

Chapter 9

Center of Mass and Linear Momentum

First Edition

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Preface

This chapter develops the mechanics of a *system* of particles. Earlier chapters described position, velocity, acceleration, force, work, and energy for individual objects or for carefully chosen mechanical systems. We now ask a different question: when many particles interact, which collective quantities allow us to describe the motion without following every internal detail?

Two ideas provide the answer. The *center of mass* gives a mass-weighted position for the system, and the *total linear momentum* gives a vector measure of its translational motion. These quantities are powerful because internal forces do not determine the motion of the center of mass of a fixed-mass system: the net external force does. Likewise, the change in total momentum is controlled by the external impulse. Collision problems then become questions about system boundaries, reference frames, external impulses, and—only when appropriate—kinetic-energy conservation.

The chapter is written for calculus-based undergraduate students in technical physics, engineering, space technologies, computer science, and related programmes. It therefore combines analytical derivations with engineering interpretation, measurement strategy, and numerical checks. The worked examples and figures are original to this chapter. Quantitative teaching data are identified as constructed or synthetic when they are not measurements.

The Guideline

Before writing a conservation equation, draw the system boundary and name the reference frame. Momentum conservation is not a property of a collision word such as “elastic” or “inelastic”; it is a consequence of negligible net external impulse over the interval of interest. Kinetic energy requires a separate audit.

Pedagogical boundary

Rotational dynamics is not required here. Angular momentum and torque are developed later. The present chapter stays with translational center-of-mass motion, linear momentum, impulse, collisions, fragmentation, and carefully bounded extensions to deformable and variable-mass systems.

Conventions and notation

Vectors are bold, for example $\mathbf{p}$ and $\mathbf{P}$. Cartesian unit vectors are $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$. A single-particle linear momentum is written $\mathbf{p} = m\mathbf{v}$, whereas the total momentum of a system is $\mathbf{P} = \sum_i \mathbf{p}_i$. Impulse is $\mathbf{J} = \int \mathbf{F} dt$. The center-of-mass position and velocity are $\mathbf{R}_{\text{CM}}$ and $\mathbf{V}_{\text{CM}}$.

Unless stated otherwise, masses are constant, velocities are measured in one specified inertial frame, and SI units are used. A force impulse has units $\text{N s} = \text{kg m s}^{-1}$, the same as momentum. Kinetic energy is frame dependent. Momentum is also frame dependent, but its conservation statement has the same form in every inertial frame when the relevant external impulse vanishes.

For two bodies, the reduced mass is

$$\mu = \frac{m_1 m_2}{m_1 + m_2}.$$

The symbol μ will not denote a friction coefficient in this chapter. The scalar symbol p is reserved elsewhere in physics for pressure; here bold $\mathbf{p}$ always means momentum.

Numerical examples retain guard digits during intermediate calculations. Rounded final values indicate the intended precision of the stipulated inputs; they do not represent experimental uncertainty unless an uncertainty is explicitly supplied.

Note
The supplied references use both “center of mass” and “centre of mass” in explanatory prose. This manuscript uses <i>center of mass</i> consistently, while retaining standard symbols and equations independent of spelling convention.

Chapter 9

Center of Mass and Linear Momentum

9.1 Center of Mass and Distributed Systems

Consider a rectangular instrument plate from which an off-center service opening is removed. Does the balance point remain at the geometric center? Which coordinates should an engineer use when placing a support or mounting the instrument? We will first construct a mass-weighted position, then return to this cutout calculation in Example 9.1.

A collection of particles can have thousands, millions, or effectively countless internal degrees of freedom. Yet its overall translational motion can often be described by one carefully chosen point: the center of mass. The center of mass is not necessarily a material point. It may lie in empty space, inside a hole, or even outside a bent object. It is a weighted average determined by the mass distribution.

Learning goals

After studying the core of this chapter, you should be able to:

- calculate the center of mass of discrete and continuous systems and interpret its location;
- derive and use $M\mathbf{A}_{\text{CM}} = \sum \mathbf{F}_{\text{ext}}$ for a fixed set of particles;
- distinguish a particle momentum from total system momentum and calculate impulse from a force–time history;
- decide when momentum conservation is justified by estimating the external impulse;
- solve one- and two-dimensional collision and fragmentation problems without assuming kinetic-energy conservation unless it is physically warranted;
- transform collision data between a laboratory frame and the center-of-mass frame;
- use momentum and energy residuals to diagnose measurements and numerical calculations;
- explain why variable-mass systems require explicit momentum-flux bookkeeping.

9.1.1 From a collection of particles to one representative position

For N particles with masses m_i at positions $\mathbf{r}_i$, define the total mass

$$M = \sum_{i=1}^N m_i \quad (9.1)$$

and the center-of-mass position

$$\mathbf{R}_{\text{CM}} = \frac{1}{M} \sum_{i=1}^N m_i \mathbf{r}_i. \quad (9.2)$$

In Cartesian components,

$$X_{\text{CM}} = \frac{\sum_i m_i x_i}{M}, \quad Y_{\text{CM}} = \frac{\sum_i m_i y_i}{M}, \quad Z_{\text{CM}} = \frac{\sum_i m_i z_i}{M}. \quad (9.3)$$

Each coordinate is a weighted mean. Large masses influence the result more strongly than small masses. If all masses are positive, each coordinate of the center of mass lies between the minimum and maximum corresponding particle coordinates, but the resulting point need not coincide with any particle.

Definition 9.1: Center of mass

For a system of particles of total mass $M > 0$, the *center of mass* is the mass-weighted position defined by Eq. (9.2). It depends on the chosen origin through its coordinates, but relative displacements and physical predictions transform consistently when the origin is translated.

A translation of the origin by a constant vector $\mathbf{a}$ changes every particle coordinate to $\mathbf{r}'_i = \mathbf{r}_i - \mathbf{a}$. Equation (9.2) then gives $\mathbf{R}'_{\text{CM}} = \mathbf{R}_{\text{CM}} - \mathbf{a}$. Thus an origin shift moves the numerical coordinate of the center of mass by exactly the same amount as every other position; it does not change where the center of mass is physically located in the body.

9.1.2 Continuous mass distributions

When a body can be modelled as continuous, divide it conceptually into elements of mass dm . The discrete sum becomes

$$\mathbf{R}_{\text{CM}} = \frac{1}{M} \int \mathbf{r} dm. \quad (9.4)$$

The mass element depends on the geometry:

$$dm = \lambda d\ell \quad \text{for a line distribution,} \quad (9.5)$$

$$dm = \sigma dA \quad \text{for a thin lamina,} \quad (9.6)$$

$$dm = \rho dV \quad \text{for a three-dimensional body,} \quad (9.7)$$

where λ , σ , and ρ are linear, areal, and volume mass densities. If the density is uniform, it often cancels between numerator and denominator.

Symmetry should be used before integration. A uniform rectangular plate centered at the origin has its center of mass at the origin; no double integral is needed. If a centered cutout shares both mirror symmetries of that rectangle, the remaining mass distribution still has both symmetries, so its center remains at the original center. The removed-area formula gives the same limiting result when the removed center equals the original center. If a shape has a mirror plane and its density shares that symmetry, the center of mass lies in the mirror plane. Multiple independent symmetries can fix the center completely.

Method

For a composite body, choose one origin and keep it for every component. Tabulate each component's signed bookkeeping mass M_k and known center $\mathbf{R}_k$, then use

$$\mathbf{R}_{\text{CM}} = \frac{\sum_k M_k \mathbf{R}_k}{\sum_k M_k}.$$

A removed region may be represented by a *negative bookkeeping mass*; this is an algebraic device, not negative physical mass.

Example 9.1: A plate with an off-center rectangular cutout

A uniform rectangular plate has width 0.60 m and height 0.40 m. The origin is at its lower-left corner. A rectangular hole of width 0.20 m and height 0.10 m is cut out. The hole center is at

(0.45, 0.30) m. Find the center of mass of the remaining plate.

Model and strategy. The plate has uniform areal density σ , so areas can replace masses. Treat the full rectangle as a positive component and the removed rectangle as a negative component.

The full plate area and center are

$$A_0 = (0.60)(0.40) = 0.240 \text{ m}^2, \quad (x_0, y_0) = (0.300, 0.200) \text{ m}.$$

The removed area is

$$A_h = (0.20)(0.10) = 0.020 \text{ m}^2, \quad (x_h, y_h) = (0.450, 0.300) \text{ m}.$$

The remaining area is $A = A_0 - A_h = 0.220 \text{ m}^2$. Therefore

$$X_{\text{CM}} = \frac{A_0 x_0 - A_h x_h}{A_0 - A_h} = \frac{(0.240)(0.300) - (0.020)(0.450)}{0.220} = 0.2864 \text{ m}, \quad (9.8)$$

$$Y_{\text{CM}} = \frac{A_0 y_0 - A_h y_h}{A_0 - A_h} = \frac{(0.240)(0.200) - (0.020)(0.300)}{0.220} = 0.1909 \text{ m}. \quad (9.9)$$

Thus

$$\mathbf{R}_{\text{CM}} = (0.286 \hat{x} + 0.191 \hat{y}) \text{ m}.$$

Interpretation. The removed material lies above and to the right of the original center, so the remaining center of mass shifts down and left. This directional prediction is an independent check on the arithmetic.

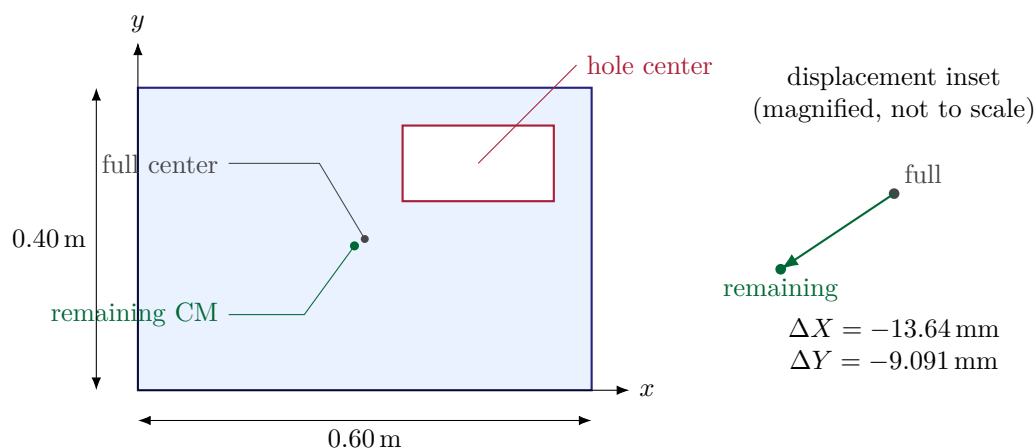


Figure 9.1: The plate and hole are drawn with equal physical axis scales; the hole measures $0.20 \text{ m} \times 0.10 \text{ m}$. Leader lines separate the nearly coincident full and remaining centers. The inset magnifies only the shift, which is directed away from the removed mass.

9.1.3 Center of mass versus geometric center and center of gravity

For a homogeneous body with sufficient geometric symmetry, the center of mass often coincides with the geometric center. If density varies, the two need not coincide. The phrase *center of gravity* refers to the point at which the gravitational torque can be represented as if the total weight acted. In a nearly uniform gravitational field over a laboratory-sized object, center of gravity and center of mass coincide to excellent approximation. Over a sufficiently extended object in a nonuniform gravitational field, the distinction can matter.

Guided checks

Check 1. Can the center of mass lie in a region containing no material? Yes. A uniform ring has its center of mass at its empty geometric center.

Check 2. If a small piece is removed from the right side of a uniform plate, which way does the center of mass move? Away from the removed mass, toward the left.

Check 3. Does changing the coordinate origin move the physical center of mass? No. It changes only its coordinate description.

9.1.4 Two masses: the weighted mean as a geometric construction

The definition of center of mass becomes more useful when its geometry is made explicit. Consider two particles on a line, with the first at x_1 and the second at $x_2 > x_1$. Let $L = x_2 - x_1$ be their separation and measure the center's displacement $d_1 = X_{\text{CM}} - x_1$ from the first particle. Substituting the weighted mean and collecting terms gives

$$d_1 = \frac{m_2}{m_1 + m_2}L, \quad d_2 = x_2 - X_{\text{CM}} = \frac{m_1}{m_1 + m_2}L, \quad (9.10)$$

so that

$$m_1 d_1 = m_2 d_2. \quad (9.11)$$

The heavier particle has the shorter distance to the center. This is sometimes called a lever-arm relation, but here it follows directly from a weighted position and does not require rotational dynamics. In a uniform gravitational field it is also the relation underlying the balance of the two weights about a support placed at the center.

For particles in three dimensions the same construction holds along the line joining their positions:

$$\mathbf{R}_{\text{CM}} = \mathbf{r}_1 + \frac{m_2}{m_1 + m_2}(\mathbf{r}_2 - \mathbf{r}_1).$$

The multiplier is between zero and one for positive masses. Thus the center is on the line segment between the particles even when that segment is tilted relative to every coordinate axis. Coordinates are a convenient calculation tool; the physical construction does not privilege an axis.

Example 9.2: Locating the center between unequal masses

Two compact masses of 1.50 kg and 4.50 kg are fixed at $x_1 = 0.200$ m and $x_2 = 1.400$ m on a very light connector. Find the center of mass and check the result using distances from the two masses.

Model and calculation. The connector's mass is neglected and the compact objects are represented by particles. Their total mass is 6.00 kg, so

$$X_{\text{CM}} = \frac{(1.50)(0.200) + (4.50)(1.400)}{6.00} = 1.100 \text{ m}.$$

The distances are $d_1 = 1.100 - 0.200 = 0.900$ m and $d_2 = 1.400 - 1.100 = 0.300$ m. Hence

$$m_1 d_1 = (1.50)(0.900) = 1.350 \text{ kg m} = m_2 d_2.$$

The center is three times closer to the mass that is three times larger. Its coordinate also lies between the two supplied coordinates, as a positive weighted mean must.

Limits and interpretation. Equal masses would put the center at the midpoint, 0.800 m. Increasing the right-hand mass while holding positions fixed would move the center toward 1.400 m. It would approach that coordinate continuously; it would not move beyond it. A connector with appreciable mass would contribute its own mass and first moment.

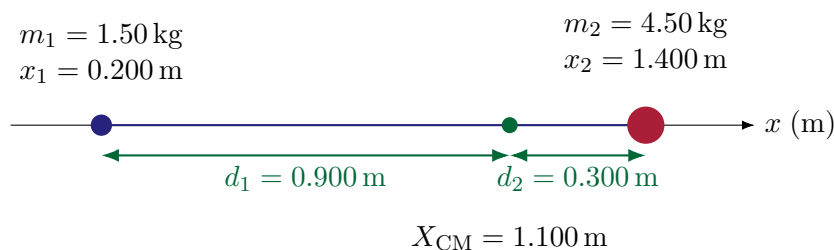


Figure 9.2: The distances in Example 9.2 use a common coordinate scale. Marker sizes distinguish the masses; their drawn areas do not encode an exact mass ratio.

9.1.5 A three-dimensional mass inventory and a balance requirement

An engineering mass inventory is a discrete center-of-mass model. Each component contributes its own mass multiplied by the coordinate of its own center. It is unnecessary to divide a uniform component into many particles if its center is already known. Conversely, using a component's mounting corner instead of its center introduces a modelling error even when the subsequent arithmetic is flawless. For an extended instrument, its listed position must represent its center of mass or be corrected to that position.

Three-dimensional balance requires three first moments, but the required constraints can be fewer than three. An assembly may need its center on the vertical axis of a test fixture while its height is unconstrained. That requirement sets $X_{\text{CM}} = Y_{\text{CM}} = 0$ and leaves Z_{CM} to be calculated. Solving a balance problem therefore begins by translating the geometric requirement into coordinate equations, rather than forcing every coordinate to vanish.

Example 9.3: Repositioning one payload to balance an assembly

An idealized spacecraft assembly has the component inventory below. All coordinates are in metres in one body-fixed Cartesian frame. Component D can be moved horizontally while its height remains $z_D = 0.500 \text{ m}$.

component	$m \text{ (kg)}$	x	y	z
A: platform	40.0	0	0	0
B: instrument	12.0	0.800	0	0.200
C: battery	8.00	-0.400	0.600	0.100
D: payload	10.0	0.200	-0.300	0.500

Calculate the original center of mass. Then determine the new horizontal coordinates of D needed to place the center on the z axis.

Original inventory. The total mass is $M = 70.0 \text{ kg}$. Computing the first moments separately gives

$$\sum mx = 8.40 \text{ kg m}, \quad \sum my = 1.80 \text{ kg m}, \quad \sum mz = 8.20 \text{ kg m}.$$

Therefore

$$\mathbf{R}_{\text{CM}} = (0.120 \hat{x} + 0.0257 \hat{y} + 0.117 \hat{z}) \text{ m}.$$

The factors of mass in the denominator turn each first moment into a position.

Balance equations. Since M remains positive and unchanged, zero horizontal coordinates require zero horizontal first moments:

$$12.0(0.800) + 8.00(-0.400) + 10.0x_D = 0,$$

$$8.00(0.600) + 10.0y_D = 0.$$

Solving yields

$$x_D = -0.640 \text{ m}, \quad y_D = -0.480 \text{ m}.$$

The payload must move by $(-0.840, -0.180, 0) \text{ m}$ from its initial position. Its height has not changed, so the vertical first moment remains 8.20 kg m and the new center is $(0, 0, 0.117) \text{ m}$.

Checks and scope. Substituting the new coordinates makes both horizontal first-moment sums exactly zero. The required move is a geometric result of the stipulated inventory. An actual design must also establish that the position is accessible and that supports, cables, fuel, and omitted structural masses are included. Being balanced on an axis does not by itself specify a desired rotational inertia about that axis.

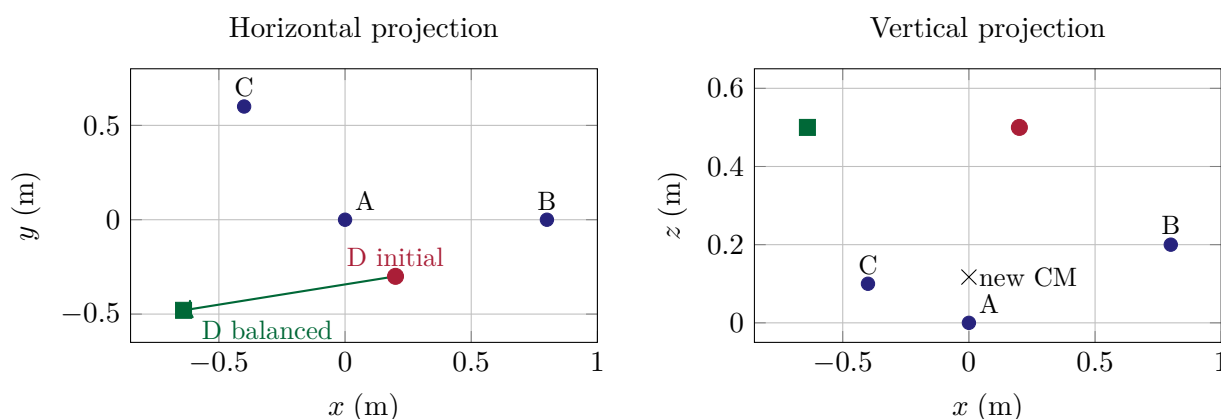


Figure 9.3: Two coordinate projections display all three dimensions of the inventory in Example 9.3. Moving D changes its horizontal first moments while preserving its vertical first moment. Component markers represent centers, not component outlines.

How an omitted mass or a placement error changes the result

The inventory also predicts sensitivity to a modelling change. Suppose the current assembly has mass M and center $\mathbf{R}$, and an omitted component of positive mass δm is found at $\mathbf{r}$. The corrected weighted mean is

$$\mathbf{R}_{\text{new}} = \frac{M\mathbf{R} + \delta m \mathbf{r}}{M + \delta m}.$$

Subtracting the original center gives the exact correction

$$\mathbf{R}_{\text{new}} - \mathbf{R} = \frac{\delta m}{M + \delta m}(\mathbf{r} - \mathbf{R}). \quad (9.12)$$

The shift points toward the omitted mass. A component placed at the existing center increases total mass without changing the center's position. A small component far away can matter more than a larger component near the center, because the first moment depends on both mass and displacement.

If $\delta m/M$ is small, the prefactor is approximately $\delta m/M$; the exact expression should be retained when that ratio is not small. This is an approximation in a dimensionless mass ratio, rather than a replacement of the body's geometry. It provides a quick estimate of whether an incomplete inventory can meet a requested placement tolerance.

A placement error in a component already included in the inventory has a different correction. Moving component k by $\delta \mathbf{r}_k$ at fixed mass changes the center by exactly $(m_k/M)\delta \mathbf{r}_k$. For the 10.0 kg payload in the 70.0 kg assembly, a 7.00 mm horizontal placement error shifts the assembly center by 1.00 mm in that direction. The other components do not need to move for this consequence

to follow. This calculation relates a component positioning tolerance to an overall balance tolerance without guessing a new center.

9.1.6 A graded rod: mass elements, normalization, and units

A continuous center-of-mass calculation has two linked integrals. The first finds total mass; the second finds a first moment. Using an unnormalized density in the moment while dividing by a guessed mass is a common error. Both integrals must describe the same material over the same interval. This is especially important when the density varies, because there is then no single constant density that can be cancelled before integration.

For a straight rod on $0 \leq x \leq L$, the position of a small segment is represented by x and its mass by $dm = \lambda(x)dx$. Thus

$$M = \int_0^L \lambda(x)dx, \quad X_{\text{CM}} = \frac{\int_0^L x\lambda(x)dx}{\int_0^L \lambda(x)dx}.$$

The density has units kg m^{-1} , so λdx has units of mass and $x\lambda dx$ has units of mass times length. This dimensional check distinguishes the first moment from the mass itself. It also explains why an integral of λ alone cannot locate the center.

Example 9.4: Center of mass of a rod with a linear density gradient

A slender rod of length $L = 1.20$ m has linear density

$$\lambda(x) = \lambda_0 \left(1 + \alpha \frac{x}{L}\right), \quad \lambda_0 = 0.800 \text{ kg m}^{-1}, \quad \alpha = 2.00.$$

Find its mass and center of mass. Derive the dependence on α before substituting the numbers.

Mass integral. The ratio x/L and the parameter α are dimensionless. Therefore the formula has the required density units everywhere:

$$M = \lambda_0 \left[x + \frac{\alpha x^2}{2L} \right]_0^L = \lambda_0 L \left(1 + \frac{\alpha}{2}\right).$$

For the stated values, $M = (0.800)(1.20)(2.00) = 1.92$ kg.

First moment and normalized result.

$$\int_0^L x dm = \lambda_0 \left[\frac{x^2}{2} + \frac{\alpha x^3}{3L} \right]_0^L = \lambda_0 L^2 \left(\frac{1}{2} + \frac{\alpha}{3} \right).$$

Dividing by the calculated mass gives

$$\frac{X_{\text{CM}}}{L} = \frac{\frac{1}{2} + \frac{\alpha}{3}}{1 + \frac{\alpha}{2}}. \quad (9.13)$$

At $\alpha = 2.00$, the first moment is 1.344 kg m, and

$$X_{\text{CM}} = \frac{1.344}{1.92} = 0.700 \text{ m}.$$

This lies to the right of the geometric midpoint, 0.600 m, because the right side contains more mass per unit length.

Independent limits. Setting $\alpha = 0$ recovers the uniform rod, $X_{\text{CM}} = L/2$. A nonnegative density on the entire rod requires $\alpha \geq -1$ for $\lambda_0 > 0$. At $\alpha = -1$ the density decreases linearly to zero and the formula gives $L/3$. As α grows without bound, the normalized density tends to one proportional to x , and the center tends to $2L/3$. These limits constrain the result more strongly than merely checking that it lies somewhere on the rod.

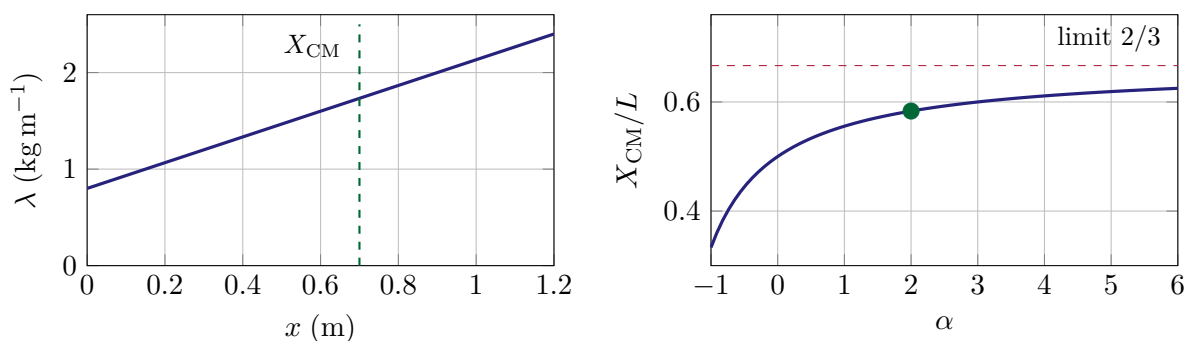


Figure 9.4: Left: the stipulated density of the graded rod. Right: its normalized center as the gradient varies through physically allowed values. The dashed limiting level is approached as $\alpha \rightarrow \infty$; it is not reached at any finite α .

The dependence on the density gradient can also be checked without plotting. Differentiating Eq. (9.13) gives

$$\frac{d(X_{\text{CM}}/L)}{d\alpha} = \frac{1}{12(1 + \alpha/2)^2} > 0 \quad (\alpha \geq -1).$$

Increasing the gradient therefore always shifts the center to the right within the physical family. The decreasing derivative explains why the plotted curve flattens rather than crossing its limiting value. At $\alpha = 2$, the derivative is $1/48$; a small change $\delta\alpha$ produces the first-order position change $\delta X_{\text{CM}} \simeq L\delta\alpha/48$. This last relation is a local sensitivity estimate, while Eq. (9.13) remains the exact result for a finite change.

The numerical rod model uses a slender-body approximation: mass distribution across the transverse section is represented by its integral, the linear density. It does not require the volume density itself to be uniform. A tapered homogeneous rod and a straight rod made from a graded material can produce the same $\lambda(x)$ and hence the same axial center of mass. Their stresses and rotational inertias may differ, but that additional information is not required by this particular weighted-position calculation.

9.1.7 Identical outlines can represent different mass distributions

A shape alone does not specify a center of mass. A thin wire laid along a semicircular arc and a filled semicircular plate share the same curved boundary, but one distributes mass along length and the other across area. Treating both as a semicircle without stating their physical model loses the information needed for the integral. Their symmetry fixes the horizontal coordinate, yet symmetry does not determine how far above the diameter either center lies.

Parameterization helps prevent this ambiguity. On a circular wire, every element lies at radius R , and the line element is $Rd\theta$. In a filled plate, the radius varies from zero to R , and the area element is $rdrd\theta$. The extra factor of r in the area element counts the increasing area of annular sectors. It must be introduced by geometry, rather than added as an arbitrary correction after an incorrect integral.

Example 9.5: Comparing a semicircular wire with a semicircular lamina

Compare the centers of mass of a uniform thin upper semicircular wire and a uniform thin upper semicircular lamina, each of radius $R = 0.600$ m. Put the origin at the midpoint of the diameter, with y positive upward.

Symmetry. Reflection through the y axis preserves each distribution, so $X_{\text{CM}} = 0$. It does not imply $Y_{\text{CM}} = 0$ because neither distribution is symmetric above and below the diameter.

Wire. Write $y = R \sin \theta$ and $dm = \lambda R d\theta$, for $0 \leq \theta \leq \pi$. Its mass and vertical first moment

are

$$M_{\text{wire}} = \lambda\pi R, \quad \int y \, dm = \lambda R^2 \int_0^\pi \sin \theta \, d\theta = 2\lambda R^2.$$

Hence

$$Y_{\text{wire}} = \frac{2R}{\pi} = 0.382 \, \text{m}.$$

Lamina. In polar coordinates, $y = r \sin \theta$ and $dm = \sigma r \, dr \, d\theta$. The mass is $M_{\text{lamina}} = \sigma\pi R^2/2$. Its first moment is

$$\int y \, dm = \sigma \int_0^\pi \sin \theta \, d\theta \int_0^R r^2 \, dr = \frac{2\sigma R^3}{3}.$$

Consequently

$$Y_{\text{lamina}} = \frac{4R}{3\pi} = 0.255 \, \text{m}.$$

Both density constants cancel only because each of the two stipulated distributions is uniform.

Physical interpretation. The wire puts all its mass on the outer arc. The plate includes material closer to the diameter, so its mean height is lower. The ratio $Y_{\text{wire}}/Y_{\text{lamina}} = 3/2$ is independent of the radius and of the two density constants. Both heights are between zero and R ; neither equals the half-height $R/2$ merely because the outline is semicircular.

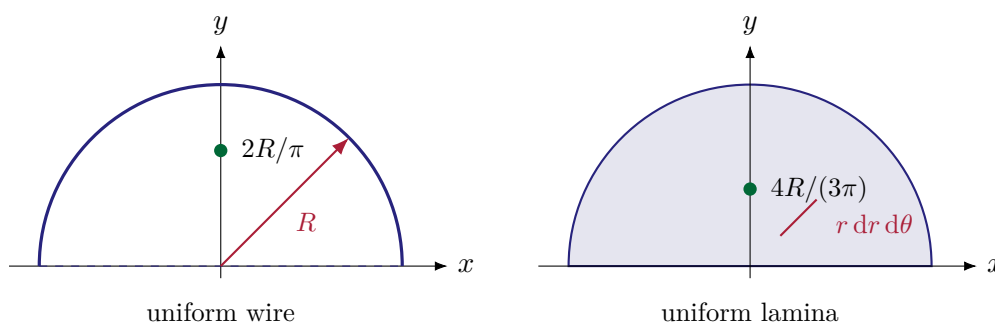


Figure 9.5: Semicircular wire and lamina use the same geometric scale. Green points show their separately derived centers. The small red segment on the right indicates the radial direction of an area element; it is schematic rather than an area drawn to differential scale.

9.1.8 Composite regions, empty space, and the convex-hull check

Composite construction reduces an integral to a small mass inventory, provided the pieces cover the material exactly once. Positive pieces can be added, removed pieces can be subtracted, and a region shared by two added shapes must be subtracted once to avoid counting it twice. The operation acts on both mass and first moment. Subtracting only an overlap's area from the denominator leaves an incorrect numerator.

