810.$$

Thus $\eta = 0.818 \pm 0.028$, or $(81.8 \pm 2.8)\%$. The residual may include internal conversion, acoustic/environmental transfer or other unmeasured stores. Neither a significant residual nor this fraction alone identifies which mechanism received it.

Problem 42: resolving a slow barrier passage

(a) At $x = 0$, $K = 9.18 - 9.00 = 0.18$ J and $v_0 = \sqrt{2(0.18)/0.80} = 0.670820$ m s⁻¹. This nonzero speed is small relative to the speeds farther downhill. (b) With $h_x = 0.025$ m, symmetry reduces the integrand values $f = \sqrt{m/2}/\sqrt{E - U}$ to the following five entries; reflect them for negative x :

$ x $ (m)	U (J)	f (s m ⁻¹)
0.000	9.000000	1.490712
0.025	8.875434	1.146012
0.050	8.506944	0.770912
0.075	7.910156	0.561248
0.100	7.111111	0.439705

Then the eight-interval estimate is

$$t_8 = \frac{0.025}{3} \{2(0.439705) + 4[2(0.561248) + 2(1.146012)] + 2[2(0.770912) + 1.490712]\} = 0.171688 \text{ s}.$$

The s/m integrand times m gives seconds. Refining to $n = 16$ gives 0.172503 s; $n = 32$ gives 0.172517 s, agreeing with direct numerical integration 0.1725174 s. The first estimate is about 0.481% low, illustrating why the narrow central peak needs refinement rather than blind use of a coarse rule. (c) Put $E = U_* + \epsilon$. Near zero,

$$E - U(x) = \epsilon + \frac{2U_*}{\ell^2} x^2 + O(x^4).$$

Over a fixed small symmetric interval $[-a, a]$, the leading time is

$$t_{\text{near}} \simeq \frac{\sqrt{m} \ell}{\sqrt{U_*}} \operatorname{arsinh} \left(\frac{a \sqrt{2U_*}}{\ell \sqrt{\epsilon}} \right).$$

As $\epsilon \downarrow 0$, the inverse hyperbolic sine grows logarithmically, so transit time diverges. At equality the integrand behaves as $1/|x|$. The threshold line touches the potential at zero kinetic energy and zero force; it describes asymptotic approach or exact unstable rest, not passage in a finite duration.

8.12 Bridge to Chapter 9

Energy methods connect states without following every force component in time, but they do not directly tell us how a system's translational motion is shared among interacting bodies. Chapter 9 introduces the centre of mass and linear momentum, quantities that remain especially powerful during interactions and collisions.

The system-boundary discipline developed here carries forward unchanged. If two bodies interact internally, their individual momenta can change while the total momentum of an appropriately isolated system remains constant. Just as an internal frictional interaction can redistribute energy between mechanical and internal forms, an internal force pair can redistribute momentum between bodies without changing the system total.

Potential energy will also return repeatedly: in rotation and gravitation, oscillations, fluids, thermal physics and electromagnetic systems. The central habit to retain is not a formula but an audit: identify the system, identify storage and transfer, state the model conditions, and check whether the quantity claimed to be conserved actually belongs to the chosen closed description.

Take-home message

Energy conservation is not a license to set every before-and-after sum equal. It is a statement about a defined system and a complete accounting of energy storage and transfer. Potential energy makes conservative interactions economical to represent; internal energy and boundary transfers keep that economy honest when the mechanical picture is incomplete.

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