Example 9.6: The center of a uniform L-shaped plate

A uniform thin L-shaped plate consists of a horizontal arm extending from $(0,0)$ to $(0.600, 0.100)$ m and a vertical arm extending from $(0,0)$ to $(0.100, 0.500)$ m. Find its center of mass and determine whether that point lies in the plate.

Nonoverlapping decomposition. Use the entire horizontal rectangle, of area $A_1 = 0.0600 \, \text{m}^2$ and center $(0.300, 0.0500)$ m. Add only the part of the vertical arm above the horizontal arm. This second rectangle has area $A_2 = 0.0400 \, \text{m}^2$ and center $(0.0500, 0.300)$ m. Their union is the

plate and they have no overlapping area. The constant areal density cancels, giving

$$X_{\text{CM}} = \frac{0.0600(0.300) + 0.0400(0.0500)}{0.1000} = 0.200 \text{ m},$$

$$Y_{\text{CM}} = \frac{0.0600(0.0500) + 0.0400(0.300)}{0.1000} = 0.150 \text{ m}.$$

Thus $\mathbf{R}_{\text{CM}} = (0.200, 0.150) \text{ m}$.

A separate decomposition check. Alternatively add the two full arms, then subtract their $0.100 \text{ m} \times 0.100 \text{ m}$ overlap. The area is $0.0600 + 0.0500 - 0.0100 = 0.1000 \text{ m}^2$. The horizontal first moment is $0.0600(0.300) + 0.0500(0.0500) - 0.0100(0.0500) = 0.0200 \text{ m}^3$; the vertical first moment is $0.0600(0.0500) + 0.0500(0.250) - 0.0100(0.0500) = 0.0150 \text{ m}^3$. Division reproduces both coordinates.

Location check. Since $X_{\text{CM}} > 0.100 \text{ m}$ and $Y_{\text{CM}} > 0.100 \text{ m}$, this point lies in the empty notch rather than in either arm. There is no contradiction: a center of mass is a weighted average position, not a required piece of material.

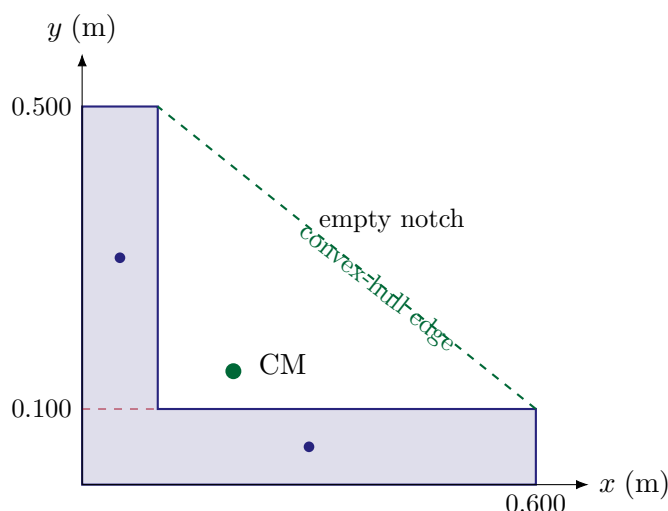


Figure 9.6: The L-shaped plate has its center in empty space. The dashed diagonal completes its convex hull, which does contain the center. Blue points are the centers of the two nonoverlapping rectangular components used in the first calculation.

The strongest general location check for positive masses is the *convex hull*: the smallest convex region containing the entire mass distribution. A region is convex if the straight segment joining any two of its points remains in the region. Since the coefficients $w_i = m_i/M$ are nonnegative and sum to one, $\mathbf{R}_{\text{CM}} = \sum_i w_i \mathbf{r}_i$ is a convex combination of the particle positions. The corresponding integral has the same property for a nonnegative continuous density.

This statement is stronger than separate bounds on x , y , and z . Those bounds locate the center in a rectangular box. The convex hull can be substantially smaller than that box. For three particles at the corners of a triangle, for example, their center is inside the triangle, not merely somewhere inside its surrounding rectangle. For an L shape, the convex hull includes some empty space, so it allows precisely the situation found above.

The convex-hull condition relies on positive physical mass. Signed masses used for a cutout do not represent a new physical distribution with negative material. They are an intermediate subtraction scheme for the remaining positive material. The final center must therefore still lie in the convex hull of that remaining material. Passing the convex-hull test is necessary, although it cannot guarantee

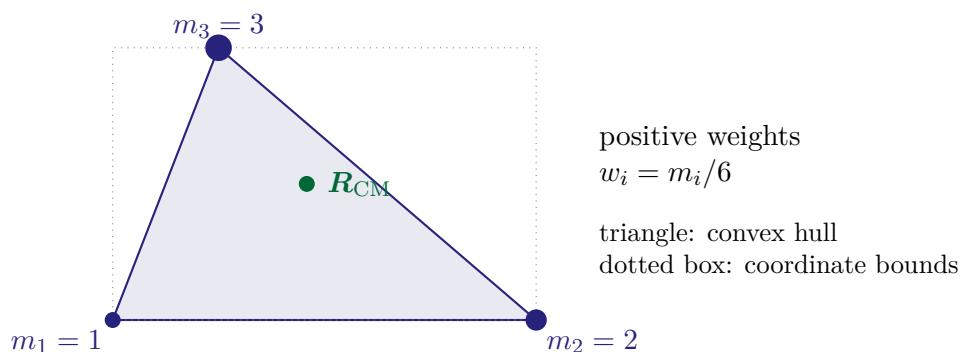


Figure 9.7: A positive weighted mean of three noncollinear positions lies in their triangular convex hull. For the displayed relative masses and coordinates $(0,0)$, $(4,0)$, and $(1,3)$ in a common length unit, the weighted mean is $(11/6, 3/2)$.

that the component inventory or density model is otherwise correct.

Symmetry is another independent check, but it must be a symmetry of the mass distribution, not just of the outer boundary. Reflect a distribution through the plane $x = 0$. If each element at x has an equal-mass counterpart at $-x$, their horizontal first moments cancel, so $X_{\text{CM}} = 0$. A reflection-symmetric outline filled with denser material on one side lacks this mass symmetry. Likewise, a hole can destroy a previously useful symmetry even though much of the original body remains.

These calculations establish where the center lies for a specified inventory. Their next use is dynamical: when the component positions change, the weighted center traces a trajectory. The motion theorem below connects that trajectory to external forces without requiring a detailed solution for every internal degree of freedom.

9.2 Motion of the Center of Mass

The center of mass is more than a geometrical average. For a fixed set of particles of constant mass, differentiate Eq. (9.2):

$$\mathbf{V}_{\text{CM}} = \frac{d\mathbf{R}_{\text{CM}}}{dt} = \frac{1}{M} \sum_i m_i \mathbf{v}_i. \quad (9.14)$$

Multiplying by the total mass gives a central identity:

$$\boxed{M\mathbf{V}_{\text{CM}} = \sum_i m_i \mathbf{v}_i \equiv \mathbf{P}.} \quad (9.15)$$

Thus the total momentum is exactly the momentum that a single fictitious particle of mass M would have if it moved with the center-of-mass velocity.

Differentiate again:

$$M\mathbf{A}_{\text{CM}} = \frac{d\mathbf{P}}{dt} = \sum_i m_i \mathbf{a}_i. \quad (9.16)$$

Applying Newton's second law to each particle gives

$$\sum_i m_i \mathbf{a}_i = \sum_i \mathbf{F}_i.$$

Split each particle force into contributions from particles inside the system and forces from outside:

$$\sum_i \mathbf{F}_i = \sum_i \mathbf{F}_{i,\text{ext}} + \sum_i \sum_{j \neq i} \mathbf{F}_{ij}. \quad (9.17)$$

For pairwise internal forces satisfying Newton's third law, $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$, the internal terms cancel in pairs. Therefore

$$\boxed{M\mathbf{A}_{\text{CM}} = \sum \mathbf{F}_{\text{ext}}.} \quad (9.18)$$

This is the translational equation of motion for the center of mass of a fixed-mass system.

Key message

Internal forces can dramatically change the motion of individual parts while leaving the total momentum unchanged. They cannot, by themselves, accelerate the center of mass of a fixed isolated system.

9.2.1 System boundaries make the theorem meaningful

Equation (9.18) depends on the chosen system. A spring force between two carts is internal if both carts and the spring are inside the boundary, but it is external if the system contains only one cart. The phrase “internal force” is therefore not an intrinsic label attached permanently to a force; it is a statement about a system boundary.

Similarly, a track force on a cart is external to the cart system. If the track is level and its normal force cancels the carts' weight vertically, the horizontal external force can still be negligible. The approximation “momentum is conserved horizontally” may then be excellent even though the full vector system experiences large vertical external forces whose net vertical impulse is essentially zero.

Example 9.7: Two carts released by an internal spring

Two carts of masses $m_1 = 2.0 \text{ kg}$ and $m_2 = 3.0 \text{ kg}$ move together at 0.80 m s^{-1} on a nearly frictionless horizontal track. A compressed spring of negligible mass between them is released. Afterwards the first cart moves at -0.70 m s^{-1} . What is the speed of the second cart if the horizontal external impulse is negligible? What happens to the center-of-mass velocity?

System and frame. The system contains both carts and the spring, observed in the laboratory frame. The spring forces are internal. Horizontal external impulse is neglected over the release interval.

Initially,

$$P_i = (2.0 + 3.0)(0.80) = 4.0 \text{ kg m s}^{-1}.$$

After release,

$$P_f = (2.0)(-0.70) + (3.0)v_2.$$

Setting $P_f = P_i$ gives

$$-1.40 + 3.0v_2 = 4.0, \quad \boxed{v_2 = 1.80 \text{ m s}^{-1}}.$$

The center-of-mass velocity before and after is

$$V_{\text{CM}} = \frac{P}{M} = \frac{4.0}{5.0} = \boxed{0.80 \text{ m s}^{-1}}.$$

Interpretation. The carts separate strongly relative to one another, but the center of mass continues at constant velocity because the net horizontal external force is negligible. Internal energy stored in the spring changes the relative kinetic energy, not the total momentum.

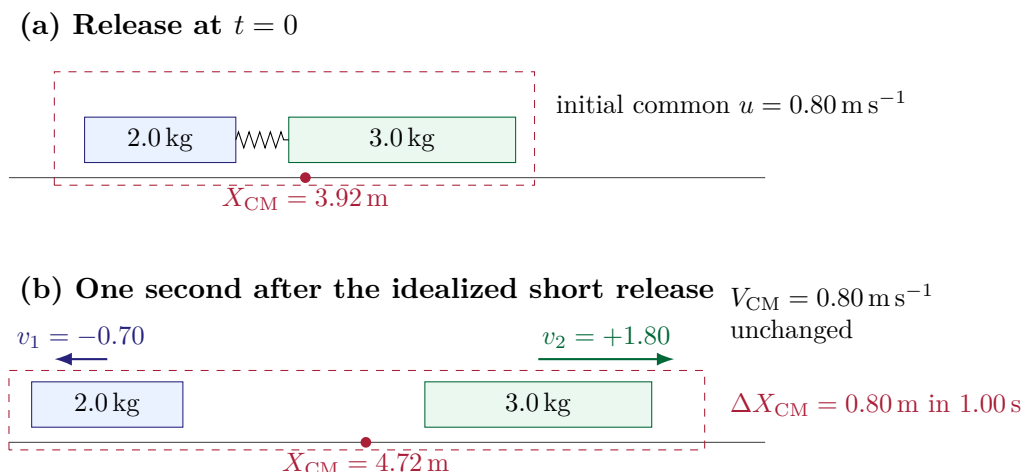


Figure 9.8: Time-consistent one-dimensional positions for Example 9.7. The dashed boundaries enclose both carts and the negligible-mass spring, so spring forces are internal. The release is approximated as instantaneous, the spring mass is negligible, and subsequent velocities are constant. One centimeter on the position axis represents one meter; cart widths are only a representation of each uniform cart. Centers move from $(2, 5.2) \text{ m}$ to $(1.3, 7.0) \text{ m}$ in one second, giving the marked weighted centers. Velocity arrow scale is 1 cm per 1 m s^{-1} ; velocity labels are in m s^{-1} .

9.2.2 When internal-force cancellation needs care

The pair-cancellation derivation is exact for the classical particle models used here when internal forces obey equal-and-opposite pair forces. Electromagnetic systems with time-dependent fields can store momentum in the fields themselves, and a purely particle-only boundary may then be incomplete. Variable-mass systems also require care because the set of particles inside the system can change. Neither issue invalidates the fixed-system theorem; both signal that the system definition has changed or omitted a momentum-carrying part.

Guided checks

Check 1. Two skaters push one another apart on frictionless ice. Can both speed up while the center of mass remains at constant velocity? Yes. Their internal forces redistribute momentum.

Check 2. If the total external force is zero at one instant, is total momentum necessarily constant for all later times? No. The condition must hold, or have negligible integrated impulse, over the interval considered.

Check 3. Can the center of mass accelerate even if no internal forces are known? Yes. Its acceleration is determined by the net external force; the detailed internal forces need not be computed.

9.2.3 Internal motion on a freely moving platform

A person can walk across a platform while the combined center of mass remains at rest. This is a particularly useful counterexample to the mistaken claim that zero external force makes every part of a system stationary. Choose an inertial laboratory coordinate x , and distinguish the person's laboratory position x_a from the platform center x_p . For person mass m and platform mass M , conservation of the combined center of mass requires

$$m \Delta x_a + M \Delta x_p = 0. \quad (9.19)$$

The distance walked relative to the platform is a different quantity:

$$s = \Delta(x_a - x_p) = \Delta x_a - \Delta x_p. \quad (9.20)$$

Solving the two equations, rather than identifying s with a laboratory displacement, gives

$$\Delta x_p = -\frac{m}{M+m}s, \quad \Delta x_a = \frac{M}{M+m}s. \quad (9.21)$$

The platform moves opposite to the person's relative motion. The denominator contains the *combined* mass because both objects contribute to the center-of-mass constraint. A displacement is sufficient to solve this problem: the detailed sequence of foot contact forces, accelerations, and walking speeds need not be known.

Example 9.8: Walking on a free platform: positions, speeds, and energy

A 60 kg person walks 3.20 m to the right relative to a 180 kg platform. Initially both are at rest. Idealize the platform as translating horizontally without rotation, and neglect horizontal external force. (a) Find the laboratory displacements. (b) During a portion of the walk, the person moves at 0.400 m s^{-1} relative to the platform. Find both laboratory velocities and the combined kinetic energy. (c) Explain the final motion when the person stops relative to the platform.

Boundary and balance. The system contains the person and platform. Their contact forces are internal. Vertical support and gravity need not vanish individually; their net horizontal impulse is negligible. Since the combined initial momentum is zero, the center of mass stays at its original horizontal position.

(a) Displacements. Equations (9.21) give

$$\Delta x_p = -\frac{60}{240}(3.20) = -0.800 \text{ m}, \quad \Delta x_a = \frac{180}{240}(3.20) = +2.40 \text{ m}.$$

The relative displacement is $2.40 - (-0.800) = 3.20 \text{ m}$, and the weighted displacement is $60(2.40) + 180(-0.800) = 0$. These are independent checks on the meaning of the two distances.

(b) Velocities and energy. Write $v_a - v_p = 0.400 \text{ m s}^{-1}$ together with $60v_a + 180v_p = 0$. Therefore

$$v_p = -0.100 \text{ m s}^{-1}, \quad v_a = +0.300 \text{ m s}^{-1}.$$

The combined translational kinetic energy is

$$K = \frac{1}{2}(180)(0.100)^2 + \frac{1}{2}(60)(0.300)^2 = \boxed{3.60 \text{ J}}.$$

It is positive although total momentum is zero. Energy supplied by the person's internal stores creates relative motion; zero momentum does not imply zero kinetic energy.

(c) Stopping. When the person is again at rest relative to the platform, both share a velocity V . Zero total momentum gives $240V = 0$, so both are at rest in the laboratory. The 3.60 J of translational kinetic energy present in the stipulated constant-speed portion must be removed during stopping, for example through internal dissipation. Neither that energy change nor the walking forces can shift the combined center of mass. The detailed energy expenditure by a real person is outside this ideal translational model.

The same relations apply to motion in a freely drifting cabin along a straight line, provided the selected system includes everything transmitting the internal forces. They also show why measuring a person's distance along the deck alone is insufficient to infer their inertial displacement. If the platform is tied to a wall, the tension is external to the person–platform system and Eq. (9.19) must be replaced by a force or impulse balance. A change in the boundary condition changes the mathematical problem.

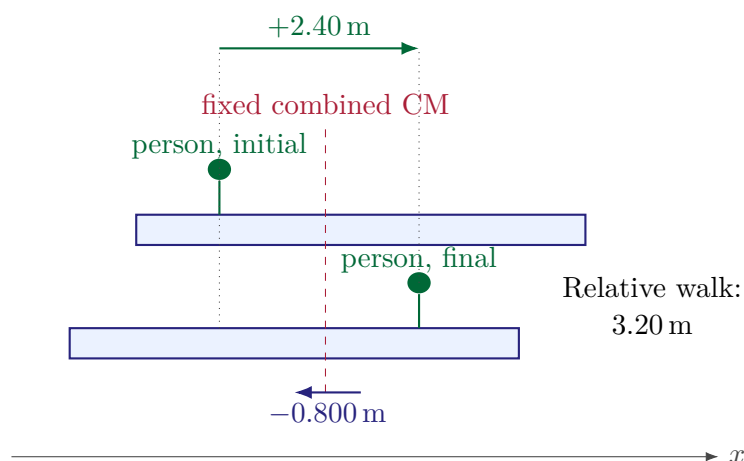


Figure 9.9: Horizontal displacement bookkeeping for Example 9.8. Initial platform and person coordinates are 2.70 and 1.00 m; final coordinates are 1.90 and 3.40 m. Both snapshots therefore give $X_{\text{CM}} = 2.275$ m. Vertical separation distinguishes snapshots and represents no vertical motion.

9.2.4 External force is determined by the selected boundary

For the person alone, the horizontal platform contact force is external and changes the person's momentum. For the platform alone, the equal-and-opposite person contact force is external. For the combined system, these forces cancel. This cancellation follows from adding two different equations, not from ignoring the forces on either body. In a horizontal model,

$$ma_a = F_{p \rightarrow a}, \quad Ma_p = -F_{p \rightarrow a} + F_{\text{ext}},$$

so $ma_a + Ma_p = F_{\text{ext}}$. If a tether exerts $F_{\text{ext}} = 12$ N to the right on the 240 kg combined system, its center-of-mass acceleration is 0.0500 m s^{-2} , whatever the person does internally. A constant force over 2.00 s changes total momentum by 24.0 kg m s^{-1} and CM speed by 0.100 m s^{-1} . These statements require no assumption that the person and platform maintain a common velocity.

The mass-weighted acceleration provides a diagnostic of an internal-motion simulation. In the free-platform model, its value must be $[ma_a(t) + Ma_p(t)]/(m + M) = 0$ at every instant, even during startup and stopping. If the person has $a_a = +0.300 \text{ m s}^{-2}$ at a particular instant, the platform must have $a_p = -0.100 \text{ m s}^{-2}$. Their individual accelerations are nonzero but their weighted sum is zero. With the stated 12 N tether force, the same diagnostic must instead return 0.0500 m s^{-2} . Internal-force cancellation is therefore a quantitative condition on the weighted sum, not on each acceleration separately.

The practical sequence is to draw the boundary, identify momentum-carrying bodies, project the external forces onto the required direction, and only then decide whether the external impulse is negligible. A large force in an irrelevant direction does not invalidate a component balance. Conversely, a small persistent horizontal force can have a substantial impulse over a long interval. The initial and final snapshots must refer to the same fixed-particle system if the simple CM theorem is used.

9.3 Momentum and Impulse

For one particle of constant mass,

$$\boxed{\mathbf{p} = m\mathbf{v}.} \quad (9.22)$$

Newton's second law may be written

$$\mathbf{F}_{\text{net}} = \frac{d\mathbf{p}}{dt}. \quad (9.23)$$

For constant mass this is equivalent to $m\mathbf{a}$, but the momentum form connects naturally to short-duration forces.

Integrating Eq. (9.23) from t_i to t_f gives

$$\mathbf{J} = \int_{t_i}^{t_f} \mathbf{F}_{\text{net}}(t) dt = \mathbf{p}_f - \mathbf{p}_i \equiv \Delta \mathbf{p}. \quad (9.24)$$

The vector $\mathbf{J}$ is the impulse. Graphically, a component of impulse is the signed area under the corresponding force–time graph.

Definition 9.2: Impulse

The impulse delivered by a force over a time interval is its time integral. It equals the momentum change caused by the net force over that interval. An average force is defined by

$$\mathbf{F}_{\text{avg}} = \frac{\mathbf{J}}{\Delta t},$$

but the average generally does not determine the peak force or the detailed force history.

9.3.1 Equal impulse does not mean equal force history

Impact mitigation often aims to produce a required momentum change while modifying the time profile of force. Extending the duration of a given impulse reduces the required average force, but a statement about peak force requires a model of the pulse shape.

Example 9.9: Two pulse shapes with the same impulse

A one-dimensional interaction delivers 14.4 N s of positive impulse. Compare (A) a triangular force pulse of duration 12 ms and peak 2.40 kN with (B) a half-sine pulse of duration 24 ms,

$$F_B(t) = F_0 \sin\left(\frac{\pi t}{T}\right), \quad 0 \leq t \leq T.$$

Find F_0 so that both impulses are equal, and compare their average and peak forces.

For pulse A,

$$J_A = \frac{1}{2}(0.012)(2400) = 14.4 \text{ N s}, \quad F_{A,\text{avg}} = \frac{14.4}{0.012} = 1.20 \text{ kN}.$$

For the half-sine pulse,

$$J_B = \int_0^T F_0 \sin\left(\frac{\pi t}{T}\right) dt = \frac{2F_0 T}{\pi}.$$

With $T = 0.024 \text{ s}$ and $J_B = 14.4 \text{ N s}$,

$$F_0 = \frac{J_B \pi}{2T} = 942 \text{ N} = 0.942 \text{ kN}.$$

Its average is

$$F_{B,\text{avg}} = \frac{14.4}{0.024} = 0.600 \text{ kN}.$$

Interpretation. Both interactions cause the same momentum change, but pulse B acts twice as long and has both lower average force and lower peak force for these chosen analytical profiles. The result is a model comparison, not a universal safety threshold.

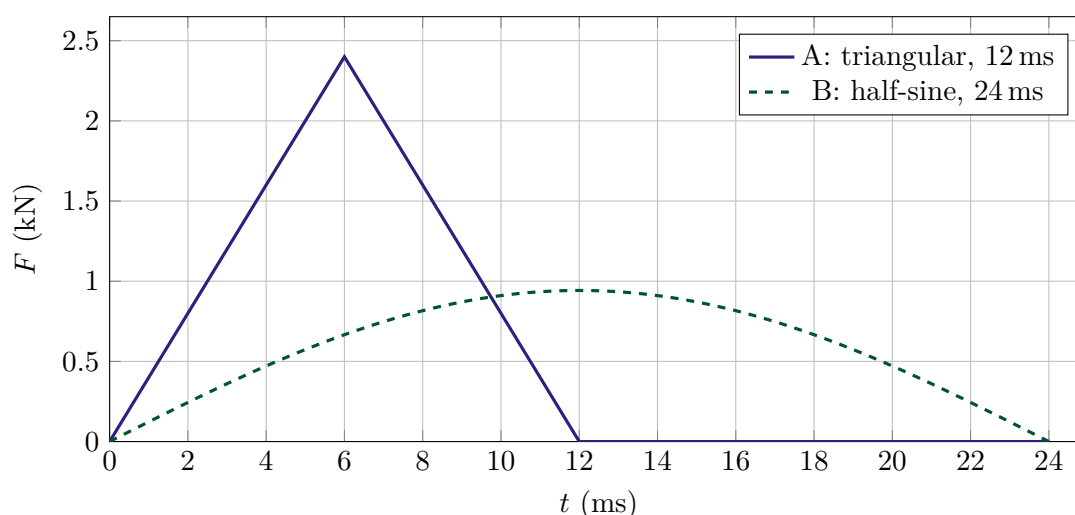


Figure 9.10: Analytical force–time models from Example 9.9. Both enclosed areas are 14.4 Ns, although duration, average force, and peak force differ. These are constructed teaching curves, not measured impact data.

9.3.2 Impulse components and sign

Impulse is a vector. A force that changes direction during the interval can give a small net impulse even when the instantaneous force magnitude is large. For planar motion,

$$J_x = \int F_x dt, \quad J_y = \int F_y dt. \quad (9.25)$$

Areas below an axis contribute negatively to the corresponding impulse component. Integrating the magnitude $|F|$ instead would answer a different question and generally would not equal the magnitude of the net impulse.

Caution

Do not use “large force” and “large impulse” as synonyms. A large force can act for a very short time and produce modest impulse; a smaller force can produce the same impulse if it acts longer. The product $F\Delta t$ is valid only for constant force or when F is explicitly the average force over the interval.

9.3.3 Signed force components and the entire momentum history

An impulse calculation should preserve the direction information in the measured or modelled force. Choose fixed coordinate axes and integrate each signed component over the same time interval. The force vector may turn, a component may reverse sign, and the object’s velocity may reverse at a different instant. None of those possibilities changes the componentwise theorem.

For an intermediate time t the impulse accumulated so far determines the momentum then:

$$\mathbf{p}(t) = \mathbf{p}(t_i) + \int_{t_i}^t \mathbf{F}_{\text{net}}(\tau) d\tau. \quad (9.26)$$

The integration variable τ distinguishes the time being integrated from the upper endpoint t . This formula provides more than an endpoint result. Its slope is the force, its value is the momentum, and dividing by a constant particle mass gives velocity. It is especially helpful when an initial momentum opposes the applied force: the object can first slow down, stop, and then reverse.

The impulse unit, Ns, equals kg m s^{-1} , because a newton is kg m s^{-2} . Impulse and momentum therefore share dimensions but play different roles: momentum describes the state at a time, while

impulse accumulates the action of a force over an interval. A finite jump in a force graph can change the slope of momentum instantaneously, but it cannot produce a finite momentum jump over zero time. In the piecewise model below, momentum remains continuous at the switching instant. An ideal instantaneous impulse is a distinct limiting model in which the force peak grows as the duration shrinks while its integral stays finite.

Example 9.10: A vector impulse with a reversing horizontal force

A particle of mass 0.500 kg starts with velocity $(-1.00, 2.00)\text{ m s}^{-1}$. During a 0.100 s interaction its net force components are

$$F_x(t) = \begin{cases} +60.0\text{ N}, & 0 \leq t < 0.0400\text{ s}, \\ -20.0\text{ N}, & 0.0400 \leq t \leq 0.100\text{ s}, \end{cases} \quad F_y(t) = +30.0\text{ N}.$$

Find the net impulse, final velocity, and instant when the horizontal velocity first becomes zero.

Signed areas. The two horizontal rectangles have opposite signs:

$$J_x = 60.0(0.0400) - 20.0(0.0600) = 1.20\text{ N s}, \quad J_y = 30.0(0.100) = 3.00\text{ N s}.$$

The impulse is $(1.20, 3.00)\text{ N s}$, with magnitude 3.23 N s and direction 68.2° counterclockwise from $+x$.

Momentum and velocity. Initially $\mathbf{p}_i = (-0.500, 1.00)\text{ kg m s}^{-1}$, so

$$\mathbf{p}_f = (0.700, 4.00)\text{ kg m s}^{-1}, \quad \boxed{\mathbf{v}_f = (1.40, 8.00)\text{ m s}^{-1}}.$$

During the first interval, $p_x(t) = -0.500 + 60.0t$ in SI units. It reaches zero at

$$\boxed{t = \frac{0.500}{60.0} = 0.00833\text{ s}}.$$

At 0.0400 s , $p_x = 1.90\text{ kg m s}^{-1}$. The later negative force decreases it to 0.700 kg m s^{-1} , without reversing its sign again.

Magnitude is a different integral. The integral of $|F_x|$ is 3.60 N s , not J_x . The integral of the full force magnitude is

$$0.0400\sqrt{60.0^2 + 30.0^2} + 0.0600\sqrt{20.0^2 + 30.0^2} = 4.85\text{ N s},$$

which exceeds $|\mathbf{J}| = 3.23\text{ N s}$. Adding scalar magnitudes removes directional cancellation and therefore cannot supply the net momentum change.

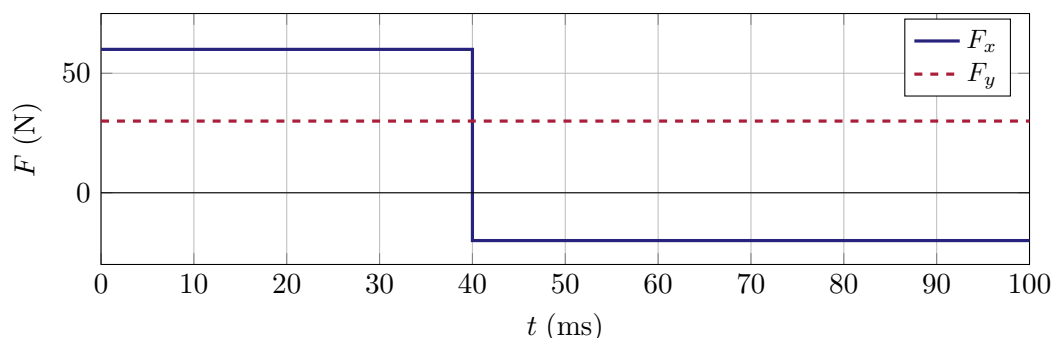


Figure 9.11: The stipulated net-force components in Example 9.10. The negative horizontal area subtracts from the positive area. The idealized vertical jumps have zero duration and contribute no extra impulse.

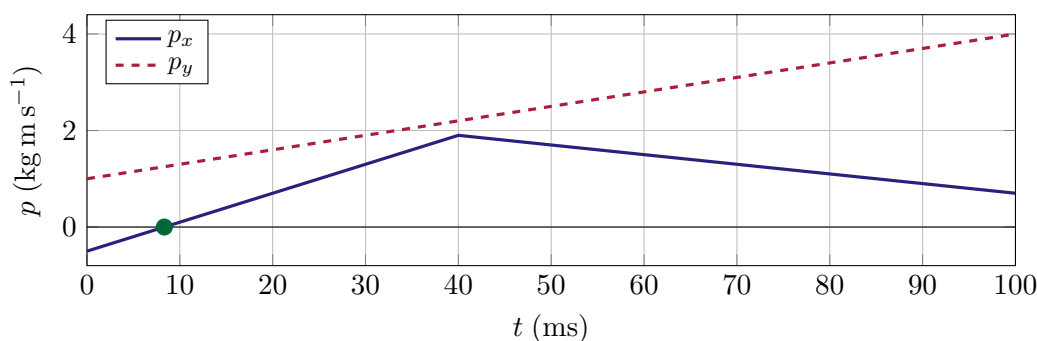


Figure 9.12: Integrating the component curves gives continuous momenta even though the force model has a discontinuity. The green point marks zero horizontal velocity; the force reversal occurs later, at 40 ms.

9.3.4 Momentum-space geometry and the integral of force magnitude

The signed pulse also has a useful vector picture. During its first interval the momentum changes by $(2.40, 1.20) \text{ kg m s}^{-1}$; during its second it changes by $(-1.20, 1.80) \text{ kg m s}^{-1}$. Drawing those increments head to tail produces the complete change $(1.20, 3.00) \text{ kg m s}^{-1}$. The intermediate vertex represents the actual momentum at 40.0 ms, namely $(1.90, 2.20) \text{ kg m s}^{-1}$.

This construction is a path in *momentum space*. Its axes have momentum units, so it is neither the spatial path of the particle nor a map of force application points. For a constant force the momentum-space segment is straight. A varying force can make a curved path because its direction determines the instantaneous direction of the path's tangent. The total impulse is the vector from the initial endpoint to the final endpoint, regardless of how the intervening path bends.

The distinction between integrating a vector and integrating its magnitude now has a geometrical explanation. Since $d\mathbf{p} = \mathbf{F}_{\text{net}} dt$, the integral $\int |\mathbf{F}_{\text{net}}| dt$ is the length of the momentum-space path. The magnitude $|\int \mathbf{F}_{\text{net}} dt|$ is the length of its endpoint chord. A path cannot be shorter than the straight chord joining its endpoints. Therefore

$$\left| \int_{t_i}^{t_f} \mathbf{F}_{\text{net}} dt \right| \leq \int_{t_i}^{t_f} |\mathbf{F}_{\text{net}}| dt. \quad (9.27)$$

Equality holds when all nonzero force contributions point in the same fixed direction. Merely keeping the force magnitude constant is insufficient; the direction must also satisfy that condition.

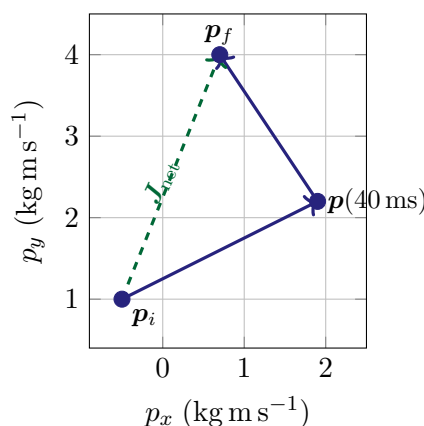


Figure 9.13: The signed pulse traces two straight segments in momentum space. Their total length is 4.85 N s, while the green endpoint chord has length 3.23 N s. These axes describe momentum components rather than particle positions.

For the displayed path, the segment lengths are $\sqrt{2.40^2 + 1.20^2} = 2.68 \text{ N s}$ and $\sqrt{(-1.20)^2 + 1.80^2} = 2.16 \text{ N s}$, reproducing the integral of force magnitude above. A particularly revealing limit consists

of equal, opposite increments: the particle returns to its original momentum, the chord has zero length, but the path has positive length. The original and final momenta then match even though acceleration occurred between them.

9.3.5 Contact impulse and net impulse during a rebound

A force sensor at a contact generally measures a particular interaction force, not the total force on the body. For a vertical bounce on a horizontal surface, let upward be positive. A contact sensor can measure the upward normal force $N(t)$, while gravity supplies the downward force mg . Neglecting air resistance, the particle equation is

$$\frac{dp_y}{dt} = N(t) - mg.$$

Integrating over the interval from first contact to separation gives

$$J_{\text{contact}} - mgT = m(v_f - v_i), \quad J_{\text{contact}} = \int_0^T N(t)dt. \quad (9.28)$$

This separates two different impulses before any approximation is made. The impulse approximation neglects mgT only when its size is acceptably small compared with the quantity being inferred. A short interval often makes that approximation good, but a duration alone does not establish its accuracy.

Example 9.11: A rebound for which the gravity correction matters

A 0.200 kg ball reaches a horizontal compliant surface at $v_i = -3.00 \text{ m s}^{-1}$ and leaves at $v_f = +2.40 \text{ m s}^{-1}$. Contact lasts $T = 0.0600 \text{ s}$. Use $g = 9.81 \text{ m s}^{-2}$. Find the net and contact impulses and the mean contact force. If a half-sine models the contact force, find its peak.

Net impulse from velocities. The velocity change is positive, despite the initially downward motion:

$$J_{\text{net}} = m(v_f - v_i) = 0.200[2.40 - (-3.00)] = 1.08 \text{ N s}.$$

Gravity acts throughout contact and contributes

$$J_g = -mgT = -(0.200)(9.81)(0.0600) = -0.11772 \text{ N s}.$$

Consequently

$$J_{\text{contact}} = J_{\text{net}} - J_g = 1.19772 \text{ N s} \simeq 1.20 \text{ N s}.$$

The mean contact force is $J_{\text{contact}}/T = 19.962 \text{ N} \simeq 20.0 \text{ N}$. The mean *net* force is $J_{\text{net}}/T = 18.0 \text{ N}$. Their difference, 1.962 N, is precisely the ball's weight.

Peak of the stipulated contact model. Let

$$N(t) = N_0 \sin(\pi t/T), \quad 0 \leq t \leq T.$$

Its impulse is $2N_0T/\pi$, so

$$N_0 = \frac{\pi J_{\text{contact}}}{2T} = 31.4 \text{ N}.$$

The net force at the peak is $31.3562 - 1.962 = 29.3942 \text{ N}$ before rounding.

Accuracy and interpretation. Inferring contact impulse directly from $m\Delta v$ would underestimate it by 0.11772 N s, or 9.83% of the true contact impulse in this example. Near first contact and separation the half-sine normal force is below the weight, so the instantaneous net force is downward even though the total net impulse is upward. The pulse is a constructed model consistent with the supplied endpoint momentum change; it does not establish a material

contact law or determine deformation. Because the net force is symmetric about $T/2$, its implied vertical displacement is

$$\Delta y = \frac{v_i + v_f}{2}T = \frac{-3.00 + 2.40}{2}(0.0600) = -0.0180 \text{ m.}$$

Thus these endpoints do not represent separation at the original center-of-mass contact height. Compliance or permanent deformation can allow different endpoint heights; an equal-height rebound from an unchanged surface would require a different, generally asymmetric force history or a developed deformation model. The impulse calculation alone does not establish contact geometry.

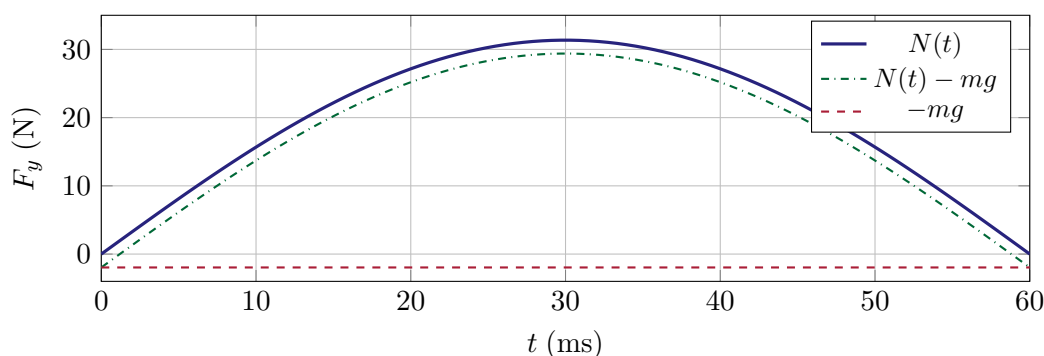


Figure 9.14: Contact and net-force models for Example 9.11. Subtracting the constant weight changes the signed area by 0.11772 Ns and produces small negative net-force intervals near the endpoints.

The desired tolerance can be converted into an explicit duration test. With the same endpoint velocities, the fractional error in the inferred contact impulse is $mgT/(J_{\text{net}} + mgT)$. Requiring it to be below 1.00% gives

$$\frac{mgT}{J_{\text{net}} + mgT} < 0.0100 \implies T < \frac{0.0100 J_{\text{net}}}{0.9900 mg} = 0.00556 \text{ s.}$$

Thus the 60.0 ms interaction fails that tolerance by a substantial margin. The test uses the specified velocity change; a different rebound speed would change the bound. A fixed statement that gravity is negligible for every collision shorter than a particular duration would therefore be unjustified.

For a force platform, the sign convention also determines which body the recorded force acts on. The platform receives the ball's downward contact force; the ball receives the platform's upward contact force. Their magnitudes match under the classical contact model, but their signed components are opposite. A data file should identify the measured body and the instrument's positive direction before its integral is inserted into a ball momentum balance.

9.3.6 Designing a pulse from a required momentum change

The impulse theorem can be used in reverse. A required initial and final velocity specifies the net impulse, but it does not uniquely specify a force history. One must also choose a duration and a pulse model. This distinction matters in impact design: momentum balance supplies a necessary integral constraint, while peak force, deformation, and admissible stress require additional physical information.

Write a one-dimensional pulse as $F(t) = Ag(t/T)$ on $0 \leq t \leq T$, where g is a dimensionless profile and A is a signed amplitude. Changing variables to $s = t/T$ gives

$$J = AT \int_0^1 g(s) ds.$$

If the profile integral is nonzero, the required amplitude is

$$A = \frac{m(v_f - v_i)}{T \int_0^1 g(s) ds}. \quad (9.29)$$

An integral with units has therefore been separated into a dimensional scale and a dimensionless shape factor. A profile with zero integral cannot deliver a nonzero net impulse at any finite amplitude; its positive and negative contributions cancel.

Example 9.12: A smooth braking pulse with a specified endpoint

A 0.800 kg cart moves initially at $+1.50 \text{ m s}^{-1}$. It is to stop after $T = 0.0800 \text{ s}$ under a stipulated net-force pulse

$$F_x(t) = -F_0 \sin^2(\pi t/T), \quad F_0 > 0.$$

Determine F_0 , the average force, and the cart's momentum throughout the interaction.

Required integral. The initial momentum is $+1.20 \text{ kg m s}^{-1}$ and the required final momentum is zero, so $J_x = -1.20 \text{ N s}$. Using $\sin^2 u = (1 - \cos 2u)/2$ gives

$$\int_0^T \sin^2(\pi t/T) dt = \frac{T}{2}.$$

Therefore $-F_0 T/2 = -1.20 \text{ N s}$ and

$$F_0 = 30.0 \text{ N}, \quad F_{\text{avg}} = -15.0 \text{ N}.$$

The peak magnitude is twice the magnitude of the mean force for this chosen profile.

Cumulative momentum. Integrating to an arbitrary time gives

$$p_x(t) = 1.20 - F_0 \left[\frac{t}{2} - \frac{T}{4\pi} \sin\left(\frac{2\pi t}{T}\right) \right]$$

in SI units. The expression equals 1.20 kg m s^{-1} at $t = 0$, $0.600 \text{ kg m s}^{-1}$ at $t = T/2$, and zero at $t = T$. Its derivative is the nonpositive stipulated force. Thus momentum decreases monotonically and the cart does not reverse before the specified stop.

A second check. Evaluating the velocity at the midpoint gives $v_x = (0.600)/(0.800) = 0.750 \text{ m s}^{-1}$. The average force over the full duration gives the same endpoint change, $(-15.0)(0.0800) = -1.20 \text{ N s}$, but replacing the pulse by that average would produce a different intermediate momentum history.

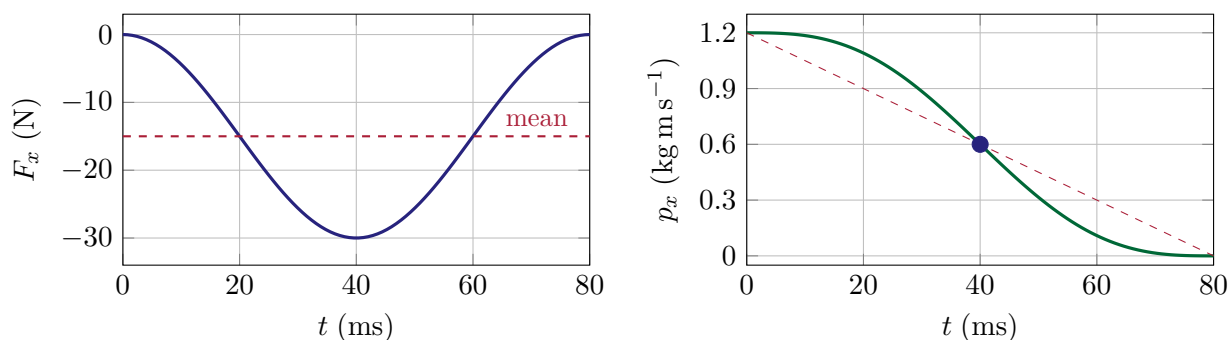


Figure 9.15: The inverse-designed braking pulse and its integrated momentum. Dashed lines show the mean force and its linear momentum history. They agree at both endpoints and at the midpoint here, while the intervening histories differ.

9.3.7 What equal area fixes, and what remains undetermined

Guided checks

Diagnostic. Two force pulses have the same signed area. Must their peaks be equal, and must the body have the same velocity at every time during them?

Complete answer. Their impulses are equal, so they give the same endpoint momentum change when applied as net forces to the same constant-mass body. Their peaks need not be equal, because the area is only one integral constraint on an entire function. Their intermediate velocities need not match, because the partial integrals up to an intermediate time can differ. If the pulses have different durations, even their mean forces differ. Matching their endpoint momentum does not establish matching deformation or stress.

A useful mathematical reason is that an added force component $q(t)$ with zero integral changes the pulse history without changing its total impulse. For instance, $q(t) = B \sin(2\pi t/T)$ satisfies $\int_0^T q(t)dt = 0$. Its partial integral is

$$\int_0^t q(\tau)d\tau = \frac{BT}{2\pi} [1 - \cos(2\pi t/T)].$$

This quantity is generally nonzero before the endpoint. Adding q therefore modifies the intermediate momentum while preserving the final momentum. The example is a mathematical demonstration of nonuniqueness, not a claim that every such modified profile is realizable by a passive contact material.

Even the displacement can depend on when the impulse is delivered. For one-dimensional constant-mass motion, integrating the cumulative momentum once more and interchanging the order of integration gives

$$x(T) - x(0) = v_i T + \frac{1}{m} \int_0^T (T - \tau) F_{\text{net}}(\tau) d\tau. \quad (9.30)$$

An early force contribution changes the velocity for a longer remaining interval and therefore carries the larger weight $T - \tau$. The endpoint momentum uses an unweighted force integral; the endpoint displacement uses this time-weighted integral. Matching one does not automatically match the other.

For example, adding the zero-integral sinusoid above changes the displacement by $BT^2/(2\pi m)$, obtained by substituting q into Eq. (9.30). If $B = 10.0 \text{ N}$, $T = 0.0800 \text{ s}$, and $m = 0.800 \text{ kg}$, that change is 0.0127 m while the final momentum change remains exactly zero. This complete calculation answers why a final velocity alone cannot reconstruct a time history or determine the position reached during an interaction.

In practice, a momentum change is often known more reliably than a peak contact force. Inferring a peak from $m\Delta v/T$ silently assumes the peak equals the mean, which holds only for a constant-force model. If the pulse shape is unmeasured, the integral and duration determine the mean and give a lower bound on the maximum magnitude, $F_{\text{max}} \geq |\mathbf{J}|/T$, but they do not determine a unique maximum. Measuring or validating the shape supplies the missing information.

The endpoint momentum measured from velocities and the force integral provide complementary information. Their comparison must use the same body, time interval, coordinate frame, and sign convention, and must include other forces such as gravity when needed. Once these choices are explicit, the same impulse bookkeeping extends naturally from one particle to a fixed collection of particles. The next section asks when the total external impulse is small enough for that system's total momentum to be conserved.

9.4 Momentum Conservation

For a fixed system of particles, Eq. (9.18) may be written

$$\frac{d\mathbf{P}}{dt} = \sum \mathbf{F}_{\text{ext}}. \quad (9.31)$$

Integrating gives the system impulse–momentum theorem:

$$\mathbf{P}_f - \mathbf{P}_i = \mathbf{J}_{\text{ext}} = \int_{t_i}^{t_f} \sum \mathbf{F}_{\text{ext}} dt. \quad (9.32)$$

If the net external impulse is zero or negligible compared with the momentum scale relevant to the required precision,

$$\mathbf{P}_f = \mathbf{P}_i. \quad (9.33)$$

Momentum conservation is componentwise. A system can conserve horizontal momentum while vertical momentum changes if there is a significant net vertical external impulse.

Key message

“Isolated” is an idealization, not a magic word. In an experiment, justify momentum conservation by identifying external forces and estimating their impulse over the interaction time. A short collision can make a modest friction force negligible even if friction matters over the longer motion before and after the collision.

9.4.1 A quantitative external-impulse test

Suppose a two-cart collision lasts 8 ms and the net horizontal parasitic force from rolling resistance and track tilt is bounded by 0.12 N. Then

$$|J_{\text{ext},x}| \lesssim (0.12)(0.008) = 9.6 \times 10^{-4} \text{ N s}.$$

If the characteristic cart momenta are 0.5 kg m s^{-1} , the maximum fractional momentum disturbance is about 0.2%. For a 0.1% momentum test, the approximation would not be sufficient; for a 2% classroom test, it may be entirely adequate. The quality of a conservation approximation is therefore tied to a stated tolerance.

Example 9.13: Two carts couple and move together

A cart of mass $m_1 = 0.80 \text{ kg}$ moves at $u_1 = +2.50 \text{ m s}^{-1}$. A second cart of mass $m_2 = 1.20 \text{ kg}$ moves at $u_2 = -0.50 \text{ m s}^{-1}$. They collide on a horizontal guide and latch together. The external horizontal impulse during the collision is negligible. Find their common final velocity.

System and frame. Both carts are the system; velocities are laboratory-frame components along x .

Initial momentum:

$$P_i = m_1 u_1 + m_2 u_2 = (0.80)(2.50) + (1.20)(-0.50) = 1.40 \text{ kg m s}^{-1}.$$

After coupling, the total mass $M = 2.00 \text{ kg}$ moves at v_f :

$$P_f = M v_f.$$

Conservation gives

$$v_f = \frac{1.40}{2.00} = +0.700 \text{ m s}^{-1}.$$

Important point. A perfectly inelastic collision means the bodies stick and share a common final velocity. That common velocity is not generally zero in the laboratory frame. It is zero only in a frame where the initial total momentum is zero.

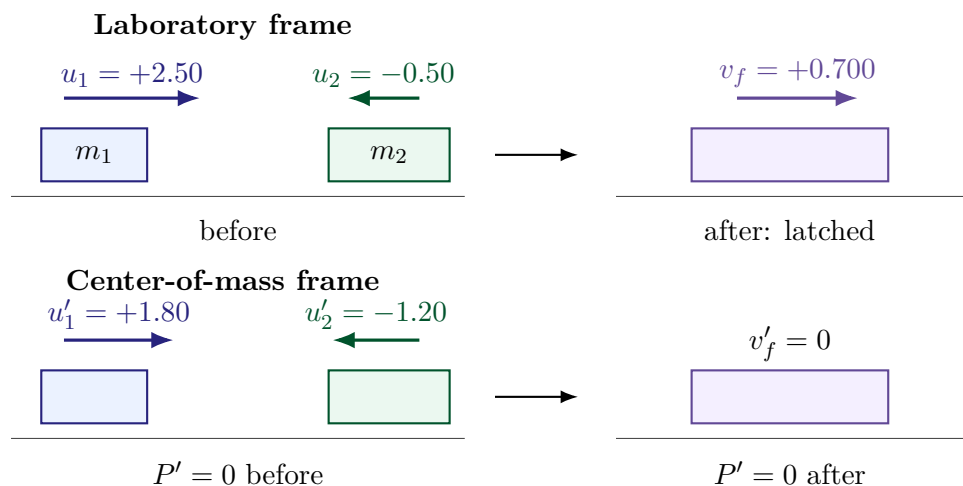


Figure 9.16: The same perfectly inelastic collision viewed in two inertial frames. The coupled carts move at $+0.700 \text{ m s}^{-1}$ in the laboratory but are at rest after collision in the center-of-mass frame. Velocity labels are in m s^{-1} ; these arrows indicate directions and have schematic lengths.

9.4.2 Reference frames and Galilean transformations

If a second inertial frame moves at constant velocity $\mathbf{U}$ relative to the laboratory, particle velocities transform as

$$\mathbf{v}'_i = \mathbf{v}_i - \mathbf{U}. \quad (9.34)$$

The total momentum becomes

$$\mathbf{P}' = \sum_i m_i (\mathbf{v}_i - \mathbf{U}) = \mathbf{P} - M\mathbf{U}. \quad (9.35)$$

Choosing $\mathbf{U} = \mathbf{V}_{\text{CM}} = \mathbf{P}/M$ gives

$$\boxed{\mathbf{P}' = 0}, \quad (9.36)$$

which defines the center-of-mass frame for the system. Momentum conservation does not mean the numerical total momentum has the same value in every frame. It means that, for a fixed isolated system, the total momentum is constant in each inertial frame.

9.5 Collisions and Kinetic Energy

A collision is an interaction occurring over a comparatively short interval during which large internal forces act between bodies. The momentum audit and the kinetic-energy audit answer different questions:

- If the net external impulse is negligible, total momentum is conserved.
- If total kinetic energy is also unchanged, the collision is elastic.
- If the bodies stick together, the collision is perfectly inelastic. Kinetic energy decreases unless the initial relative velocity is zero.

Definition 9.3: Elastic and inelastic collisions

An *elastic collision* conserves the total kinetic energy of the chosen isolated system in addition to conserving momentum. An *inelastic collision* conserves momentum when external impulse is negligible but does not conserve kinetic energy. In a *perfectly inelastic collision*, the colliding bodies move together with a common final velocity.

The “lost kinetic energy” in an inelastic collision is not lost from total energy. It is transformed into internal energy, deformation, sound, thermal energy, or other forms. For a passive inelastic impact with no release of stored energy, translational kinetic energy decreases. An event that releases internal energy can instead increase it and is often called superelastic; momentum and total-energy balances still apply separately.

9.5.1 Perfectly inelastic collision and reduced mass

For two particles moving along one line with initial velocities u_1, u_2 , conservation of momentum gives the common final velocity

$$V = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2}. \quad (9.37)$$

The initial and final kinetic energies are

$$K_i = \frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2, \quad K_f = \frac{1}{2}(m_1 + m_2)V^2.$$

Direct algebra gives

$$K_i - K_f = \frac{1}{2}\mu(u_1 - u_2)^2, \quad \mu = \frac{m_1 m_2}{m_1 + m_2}. \quad (9.38)$$

The loss depends only on the relative speed, not on the laboratory’s translational motion. This frame-independent difference will reappear naturally in the advanced center-of-mass energy decomposition.

For Example 9.13,

$$K_i = \frac{1}{2}(0.80)(2.50)^2 + \frac{1}{2}(1.20)(0.50)^2 = 2.65 \text{ J},$$

whereas

$$K_f = \frac{1}{2}(2.00)(0.700)^2 = 0.490 \text{ J}.$$

Thus $K_i - K_f = 2.16 \text{ J}$. Equation (9.38) gives the same result because $\mu = 0.480 \text{ kg}$ and $|u_1 - u_2| = 3.00 \text{ m s}^{-1}$.

9.5.2 One-dimensional elastic collisions

For an isolated elastic collision in one dimension,

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2, \quad (9.39)$$

$$\frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2 = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2. \quad (9.40)$$

For the nontrivial separating collision branch, subtracting and factoring these equations yields the relative-speed relation. The unchanged-velocity algebraic solution also satisfies both conservation laws but does not describe the separating outcome of initially closing bodies:

$$v_1 - v_2 = -(u_1 - u_2). \quad (9.41)$$

Thus the relative velocity reverses in a one-dimensional elastic collision. Combining with momentum conservation gives

$$v_1 = \frac{m_1 - m_2}{m_1 + m_2} u_1 + \frac{2m_2}{m_1 + m_2} u_2, \quad (9.42)$$

$$v_2 = \frac{2m_1}{m_1 + m_2} u_1 + \frac{m_2 - m_1}{m_1 + m_2} u_2. \quad (9.43)$$

These formulas are consequences of the two conservation equations, not replacements for thinking about the physical system.

Example 9.14: Same incoming carts: sticking versus elastic collision

Use the same initial state as Example 9.13: $m_1 = 0.80 \text{ kg}$, $m_2 = 1.20 \text{ kg}$, $u_1 = +2.50 \text{ m s}^{-1}$, $u_2 = -0.50 \text{ m s}^{-1}$.

Perfectly inelastic case. The common final velocity is $+0.700 \text{ m s}^{-1}$ and $K_f = 0.490 \text{ J}$.

Elastic case. Equations (9.42)–(9.43) give

$$v_1 = -1.10 \text{ m s}^{-1}, \quad v_2 = +1.90 \text{ m s}^{-1}.$$

The final momentum is

$$(0.80)(-1.10) + (1.20)(1.90) = 1.40 \text{ kg m s}^{-1},$$

and the final kinetic energy is

$$K_f = \frac{1}{2}(0.80)(1.10)^2 + \frac{1}{2}(1.20)(1.90)^2 = 2.65 \text{ J} = K_i.$$

Center-of-mass interpretation. Since $V_{\text{CM}} = 0.700 \text{ m s}^{-1}$, the incoming velocities in the CM frame are $u'_1 = +1.80$ and $u'_2 = -1.20 \text{ m s}^{-1}$. In the elastic outcome they become $v'_1 = -1.80$ and $v'_2 = +1.20 \text{ m s}^{-1}$: each reverses. In the perfectly inelastic outcome both become zero in the CM frame.

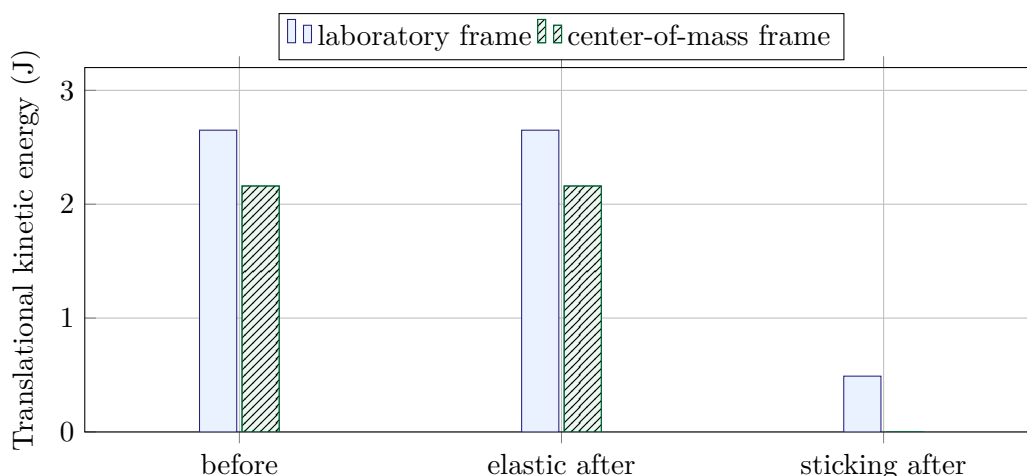


Figure 9.17: Same incoming state, two possible outcomes, and two inertial frames for Example 9.14. Laboratory kinetic energies are 2.65, 2.65, and 0.490 J; CM-frame energies are 2.16, 2.16, and zero. The sticking decrease is 2.16 J in both frames. The 0.490 J CM translation present in every laboratory state cannot be removed by internal forces.

9.5.3 Coefficient of restitution as a model parameter

For some one-dimensional impact models, a coefficient of restitution e_r is defined by

$$e_r = -\frac{v_1 - v_2}{u_1 - u_2}, \quad (9.44)$$

for closing and separating relative velocities along the impact line. For a passive one-dimensional normal-impact model, $0 \leq e_r \leq 1$: $e_r = 1$ describes elastic separation and $e_r = 0$ gives zero relative separating velocity. Permanent sticking requires an additional adhesion or latch condition. In more than one dimension zero normal restitution does not remove tangential relative motion. Stored-energy release can produce $e_r > 1$. Real values may depend on speed, material, geometry, temperature, and deformation history. The symbol is a phenomenological model parameter, not a new conservation law.

Caution

Do not infer “elastic” merely because two bodies separate after impact, and do not infer “perfectly inelastic” merely because kinetic energy decreases. Perfectly inelastic means the bodies share a common final velocity. Elastic means the total kinetic energy is unchanged in the chosen isolated system.

9.5.4 What is hidden when the elastic equations are factored?

The relative-speed reversal used above deserves a careful derivation because division can silently remove a mathematical branch. Rearrange momentum conservation and factor the energy equation:

$$m_1(u_1 - v_1) = m_2(v_2 - u_2), \quad (9.45)$$

$$m_1(u_1 - v_1)(u_1 + v_1) = m_2(v_2 - u_2)(v_2 + u_2). \quad (9.46)$$

If $u_1 - v_1 \neq 0$, divide the second equation by the first to obtain $u_1 + v_1 = v_2 + u_2$. Thus $v_2 - v_1 = u_1 - u_2$: the separating speed equals the approaching speed. But if $u_1 - v_1 = 0$, momentum conservation also gives $v_2 = u_2$. This is a second algebraic solution: neither particle’s velocity changes. Conservation laws alone do not require an encounter. The unchanged-velocity solution is appropriate to objects that miss one another or do not interact; it is not the separating-contact solution for two initially approaching impenetrable bodies.

For an actual one-dimensional contact, choose the ordering and the sign of the approach. If body 1 lies to the left of body 2 and $u_1 > u_2$, contact followed by separation requires $v_2 \geq v_1$. The unchanged branch would retain $u_1 > u_2$ and continued closure, so it is excluded by the contact condition. When $u_1 = u_2$, there is no initial relative approach and no nontrivial ordinary collision to solve. This limiting case also explains why a restitution ratio should not be formed with a zero denominator.

9.5.5 A complete one-dimensional restitution model

For the stated left-to-right approach, let $g = u_1 - u_2 > 0$ and $e_r \geq 0$. Combining momentum conservation with $v_2 - v_1 = e_r g$ gives

$$v_1 = \frac{m_1 u_1 + m_2 u_2 - m_2 e_r (u_1 - u_2)}{m_1 + m_2}, \quad (9.47)$$

$$v_2 = \frac{m_1 u_1 + m_2 u_2 + m_1 e_r (u_1 - u_2)}{m_1 + m_2}. \quad (9.48)$$

These expressions work for arbitrary signed incoming laboratory velocities, provided there is a closing contact along the selected line. The CM velocity V is unchanged; the relative final velocities

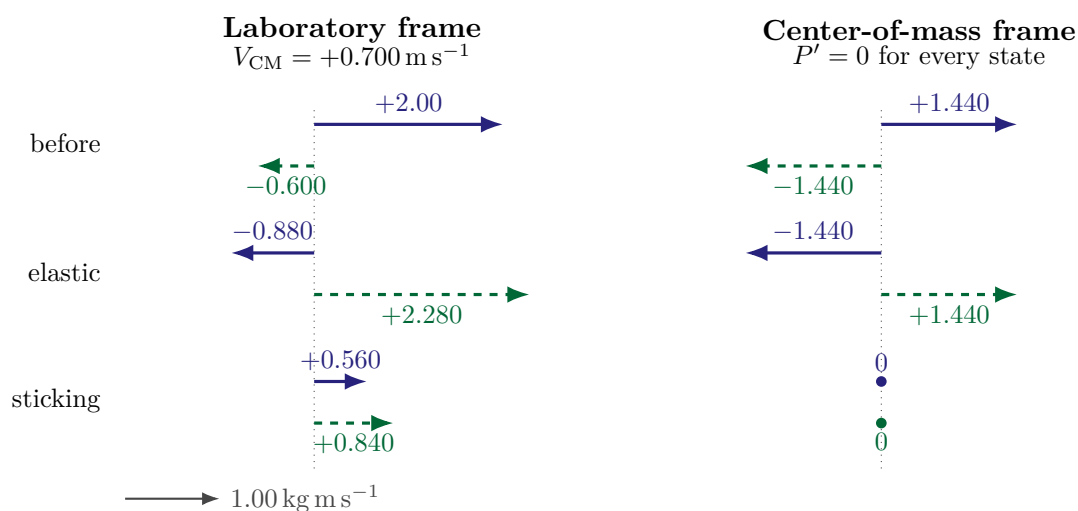


Figure 9.18: Quantitative momentum arrows for the retained 0.800 and 1.200 kg carts: incoming velocities $+2.50$ and -0.50 m s^{-1} . Solid arrows denote cart 1; dashed arrows, cart 2. Every nonzero arrow uses the same declared scale, $1.25 \text{ cm per } 1.00 \text{ kg m s}^{-1}$. Laboratory sums are $+1.40 \text{ kg m s}^{-1}$; CM sums vanish. Row spacing and separate arrow origins organize the vectors and are not cart positions. Dots represent zero momentum.

are $v_1 - V = -m_2 e_r g / (m_1 + m_2)$ and $v_2 - V = m_1 e_r g / (m_1 + m_2)$. Therefore

$$K_i - K_f = \frac{1}{2} \mu (1 - e_r^2) (u_1 - u_2)^2. \quad (9.49)$$

One can verify this by direct substitution; the advanced CM-energy decomposition explains why it is so compact. A passive translational impact model ordinarily has $0 \leq e_r \leq 1$. A measured $e_r > 1$ can describe release of stored internal energy, and would give a kinetic-energy increase. It must be justified by that energy source. In more than one dimension, restitution normally constrains the normal contact component; a tangential force model is then also needed. Applying this one-dimensional formula to the full magnitudes of oblique velocities is generally incorrect.

Example 9.15: A partially inelastic head-on impact

Two carts approach on a straight frictionless track. Their masses and incoming velocities are $m_1 = 1.50 \text{ kg}$, $m_2 = 2.50 \text{ kg}$, $u_1 = +4.00 \text{ m s}^{-1}$, and $u_2 = -1.00 \text{ m s}^{-1}$. The restitution coefficient is $e_r = 0.600$. Find both final velocities, verify separation and momentum conservation, and determine the translational kinetic-energy decrease.

Model. Both carts form the system; the horizontal external impulse is negligible. The first cart is initially to the left, so the approach speed is $g = 4.00 - (-1.00) = 5.00 \text{ m s}^{-1}$. The conserved momentum and CM velocity are

$$P = 1.50(4.00) + 2.50(-1.00) = 3.50 \text{ kg m s}^{-1}, \quad V = 3.50/4.00 = 0.875 \text{ m s}^{-1}.$$

Using Eqs. (9.47)–(9.48),

$$v_1 = -1.00 \text{ m s}^{-1}, \quad v_2 = +2.00 \text{ m s}^{-1}.$$

Their separating speed is $v_2 - v_1 = 3.00 \text{ m s}^{-1} = 0.600g$, so the directions are consistent with contact and separation. Their final momentum is $1.50(-1.00) + 2.50(2.00) = 3.50 \text{ kg m s}^{-1}$. The incoming and outgoing energies are

$$K_i = \frac{1}{2}(1.50)(4.00)^2 + \frac{1}{2}(2.50)(1.00)^2 = 13.25 \text{ J},$$

$$K_f = \frac{1}{2}(1.50)(1.00)^2 + \frac{1}{2}(2.50)(2.00)^2 = 5.75 \text{ J}.$$

Hence $K_i - K_f = 7.50 \text{ J}$. Independently, $\mu = (1.50)(2.50)/4.00 = 0.9375 \text{ kg}$ gives $\frac{1}{2}(0.9375)(1 - 0.600^2)(5.00)^2 = 7.50 \text{ J}$. The bodies separate, but the collision is inelastic. In this example the coefficient describes the retained *relative speed*; its square describes the retained relative kinetic-energy fraction, $e_r^2 = 0.360$.

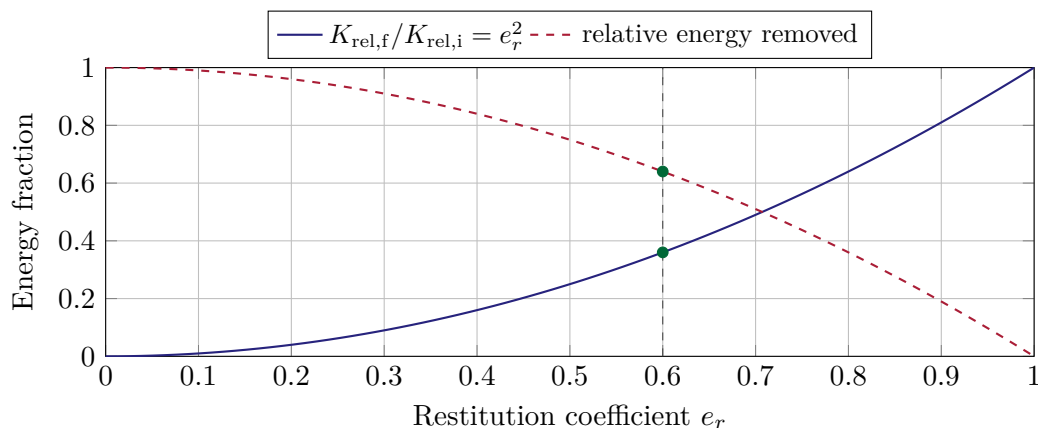


Figure 9.19: Restitution partitions the initial two-body relative translational energy, while leaving the CM translational term unchanged in an isolated impact. The marked points correspond to Example 9.15. These curves are model predictions, not a measured material response.

A momentum arrow and an energy bar also encode different mathematical quantities. Momentum is signed along a chosen axis, whereas translational kinetic energy is nonnegative and depends on the square of speed. Opposite momentum arrows can cancel without either energy vanishing. In the scaled laboratory and CM panels, changing frame subtracts a different momentum from each cart because their masses differ. The combined frame shift subtracts MV_{CM} from total momentum, making the CM-frame sum zero. Individual CM momenta are therefore equal and opposite in a two-body system, even when the laboratory momenta and the body speeds are unequal. Comparing arrows labelled as velocities with arrows labelled as momenta would obscure this distinction; their units and declared scales must be kept explicit.

9.5.6 Mass ratio controls elastic energy transfer

For a target initially at rest, define the positive mass ratio $r = m_2/m_1$. The elastic collision formulas simplify to

$$\frac{v_1}{u_1} = \frac{1-r}{1+r}, \quad \frac{v_2}{u_1} = \frac{2}{1+r}. \quad (9.50)$$

The projectile rebounds when $r > 1$, stops when $r = 1$, and continues forward when $r < 1$. The target may move faster than the original projectile when it is lighter; this is fully compatible with energy conservation because its mass is smaller. The fraction of initial kinetic energy delivered to the target is

$$\eta = \frac{\frac{1}{2}m_2v_2^2}{\frac{1}{2}m_1u_1^2} = \frac{4r}{(1+r)^2}. \quad (9.51)$$

Differentiation gives $d\eta/dr = 4(1-r)/(1+r)^3$. The maximum is therefore $\eta = 1$ at equal masses. Since $\eta(r) = \eta(1/r)$, reciprocal mass ratios give the same transfer fraction even though the projectile's final direction differs. The limits $r \rightarrow 0$ and $r \rightarrow \infty$ both give vanishing transferred energy, for different physical reasons.

Example 9.16: Choosing a target mass for energy transfer

A 0.200 kg cart moves at 6.00 m s^{-1} and strikes a stationary 0.800 kg cart in an ideal one-dimensional elastic collision. (a) Find both velocities and the energy delivered to the target. (b) Find the target-mass interval giving at least 90% energy transfer for the same projectile.

(a) **Outcome and checks.** The ratio is $r = 4$, so

$$v_1 = -3.60 \text{ m s}^{-1}, \quad v_2 = +2.40 \text{ m s}^{-1}.$$

The target receives $K_{2f} = \frac{1}{2}(0.800)(2.40)^2 = 2.304 \text{ J}$. Initially $K_i = \frac{1}{2}(0.200)(6.00)^2 = 3.600 \text{ J}$, so $\eta = 2.304/3.600 = 0.640$. The projectile retains 1.296 J; the energies sum to 3.600 J. Momentum is $0.200(-3.60) + 0.800(2.40) = 1.200 \text{ kg m s}^{-1}$, equal to its incoming value.

(b) **Required mass interval.** Requiring $4r/(1+r)^2 \geq 0.900$ gives $9r^2 - 22r + 9 \leq 0$. Thus

$$\frac{11 - 2\sqrt{10}}{9} \leq r \leq \frac{11 + 2\sqrt{10}}{9}, \quad 0.51949 \leq r \leq 1.92495.$$

Multiplying by the projectile mass gives

$$0.1039 \text{ kg} \leq m_2 \leq 0.3850 \text{ kg}.$$

Equal masses lie inside this interval and produce complete transfer: $v_1 = 0$, $v_2 = u_1$. The result concerns ideal translational energy transfer, not contact damage or force amplitude. Those require additional information about deformation and interaction time.

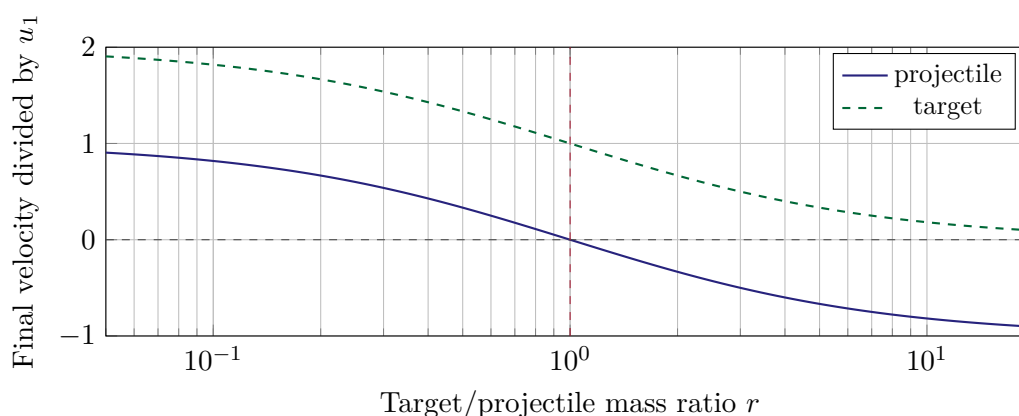


Figure 9.20: Elastic velocities for an initially stationary target. At $r = 1$, the projectile stops and the target takes its velocity. For a heavy target, the projectile rebounds; for a light target, the target speed approaches $2u_1$ while its energy remains small. A logarithmic mass-ratio axis displays both limits.

9.5.7 Interpreting restitution measurements and collision limits

To infer restitution experimentally, velocities should be measured just before first contact and just after final separation. Using velocities measured long before or after the event can include rolling resistance, track tilt, or other external impulses. The ratio then mixes contact behaviour with free motion and need not represent the contact itself. Consistency also requires the same inertial frame, contact direction, and velocity sign convention for all four velocity measurements. Transforming both incoming velocities and both outgoing velocities by the same Galilean shift leaves each relative velocity unchanged, so the inferred one-dimensional coefficient is frame independent.

The coefficient does not predict the duration or peak force. An impact with a prescribed pair of initial and final velocities fixes the impulse on each body, but many different contact-force histories

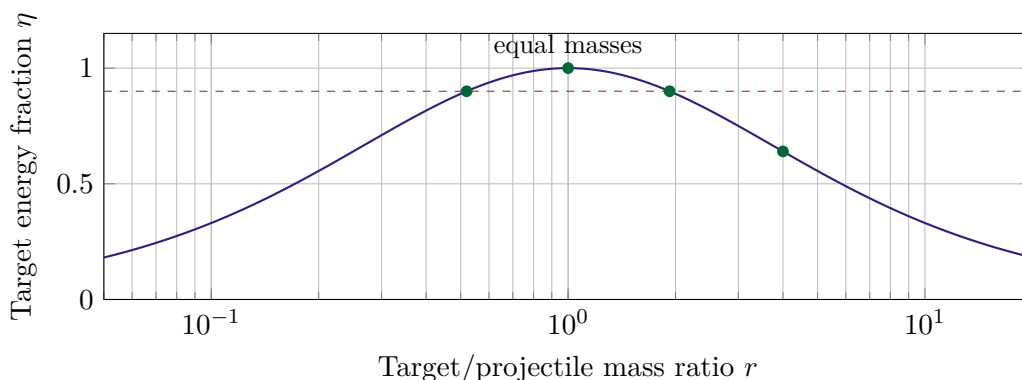


Figure 9.21: Energy-transfer fraction from Eq. (9.51). The intersection points with $\eta = 0.900$ delimit the mass interval in Example 9.16. The point at $r = 4$ represents the specified 0.800 kg target.

integrate to that same impulse. A short stiff pulse and a longer compliant pulse may yield the same restitution while imposing very different peak loads. Similarly, equal mass is optimal for the ideal energy-transfer fraction derived here, but this does not by itself select a practical protective buffer. The buffer may be intended to dissipate energy rather than return it, and its mass may be only one of several constraints.

Check the extreme cases before trusting a numerical routine. At $e_r = 0$, the two formulas must give the same CM velocity. At $e_r = 1$, they must reproduce the elastic formulas. For equal masses and a stationary target, the elastic outcome exchanges velocities. For an increasingly massive stationary target, the target's speed tends to zero while the projectile's elastic speed approaches its incoming magnitude with reversed sign. These limits test different parts of the algebra: common velocity, retained relative energy, symmetry, and inertia. A routine that satisfies only momentum can still assign an impossible energy balance or a continued-closing state after the alleged impact.

9.5.8 A ballistic pendulum requires two different balances

A projectile embedded in a suspended catcher is a useful test of model selection. During the brief impact, horizontal external impulse can be negligible while translational kinetic energy is strongly reduced. During the following swing, gravity and the suspension exert external forces, so momentum is not conserved for projectile plus catcher. In an ideal swing with negligible dissipation, mechanical energy is conserved. The two intervals must be joined at the *immediately-after-impact* speed, not collapsed into one conservation equation.

Example 9.17: Reconstructing an incoming speed from a pendulum rise

A 12.0 g projectile embeds in a 1.988 kg catcher initially at rest at the bottom of a pendulum. The combined mass can be idealized as a point at distance $L = 1.20$ m from the pivot. It subsequently rises to 30.0° from the vertical. Use $g = 9.81 \text{ m s}^{-2}$. (a) Find the speed immediately after impact and the projectile's initial speed. (b) Calculate the kinetic energy removed during capture. (c) State which conservation law applies in each stage.

(a) Swing first, impact second. The center rises through

$$h = L(1 - \cos 30.0^\circ) = 0.16077 \text{ m}.$$

Between the bottom just after impact and the turning point, ideal mechanical-energy conservation gives $\frac{1}{2}(M + m)V^2 = (M + m)gh$. Hence

$$V = \sqrt{2gh} = 1.7760 \text{ m s}^{-1}.$$

During the short capture, the string is initially vertical and the horizontal support impulse is assumed negligible. Horizontal momentum then gives $mu = (M + m)V$, so

$$u = \frac{2.000}{0.0120}(1.7760) = 296.0 \text{ m s}^{-1}.$$

This speed is an inference within the stipulated point-catcher, embedding, and ideal-swing model. It would require correction if catcher rotation or dissipative swing losses were significant.

(b) Energy audit. The incoming energy is $K_i = \frac{1}{2}(0.0120)(296.0057)^2 = 525.716 \text{ J}$. Immediately after capture it is $K_c = \frac{1}{2}(2.000)(1.776034)^2 = 3.15430 \text{ J}$, and therefore

$$K_i - K_c = 522.562 \text{ J}.$$

Only $K_c/K_i = m/(M + m) = 0.00600$, or 0.600%, remains as translational kinetic energy. At the turning point this remaining energy appears as gravitational potential energy. The large impact loss is not supplied to the pendulum's gravitational rise.

(c) Conservation laws. Capture uses horizontal momentum, with kinetic-energy loss allowed. Swing uses mechanical energy, with momentum changing because of external suspension and gravitational forces. Conserving mechanical energy across the capture would underestimate u severely; conserving momentum through the swing would also be physically unjustified.

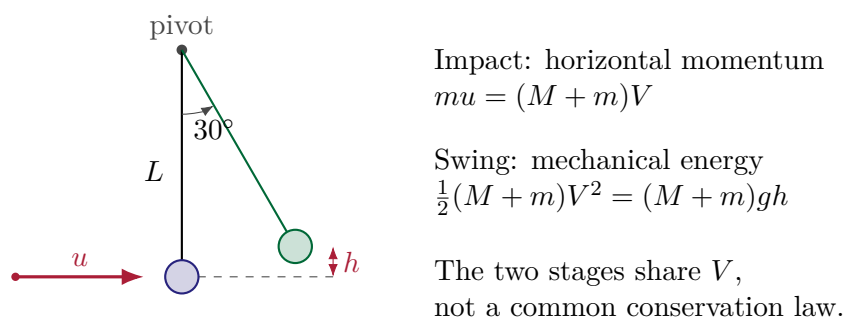


Figure 9.22: Point-catcher ballistic pendulum for Example 9.17. The angle and rise satisfy $h = L(1 - \cos \theta)$. The projectile arrow indicates incoming motion; the turning-point catcher has zero speed. The collision is treated as short enough that horizontal external impulse is negligible.

9.5.9 What the pendulum measurement actually constrains

The measured turning angle determines the gravitational rise of the stipulated point catcher, and therefore its immediately-after-impact kinetic energy only if the intervening swing conserves mechanical energy. A real swing that loses energy to bearing friction or air resistance reaches a smaller angle than the ideal swing with the same initial speed. Applying the ideal inversion to that smaller angle would underestimate the incoming projectile speed. An observed angle is therefore not an independent test of the capture model unless the catcher geometry and swing losses are also characterized.

For a fixed pendulum length, the inverse expression is proportional to $\sqrt{1 - \cos \theta}$. Its logarithmic angle sensitivity is $(1/2) \sin \theta / (1 - \cos \theta) = (1/2) \cot(\theta/2)$, with angle changes measured in radians. At 30° , this factor is about 1.866 per radian, so an angle uncertainty of 1° contributes approximately 3.26% relative speed uncertainty in a linear uncertainty estimate. At very small rise angles the relative sensitivity is larger. This is one reason to separate arithmetic precision from the precision justified by an actual measurement. The numbers in the worked example are stipulated inputs, not uncertainty estimates.

9.6 Two-Dimensional Collisions and Fragmentation

In two or three dimensions, momentum conservation is still a vector equation:

$$\mathbf{P}_i = \mathbf{P}_f. \quad (9.52)$$

In a plane this supplies two scalar equations,

$$P_{x,i} = P_{x,f}, \quad P_{y,i} = P_{y,f}, \quad (9.53)$$

but the number of unknown speed and direction variables can exceed two. A collision statement may therefore require an additional condition: elastic energy conservation, a known scattering angle, a known final speed, a geometric constraint, or a measured trajectory.

9.6.1 Momentum polygons are not trajectory diagrams

A vector polygon represents momentum addition in an abstract vector space. Its arrows may be moved head-to-tail without implying that the objects followed those geometric paths in physical space. Confusing a momentum triangle with a spatial trajectory is a frequent source of incorrect angle relations.

Example 9.18: Planar fragmentation of a moving module

A parent body of mass 5.0 kg moves with velocity

$$\mathbf{v}_0 = (4.0 \hat{x} + 1.0 \hat{y}) \text{ m s}^{-1}.$$

It separates rapidly into fragments of masses 2.0 kg and 3.0 kg. During the short separation interval the external impulse is negligible. The 2.0 kg fragment is measured to have

$$\mathbf{v}_A = (7.0 \hat{x} + 3.0 \hat{y}) \text{ m s}^{-1}.$$

Find the velocity of the other fragment and the change in total kinetic energy.

Initial momentum is

$$\mathbf{P}_i = 5.0(4.0, 1.0) = (20, 5) \text{ kg m s}^{-1}.$$

Fragment A has

$$\mathbf{p}_A = 2.0(7.0, 3.0) = (14, 6) \text{ kg m s}^{-1}.$$

Hence

$$\mathbf{p}_B = \mathbf{P}_i - \mathbf{p}_A = (6, -1) \text{ kg m s}^{-1},$$

so

$$\mathbf{v}_B = (2.00 \hat{x} - 0.333 \hat{y}) \text{ m s}^{-1}.$$

The initial kinetic energy is

$$K_i = \frac{1}{2}(5.0)(4.0^2 + 1.0^2) = 42.5 \text{ J}.$$

The final kinetic energy is

$$K_f = \frac{1}{2}(2.0)(7.0^2 + 3.0^2) + \frac{1}{2}(3.0) \left(2.0^2 + \frac{1}{3^2} \right) = 64.17 \text{ J}.$$

Thus

$$\Delta K = +21.67 \text{ J}.$$

Interpretation. The increased translational kinetic energy must come from another energy store of the parent system, such as internal elastic, chemical, or explosive energy. Momentum conservation alone does not require kinetic-energy conservation.

Energy-budget test. Suppose an independent idealized mechanism can release only 20.0 J, with no other energy supply and no reduction of some additional energy store. The proposed final state requires 21.67 J, so it is incompatible with that budget by 1.67 J. Passing a momentum check is necessary here but does not establish physical feasibility. If some energy remains as deformation or vibration, the necessary released energy is still larger.

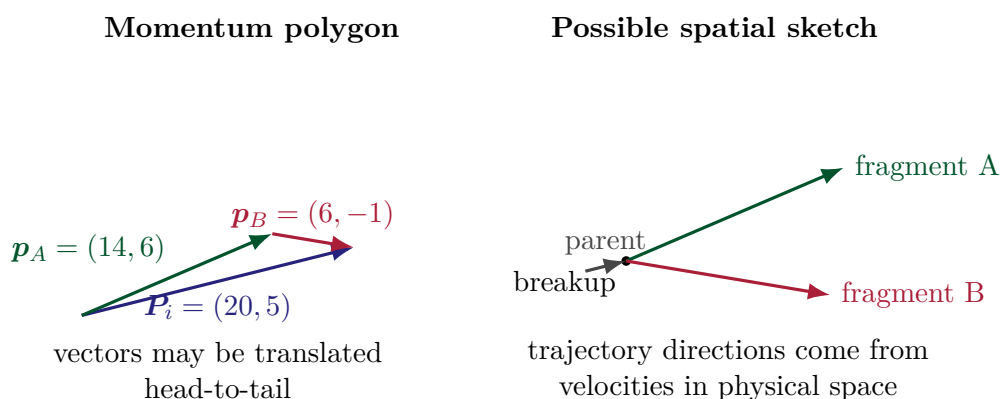


Figure 9.23: Momentum-vector bookkeeping for Example 9.18 compared with a spatial trajectory sketch. The momentum polygon is an addition diagram; its arrow placement is not a record of where the fragments travelled. Momentum components are in kg m s^{-1} ; one drawn centimeter represents 5 Ns before the common display scale of 0.9. Spatial arrow directions match the fragment velocities, but spatial lengths are schematic.

9.6.2 Scattering with an initially stationary target

Suppose a projectile of mass m_1 approaches a stationary target m_2 . After a planar collision, let θ_1 and θ_2 be nonnegative angle magnitudes above and below the incident $+x$ direction, respectively. The transverse signs are therefore opposite. Momentum conservation gives

$$p_i = p_{1f} \cos \theta_1 + p_{2f} \cos \theta_2, \quad (9.54)$$

$$0 = p_{1f} \sin \theta_1 - p_{2f} \sin \theta_2. \quad (9.55)$$

These two equations alone do not determine four unknowns $p_{1f}, p_{2f}, \theta_1, \theta_2$. If the collision is also known to be elastic, kinetic-energy conservation supplies a third equation; another measured or specified quantity is still required in the general unequal-mass case. Problem statements must be counted before algebra begins.

For equal masses in an ideal elastic collision with the target initially at rest, a special result follows: the outgoing velocity vectors are perpendicular when both are nonzero. That result depends on equal masses, an initially stationary target, elasticity, and nonrelativistic mechanics; it is not a universal property of two-dimensional collisions.

Guided checks

Check 1. A firework explodes at the top of its trajectory. Is momentum conserved during the explosion? Approximately, if the explosion time is short enough that gravitational impulse during the event is negligible.

Check 2. Can kinetic energy increase in a fragmentation event while momentum is conserved? Yes. Internal energy can be converted to kinetic energy.

Check 3. If P_x is conserved but P_y is not, must the full momentum vector be conserved? No. Vector conservation requires every component to satisfy the required condition.

9.6.3 Why equal-mass elastic outgoing velocities are perpendicular

Let two equal masses undergo a planar elastic collision with one initially stationary. Dividing momentum conservation by their common mass gives $\mathbf{u} = \mathbf{v}_1 + \mathbf{v}_2$. Squaring this vector equation yields

$$u^2 = v_1^2 + v_2^2 + 2\mathbf{v}_1 \cdot \mathbf{v}_2.$$

Energy conservation independently gives $u^2 = v_1^2 + v_2^2$. Comparing the two equations leaves $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$. If both outgoing velocities are nonzero, their directions are perpendicular. If one is zero, the dot product still vanishes, but there is no defined angle for that zero vector. This special endpoint must not be described as two visible tracks meeting at 90° .

Example 9.19: Equal masses with a measured deflection

A 0.400 kg projectile moves at 5.00 m s^{-1} along $+x$ toward an equal stationary target. The collision is elastic and planar. The projectile's outgoing direction is measured to be 35.0° above $+x$. Find the two outgoing speeds and the target's direction, and verify both conserved quantities.

The two nonzero outgoing velocities are perpendicular. The target is below the incident direction, so its angle is $-(90.0^\circ - 35.0^\circ) = -55.0^\circ$. Let their positive speeds be v_1, v_2 . The y balance is $v_1 \sin 35^\circ - v_2 \cos 35^\circ = 0$; the x balance is $5.00 = v_1 \cos 35^\circ + v_2 \sin 35^\circ$. Solving gives

$$v_1 = 5.00 \cos 35.0^\circ = 4.096 \text{ m s}^{-1}, \quad v_2 = 5.00 \sin 35.0^\circ = 2.868 \text{ m s}^{-1}.$$

The component velocities are

$$\mathbf{v}_1 = (3.3551, 2.3492) \text{ m s}^{-1}, \quad \mathbf{v}_2 = (1.6449, -2.3492) \text{ m s}^{-1},$$

whose sum is $(5.0000, 0) \text{ m s}^{-1}$. Multiplying by 0.400 kg verifies momentum. Their kinetic energies are 3.3551 J and 1.6449 J, summing to the incoming 5.0000 J. The measured angle supplies the remaining information beyond conservation: equal mass and elasticity alone allow a continuous family of deflections.

9.6.4 Specified outgoing directions make a planar impact solvable

If a projectile initially moving along $+x$ strikes a stationary target and the positive outgoing angles are θ_1 above and θ_2 below that direction, the component balances can be solved for the momentum magnitudes without assuming elasticity:

$$p_1 = \frac{p_i \sin \theta_2}{\sin(\theta_1 + \theta_2)}, \quad p_2 = \frac{p_i \sin \theta_1}{\sin(\theta_1 + \theta_2)}. \quad (9.56)$$

The denominators require a nondegenerate angle configuration. After calculating the velocities, an energy audit tests whether the stipulated directions could arise from a passive collision. If the calculated final translational energy exceeds the initial energy, there must be an internal-energy source or a failure in the assumed boundary or measurements. A momentum solution is not automatically a physically admissible passive-impact solution.

Example 9.20: A planar spacecraft buffer encounter

In a horizontal plane in deep space, a 200 kg test vehicle approaches a stationary 300 kg docking target at 1.20 m s^{-1} . They contact buffers and separate, without latching. External impulse and rotational energy changes are negligible in the stipulated translational model. The outgoing vehicle direction is 30.0° above the incident direction and the target direction is 20.0° below

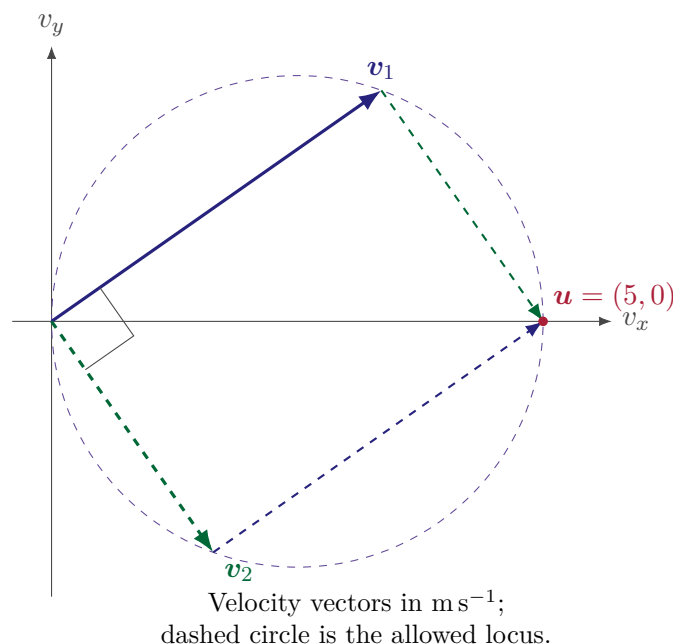


Figure 9.24: Velocity-space geometry for equal-mass elastic scattering. Either outgoing vector lies on the circle with incident velocity as a diameter. The vectors sum to the incident velocity and meet at a right angle when both are nonzero. This is a vector diagram, not a spatial trajectory map.

it. (a) Find both speeds. (b) Find the translational kinetic-energy decrease. (c) Compare a perfectly inelastic latching outcome from the same incoming state.

(a) Component solution. The incoming momentum is $p_i = 200(1.20) = 240 \text{ kg m s}^{-1}$. Equation (9.56) gives

$$p_1 = 107.154 \text{ kg m s}^{-1}, \quad p_2 = 156.649 \text{ kg m s}^{-1},$$

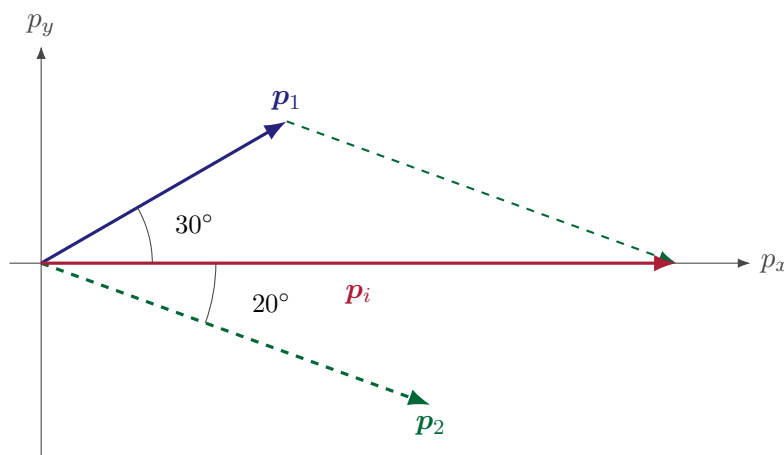
so $v_1 = 0.5358 \text{ m s}^{-1}$ and $v_2 = 0.5222 \text{ m s}^{-1}$. Their momenta have components $(92.798, 53.577)$ and $(147.202, -53.577) \text{ kg m s}^{-1}$. Their sum is $(240.000, 0)$, providing a direct vector check.

(b) Energy audit. Initially $K_i = \frac{1}{2}(200)(1.20)^2 = 144.000 \text{ J}$. After separation,

$$K_f = \frac{p_1^2}{2(200)} + \frac{p_2^2}{2(300)} = 69.603 \text{ J}.$$

Therefore $K_i - K_f = 74.397 \text{ J}$. This positive decrease is compatible with a passive inelastic buffer impact. The outgoing directions differ by 50.0° , illustrating that perpendicularity is not a general planar-collision rule. A restitution coefficient cannot be inferred from speed magnitudes alone without the contact-normal direction and an appropriate tangential model.

(c) Latching comparison. A common final velocity would be $\mathbf{V} = (240/500, 0) = (0.480, 0) \text{ m s}^{-1}$, giving $K_{\text{latch}} = \frac{1}{2}(500)(0.480)^2 = 57.600 \text{ J}$. The latching decrease is therefore 86.400 J , larger than for the separating outcome. Both outcomes have the same CM velocity; the separating vehicles retain $69.603 - 57.600 = 12.003 \text{ J}$ of relative translational energy. The idealized latching case removes all of it.



Momentum components in kg m s^{-1}

Figure 9.25: Scaled momentum polygon for Example 9.20. The target momentum is repeated by a translated dashed arrow to show head-to-tail addition. Transverse components cancel; the outgoing directions were supplied as measurements and elasticity was not assumed.

9.6.5 Angle conventions and incomplete planar information

The two supplied outgoing directions in the buffer problem play a different role from an elastic-energy condition. They close the two-component momentum problem, after which the kinetic energy is calculated and assessed. If only one angle had been supplied and the collision were merely described as inelastic, the outgoing state would remain underdetermined. There would be two component equations for two speeds and a second angle. Labelling an event inelastic only says that translational kinetic energy is not conserved; it supplies no numerical equation for the amount converted internally.

An angle should specify both its reference axis and its side of that axis. Two unsigned angles without their sides can lead to a false transverse balance. For an initially axial momentum and two nonzero outgoing transverse components, those components must cancel under the stated isolated-system assumptions. Angles on the same side cannot accomplish that cancellation. Finally, the divergence of Eq. (9.56) near a vanishing denominator flags degenerate or poorly conditioned input geometry. Large calculated momenta from nearly cancelling directions require an energy source or contradict a passive collision. Geometry alone cannot supply that source.

9.7 Engineering, Computational, and Measurement Connections

Momentum methods are useful because many technical interactions occur over short times or involve internal forces too complicated to model directly. The method is strongest when its limits are explicit: a momentum balance can robustly connect measured velocities and impulses even when the detailed deformation force is unknown, but it does not by itself predict damage, material failure, or a collision force history.

9.7.1 Impact mitigation: changing the force history

Seat restraints, packaging, landing systems, bumpers, compliant mounts, and sports equipment all exploit the relation $\mathbf{J} = \Delta\mathbf{p}$. If a prescribed momentum change is delivered over a longer time, the required average force magnitude is smaller. Real design, however, depends on distributed stresses, deformation, human tolerance, material constitutive laws, and peak/local loads. Example 9.9 therefore illustrates a mechanical principle rather than supplying a safety specification.

A useful design sequence is:

1. determine the required change in momentum from the system-level motion;
2. estimate the admissible interaction duration and displacement;
3. select a force–time or force–displacement model;
4. verify peak force, energy absorption, stress, and geometric constraints separately.

The first step is a momentum problem; the later steps require additional physics and engineering models.

9.7.2 A computational collision audit

A compact program can calculate total momentum, kinetic energy, and center-of-mass velocity before and after a proposed collision. The code should not decide whether conservation applies; that is a modelling decision based on the physical boundary and external impulses.

Listing 9.1: Minimal one-dimensional collision audit. The program checks stated data; it does not decide whether the system is isolated.

```
def audit(masses, before, after):
    if not (len(masses) == len(before) == len(after)):
        raise ValueError("one before/after velocity per mass required")
    if not masses or any(m <= 0 for m in masses):
        raise ValueError("masses must be positive")
    M = sum(masses)
    P_i = sum(m*v for m, v in zip(masses, before))
    P_f = sum(m*v for m, v in zip(masses, after))
    K_i = sum(0.5*m*v*v for m, v in zip(masses, before))
    K_f = sum(0.5*m*v*v for m, v in zip(masses, after))
    return {
        "P_i": P_i, "P_f": P_f, "dP": P_f-P_i,
        "K_i": K_i, "K_f": K_f, "dK": K_f-K_i,
        "Vcm_i": P_i/M, "Vcm_f": P_f/M
    }

print(audit([0.8, 1.2], [2.5, -0.5], [-1.1, 1.9]))
# Exact model data give P_i=P_f=1.4 kg m/s and K_i=K_f=2.65 J.
```

Useful validation tests include:

- translating to the center-of-mass frame and checking that total momentum becomes zero;
- transforming the output back to the laboratory frame and recovering the original velocities;
- swapping particle labels and confirming that system totals are unchanged;
- multiplying every mass by the same factor and verifying that velocities from an ideal two-body elastic-collision formula are unchanged while momentum and energy scale with mass;
- checking the exactly soluble limits $m_1 = m_2$ and $m_2 \gg m_1$.

9.7.3 Numerical impulse from sampled force data

Experimental force signals are often sampled at discrete times. With samples (t_k, F_k) , the trapezoidal approximation is

$$J \approx \sum_{k=0}^{N-1} \frac{F_k + F_{k+1}}{2} (t_{k+1} - t_k). \quad (9.57)$$

The sampling interval must resolve the pulse. A converged integral from under-resolved data is impossible: numerical quadrature cannot reconstruct an unmeasured narrow peak without a justified signal model. Baseline offsets also integrate into spurious impulse, so a pre-impact zero-force interval is valuable.

For the analytical half-sine pulse of Example 9.9, a numerical integration can be checked against the exact result $2F_0T/\pi$. Refining the time step should make the numerical value approach 14.4 N s. This benchmark checks the integration code independently of any collision experiment.

9.7.4 Measurement connection: low-speed cart collisions

Consider a proposed low-speed cart experiment with photogates or motion sensors. Measure the masses, pre-collision velocities, post-collision velocities, and an estimate of the collision duration. Before declaring momentum conservation, estimate the largest plausible horizontal external impulse from track tilt, rolling resistance, cables, or sensor interactions.

The following table is a *synthetic teaching dataset*, not a laboratory measurement. It is generated near an ideal elastic collision but includes small plausible deviations for analysis practice.

Table 9.1: Synthetic low-speed cart data for a momentum audit. Quoted uncertainties are stipulated independent standard uncertainties for the exercise.

Quantity	Cart 1	Cart 2
Mass m (kg)	0.500 ± 0.002	0.750 ± 0.002
Before velocity u (m s^{-1})	$+0.600 \pm 0.006$	-0.100 ± 0.006
After velocity v (m s^{-1})	-0.238 ± 0.006	$+0.457 \pm 0.006$

Using the central values,

$$P_i = (0.500)(0.600) + (0.750)(-0.100) = 0.2250 \text{ kg m s}^{-1}, \quad (9.58)$$

$$P_f = (0.500)(-0.238) + (0.750)(0.457) = 0.22375 \text{ kg m s}^{-1}. \quad (9.59)$$

The momentum residual is

$$\Delta P = P_f - P_i = -1.25 \times 10^{-3} \text{ kg m s}^{-1}. \quad (9.60)$$

The masses are reused before and after the event, so the two momentum estimates are correlated. Their individual uncertainties, computed from the independent input measurements, are approximately

$$\sigma_{P_i} = 5.54 \times 10^{-3} \text{ kg m s}^{-1}, \quad \sigma_{P_f} = 5.51 \times 10^{-3} \text{ kg m s}^{-1},$$

but the residual must be propagated directly from

$$\Delta P = m_1(v_1 - u_1) + m_2(v_2 - u_2).$$

If the two masses and four velocities are mutually independent measurements,

$$\begin{aligned} \sigma_{\Delta P}^2 &= (v_1 - u_1)^2 \sigma_{m_1}^2 + (v_2 - u_2)^2 \sigma_{m_2}^2 \\ &\quad + m_1^2(\sigma_{u_1}^2 + \sigma_{v_1}^2) + m_2^2(\sigma_{u_2}^2 + \sigma_{v_2}^2), \end{aligned} \quad (9.61)$$

which gives $\sigma_{\Delta P} = 7.91 \times 10^{-3} \text{ N s}$. Adding $\sigma_{P_i}^2$ and $\sigma_{P_f}^2$ would incorrectly discard the covariance caused by the same measured masses appearing twice.

The observed residual is only about 0.16 standard deviations from zero under these stipulated assumptions. It is therefore consistent with zero external impulse at this precision, but that statement is not proof that every systematic effect has been controlled.

The corresponding kinetic energies are

$$K_i = 0.09375 \text{ J}, \quad K_f = 0.09248 \text{ J}.$$

The central-value decrease is about 1.36%. For the same independent input measurements, propagate the energy difference directly:

$$\Delta K = \frac{1}{2} \sum_i m_i (v_i^2 - u_i^2), \quad \sigma_{\Delta K}^2 = \sum_i \left[\frac{1}{4} (v_i^2 - u_i^2)^2 \sigma_{m_i}^2 + m_i^2 (v_i^2 \sigma_{v_i}^2 + u_i^2 \sigma_{u_i}^2) \right].$$

This yields $\Delta K = -1.271 \times 10^{-3} \text{ J}$ and $\sigma_{\Delta K} = 2.883 \times 10^{-3} \text{ J}$. The central decrease is only 0.44 standard uncertainties from zero, so the synthetic observations do not distinguish this event from elastic at the stipulated precision. A common timing calibration error would correlate velocities and require extra covariance terms; the independence model is a teaching assumption.

Method

A defensible collision report should state: the system boundary; laboratory axes and sign convention; masses and velocity estimates; interaction interval; external-impulse estimate; $\Delta \mathbf{P}$ with uncertainty; ΔK with uncertainty or a justified qualitative statement; and known systematic limitations. Report the measured residual rather than replacing it by zero because momentum is expected to be conserved.

9.7.5 A controlled sampling study and a conservation residual

The exact half-sine model is useful because it distinguishes a numerical error from an experimental one. Let N equal subintervals cover the full known duration $T = 24 \text{ ms}$, including both zero endpoints. The composite trapezoidal estimate is

$$J_N = \frac{T}{N} \left[\frac{F(0) + F(T)}{2} + \sum_{k=1}^{N-1} F(kT/N) \right].$$

For $F(t) = F_0 \sin(\pi t/T)$, this becomes

$$J_N = \frac{F_0 T}{N} \sum_{k=1}^{N-1} \sin\left(\frac{k\pi}{N}\right) = \frac{F_0 T}{N} \cot\left(\frac{\pi}{2N}\right).$$

The finite sine sum follows by summing a geometric series in $\exp(i\pi/N)$ and taking its imaginary part; alternatively, pairing angles and using the sine addition identity gives the same result. Because the exact integral is $J = 2F_0 T/\pi$, the ratio of the sampled estimate to the exact impulse is

$$\frac{J_N}{J} = z \cot z, \quad z = \frac{\pi}{2N}. \quad (9.62)$$

Near zero, $\sin z = z - z^3/6 + \dots$ and $\cos z = 1 - z^2/2 + \dots$, so $z \cot z = 1 - z^2/3 + \dots$. Therefore the leading relative error is $-\pi^2/(12N^2)$. This is a convergence statement for samples of a known smooth model, not a claim that interpolating a sparse measured pulse discovers its unobserved shape.

Figure 9.26 shows why the trapezoidal rule underestimates this concave positive pulse: straight chords lie below the sine. Doubling N reduces the asymptotic error by about four, as the table shows. This checks both the algorithm and the published values against an exact integral. It cannot assess a force-sensor bandwidth or contact model without their actual measurements.

A constant offset b contributes bT to an impulse regardless of how fine the quadrature is. In this model, an unremoved 20 N offset adds 0.480 N s, or 3.33% of the true impulse. A baseline estimated

Table 9.2: Controlled trapezoidal refinement of the analytical 14.4 Ns half-sine. The samples include the entire pulse.

N	Interval (ms)	J_N (N s)	Relative error (%)
4	6.000	13.652	−5.19
8	3.000	14.214	−1.29
16	1.500	14.354	−0.321
32	0.750	14.388	−0.0803

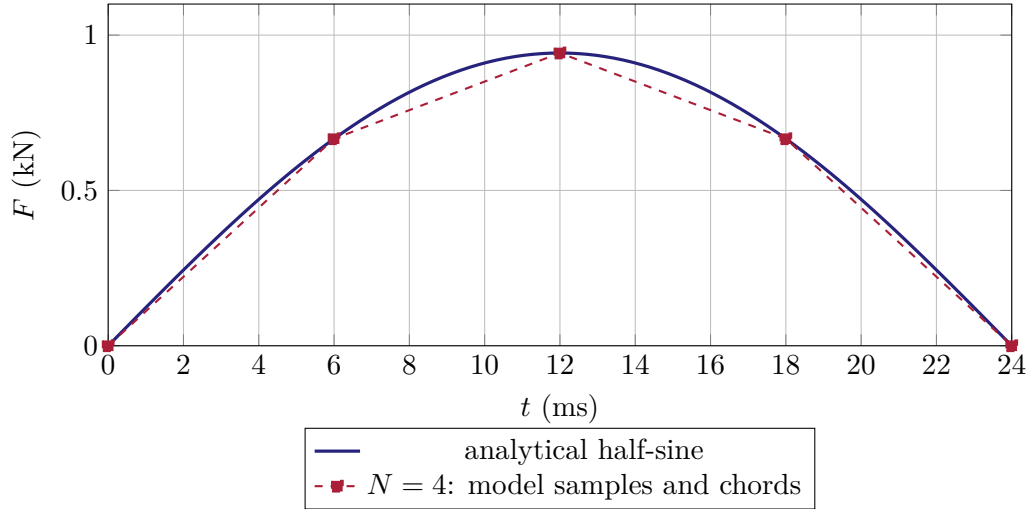


Figure 9.26: Quantitative samples of a stipulated analytical model. The sampling markers are synthetic; the dashed chords illustrate the four-subinterval quadrature, not an independently measured continuous force.

from a pre-trigger interval can remove a stable offset; slow drift requires a separate model and uncertainty. Missing pulse tails create another error: even perfectly integrated recorded samples omit any force outside the observation window. Improving the integration formula alone addresses neither effect.

For the synthetic cart data, it is useful to graph the residual separately from the much larger before/after momenta. Figure 9.27 places zero on the same physical axis as ΔP and its stipulated standard uncertainty. The interval shown is one standard uncertainty, not a hard bound. Its overlap with zero expresses consistency under the adopted error model; it does not demonstrate an exact law from finite data or certify a proposed apparatus.

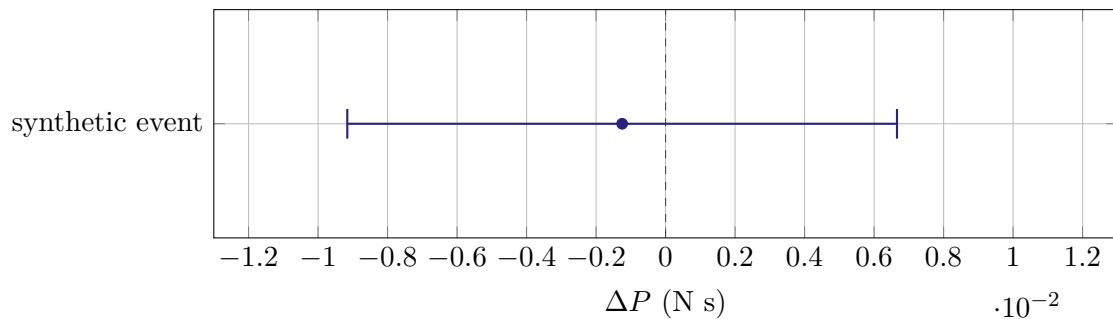


Figure 9.27: The central residual -0.00125 Ns and one standard uncertainty 0.007908854 Ns for the synthetic cart event. The shared masses are propagated directly through the residual. The dashed vertical line denotes zero.

Guided checks

A proposed experiment and its worked evaluation. Record cart masses, four velocity components along the track, contact duration, and a bound on the external force. Compute the residual from the signed measurements, propagate their stated uncertainty model, and compare the external-impulse bound with the precision sought. The reference interpretation for Table 9.1 is that both momentum and elastic energy are statistically consistent with the model at the stipulated precision, while possible systematic effects remain an experimental question. A satisfactory student report gives the system, axes, raw values, formulas, assumptions about correlation, numerical residuals, and limits. A report that replaces the residual with zero, or declares energy conservation from momentum alone, does not meet the criterion.

9.7.6 Space-technologies perspective

Momentum bookkeeping appears throughout space engineering: separation events, docking, deployment, propellant expulsion, impact and debris analysis, attitude-control thruster firings, and momentum management. A spacecraft plus expelled propellant can be treated as a closed system only if both are included; the spacecraft alone is an open, variable-mass subsystem. That distinction motivates the advanced rocket analysis in Section 9.9.

9.8 Chapter Summary and a Working Formula Sheet**Summary**

The center of mass is a mass-weighted position. For a fixed-mass particle system, total momentum is $\mathbf{P} = M\mathbf{V}_{\text{CM}}$ and the center of mass obeys $M\mathbf{A}_{\text{CM}} = \sum \mathbf{F}_{\text{ext}}$. Impulse is the time integral of force and equals the momentum change. Momentum conservation is justified when the net external impulse is negligible over the selected interval. Collision type and kinetic-energy behaviour require a separate energy audit. Perfectly inelastic means common final velocity, not zero final velocity in every frame. Multidimensional momentum conservation supplies one scalar equation per component, and additional physical information may be required. System boundaries and reference frames are part of the physics, not bookkeeping afterthoughts.

Formula summary**Center of mass and total momentum**

$$\mathbf{R}_{\text{CM}} = \frac{1}{M} \sum_i m_i \mathbf{r}_i, \quad \mathbf{R}_{\text{CM}} = \frac{1}{M} \int \mathbf{r} \, dm,$$

$$\mathbf{P} = \sum_i m_i \mathbf{v}_i = M\mathbf{V}_{\text{CM}}, \quad M\mathbf{A}_{\text{CM}} = \sum \mathbf{F}_{\text{ext}}.$$

Impulse and momentum balance

$$\mathbf{J} = \int_{t_i}^{t_f} \mathbf{F} \, dt = \Delta \mathbf{p}, \quad \mathbf{P}_f - \mathbf{P}_i = \mathbf{J}_{\text{ext}}.$$

If $\mathbf{J}_{\text{ext}} \simeq 0$, then $\mathbf{P}_f \simeq \mathbf{P}_i$.

Perfectly inelastic two-body collision

$$\mathbf{V} = \frac{m_1 \mathbf{u}_1 + m_2 \mathbf{u}_2}{m_1 + m_2}, \quad K_i - K_f = \frac{1}{2} \mu |\mathbf{u}_1 - \mathbf{u}_2|^2, \quad \mu = \frac{m_1 m_2}{m_1 + m_2}.$$

One-dimensional elastic collision

$$v_1 - v_2 = -(u_1 - u_2),$$
$$v_1 = \frac{m_1 - m_2}{m_1 + m_2}u_1 + \frac{2m_2}{m_1 + m_2}u_2, \quad v_2 = \frac{2m_1}{m_1 + m_2}u_1 + \frac{m_2 - m_1}{m_1 + m_2}u_2.$$

Before accepting a momentum solution, ask five questions: What is the system? Which frame defines the velocities? Which forces are external? Is their impulse negligible over the chosen interval? Has kinetic energy been audited independently rather than silently assumed to be conserved?

9.9 Advanced Topics for Technical and Engineering Students

The core chapter treats fixed sets of particles and uses momentum balances directly. This section develops three extensions useful in engineering and technical physics: the decomposition of kinetic energy into center-of-mass and internal motion, a more systematic interpretation of inelastic energy changes, and open-system momentum bookkeeping for a variable-mass rocket. The additional mathematical tools are vector expansions, differential limits, and logarithmic integration. These topics are optional enrichment and are not prerequisites for the core exercises.

9.9.1 Kinetic energy relative to the center of mass

Write each particle velocity as

$$\mathbf{v}_i = \mathbf{V}_{\text{CM}} + \mathbf{v}'_i, \quad (9.63)$$

where $\mathbf{v}'_i$ is measured in the CM frame. By definition,

$$\sum_i m_i \mathbf{v}'_i = 0. \quad (9.64)$$

Then the total kinetic energy is

$$K = \sum_i \frac{1}{2} m_i |\mathbf{V}_{\text{CM}} + \mathbf{v}'_i|^2 \quad (9.65)$$

$$= \frac{1}{2} M V_{\text{CM}}^2 + \mathbf{V}_{\text{CM}} \cdot \sum_i m_i \mathbf{v}'_i + \sum_i \frac{1}{2} m_i v_i'^2. \quad (9.66)$$

The cross term vanishes by Eq. (9.64), so

$$K = \frac{1}{2} M V_{\text{CM}}^2 + K_{\text{rel}}, \quad K_{\text{rel}} = \sum_i \frac{1}{2} m_i v_i'^2. \quad (9.67)$$

A frame moving at constant velocity $\mathbf{U}$ changes the total kinetic energy to $K' = K - \mathbf{U} \cdot \mathbf{P} + \frac{1}{2} M U^2$. Thus, when $\mathbf{P}_f = \mathbf{P}_i$ and M is unchanged, $K_f' - K_i' = K_f - K_i$: the sticking loss is invariant even though each energy is frame dependent.

The first term is the kinetic energy of translational CM motion; the second measures motion relative to the center of mass.

For an isolated collision, $\mathbf{P} = M \mathbf{V}_{\text{CM}}$ is constant, so the CM translational term is unchanged. Any collision-induced change in total kinetic energy therefore occurs entirely in K_{rel} .

Theoretical point 9.1: Two-body relative kinetic energy

For two particles,

$$K_{\text{rel}} = \frac{1}{2} \mu |\mathbf{v}_1 - \mathbf{v}_2|^2, \quad \mu = \frac{m_1 m_2}{m_1 + m_2}. \quad (9.68)$$

This follows by writing the two CM-frame velocities in terms of the relative velocity. It explains why Eq. (9.38) depends only on the initial relative speed: a perfectly inelastic collision removes all two-body CM-frame translational motion.

Example 9.21: Frame-dependent kinetic energy, frame-independent inelastic loss

For the carts of Examples 9.13 and 9.14, the laboratory-frame initial kinetic energy is 2.65 J

and the CM speed is 0.700 m s^{-1} . The translational CM energy is

$$K_{\text{CM}} = \frac{1}{2}(2.00)(0.700)^2 = 0.490 \text{ J}.$$

Therefore the initial relative kinetic energy is

$$K_{\text{rel,i}} = 2.65 - 0.490 = 2.16 \text{ J}.$$

After the carts stick, $K_{\text{rel,f}} = 0$, so the loss is 2.16 J. In the CM frame the total initial kinetic energy is already 2.16 J and the final kinetic energy is zero. Although the absolute kinetic energies differ between frames, their inelastic decrease is the same for these Galilean-related inertial frames because the conserved CM translational term cancels from the difference.

9.9.2 Deformable systems and internal energy

A system can conserve momentum while converting organized translational kinetic energy into internal modes. A spring, deformable bumper, granular aggregate, or molecular system can store energy in elastic deformation, vibration, rotation, thermal motion, fracture surfaces, or sound. Momentum conservation constrains the vector sum of motions; it does not specify how energy is partitioned internally.

For a perfectly elastic macroscopic collision, the bodies may deform substantially during contact and later recover that energy, leaving the same total translational kinetic energy after separation. “Elastic” describes the initial-to-final energy balance, not an assertion that the bodies remained undeformed at every intermediate instant.

9.9.3 Variable-mass systems: why a subsystem needs momentum flux

Equation (9.18) was derived for a fixed set of particles. A rocket is different if we choose only the material still attached to the vehicle as the system: mass continuously crosses the boundary. One must account explicitly for the momentum carried through that boundary.

Consider one-dimensional motion in an inertial frame. At time t a rocket has mass m and velocity v . During a short interval, its mass changes by $dm < 0$ and its velocity changes by dv . The expelled mass is $-dm > 0$. Let the exhaust leave with speed $u_e > 0$ relative to the rocket, backward along the line of motion. To first order its inertial-frame velocity is $v - u_e$.

Neglect external impulse during the short interval. Momentum before is mv . Momentum after is

$$(m + dm)(v + dv) + (-dm)(v - u_e) = mv + m dv + u_e dm + \text{second-order terms.} \quad (9.69)$$

Equating before and after gives

$$\boxed{m dv = -u_e dm.} \quad (9.70)$$

Because $dm < 0$, a backward exhaust produces $dv > 0$.

Integrating from (m_i, v_i) to (m_f, v_f) with constant effective exhaust speed,

$$\int_{v_i}^{v_f} dv = -u_e \int_{m_i}^{m_f} \frac{dm}{m}, \quad (9.71)$$

$$\boxed{\Delta v = v_f - v_i = u_e \ln \left(\frac{m_i}{m_f} \right)} \quad (9.72)$$

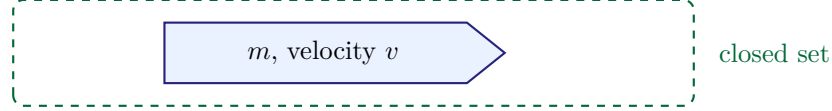
This is the ideal rocket equation for the stated one-dimensional assumptions.

If external force F_{ext} acts and the positive mass-outflow rate is $\dot{m}_e = -\dot{m} > 0$, the corresponding one-dimensional equation is

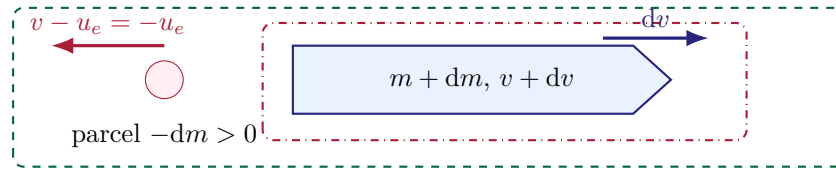
$$\boxed{m \frac{dv}{dt} = F_{\text{ext}} + u_e \dot{m}_e,} \quad (9.73)$$

with signs defined by the chosen positive direction and the assumption that u_e is backward exhaust speed relative to the vehicle.

(a) At t : fixed material selected



(b) At $t + dt$: same material plus open vehicle boundary



dash-dot: open vehicle

dashed: same closed material

Figure 9.28: Schematic differential rocket bookkeeping in the instantaneously comoving inertial frame, where the initial vehicle velocity is $v = 0$. Therefore the expelled parcel points backward in this drawing. In an arbitrary laboratory frame $v - u_e$ can have either sign. The dashed boundary follows the fixed selected material; the dash-dot boundary excludes the expelled parcel and defines an open vehicle subsystem. Arrow lengths and distances are schematic.

Example 9.22: Ideal rocket mass-flow audit

A spacecraft in deep space is idealized as one-dimensional, with negligible external force. Its mass decreases from $m_i = 1200 \text{ kg}$ to $m_f = 900 \text{ kg}$ while the effective exhaust speed is constant at $u_e = 2200 \text{ m s}^{-1}$. Estimate the ideal velocity increment. If propellant is expelled at 3.0 kg s^{-1} , also find the ideal thrust magnitude and burn duration under these assumptions.

The ideal rocket equation gives

$$\Delta v = (2200) \ln \left(\frac{1200}{900} \right) = 632.9 \text{ m s}^{-1}.$$

Thus

$$\boxed{\Delta v \simeq 633 \text{ m s}^{-1}.$$

The ideal thrust magnitude associated with a constant mass-outflow rate is

$$T = u_e \dot{m}_e = (2200)(3.0) = 6.60 \times 10^3 \text{ N}.$$

The expelled propellant mass is 300 kg , so the burn duration is

$$\Delta t = \frac{300}{3.0} = 100 \text{ s}.$$

Checks and limits. When $m_f = m_i$, the logarithm vanishes and the predicted velocity increment is zero. Reversing the exhaust direction reverses the thrust term rather than changing the sign of an arbitrarily named positive outflow rate. The logarithm is dimensionless, and $m_i/m_f > 1$ gives positive Δv . The calculation neglects gravity, drag, pressure-thrust corrections, changing exhaust performance, staging, and relativistic effects. It is an ideal momentum-flux model, not a mission-performance prediction.

9.9.4 Why $F = ma$ can be misused for changing mass

Writing $\mathbf{F} = m\mathbf{a}$ for the rocket alone and then inserting a changing $m(t)$ without a momentum-flux term is incomplete. Conversely, differentiating $m\mathbf{v}$ as $m\dot{\mathbf{v}} + \dot{m}\mathbf{v}$ is not automatically the correct rocket equation for the open subsystem, because the crossing mass carries momentum at a velocity different from the rocket velocity. The flux velocity must be specified. Equation (9.73) is a consequence of that careful boundary accounting.

Caution

Variable-mass equations are sign-convention sensitive. State whether $\dot{m}$ is the rocket's mass rate (negative during expulsion) or whether $\dot{m}_e$ is a positive outflow rate. Also state whether exhaust velocity is measured in the inertial frame or relative to the vehicle. Never combine formulas that use different conventions without translating them.

9.9.5 Computational check: frames and kinetic energy

For any finite fixed-mass dataset, a robust numerical check is to compute

$$\mathbf{V}_{\text{CM}} = \frac{\sum_i m_i \mathbf{v}_i}{M}, \quad K_{\text{rel}} = \sum_i \frac{1}{2} m_i |\mathbf{v}_i - \mathbf{V}_{\text{CM}}|^2,$$

and verify numerically that Eq. (9.67) reproduces the direct total kinetic energy. For an isolated elastic collision, both $\mathbf{V}_{\text{CM}}$ and K_{rel} should remain constant. For a perfectly inelastic collision, $K_{\text{rel,f}} = 0$. These checks catch sign and frame mistakes that a single momentum residual can miss.

Take-home message

The deeper unifying idea of this chapter is not a collection of collision formulas. It is a hierarchy of balances. For a fixed system, external force controls center-of-mass acceleration and external impulse controls total-momentum change. Kinetic energy requires its own balance. For an open system, momentum carried across the boundary must be included explicitly.

9.9.6 A lower bound on kinetic energy at fixed momentum

The CM decomposition has a useful consequence: for a fixed total mass and momentum,

$$K \geq \frac{P^2}{2M}. \quad (9.74)$$

Every relative-motion term is nonnegative. Equality is possible only when all particles share the common CM velocity. A perfectly inelastic two-body collision therefore produces the minimum possible translational kinetic energy consistent with the conserved momentum. It does not remove CM translation, which cannot vanish unless the total momentum vanishes in the selected frame. This lower bound is also an immediate diagnostic for a proposed numerical result: a state with $K < P^2/(2M)$ cannot be assembled from ordinary positive masses and real nonrelativistic velocities.

For the partial-impact example, $M = 4.00 \text{ kg}$ and $P = 3.50 \text{ kg m s}^{-1}$, so the bound is $K_{\text{CM}} = 1.53125 \text{ J}$. The incoming relative energy is $13.25 - 1.53125 = 11.71875 \text{ J}$; the outgoing relative energy is $5.75 - 1.53125 = 4.21875 \text{ J}$. Their ratio is $0.360 = e_r^2$. The separating collision retains some relative motion, while a common-velocity outcome would attain the bound.

Under an inertial frame translation at velocity $\mathbf{w}$, the transformed velocities are $\mathbf{v}_i^* = \mathbf{v}_i - \mathbf{w}$. Expanding the squares gives

$$K^* = K - \mathbf{w} \cdot \mathbf{P} + \frac{1}{2} M w^2. \quad (9.75)$$

For an isolated collision with unchanged M and $\mathbf{P}$, the boost terms are the same before and after. Thus $K_i^* - K_f^* = K_i - K_f$, although neither kinetic energy separately is frame invariant. The subtraction must be made between states of the same fixed system. A rocket-only subsystem does not satisfy those fixed-mass conditions and cannot inherit this cancellation without including the expelled material.

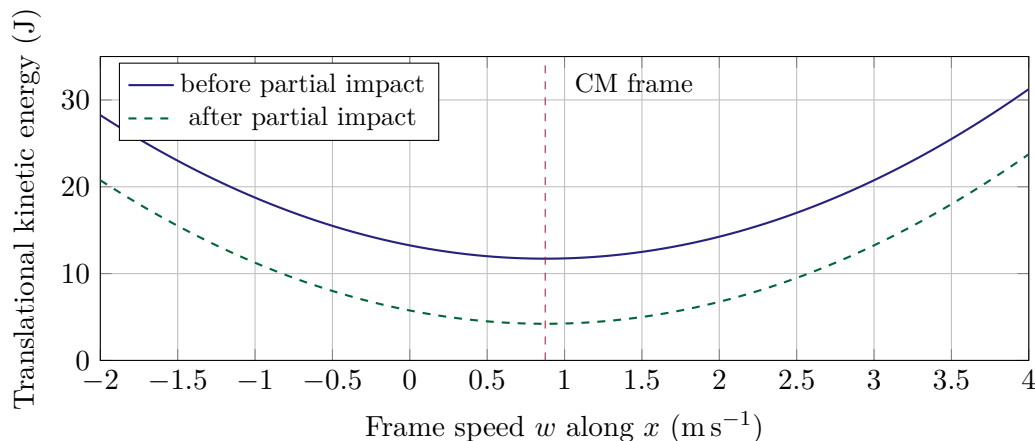


Figure 9.29: Frame-dependent kinetic energies for Example 9.15. Both parabolas reach their minima at $w = V_{\text{CM}} = 0.875 \text{ m s}^{-1}$ and are separated by 7.50 J everywhere. Their minima are the initial and final relative kinetic energies.

9.9.7 Inverse rocket sizing and the cost of carrying spent structure

The ideal rocket equation can be used in reverse. If the final attached mass is specified and a velocity increment is required, the initial mass and propellant requirement are

$$m_i = m_f \exp(\Delta v / u_e), \quad m_{\text{prop}} = m_f [\exp(\Delta v / u_e) - 1]. \quad (9.76)$$

The exponential dependence is the important interpretation: doubling the requested velocity increment does not merely double the initial-to-final mass ratio. Here m_f includes everything still attached at burnout, such as payload, tanks, engine, and residual material. It is not the payload alone unless every other retained mass has explicitly been removed.

Staging can improve the ideal velocity increment by discarding structure after its associated propellant is exhausted. A discarded stage with the same instantaneous velocity as the remaining vehicle produces no impulsive speed change in this idealization. Its benefit appears in the next burn, which accelerates a smaller attached mass. Each burn has its own initial mass, final mass, and effective exhaust speed. Taking one logarithm of the original launch mass divided by the final payload mass would incorrectly count jettisoned dry structure as propellant expelled at exhaust speed.

Example 9.23: Inverse sizing and a two-stage deep-space comparison

All motion is along one horizontal line in deep space, with external forces neglected and nonrelativistic exhaust. (a) A single-stage vehicle must gain 1800 m s^{-1} , has 700 kg attached at burnout, and uses constant effective exhaust speed 2400 m s^{-1} . Find initial mass, propellant mass, and burn time at 2.00 kg s^{-1} outflow. (b) Independently consider a two-stage stack with 400 kg payload, upper-stage dry mass 150 kg and propellant 600 kg , and lower-stage dry mass 250 kg and propellant 1000 kg . The lower and upper effective exhaust speeds are 2200 and 2800 m s^{-1} . Find its ideal velocity increment. (c) Compare retaining the empty lower structure throughout the upper burn.

(a) Inverse sizing. The required mass ratio is $\exp(1800/2400) = \exp(0.750) = 2.11700$. Therefore

$$\boxed{m_i = 1481.90 \text{ kg}}, \quad \boxed{m_{\text{prop}} = 781.90 \text{ kg}}, \quad \boxed{t_b = 390.95 \text{ s}}.$$

The thrust is $T = u_e \dot{m}_e = 4800 \text{ N}$. Acceleration rises from $T/m_i = 3.239 \text{ m s}^{-2}$ to $T/m_f = 6.857 \text{ m s}^{-2}$ as the attached mass decreases. Constant thrust therefore does not imply constant acceleration.

(b) Staged mass ledger. The initial mass is 2400 kg. After the lower propellant burns, 1400 kg remains. Jettisoning its 250 kg dry structure leaves 1150 kg for the upper burn; after that burn the attached mass is 550 kg, comprising payload and upper dry structure. Thus

$$\Delta v_1 = 2200 \ln(2400/1400) = 1185.79 \text{ m s}^{-1},$$

$$\Delta v_2 = 2800 \ln(1150/550) = 2065.28 \text{ m s}^{-1},$$

$$\boxed{\Delta v_{\text{staged}} = 3251.07 \text{ m s}^{-1}}.$$

Subsequently discarding the upper dry structure without a relative separation speed leaves the 400 kg payload at the same velocity; it adds no rocket-equation increment.

(c) Retained lower structure. The first burn is identical, but the upper burn now starts at 1400 kg and finishes at 800 kg. Hence

$$\Delta v_{\text{retained}} = 1185.79 + 2800 \ln(1400/800) = \boxed{2752.72 \text{ m s}^{-1}}.$$

Ideal staging gains $\boxed{498.35 \text{ m s}^{-1}}$ with the stipulated identical propellant masses and exhaust speeds. This comparison includes no gravity loss, drag, separation impulse, pressure-thrust correction, or change in propulsion performance. It isolates the momentum consequence of accelerating less spent structure during the second burn.

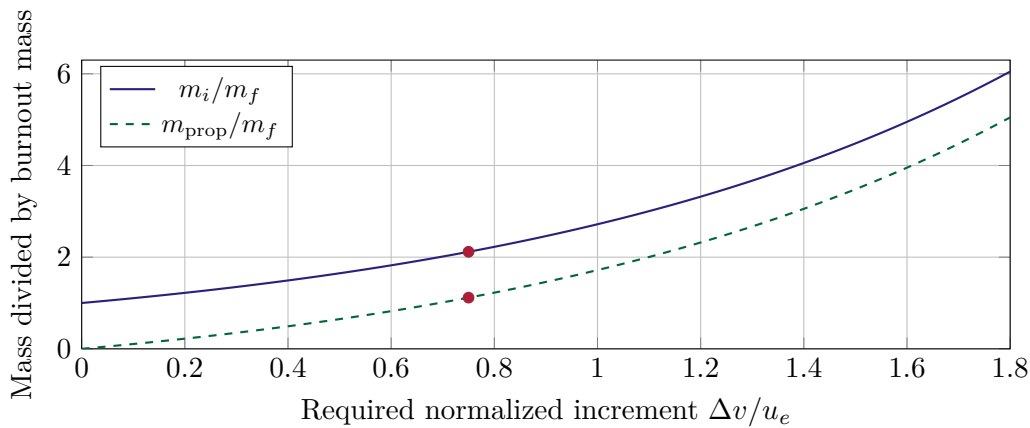


Figure 9.30: Inverse ideal-rocket sizing. The two ratios differ by one because initial mass is burnout mass plus expelled propellant. Points represent the single-stage calculation in Example 9.23. The curves assume constant effective exhaust speed and no external force.

The ideal staging curve also distinguishes a mass discontinuity from an impulse. At the jettison point, the detached lower structure continues with the velocity it had immediately beforehand. The momentum of the *remaining attached subsystem* decreases because material has been removed from that subsystem, even though its velocity does not change. For the fixed system containing both pieces, total momentum is unchanged. Giving the stage a prescribed backward separation speed would be a different problem, with a recoil increment that must be calculated from its own momentum balance.

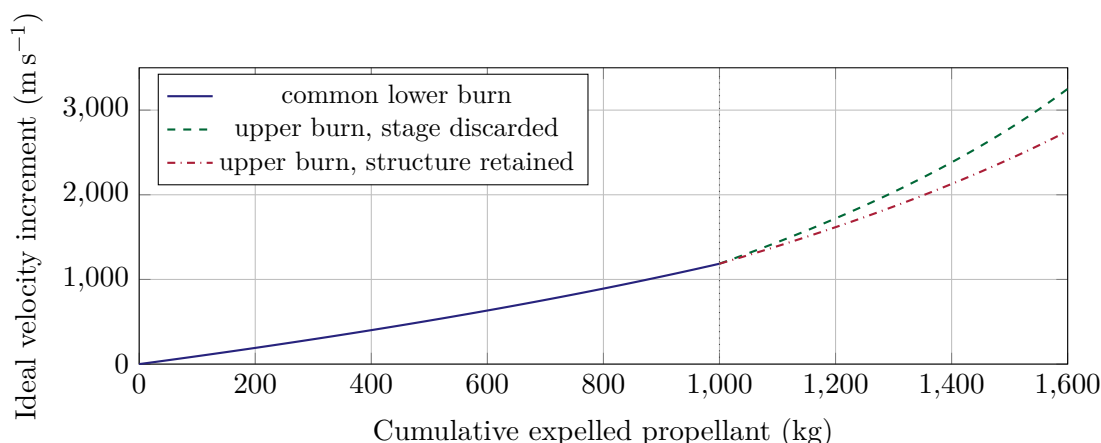


Figure 9.31: Constructed velocity-increment curves for the two-stage mass ledger. At 1000 kg expelled propellant, dry-stage jettison changes attached mass but causes no velocity jump in the idealization. The steeper subsequent staged curve results from the smaller mass accelerated during the upper burn.

The horizontal deep-space model isolates propulsion from trajectory dynamics. If an external force were included, its contribution would be integrated as $F_{\text{ext}}(t)/m(t)$ over the burn, alongside the exhaust contribution. The burn duration then matters even when initial and final masses are unchanged. Under the zero-external-force assumptions used here, it affects the time history of thrust and acceleration but not the ideal total velocity increment for a given mass ratio and constant exhaust speed. This is a consequence of integrating the momentum-flux equation, not a claim that burn scheduling never matters in a mission.

An inverse calculation is only as meaningful as its mass ledger. Every attached dry component belongs in both ends of a burn unless it is actually removed during that burn, which would require a separate boundary model. A discarded component belongs neither in the next burn's initial mass nor in that burn's propellant expenditure. These distinctions make the logarithmic formula operational: it is a momentum-flux integral for a specified burn, rather than a rule for any arbitrary decrease in vehicle mass. The same discipline of defining the system and the interval that made the collision problems reliable is essential for variable-mass motion.

9.10 Exercises and Problems

Unless stated otherwise, use SI units, treat stated particles or carts as a fixed system during the interaction, and work in the laboratory frame. State the system and sign convention before using momentum conservation. Problems 35–41 use material from Section 9.9.

A. Center of mass and system motion

1. A uniform plate has a circular hole cut near its upper-right corner. Without calculation, state the direction in which the center of mass of the remaining plate moves relative to the original geometric center. Explain why treating the hole as negative bookkeeping mass gives the correct direction.
2. Three point masses are located in a plane: 2.0 kg at (0,0), 3.0 kg at (4.0,0) m, and 5.0 kg at (0,2.0) m. Find $\mathbf{R}_{\text{CM}}$.
3. A thin rod occupies $0 \leq x \leq L$ and has linear density $\lambda(x) = \lambda_0(1 + x/L)$. Find its center of mass. Check that your answer lies to the right of $L/2$.
4. A uniform semicircular wire of radius R occupies the upper half of the circle centered at the

origin. By symmetry, what center-of-mass component must vanish? Set up, but do not evaluate, a one-variable integral for the other component using angle θ measured from the $+x$ axis.

5. Two carts of masses 1.5 kg and 2.5 kg are initially at rest on a horizontal low-friction track with a compressed spring between them. After release, the first cart moves at -2.0 m s^{-1} . Find the second cart's velocity and verify the center-of-mass velocity.
6. A 5.0 kg system has initial center-of-mass velocity -0.20 m s^{-1} along x . A constant net external force $+4.0 \text{ N}$ acts for 3.0 s. Find the final total momentum and final center-of-mass velocity.
7. A wrench is thrown inside a freely drifting spacecraft cabin and caught by another astronaut. If the system is the astronauts, wrench, and spacecraft and external force is negligible, describe the center-of-mass motion during the throw and catch. Does the answer imply that every component moves uniformly?
8. A student computes the center of mass of a ring and rejects the result because it lies in empty space. Diagnose the error in the student's reasoning.

B. Impulse and momentum balance

9. A 0.150 kg ball changes velocity from $(20 \hat{x}) \text{ m s}^{-1}$ to $(-10 \hat{x} + 12 \hat{y}) \text{ m s}^{-1}$. Find the impulse vector, its magnitude, and its direction measured counterclockwise from $+x$.
10. A triangular force pulse rises from zero to 3.0 kN and returns to zero over a total duration of 6.0 ms. Find its impulse and average force.
11. A force is $F(t) = F_0(1 - t/T)$ for $0 \leq t \leq T$ and zero otherwise, with $F_0 = 600 \text{ N}$ and $T = 50 \text{ ms}$. Find the impulse. What constant force acting for the same interval would give the same impulse?
12. A collision system has characteristic momentum 8.0 kg m s^{-1} . A conservative upper bound on the net external impulse during the collision is 0.080 N s . Express the bound as a percentage of the momentum scale. Would neglecting it be adequate for a claimed 0.2% conservation test?
13. Explain why integrating $|F_x(t)|$ instead of $F_x(t)$ generally does not give $|\Delta p_x|$. Construct a simple pulse with equal positive and negative rectangular parts to illustrate the difference.
14. Sampled force data are recorded at $t = (0, 2, 4, 6, 8) \text{ ms}$ with $F = (0, 400, 800, 400, 0) \text{ N}$. Use the trapezoidal rule to estimate the impulse.

C. One-dimensional collisions

15. A 0.60 kg cart moving at $+3.0 \text{ m s}^{-1}$ sticks to a 0.90 kg cart initially at rest. Find their common velocity, initial and final kinetic energies, and the kinetic-energy decrease.
16. Two equal masses collide elastically in one dimension with initial velocities $+2.0$ and -1.0 m s^{-1} . Find the final velocities. Explain the result in the center-of-mass frame.
17. A 1.0 kg cart moving at $+4.0 \text{ m s}^{-1}$ collides elastically with a stationary 3.0 kg cart. Find both final velocities and verify momentum and kinetic-energy conservation.
18. A 2.0 kg cart moving at $+5.0 \text{ m s}^{-1}$ hits a stationary 3.0 kg cart. The one-dimensional coefficient of restitution is $e_r = 0.50$. Find both final velocities using momentum conservation and the relative-speed definition of e_r .
19. A 0.20 kg projectile moving at 10 m s^{-1} embeds in a stationary 0.80 kg block. Find the common velocity and the translational kinetic-energy decrease. Where has the missing kinetic energy gone in a complete energy account?

20. A textbook statement says, “In a perfectly inelastic collision all objects are motionless after the collision.” Give a counterexample and replace the statement by a correct frame-independent definition.

D. Two-dimensional motion, fragmentation, and frames

21. A system initially at rest fragments into three pieces. Piece A has mass 1.0 kg and velocity $(4, 0) \text{ m s}^{-1}$. Piece B has mass 2.0 kg and velocity $(-1, 2) \text{ m s}^{-1}$. Piece C has mass 3.0 kg. Find the velocity of C if external impulse is negligible.
22. An equal-mass elastic collision occurs in a plane. The target is initially at rest and the incident particle moves at 5.0 m s^{-1} along $+x$. Afterward the incident particle has speed 3.0 m s^{-1} . Find the other particle’s speed and show that the two outgoing velocity vectors are perpendicular. Determine one consistent pair of scattering angles.
23. Two particles have masses 2.0 and 1.0 kg and velocities $+3.0$ and -1.0 m s^{-1} along x . Find V_{CM} and the two velocities in the CM frame. Verify that their CM-frame momenta sum to zero.
24. A two-dimensional collision problem specifies the two masses and both initial velocity vectors. After collision it gives no final speed or angle and merely says the collision is inelastic. Explain why momentum conservation alone is insufficient to determine both final velocity vectors uniquely.
25. A projectile fragments while airborne. During a 4.0 ms breakup, the total mass is 2.0 kg. Estimate the gravitational impulse magnitude and compare it with an internal-fragment momentum scale of 6.0 kg m s^{-1} . Is neglecting gravity during the breakup reasonable at the percent level? Use $g = 9.81 \text{ m s}^{-2}$.
26. Explain why a momentum triangle may be redrawn head-to-tail without implying that the fragments physically travelled along the sides of that triangle.

E. Measurement, computation, and modelling

27. For the synthetic force samples in Problem 14, suppose every force sample has an unknown constant baseline offset of $+20 \text{ N}$. What spurious impulse does the offset add over 8 ms? Why can a zero-force pre-trigger interval help?
28. Reproduce the momentum residual for Table 9.1. Compare $|\Delta P|$ with the stated propagated standard uncertainty $\sigma_{\Delta P} = 7.91 \times 10^{-3} \text{ kg m s}^{-1}$. Does the central residual alone establish a real momentum loss?
29. A cart collision lasts 12 ms. The largest plausible parasitic horizontal force is 0.15 N, and the characteristic momentum is 0.90 kg m s^{-1} . Estimate the maximum fractional external-impulse effect.
30. Write pseudocode for a two-dimensional momentum audit that accepts arrays of masses and before/after velocity pairs and returns ΔP_x , ΔP_y , ΔK , and $\mathbf{V}_{\text{CM}}$ before and after. State at least three physics-based validation tests.
31. A force sensor samples a narrow impact pulse every 5 ms, but an independent high-speed camera shows that contact lasts only 3 ms. Explain why refining the trapezoidal integration formula on the existing samples cannot recover the true impulse reliably.
32. A simulation of an isolated two-particle collision reports $P_i = P_f$ to machine precision but $V_{\text{CM},f} \neq V_{\text{CM},i}$. List two possible coding or reporting errors if the total mass is constant.

- 33.** A cart experiment finds $P_f - P_i = 0.003 \text{ kg m s}^{-1}$ with a standard uncertainty of $0.002 \text{ kg m s}^{-1}$. Separately, the estimated possible systematic external impulse is 0.004 N s . Explain why quoting the residual as “evidence of momentum violation” would be unjustified.
- 34.** A 12 kg package moving at 3.0 m s^{-1} must be brought to rest. Compare the average force magnitudes for stopping times 0.060 s and 0.240 s . State why this comparison alone is insufficient for a packaging design approval.

F. Advanced challenges

- 35.** For two particles of masses 2.0 and 1.0 kg with velocities $+3.0$ and -1.0 m s^{-1} , calculate the total kinetic energy, the CM translational kinetic energy, and K_{rel} . Verify $K_{\text{rel}} = \frac{1}{2}\mu(v_1 - v_2)^2$.
- 36.** Starting from $\mathbf{v}_i = \mathbf{V}_{\text{CM}} + \mathbf{v}'_i$, derive Eq. (9.67) and identify exactly why the cross term vanishes.
- 37.** Derive Eq. (9.38) for a two-body perfectly inelastic collision by using the CM kinetic-energy decomposition rather than expanding the laboratory-frame energies directly.
- 38.** An ideal rocket in deep space has $m_i = 1000 \text{ kg}$, $m_f = 700 \text{ kg}$, and $u_e = 2500 \text{ m s}^{-1}$. Find its ideal Δv .
- 39.** What mass ratio m_i/m_f is required for an ideal $\Delta v = 1800 \text{ m s}^{-1}$ if $u_e = 3000 \text{ m s}^{-1}$?
- 40.** At one instant a rocket of mass 500 kg expels mass at a positive outflow rate 4.0 kg s^{-1} with effective exhaust speed 2000 m s^{-1} . A 1000 N external drag force acts opposite the motion. Find the instantaneous acceleration in the one-dimensional ideal model.
- 41.** A vertical ideal rocket burns from 1000 kg to 800 kg in 40 s with constant $u_e = 2500 \text{ m s}^{-1}$. Neglect drag but include constant downward gravity $g = 9.81 \text{ m s}^{-2}$. Derive and evaluate the speed change during the burn, assuming the upward direction is positive.

9.11 Complete Solutions to All Exercises and Problems

These solutions retain the numbering and all parameters of Problems 1–41. Every numerical calculation uses SI units internally. A positive component means motion or impulse in the declared positive coordinate direction; a negative component is information, not an error. “Isolated during an interaction” means that the net external impulse is negligible to the accuracy being claimed. It does not require each external force to vanish separately.

For conceptual items, the answer includes the physical reasoning and the limits of the statement. For the computational item, the pseudocode and validation examples constitute an exemplar; a different implementation is acceptable if it gives the same defined outputs and passes equivalent physical checks. Extra displayed digits are retained in intermediate calculations to make the numerical audit reproducible, rather than to imply greater measurement precision.

Problem 1: the direction of a hole-induced shift

Choose the origin at the geometric center of the original uniform plate, so that its original center of mass is $\mathbf{R}_0 = \mathbf{0}$. Let the original mass be M_0 , and let the removed circular piece have mass m_h and center $\mathbf{r}_h$. The hole lies toward the upper right, so both components of $\mathbf{r}_h$ are positive when $+x$ points right and $+y$ points up.

Mass moments add linearly. Subtracting the removed piece’s moment from the original plate’s moment gives

$$(M_0 - m_h)\mathbf{R}_{\text{rem}} = M_0\mathbf{R}_0 - m_h\mathbf{r}_h = -m_h\mathbf{r}_h.$$

Because $M_0 - m_h > 0$, both components of $\mathbf{R}_{\text{rem}}$ are negative. The remaining center of mass therefore shifts **down and left**, away from the removed upper-right material. The magnitude cannot be calculated without the masses and geometry, but the direction follows without them.

The negative mass is bookkeeping: no part of the actual plate has negative mass. It removes the correct mass and its first moment from a known complete object. Keeping the negative moment but failing to subtract m_h from the denominator would give an incorrect displacement magnitude. As a check, a hole centered at the original center would have $\mathbf{r}_h = \mathbf{0}$ and would produce no shift.

Problem 2: three point masses in a plane

The system comprises all three particles in the given coordinate frame. Its total mass is

$$M = 2.0 + 3.0 + 5.0 = 10.0 \text{ kg.}$$

Use the same mass weighting for each coordinate, rather than averaging the coordinates without weights:

$$X_{\text{CM}} = \frac{(2.0)(0) + (3.0)(4.0) + (5.0)(0)}{10.0} = 1.20 \text{ m,}$$

$$Y_{\text{CM}} = \frac{(2.0)(0) + (3.0)(0) + (5.0)(2.0)}{10.0} = 1.00 \text{ m.}$$

Thus

$$\mathbf{R}_{\text{CM}} = (1.20\hat{x} + 1.00\hat{y}) \text{ m.}$$

Both numerator components have units kg m; division by kilograms gives meters. A stronger check than checking the coordinate ranges is to compute the moments about the answer. In the x direction, $2(-1.2) + 3(2.8) + 5(-1.2) = 0$; in the y direction, $2(-1) + 3(-1) + 5(1) = 0$. The weighted position deviations sum to zero, as the definition requires. The center lies within the triangle formed by the positive masses; it need not lie on one of the particles.

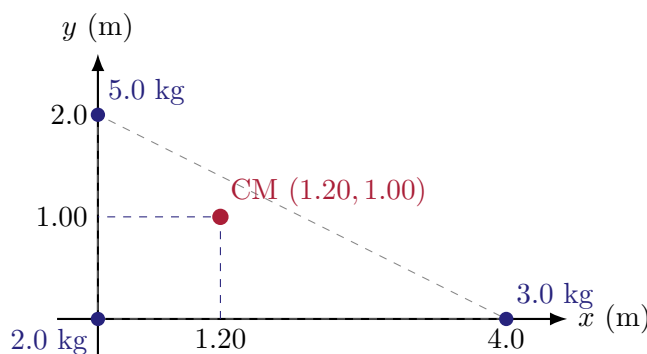


Figure 9.32: Solution geometry for Problem 2. Position coordinates are to scale; equal marker sizes identify ideal point particles and do not represent their masses. The dashed triangle is the convex hull of the three positive masses.

Problem 3: a rod with increasing linear density

The rod extends from 0 to L , with $L > 0$ and $\lambda_0 > 0$. Its mass element is $dm = \lambda_0(1 + x/L)dx$. The denominator must be integrated as well as the first moment because the density is not constant:

$$M = \lambda_0 \int_0^L \left(1 + \frac{x}{L}\right) dx = \lambda_0 \left(L + \frac{L}{2}\right) = \frac{3}{2}\lambda_0 L,$$

$$\int x dm = \lambda_0 \int_0^L \left(x + \frac{x^2}{L}\right) dx = \lambda_0 \left(\frac{L^2}{2} + \frac{L^2}{3}\right) = \frac{5}{6}\lambda_0 L^2.$$

Consequently

$$X_{\text{CM}} = \frac{(5/6)\lambda_0 L^2}{(3/2)\lambda_0 L} = \frac{5}{9}L.$$

The overall density scale cancels, but its variation with position does not. Increasing λ_0 multiplies every element's mass equally and therefore changes the total mass without moving the center.

The result satisfies $0 < X_{\text{CM}} < L$, and $5L/9 - L/2 = L/18 > 0$. This rightward shift is physically expected because equal-length intervals on the right contain more mass. Replacing the nonuniform density by its average in the first moment would incorrectly put the center at $L/2$; an average density preserves total mass but not necessarily the first moment.

Problem 4: setting up the semicircular-wire integral

The mass is uniformly distributed along a *wire*, so the appropriate element is arc length, not area. Points at angles θ and $\pi - \theta$ have equal mass, opposite x coordinates, and equal y coordinates. Reflection symmetry about the y axis therefore requires

$$X_{\text{CM}} = 0.$$

Let the constant linear density be λ . With angle measured from $+x$,

$$x = R \cos \theta, \quad y = R \sin \theta, \quad ds = R d\theta, \quad dm = \lambda R d\theta, \quad 0 \leq \theta \leq \pi.$$

The total mass is $M = \lambda\pi R$, and the requested one-variable setup is

$$Y_{\text{CM}} = \frac{1}{\lambda\pi R} \int_0^\pi (R \sin \theta) \lambda R d\theta = \frac{R}{\pi} \int_0^\pi \sin \theta d\theta.$$

The integral is deliberately left unevaluated, as requested. Its integrand and limits show that $Y_{\text{CM}} > 0$ and $Y_{\text{CM}} < R$: positive wire mass lies above the x axis, but not all of it is at the highest point. An integral over a filled semicircular lamina would describe a different mass distribution and would give a different answer.

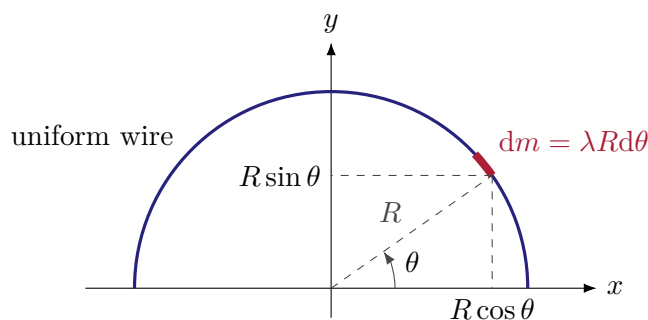


Figure 9.33: The angular parametrization used in Problem 4. The highlighted arc identifies a wire element; the interior carries no mass in this model. Reflection across the vertical axis establishes the vanishing horizontal moment before integration.

Problem 5: spring release from rest

Choose both carts and the spring as the system and $+x$ to the right. The spring force is internal to that system. On the ideal horizontal low-friction track, the horizontal external impulse is negligible, and the initial total momentum is zero. After release,

$$0 = P_f = (1.5)(-2.0) + (2.5)v_2.$$

Solving gives

$$v_2 = +1.20 \text{ m s}^{-1}.$$

The two cart momenta are -3.0 and $+3.0 \text{ kg m s}^{-1}$. The heavier cart therefore travels more slowly; opposite velocities of equal magnitude would not conserve momentum for these unequal masses.

The center-of-mass velocity follows independently from its definition:

$$V_{\text{CM}} = \frac{1.5(-2.0) + 2.5(1.20)}{1.5 + 2.5} = 0 \text{ m s}^{-1}.$$

The center of mass remains stationary in this initial laboratory frame, even while both carts move. Their final total kinetic energy is $\frac{1}{2}(1.5)(2.0)^2 + \frac{1}{2}(2.5)(1.20)^2 = 4.80 \text{ J}$. It comes from the initially compressed spring's stored energy; momentum conservation does not require this kinetic energy to have been present initially. The spring's mass is implicitly negligible or included consistently in the stated cart-system idealization.

Problem 6: changing total momentum by an external impulse

The fixed-mass system has $M = 5.0 \text{ kg}$ and initial center-of-mass velocity $V_i = -0.20 \text{ m s}^{-1}$. First convert this collective velocity into total momentum:

$$P_i = MV_i = (5.0)(-0.20) = -1.0 \text{ kg m s}^{-1}.$$

The specified force is already the *net* external force, so its impulse is

$$J_{\text{ext}} = F_{\text{ext}}\Delta t = (+4.0)(3.0) = +12 \text{ N s}.$$

Using $P_f = P_i + J_{\text{ext}}$,

$$P_f = +11 \text{ kg m s}^{-1}, \quad V_f = \frac{11}{5.0} = +2.20 \text{ m s}^{-1}.$$

The force first removes the negative momentum and then produces positive momentum; a positive impulse does not mean that the initial momentum was positive. As an independent check, $a_{\text{CM}} = F_{\text{ext}}/M = 0.80 \text{ m s}^{-2}$, so $V_f = -0.20 + (0.80)(3.0) = 2.20 \text{ m s}^{-1}$. No assumptions about relative motions inside the system are needed, because the center-of-mass equation accounts for their cancelling internal forces. The equality $1 \text{ N s} = 1 \text{ kg m s}^{-1}$ verifies the units in the momentum balance.

Problem 7: throwing and catching a wrench in a drifting cabin

Use the astronauts, wrench, and spacecraft together as the fixed-particle system, and describe the motion in an inertial frame. With negligible net external force,

$$\frac{d\mathbf{P}}{dt} = \mathbf{0}, \quad \mathbf{V}_{\text{CM}} = \frac{\mathbf{P}}{M} = \text{constant}, \quad \mathbf{R}_{\text{CM}}(t) = \mathbf{R}_{\text{CM}}(0) + \mathbf{V}_{\text{CM}}t.$$

The system center of mass follows the same straight, uniform drift through the throw, free flight, and catch. In the initial CM frame it stays at rest.

This result does *not* make every component move uniformly. The throwing astronaut transfers momentum to the wrench and receives the opposite impulse. Forces transmitted through contact can also change the spacecraft's or another astronaut's motion. At the catch, the receiving astronaut accelerates and the wrench decelerates; their interaction is internal to the chosen complete system. Kinetic energy can change through muscular work and through dissipation in the catch without changing total momentum.

Using the wrench alone as the system would give a different statement: the astronaut's force on it is external, so its momentum changes during the throw and catch. A coordinate frame tied to an accelerating cabin is also not automatically inertial. The collective conservation law should therefore be applied before translating the result into cabin-relative observations.

Problem 8: a center of mass in empty space

The student's error is assuming that the center of mass must be a material point of the object. It is a mass-weighted mean position, defined from the whole distribution. For a uniform ring centered at the origin, each mass element at $\mathbf{r}$ has an equal partner at $-\mathbf{r}$. Their moments cancel, so

$$\int \mathbf{r} dm = \mathbf{0}, \quad \boxed{\mathbf{R}_{\text{CM}} = \mathbf{0}}.$$

The origin is empty for an ideal thin ring, but that does not affect the averaging operation or the dynamics of the center of mass.

For positive masses, the center lies in the distribution's *convex hull*, which includes points enclosed by the ring. That is the relevant geometric restriction; belonging to the material itself is not required. The same reasoning allows the center of mass of a bent wire or hollow shell to lie away from its material.

One can still use $M\mathbf{a}_{\text{CM}} = \mathbf{F}_{\text{ext}}$ even when no particle occupies the center. The equation describes the weighted acceleration of the system, not a claim that an actual particle at the center experiences all external forces. Nonuniform mass around the ring could shift the center, so the cancellation argument depends on the stated symmetry.

Problem 9: a vector impulse and its direction

The system for the momentum change is the 0.150 kg ball. Use the stated laboratory axes and keep the components signed:

$$\begin{aligned} \Delta \mathbf{v} &= \mathbf{v}_f - \mathbf{v}_i = (-10 - 20)\hat{\mathbf{x}} + (12 - 0)\hat{\mathbf{y}} \\ &= (-30\hat{\mathbf{x}} + 12\hat{\mathbf{y}}) \text{ m s}^{-1}, \\ \mathbf{J}_{\text{net}} &= m\Delta \mathbf{v} = \boxed{(-4.50\hat{\mathbf{x}} + 1.80\hat{\mathbf{y}}) \text{ N s}}. \end{aligned}$$

The magnitude is

$$J = \sqrt{(-4.50)^2 + (1.80)^2} = 4.84665 \text{ N s} \simeq \boxed{4.85 \text{ N s}}.$$

Because $J_x < 0$ and $J_y > 0$, the direction lies in quadrant II. A quadrant-aware inverse tangent gives

$$\boxed{\theta = \text{atan2}(1.80, -4.50) = 158.2^\circ}$$

counterclockwise from $+x$. Taking only $\arctan(J_y/J_x)$ would yield a negative principal angle and miss the correct quadrant.

The impulse points along the change in velocity, not along either the initial or final velocity alone. The initial and final momenta are $(3.00, 0)$ and $(-1.50, 1.80) \text{ kg m s}^{-1}$; adding the computed impulse to the former reproduces the latter. Without a contact duration, the average force cannot be determined. The impulse calculated here is the net impulse on the ball over the specified velocity change.

Problem 10: the area of a triangular force pulse

Take the force's direction as positive throughout the pulse. The impulse is the signed area under the force–time graph. Convert both prefixes before multiplication:

$$F_{\text{max}} = 3.0 \text{ kN} = 3000 \text{ N}, \quad T = 6.0 \text{ ms} = 0.0060 \text{ s}.$$

The triangular graph has base T and height F_{max} , hence

$$\boxed{J = \frac{1}{2}TF_{\text{max}} = \frac{1}{2}(0.0060)(3000) = 9.0 \text{ N s}}.$$

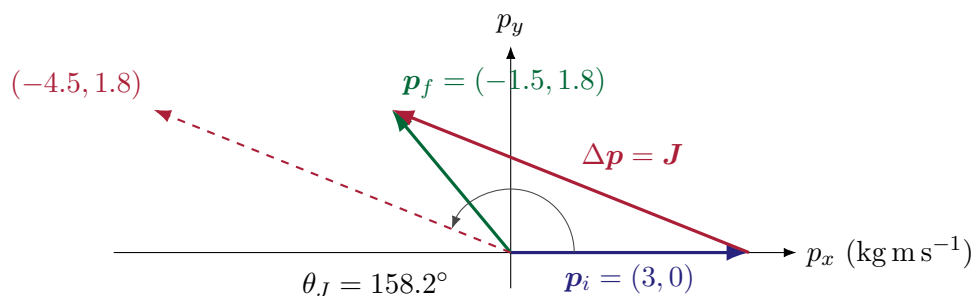


Figure 9.34: Vector subtraction for Problem 9. The solid red vector joins the tip of $\mathbf{p}_i$ to the tip of $\mathbf{p}_f$. Its dashed translated copy has the same magnitude and direction and makes the stated polar angle explicit. All vector components use the same scale.

The average force over the full pulse is defined by $F_{\text{avg}}T = J$:

$$F_{\text{avg}} = \frac{9.0}{0.0060} = 1500 \text{ N} = 1.50 \text{ kN}.$$

Its value is half the peak, not equal to the peak. A rectangular pulse with height 1.50 kN and the same duration transfers the same momentum, although its instantaneous load history differs.

The result does not depend on where within the interval the triangle reaches its maximum: two straight sides enclosing the same total base and height have the same area. It does depend on the assumption of a triangular shape. These data determine momentum transfer, but they do not by themselves determine work, which would also require displacement or velocity information during contact.

Problem 11: a linearly decreasing force

The force acts along the chosen positive axis and has the specified history only for $0 \leq t \leq T$. Integrate that history rather than using its initial value as a constant:

$$J = \int_0^T F_0 \left(1 - \frac{t}{T}\right) dt = F_0 \left[t - \frac{t^2}{2T} \right]_0^T = \frac{1}{2} F_0 T.$$

With $T = 50 \text{ ms} = 0.050 \text{ s}$,

$$J = \frac{1}{2}(600)(0.050) = 15.0 \text{ N s}.$$

A constant force over the same interval gives the same impulse when

$$F_c T = J, \quad F_c = \frac{J}{T} = \frac{F_0}{2} = 300 \text{ N}.$$

The graph is a right triangle with height 600 N and width 0.050 s, so an independent area check gives the same 15.0 N s. The average is also the mean of the endpoint force values because the force is linear in time.

Equal impulse means equal final momentum change for the same initial state and mass. It does not mean that the two forces generate identical velocity or displacement at every intermediate time. A material's response to the peak force can therefore differ even when the transferred momentum is identical.

Problem 12: testing the neglect of external impulse

The comparison must use consistent units: $1 \text{ N s} = 1 \text{ kg m s}^{-1}$. Relative to the stated characteristic momentum $P_* = 8.0 \text{ kg m s}^{-1}$, the external-impulse bound is

$$\frac{|J_{\text{ext}}|_{\text{max}}}{P_*} = \frac{0.080}{8.0} = 0.010, \quad \boxed{\text{maximum fractional effect} = 1.0\%}.$$

The claimed conservation-test accuracy is 0.2%, a factor of five smaller. The bound is therefore too large to justify ignoring external impulse at that target precision. One must either measure and correct the external contribution, establish a tighter bound, or weaken the claimed accuracy.

An upper bound is not a prediction that the actual impulse equals 0.080 N s . It says that the information given permits an effect that large. A test cannot exclude such a permitted effect merely because its central momentum residual happens to be small. The test's uncertainty budget should also include measurement uncertainties and other systematic effects.

The chosen normalization is a characteristic interaction momentum, as specified in the question. Dividing by an exactly zero initial total momentum in a symmetric collision would produce an unusable percentage and would not make the external impulse physically unimportant. The absolute impulse and an appropriate nonzero scale must both be stated.

Problem 13: signed impulse versus integrated force magnitude

The one-dimensional momentum equation is $\Delta p_x = \int F_x(t) dt$, where F_x is the net force. Its sign matters because an impulse in one direction can undo an earlier impulse in the other. Construct the following concrete net-force pulse, with times measured in seconds:

$$F_x(t) = \begin{cases} +100 \text{ N}, & 0 \leq t < 0.010, \\ -100 \text{ N}, & 0.010 \leq t < 0.020, \\ 0, & \text{otherwise.} \end{cases}$$

The two signed rectangular areas are $+1.0$ and -1.0 N s , so

$$\boxed{\Delta p_x = 0, \quad |\Delta p_x| = 0.}$$

In contrast, the integral of magnitude adds the positive areas:

$$\boxed{\int |F_x| dt = 1.0 + 1.0 = 2.0 \text{ N s.}}$$

Thus the magnitude of the integral and the integral of the magnitude are different operations. They satisfy $|\int F_x dt| \leq \int |F_x| dt$; equality holds when the nonzero force keeps one sign, apart from exceptional zero-area cases.

The particle's momentum can change during the first half and return to its original value during the second half. Zero net impulse does not imply zero force throughout the event. Work can be positive during one part of the pulse and negative during another. For this one-dimensional fixed-mass particle, the net work over the full pulse is zero because its initial and final velocities, and therefore kinetic energies, agree.

Problem 14: integrating the sampled force

The trapezoidal rule treats each adjacent pair of samples as the endpoints of a linear segment. Convert the uniform separation to $h = 2 \text{ ms} = 0.002 \text{ s}$ before integrating:

$$\begin{aligned} J_{\text{trap}} &= h \left[\frac{0 + 400}{2} + \frac{400 + 800}{2} + \frac{800 + 400}{2} + \frac{400 + 0}{2} \right] \\ &= 0.002(200 + 600 + 600 + 200) = \boxed{3.20 \text{ N s}}. \end{aligned}$$

The interval contributions are 0.40, 1.20, 1.20, and 0.40 N s, respectively. Summing the two large central contributions and two small endpoint contributions provides an arithmetic check. The average over the full 0.008 s record is $3.20/0.008 = 400$ N.

For the piecewise-linear interpolant shown below, the area is exactly a triangle of height 800 N and duration 0.008 s, giving the same value. For an actual measured force, however, the samples alone do not prove that the force was linear between readings. This is a numerical estimate of the real impulse, with accuracy dependent on temporal resolution, calibration, baseline stability, and the force shape. Adding interpolated points to the same line segments changes neither the data nor their integrated area.

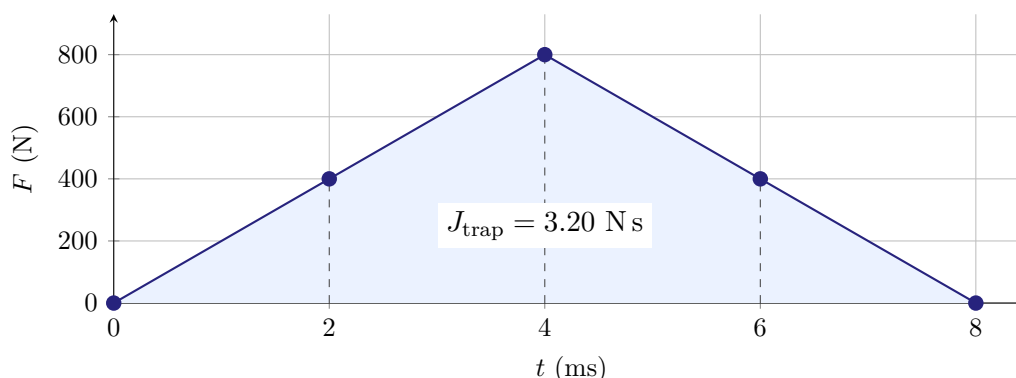


Figure 9.35: The five samples in Problem 14 and their linear interpolant. Shading marks the integrated area; millisecond axis labels are converted to seconds in the numerical calculation. The joining segments express the quadrature model, rather than additional measurements.

Problem 15: two carts sticking together

Use both carts as the system, with $+x$ in the moving cart's initial direction. Neglect the net external impulse during contact. Sticking supplies the additional final-state condition $v_1 = v_2 = v_f$. Momentum conservation then gives

$$(0.60)(3.0) + (0.90)(0) = (0.60 + 0.90)v_f, \quad \boxed{v_f = +1.20 \text{ m s}^{-1}}.$$

Kinetic energy is not conserved in the perfectly inelastic collision. Compute it from the actual before and after speeds:

$$\begin{aligned} K_i &= \frac{1}{2}(0.60)(3.0)^2 = \boxed{2.70 \text{ J}}, \\ K_f &= \frac{1}{2}(1.50)(1.20)^2 = \boxed{1.08 \text{ J}}, \\ K_i - K_f &= \boxed{1.62 \text{ J}}. \end{aligned}$$

The final momentum $1.50(1.20) = 1.80 \text{ kg m s}^{-1}$ equals the initial momentum. The common speed lies between the initial speeds 0 and 3.0 m s^{-1} , as a positive-mass weighted mean should.

An independent energy check uses the reduced mass $\mu = (0.60)(0.90)/1.50 = 0.36 \text{ kg}$: the initial relative kinetic energy is $\frac{1}{2}\mu(3.0 - 0)^2 = 1.62 \text{ J}$. It vanishes on sticking while the CM translational kinetic energy remains. In a complete energy account the decrease becomes deformation and other internal energy, with possible sound emission; it is not a disappearance of total energy.

Problem 16: equal masses in a one-dimensional elastic collision

Let both particles have mass m , and label them by identity throughout the collision. In an isolated elastic collision the final velocities obey

$$v_1 + v_2 = 2.0 - 1.0 = 1.0, \quad v_2 - v_1 = u_1 - u_2 = 3.0 \text{ m s}^{-1}.$$

The second relation expresses reversal of relative velocity for an elastic one-dimensional encounter. Solving the sum and difference gives

$$v_1 = -1.0 \text{ m s}^{-1}, \quad v_2 = +2.0 \text{ m s}^{-1}.$$

The equal masses exchange their laboratory velocities, rather than each simply reversing its own laboratory velocity.

In the CM frame, $V_{\text{CM}} = (2.0 - 1.0)/2 = 0.50 \text{ m s}^{-1}$. Initially, $u'_1 = +1.50$ and $u'_2 = -1.50 \text{ m s}^{-1}$. Elastic energy conservation and zero total CM momentum require these CM velocities to reverse at separation: $v'_1 = -1.50$ and $v'_2 = +1.50 \text{ m s}^{-1}$. Adding 0.50 to each recovers the stated laboratory answers.

Both the initial and final momentum are $m(1.0 \text{ m s}^{-1})$. The sum of squared speeds is $2.0^2 + (-1.0)^2 = 5.0 \text{ m}^2 \text{ s}^{-2}$ before and after, so the kinetic energy is also unchanged. The exchange result assumes a genuine collision followed by separation.

Problem 17: a light cart rebounding from a heavier cart

Take both carts as an isolated interaction system and $+x$ along the incident motion. Momentum conservation and elastic relative-speed reversal give

$$v_1 + 3v_2 = 4, \quad v_2 - v_1 = 4,$$

where velocities are in m s^{-1} and the masses in the first equation are in kilograms. Substituting $v_2 = v_1 + 4$ into the first equation yields $4v_1 + 12 = 4$, so

$$v_1 = -2.0 \text{ m s}^{-1}, \quad v_2 = +2.0 \text{ m s}^{-1}.$$

The negative sign means that the lighter cart rebounds. Check both conserved quantities explicitly:

$$\begin{aligned} P_i &= (1.0)(4.0) = 4.0 \text{ kg m s}^{-1}, \\ P_f &= (1.0)(-2.0) + (3.0)(2.0) = 4.0 \text{ kg m s}^{-1}, \\ K_i &= \frac{1}{2}(1.0)(4.0)^2 = 8.0 \text{ J}, \\ K_f &= \frac{1}{2}(1.0)(-2.0)^2 + \frac{1}{2}(3.0)(2.0)^2 = 2.0 + 6.0 = 8.0 \text{ J}. \end{aligned}$$

The center-of-mass speed is $4.0/4.0 = 1.0 \text{ m s}^{-1}$. The initial CM velocities are $+3.0$ and -1.0 , while the final ones are -3.0 and $+1.0 \text{ m s}^{-1}$, providing a third check. Treating the heavy cart as an immovable wall would be inappropriate here: its finite mass and recoil are essential to both conservation balances.

Problem 18: restitution smaller than unity

Take $+x$ along the 2.0 kg cart's initial motion and include both carts in the system. Conservation of momentum supplies

$$2v_1 + 3v_2 = 2(5.0) = 10.$$

The incident closing speed is $u_1 - u_2 = 5.0 \text{ m s}^{-1}$. The positive coefficient of restitution relates it to the final separation speed:

$$e_r = \frac{v_2 - v_1}{u_1 - u_2} = 0.50, \quad v_2 - v_1 = 2.50 \text{ m s}^{-1}.$$

Substitution of $v_2 = v_1 + 2.50$ gives $5v_1 + 7.50 = 10$, hence

$$v_1 = +0.50 \text{ m s}^{-1}, \quad v_2 = +3.00 \text{ m s}^{-1}.$$

Both carts move right, but the second moves faster and therefore separates from the first. Their final momentum $2(0.50) + 3(3.00) = 10 \text{ kg m s}^{-1}$ agrees with the initial value.

An energy audit, although not an additional conservation equation, is useful. Initially $K_i = 25.0 \text{ J}$; finally $K_f = 0.25 + 13.50 = 13.75 \text{ J}$. The decrease is 11.25 J . With $\mu = 6/5 = 1.20 \text{ kg}$, the formula $\frac{1}{2}\mu(1 - e_r^2)(u_1 - u_2)^2$ gives the same decrease. The final relative kinetic energy is $e_r^2 = 0.25$ times its initial value; the conserved CM kinetic energy is 10.0 J .

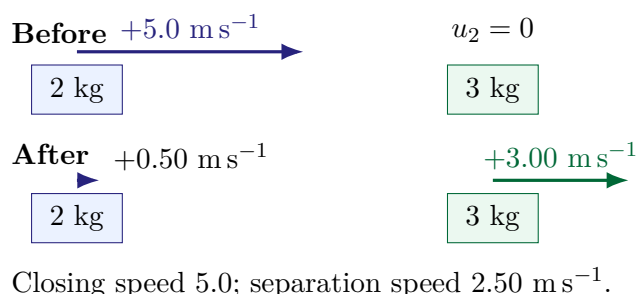


Figure 9.36: Velocity audit for Problem 18. Arrow lengths use a common scale of $0.60 \text{ cm per m s}^{-1}$; cart locations and sizes are schematic. The ratio of separation to closing speed is 0.50 , even though both final velocities have positive signs.

Problem 19: a projectile embedding in a block

Use the projectile and block together as the system, and take the initial projectile direction as positive. Their collision is perfectly inelastic because the projectile remains embedded. Over the short interaction, neglect the net external impulse. The common final speed is obtained from momentum, not kinetic-energy conservation:

$$(0.20)(10) + (0.80)(0) = (1.00)v_f, \quad \boxed{v_f = +2.0 \text{ m s}^{-1}.}$$

The initial and final translational kinetic energies are

$$K_i = \frac{1}{2}(0.20)(10)^2 = 10 \text{ J}, \quad K_f = \frac{1}{2}(1.00)(2.0)^2 = 2.0 \text{ J}.$$

The translational decrease is therefore

$$\boxed{K_i - K_f = 8.0 \text{ J}.}$$

The conserved total momentum is 2.0 kg m s^{-1} . The reduced mass is $\mu = 0.16 \text{ kg}$; the check $\frac{1}{2}\mu(10)^2 = 8.0 \text{ J}$ again matches the lost relative-motion energy.

The missing translational kinetic energy becomes internal energy, plastic deformation, vibrations, and possibly sound or other energy transferred to the surroundings. A complete energy account includes these terms; it does not assert that energy is destroyed. The relative importance of the channels cannot be inferred from masses and velocities alone. If appreciable external work occurs during the event, it must also be included in that energy account and may require relaxing the isolated-collision idealization.

Problem 20: what perfectly inelastic means

Problem 15 is a direct counterexample: the carts stick and then move together at $+1.20 \text{ m s}^{-1}$. Their initial nonzero total momentum cannot become zero without an external impulse. The quoted claim therefore fails in the laboratory frame even for a standard isolated sticking collision.

For the particle or translational-body model used here, a correct definition is: **in a perfectly inelastic collision, the colliding objects remain together and have the same final velocity.**

Their relative velocity vanishes. In one-dimensional restitution language, the final separation speed is zero and $e_r = 0$.

This definition is independent of a uniform change of inertial frame. If a new frame moves at velocity $\mathbf{w}$, then

$$\mathbf{v}'_1 - \mathbf{v}'_2 = (\mathbf{v}_1 - \mathbf{w}) - (\mathbf{v}_2 - \mathbf{w}) = \mathbf{v}_1 - \mathbf{v}_2 = \mathbf{0}.$$

Their common velocity need not itself be zero. It is zero specifically in their CM frame, or in a laboratory where the initial total momentum was zero and external impulse was negligible. For extended objects that stick and rotate, different material points can have rotational velocities; the chapter's common-velocity statement refers to the stipulated translational collision model. Perfectly inelastic is a statement about relative motion and sticking, not the destruction of all laboratory kinetic energy.

Problem 21: the third fragment's velocity

Choose all three fragments as the system and keep the stated laboratory axes. The system is initially at rest, so its initial momentum is zero. With negligible external impulse,

$$\mathbf{p}_A + \mathbf{p}_B + \mathbf{p}_C = \mathbf{0}.$$

The known fragment momenta are

$$\mathbf{p}_A = (1.0)(4, 0) = (4, 0), \quad \mathbf{p}_B = (2.0)(-1, 2) = (-2, 4)$$

in kg m s^{-1} . Their sum is $(2, 4)$, so C must carry the opposite momentum:

$$\mathbf{p}_C = (-2, -4) \text{ kg m s}^{-1}, \quad \boxed{\mathbf{v}_C = \left(-\frac{2}{3}, -\frac{4}{3}\right) \text{ m s}^{-1}.$$

Numerically its components are approximately $(-0.667, -1.333) \text{ m s}^{-1}$. Adding the three x components gives $4 - 2 - 2 = 0$, and adding the three y components gives $0 + 4 - 4 = 0$. Both component checks are necessary.

The third fragment moves down and left. Its mass determines its velocity for the required momentum: a larger C mass would reduce its speed but would not change the required momentum vector. Kinetic energy need not equal the initially zero translational kinetic energy; fragmentation can convert stored internal energy into motion. The given momentum balance alone does not determine the energy source or the fragment trajectories after breakup.

Problem 22: the planar equal-mass elastic collision

Let the equal masses be m . The incident velocity is $\mathbf{u} = (5.0, 0) \text{ m s}^{-1}$ and the target is initially at rest. Dividing momentum conservation by m gives

$$\mathbf{u} = \mathbf{v}_1 + \mathbf{v}_2.$$

Elasticity supplies $u^2 = v_1^2 + v_2^2$. Therefore

$$\boxed{v_2 = \sqrt{5.0^2 - 3.0^2} = 4.0 \text{ m s}^{-1}.$$

Squaring the vector momentum equation gives $u^2 = v_1^2 + v_2^2 + 2\mathbf{v}_1 \cdot \mathbf{v}_2$. Comparison with the energy equation shows that

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0.$$

Both speeds are nonzero, so the two outgoing velocity vectors are perpendicular.

For one consistent orientation, choose particle 1 above the x axis. Dotting the momentum equation with $\mathbf{v}_1$ gives $uv_1 \cos \theta_1 = v_1^2$, hence $\cos \theta_1 = 3/5$. Similarly $\cos \theta_2 = 4/5$, with particle 2 below the axis. Thus

$$\theta_1 = +53.13^\circ, \quad \theta_2 = -36.87^\circ.$$

The resulting vectors are $\mathbf{v}_1 = (1.8, 2.4)$ and $\mathbf{v}_2 = (3.2, -2.4) \text{ m s}^{-1}$. Their sum is $(5.0, 0)$, their squared lengths are 9 and 16, and their dot product is $5.76 - 5.76 = 0$. Reflection across the x axis gives a second equally valid pair of signs. These angles are kinematic possibilities; the impact geometry determines which scattering branch occurs.

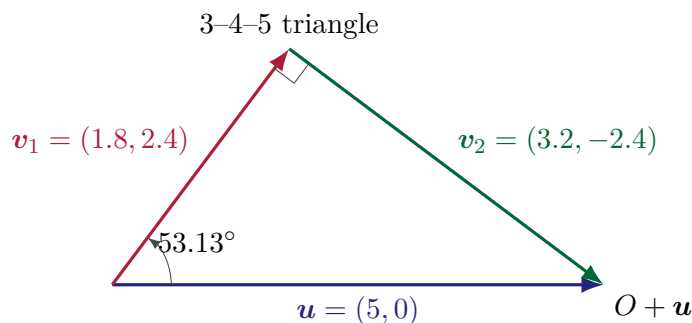


Figure 9.37: Head-to-tail velocity geometry for Problem 22, drawn to a common scale. The side lengths are 3, 4, and 5 m s^{-1} . The right-angle marker at the join expresses perpendicular outgoing velocities; it does not describe a physical corner in either particle's trajectory.

Problem 23: transformation to the center-of-mass frame

Use $+x$ along the 2.0 kg particle's initial velocity. The total mass and momentum are

$$M = 2.0 + 1.0 = 3.0 \text{ kg}, \quad P = (2.0)(3.0) + (1.0)(-1.0) = 5.0 \text{ kg m s}^{-1}.$$

The CM frame moves at

$$V_{\text{CM}} = \frac{5}{3} \text{ m s}^{-1}.$$

Subtract that *same* frame velocity from each particle's laboratory velocity:

$$v'_1 = 3 - \frac{5}{3} = \frac{4}{3} \text{ m s}^{-1}, \quad v'_2 = -1 - \frac{5}{3} = -\frac{8}{3} \text{ m s}^{-1}.$$

The CM momenta are $p'_1 = 2(4/3) = 8/3$ and $p'_2 = 1(-8/3) = -8/3 \text{ kg m s}^{-1}$, so their sum is exactly zero. This is the defining check for a correctly computed CM frame.

The equal-and-opposite momenta do not imply equal-and-opposite velocities because the masses differ. The lighter particle has twice the CM speed in magnitude. The relative velocity is unchanged by the transformation: $v'_1 - v'_2 = 4/3 + 8/3 = 4.0 \text{ m s}^{-1} = 3.0 - (-1.0)$. Adding V_{CM} back to each CM velocity returns the given laboratory velocities, supplying a reversible arithmetic check. No assumption of a collision, or of energy conservation during one, is needed to define this frame.

Problem 24: insufficient information in a planar collision

For two particles moving in a plane, the unknown final state contains four velocity components: $v_{1x}, v_{1y}, v_{2x}, v_{2y}$. If external impulse is negligible, momentum conservation gives only two scalar equalities,

$$m_1 v_{1x} + m_2 v_{2x} = P_{ix}, \quad m_1 v_{1y} + m_2 v_{2y} = P_{iy}.$$

For any selected vector $\mathbf{v}_1$, momentum alone supplies

$$\mathbf{v}_2 = \frac{\mathbf{P}_i - m_1 \mathbf{v}_1}{m_2},$$

which displays the remaining family of possible answers explicitly.

The word “inelastic” does not specify a final kinetic energy, a restitution coefficient, or an impact direction. In an ordinary passive collision it restricts the allowed kinetic-energy decrease, but that inequality does not choose a unique member of the family. Even supplying a numerical energy decrease generally leaves an angular freedom that requires additional information.

Suitable extra information could be a measured outgoing velocity vector, sufficient outgoing speed and angle data, or a validated contact model with its impact normal and restitution law. If the objects are explicitly said to *stick*, the common final velocity is $\mathbf{P}_i/(m_1 + m_2)$; that is a stronger condition than merely being inelastic. Assigning it without the sticking condition would invent information that the problem does not provide.

Problem 25: gravitational impulse during fragmentation

Use the entire 2.0 kg fragmenting projectile as the system. Internal fragmentation forces cancel in its total momentum balance, but gravity is external. With $+y$ upward and breakup duration $\Delta t = 4.0 \text{ ms} = 0.0040 \text{ s}$,

$$\mathbf{J}_g = -Mg\Delta t \hat{\mathbf{y}}, \quad J_g = (2.0)(9.81)(0.0040) = 0.07848 \text{ N s}.$$

Therefore

$$J_g \simeq 7.85 \times 10^{-2} \text{ N s}, \quad \frac{J_g}{6.0} = 0.01308 = 1.31\%.$$

The force is small only in the sense of its integrated effect relative to the stated fragment momentum scale. Its contribution points downward regardless of the internal fragment directions.

If “at the percent level” means a strict 1% error limit, the 1.31% estimate is already too large for unqualified neglect. For an approximate description accurate to a few percent, neglecting it can be reasonable. The calculation therefore supports a conditional approximation, not a universal yes. A sub-percent audit should include the gravitational impulse explicitly.

After the breakup, gravity continues to act on all pieces. Momentum conservation with gravity omitted over a short breakup interval does not justify treating the subsequent trajectories as force-free. The CM follows the appropriate gravitational projectile motion throughout, provided no other external force becomes important.

Problem 26: vector diagrams versus trajectories

A momentum vector records a magnitude and direction for a chosen particle at a chosen instant. To add vectors, they may be translated parallel to themselves so that one tail is placed at the preceding tip. Such a translation changes neither component and therefore changes neither the momentum nor its sum with other vectors.

For example, a closed triangle can express $\mathbf{p}_A + \mathbf{p}_B + \mathbf{p}_C = \mathbf{0}$. Its sides are scaled representations of three momenta, not spatial paths traversed in sequence. Different fragments can leave the same breakup location simultaneously; their separate trajectories are obtained by integrating their velocities and accelerations from their actual initial positions.

Momentum and displacement even have different units: the former is kg m s^{-1} , the latter meters. A drawing’s scale bar or labels should indicate which vector space is represented. For unequal masses, equal-length momentum arrows would correspond to unequal speed magnitudes, which further prevents interpreting their lengths as distances travelled over an equal time.

The head-to-tail geometry in the solution to Problem 22 is thus a kinematic addition construction. It is useful for proving the perpendicularity of the outgoing velocity vectors, but it does not imply that either particle turns a right-angle corner at the triangle’s join. Physical trajectories require positions and elapsed time in addition to momenta.

Problem 27: an unknown sensor baseline

Let the recorded force be $F_{\text{rec}}(t) = F_{\text{true}}(t) + b$, with the stipulated constant $b = +20 \text{ N}$. Integration over the $8 \text{ ms} = 0.008 \text{ s}$ window gives

$$J_{\text{rec}} = J_{\text{true}} + bT, \quad J_{\text{offset}} = (20)(0.008) = +0.160 \text{ N s}.$$

The positive sign is important: this baseline adds a positive apparent impulse even where the physical force is zero. The trapezoidal rule integrates a constant offset exactly, so using more intervals cannot remove the error. If the 3.20 N s value in Problem 14 represents the zero-baseline signal, adding this offset would make the recorded result 3.36 N s , a 5% upward bias.

A pre-trigger interval in which there is independently known to be no applied force allows an estimate of the baseline. Subtracting its mean from all recorded force samples removes a stable offset before integration. The baseline estimate has its own uncertainty, which contributes approximately $T\sigma_b$ to the impulse uncertainty for a constant shared offset.

This correction requires the baseline to remain stable during the event and the pre-trigger state to be truly force-free. Drift, sensor ringing, real preload, or a contact force already present before the nominal trigger cannot be repaired by assuming an arbitrary zero. They need a separately justified calibration or signal model.

Problem 28: the synthetic cart momentum residual

Use the two carts and the common laboratory axis in Table 9.1. The central values give

$$\begin{aligned} P_i &= (0.500)(0.600) + (0.750)(-0.100) = 0.22500 \text{ kg m s}^{-1}, \\ P_f &= (0.500)(-0.238) + (0.750)(0.457) = 0.22375 \text{ kg m s}^{-1}, \\ \Delta P &= P_f - P_i = \boxed{-1.25 \times 10^{-3} \text{ kg m s}^{-1}}. \end{aligned}$$

The sign is a small central-value decrease along $+x$, not by itself evidence of a physical momentum loss.

The same measured masses enter both before and after momenta. Propagate directly through $\Delta P = m_1(v_1 - u_1) + m_2(v_2 - u_2)$, treating the two mass errors and four velocity errors as the independent input uncertainties specified in the table. Then

$$\begin{aligned} \sigma_{\Delta P}^2 &= (v_1 - u_1)^2 \sigma_{m_1}^2 + (v_2 - u_2)^2 \sigma_{m_2}^2 \\ &\quad + m_1^2(\sigma_{v_1}^2 + \sigma_{u_1}^2) + m_2^2(\sigma_{v_2}^2 + \sigma_{u_2}^2) \\ &= [(-0.838)^2 + (0.557)^2](0.002)^2 + 2[(0.500)^2 + (0.750)^2](0.006)^2 \\ &= 6.2549972 \times 10^{-5} \text{ kg}^2 \text{ m}^2 \text{ s}^{-2}. \end{aligned}$$

Consequently $\sigma_{\Delta P} = 0.007908854 \text{ kg m s}^{-1}$, which rounds to the corrected stated value 7.91×10^{-3} . The comparison is

$$\boxed{\frac{|\Delta P|}{\sigma_{\Delta P}} = 0.158 \simeq 0.16.}$$

The central residual is much smaller than one standard uncertainty and is consistent with zero at this precision. An experimental momentum-loss claim would also require an external-impulse model and control of correlated calibration errors. Adding independently propagated before/after momentum variances while overlooking their shared masses is not the correct residual propagation for this dataset.

Problem 29: a bound on parasitic force

The force is not given as a measured time history, so use its largest plausible magnitude to bound the external impulse over the collision. With $T = 12 \text{ ms} = 0.012 \text{ s}$,

$$|J_{\text{ext}}| \leq F_{\text{max}} T = (0.15)(0.012) = \boxed{1.8 \times 10^{-3} \text{ N s}}.$$

Relative to the specified characteristic momentum 0.90 kg m s^{-1} ,

$$\boxed{\frac{|J_{\text{ext}}|_{\text{max}}}{P_*} = \frac{0.0018}{0.90} = 0.0020 = 0.20\%}.$$

This is a conservative magnitude bound: if the force is smaller for part of the interval or changes sign, the net impulse can be less. Its direction cannot be determined from a magnitude bound alone.

Neglecting this contribution may be reasonable for a several-percent demonstration, but it cannot establish a conservation test much more accurate than 0.20% unless the bound is improved or the contribution is measured. Measurement uncertainty must be considered alongside the bound. The duration here is the interaction window; if before/after velocities are measured over a much longer interval, the possible parasitic impulse over that longer interval must also be included in the relevant balance. A small contact-time impulse does not automatically characterize the entire measurement procedure.

Problem 30: an exemplar two-dimensional audit algorithm

Assume a fixed list of positive masses, with one before and one after velocity pair for every listed particle. All velocities must be expressed in the same inertial frame and unit system. The requested residual convention is final minus initial, including $\Delta K = K_f - K_i$, which is negative when translational kinetic energy decreases.

Listing 9.2: Pseudocode exemplar for Problem 30. The same masses are used before and after; dot products use both Cartesian components.

```
function audit(m, u, v):
    require length(m) > 0
    require length(u) == length(m) == length(v)
    require every mass finite and positive
    require every u[i], v[i] a finite two-component vector
    M = sum(m)
    Pi = [0, 0]; Pf = [0, 0]
    Ki = 0; Kf = 0
    for i in 0 .. length(m)-1:
        Pi += m[i] * u[i]
        Pf += m[i] * v[i]
        Ki += 0.5 * m[i] * dot(u[i], u[i])
        Kf += 0.5 * m[i] * dot(v[i], v[i])
    Vi = Pi / M; Vf = Pf / M
    dP = Pf - Pi
    dK = Kf - Ki
    return dP[0], dP[1], dK, Vi, Vf
```

Three complementary physical tests are sufficient to expose many mistakes. First, use Problem 17 with masses (1, 3), before velocities ((4, 0), (0, 0)), and after velocities ((-2, 0), (2, 0)). The algorithm must return $\Delta \mathbf{P} = (0, 0)$, $\Delta K = 0$, and both CM velocities $(1, 0) \text{ m s}^{-1}$. This checks signed speeds, mass weights, and squared speeds.

Second, use the sticking collision of Problem 15. It must return $\Delta \mathbf{P} = (0, 0)$, $\Delta K = -1.62 \text{ J}$, and the unchanged CM velocity $(1.20, 0) \text{ m s}^{-1}$. An algorithm that wrongly demands $\Delta K = 0$ for every isolated collision would fail this valid physical case.

Third, use the planar vectors from Problem 22 with equal unit masses. Before velocities $((5, 0), (0, 0))$ and after velocities $((1.8, 2.4), (3.2, -2.4))$ must give zero momentum residual in *both* components and zero energy change. Both CM velocities must be $(2.5, 0) \text{ m s}^{-1}$.

Further checks include permuting particle labels, which must leave every collective output unchanged, and adding the same frame shift to all velocities. For a new frame with velocity $\mathbf{w}$, the CM velocities change by $-\mathbf{w}$, $\Delta \mathbf{P}$ is invariant for fixed mass, and $\Delta K' = \Delta K - \mathbf{w} \cdot \Delta \mathbf{P}$. Energy residual invariance should therefore be required only when $\Delta \mathbf{P} = \mathbf{0}$. Independently verify the identity $\Delta \mathbf{P} = M(\mathbf{V}_f - \mathbf{V}_i)$ for every input, even when the residual is deliberately nonzero. A satisfactory computational answer defines its outputs, states assumptions, and demonstrates physical tests; syntax alone is not a validation.

Problem 31: a pulse shorter than the sampling interval

The force samples are spaced by 5 ms, but contact lasts only 3 ms. Depending on the sampling phase, the entire pulse can fall between two samples. In that case all recorded values could be near zero even though the true force has a finite area. Alternatively, a single sample could land near the peak, with no neighboring samples resolving its width or shape.

Trapezoidal integration, higher-order quadrature, and finer interpolation of those *existing* samples cannot determine what was never measured. Many narrow force histories with different areas can agree at the same sparse sampling times. A numerically stable answer from one chosen interpolant is consequently not evidence that the true impulse has been recovered.

Reliable recovery requires additional information: faster sampling with suitable sensor bandwidth and antialiasing, or an independently validated force-pulse model with enough parameters determined by other measurements. The camera's contact duration identifies the under-resolution but does not by itself specify the force amplitude or its time dependence. Separately measured before/after momentum could provide an independent impulse estimate, subject to external-force accounting.

The appropriate response is therefore to improve the acquisition or state a model-dependent result and its uncertainty. Refining the integration formula alone cannot fix this physical information deficit.

Problem 32: inconsistent momentum and CM velocity in code

For the same particle set with constant total mass, the identities

$$\mathbf{V}_{\text{CM},i} = \frac{\mathbf{P}_i}{M}, \quad \mathbf{V}_{\text{CM},f} = \frac{\mathbf{P}_f}{M}$$

make unchanged momentum and changed CM velocity mathematically inconsistent, apart from stated rounding or numerical tolerance. It is not a new physical effect.

One possible coding error is computing the CM velocity as the unweighted arithmetic mean of velocities. That coincides with the proper result only for equal masses; unequal-mass momentum exchange can change the unweighted mean while conserving momentum. Another is using different particle selections or total masses in the two calculations, such as dropping a particle after a merge or dividing a two-particle momentum by one cart's mass.

Reporting errors can produce the same symptom: momentum in the laboratory frame but CM velocity in a moving frame, meters per second in one output and centimeters per second in another, a stale before-state array reused in only one sum, or reporting scalar speed where a signed component is intended. With fixed M and exactly identical momentum vectors, even the CM speed must be identical.

Audit the invariant residual $\mathbf{P} - M\mathbf{V}_{\text{CM}}$ at each recorded state, and print the mass and frame convention used by each calculation. This identifies the inconsistency directly without relying on a collision-specific model.

Problem 33: a small residual is not a conservation-law violation

The observed central residual is $\Delta P = +0.003 \text{ kg m s}^{-1}$, with standard uncertainty $\sigma = 0.002 \text{ kg m s}^{-1}$. The standardized residual is

$$\frac{\Delta P}{\sigma} = 1.5.$$

This is a modest fluctuation relative to the stated uncertainty, not strong evidence of a nonzero underlying discrepancy. A precise significance calculation would require the error distribution and correlations; a standard uncertainty alone does not establish them.

More directly, the possible systematic external impulse is 0.004 N s , larger than the central momentum residual 0.003 N s . Ordinary momentum balance predicts $\Delta P = J_{\text{ext}}$, so an external contribution allowed by the experiment could explain the result without any failure of momentum conservation. An isolated-system claim has not been established.

Do not add the systematic bound in quadrature as if it were automatically an independent Gaussian standard uncertainty. It must be characterized according to its actual meaning: a bounded unknown impulse, a measured correction with uncertainty, or a statistical error model. A defensible report states the residual, its measurement uncertainty, and the separately estimated external-impulse range. It can conclude that the present observations do not establish a momentum-violation claim and identify better isolation and calibration as the next experimental requirements.

Problem 34: stopping a package over two durations

Use the package as the system and $+x$ along its initial velocity. Bringing 12 kg from 3.0 m s^{-1} to rest requires

$$\Delta p = 12(0 - 3.0) = -36 \text{ kg m s}^{-1}.$$

The average net force is $F_{\text{avg}} = \Delta p/T$. Thus

$$\begin{aligned} T = 0.060 \text{ s} : \quad F_{\text{avg}} &= -600 \text{ N}, \\ T = 0.240 \text{ s} : \quad F_{\text{avg}} &= -150 \text{ N}. \end{aligned}$$

The requested magnitudes are $\boxed{600 \text{ N and } 150 \text{ N}}$. The longer time is four times the shorter time, so the required average magnitude is one-quarter as large. In both cases the same 36 N s stopping impulse must be delivered.

This comparison alone cannot approve a packaging design. Average force does not determine peak force, contact stress, acceleration spikes, load distribution, or the response of fragile internal components. A practical model also needs available cushioning stroke, force–deformation behavior, orientation, rebound, fatigue, environmental conditions, and a specified failure criterion. The initial translational kinetic energy is $\frac{1}{2}(12)(3.0)^2 = 54 \text{ J}$, which must be transferred or dissipated consistently with the actual stopping displacement. A longer allowed time is helpful but does not certify that a particular material or geometry can safely provide it.

Problem 35: separating CM and relative kinetic energy

For the given masses and velocities, the laboratory kinetic energy is

$$K = \frac{1}{2}(2.0)(3.0)^2 + \frac{1}{2}(1.0)(-1.0)^2 = \boxed{9.50 \text{ J}}.$$

As in Problem 23, $M = 3.0 \text{ kg}$ and $V_{\text{CM}} = 5/3 \text{ m s}^{-1}$. The CM translational part is

$$K_{\text{CM}} = \frac{1}{2} M V_{\text{CM}}^2 = \frac{1}{2} (3) \frac{25}{9} = \boxed{\frac{25}{6} \text{ J} \simeq 4.17 \text{ J}}.$$

Subtracting gives

$$K_{\text{rel}} = K - K_{\text{CM}} = \frac{19}{2} - \frac{25}{6} = \frac{16}{3} \text{ J} \simeq 5.33 \text{ J}.$$

The reduced mass is $\mu = m_1 m_2 / M = 2/3 \text{ kg}$ and the relative speed is $3.0 - (-1.0) = 4.0 \text{ m s}^{-1}$. Hence

$$\frac{1}{2} \mu (v_1 - v_2)^2 = \frac{1}{2} \left(\frac{2}{3} \right) (4)^2 = \frac{16}{3} \text{ J},$$

which independently verifies the result. Directly summing the CM-frame energies of speeds $4/3$ and $-8/3$ gives the same value. The nonnegative relative part describes motion within the system. The CM part depends on the inertial frame, while the relative part does not change under a uniform frame-velocity shift.

Problem 36: deriving the kinetic-energy decomposition

Take a finite system of fixed positive masses in a nonrelativistic inertial frame. Its center-of-mass velocity is $\mathbf{V}_{\text{CM}} = M^{-1} \sum_i m_i \mathbf{v}_i$. Define $\mathbf{v}'_i = \mathbf{v}_i - \mathbf{V}_{\text{CM}}$ and start with the laboratory kinetic energy:

$$\begin{aligned} K &= \sum_i \frac{1}{2} m_i |\mathbf{V}_{\text{CM}} + \mathbf{v}'_i|^2 \\ &= \frac{1}{2} \left(\sum_i m_i \right) |\mathbf{V}_{\text{CM}}|^2 + \mathbf{V}_{\text{CM}} \cdot \sum_i m_i \mathbf{v}'_i + \sum_i \frac{1}{2} m_i |\mathbf{v}'_i|^2. \end{aligned}$$

The cross term vanishes because the *mass-weighted* CM-frame velocity sum is zero:

$$\sum_i m_i \mathbf{v}'_i = \sum_i m_i \mathbf{v}_i - \left(\sum_i m_i \right) \mathbf{V}_{\text{CM}} = \mathbf{P} - M \frac{\mathbf{P}}{M} = \mathbf{0}.$$

Therefore

$$K = \frac{1}{2} M |\mathbf{V}_{\text{CM}}|^2 + K_{\text{rel}}, \quad K_{\text{rel}} = \sum_i \frac{1}{2} m_i |\mathbf{v}'_i|^2.$$

No cancellation of the individual CM velocities is assumed without their mass weights, and no collision or elasticity assumption is involved. The equality is an instantaneous kinematic identity. In an isolated collision the first term is constant because total momentum and mass are constant; changes in translational kinetic energy then appear entirely in K_{rel} .

Both terms have energy units and are nonnegative. If every particle has the same velocity, all $\mathbf{v}'_i$ vanish and the whole energy is collective translation. In the CM frame the collective term vanishes and only the relative-motion energy remains. These limiting cases provide independent checks on the decomposition.

Problem 37: perfectly inelastic loss from relative motion

Consider two particles with masses m_1, m_2 , initial velocities $\mathbf{u}_1, \mathbf{u}_2$, and negligible external impulse during the sticking event. Their total mass $M = m_1 + m_2$ and CM velocity remain fixed. The decomposition derived in Problem 36 therefore gives

$$K_i - K_f = K_{\text{rel},i} - K_{\text{rel},f}.$$

After sticking, both velocities equal $\mathbf{V}_{\text{CM}}$, so both CM-frame velocities are zero and $K_{\text{rel},f} = 0$.

To express the initial relative energy in terms of the initial velocity difference, define $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$. Subtracting $\mathbf{V}_{\text{CM}} = (m_1\mathbf{u}_1 + m_2\mathbf{u}_2)/M$ from each initial velocity yields

$$\mathbf{u}'_1 = \frac{m_2}{M}\mathbf{w}, \quad \mathbf{u}'_2 = -\frac{m_1}{M}\mathbf{w}.$$

Then

$$\begin{aligned} K_{\text{rel},i} &= \frac{1}{2} \left[m_1 \frac{m_2^2}{M^2} + m_2 \frac{m_1^2}{M^2} \right] |\mathbf{w}|^2 \\ &= \frac{1}{2} \frac{m_1 m_2 (m_1 + m_2)}{M^2} |\mathbf{w}|^2 = \frac{1}{2} \mu |\mathbf{u}_1 - \mathbf{u}_2|^2, \end{aligned}$$

where $\mu = m_1 m_2 / (m_1 + m_2)$. Hence

$$K_i - K_f = \frac{1}{2} \mu |\mathbf{u}_1 - \mathbf{u}_2|^2.$$

This derivation never expands the two laboratory energies directly. It shows why the loss is independent of a uniform frame shift and why common initial motion gives zero loss. It is the maximum decrease of translational kinetic energy compatible with the same isolated two-body momentum and a common final velocity, because the final relative-motion energy reaches its lower bound zero. A complete energy account retains the corresponding internal-energy increase.

Problem 38: an ideal deep-space rocket burn

The rocket alone is an open subsystem, because exhaust crosses its boundary. In the one-dimensional ideal model, backward exhaust has constant relative speed $u_e > 0$ and the rocket mass decreases. With no external force,

$$m \, dv = -u_e \, dm, \quad \Delta v = u_e \ln \left(\frac{m_i}{m_f} \right).$$

Insert the given masses and exhaust speed:

$$\begin{aligned} \Delta v &= (2500) \ln \left(\frac{1000}{700} \right) \\ &= (2500)(0.356674944) = 891.687 \, \text{m s}^{-1}. \end{aligned}$$

Thus

$$\Delta v \simeq 8.92 \times 10^2 \, \text{m s}^{-1}.$$

The logarithm's argument is a dimensionless ratio of masses, and multiplying by u_e supplies the required velocity units. Because $m_i > m_f$, the logarithm is positive: expelling mass backward increases the forward velocity.

This is a velocity increment, not an absolute final velocity unless the initial velocity is specified. The burned mass is 300 kg, but no outflow rate is supplied, so burn duration and instantaneous thrust cannot be calculated. The model neglects external forces and changes in exhaust performance. Using $F = d(mv)/dt$ for the rocket alone without momentum flux would give an incorrect balance; the carried-away exhaust momentum is precisely what produces the logarithmic relation.

Problem 39: solving for the required mass ratio

Under the same force-free one-dimensional assumptions, the ideal rocket equation is

$$\Delta v = u_e \ln r, \quad r = \frac{m_i}{m_f}.$$

Divide by the positive exhaust speed and exponentiate:

$$\ln r = \frac{1800}{3000} = 0.600, \quad \boxed{r = e^{0.600} = 1.82212 \simeq 1.82.}$$

Both velocities must use the same units, so their quotient is dimensionless. The result exceeds one, as it must for a rocket that expels propellant while gaining forward velocity. Substitution gives $3000 \ln(1.82212) = 1800 \text{ m s}^{-1}$, verifying the inversion.

The ratio determines a propellant fraction but not an absolute mass scale:

$$\frac{m_i - m_f}{m_i} = 1 - \frac{1}{r} = 1 - e^{-0.600} = 0.45119.$$

About 45.1% of the initial mass must be expelled in this ideal model; the retained mass is about 54.9%. The retained mass must include whatever vehicle structure, payload, and unexpelled material the model assigns to m_f .

The ideal ratio does not include gravity or drag losses, reserve propellant, structural changes, or a varying exhaust speed. An actual mission requirement could consequently demand a larger mass ratio or different propulsion architecture. Those engineering adjustments cannot be inferred from the stated ideal numbers alone.

Problem 40: instantaneous acceleration with drag

Take the rocket's current forward direction as $+x$. The stated 4.0 kg s^{-1} is a *positive outflow rate*, so the rocket's own mass rate is $\dot{m} = -4.0 \text{ kg s}^{-1}$. Backward exhaust with relative speed $u_e = 2000 \text{ m s}^{-1}$ produces positive ideal thrust

$$T = u_e \dot{m}_e = (2000)(4.0) = 8000 \text{ N}.$$

The external drag is $F_{\text{ext}} = -1000 \text{ N}$. Including the momentum flux in the open-system balance gives

$$ma = F_{\text{ext}} + T = -1000 + 8000 = 7000 \text{ N}.$$

At the specified instantaneous mass $m = 500 \text{ kg}$,

$$\boxed{a = \frac{7000}{500} = +14.0 \text{ m s}^{-2}.$$

The units check because $u_e \dot{m}_e$ has units kg m s^{-2} , or newtons. The no-drag acceleration would be 16.0 m s^{-2} ; the drag contribution is -2.0 m s^{-2} , giving an independent sum check.

The positive acceleration does not mean drag vanished: thrust is simply larger than drag. The answer is instantaneous. If the burn continues, the changing mass and possibly changing drag must be used to integrate the velocity; multiplying this one acceleration by an arbitrary burn time would not generally be justified. A sign mistake that makes a positive outflow produce negative thrust reverses the actual momentum-flux physics.

Problem 41: a vertical burn with constant gravity

Take upward as positive. The ideal one-dimensional momentum-flux equation with gravity and no drag is

$$m \frac{dv}{dt} = u_e \dot{m}_e - mg, \quad \dot{m}_e = -\frac{dm}{dt} > 0.$$

Divide by m and integrate over the $T = 40$ s burn:

$$\begin{aligned} dv &= -u_e \frac{dm}{m} - g dt, \\ \Delta v &= -u_e \int_{1000}^{800} \frac{dm}{m} - g \int_0^{40} dt \\ &= u_e \ln \left(\frac{m_i}{m_f} \right) - gT. \end{aligned}$$

The propulsive increment is $2500 \ln(1000/800) = 557.858878 \text{ m s}^{-1}$, and the gravity contribution is $-(9.81)(40) = -392.4 \text{ m s}^{-1}$. Therefore

$$\boxed{\Delta v = +165.458878 \text{ m s}^{-1} \simeq +165.5 \text{ m s}^{-1}.}$$

The equation determines the signed change in upward velocity. If the rocket begins ascending and continues to ascend, this is also its speed increase. Without the initial velocity, the absolute final velocity and the change in the scalar speed for an initially descending case are not separately determined.

Only the endpoints of $m(t)$ are needed for the integrated propulsive term when u_e is constant. To draw one illustrative time history, choose constant outflow $\dot{m}_e = (1000 - 800)/40 = 5 \text{ kg s}^{-1}$, so $m(t) = 1000 - 5t$. Then

$$v(t) - v(0) = 2500 \ln \left(\frac{1000}{1000 - 5t} \right) - 9.81t.$$

The corresponding acceleration rises from $2500(5)/1000 - 9.81 = 2.69$ to $2500(5)/800 - 9.81 = 5.815 \text{ m s}^{-2}$; positive acceleration throughout this exemplar is consistent with the positive net increment. A different burn-rate history would change the curve while leaving the stated endpoint increment unchanged under the same constant- g , constant- u_e assumptions.

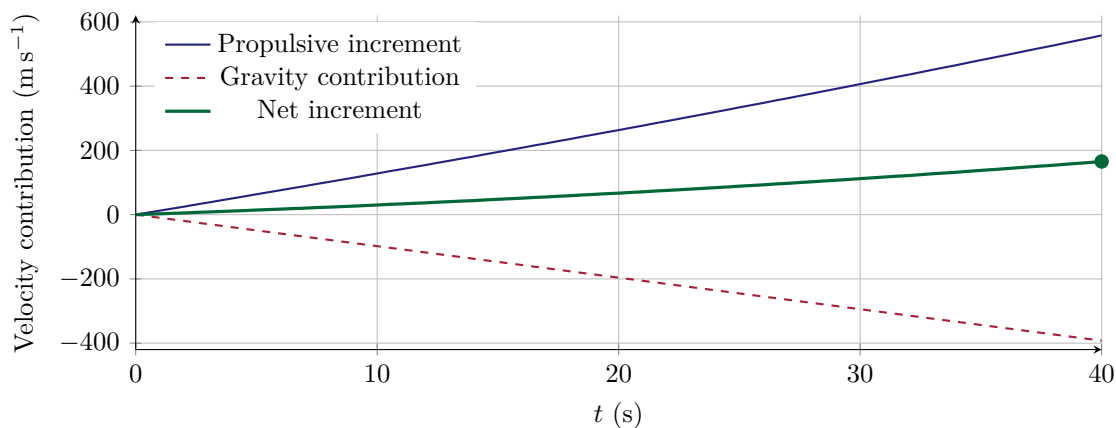


Figure 9.38: An explicit constant-outflow exemplar for Problem 41. The positive propulsive and negative gravitational velocity increments sum to the green curve. At 40 s their sum is 165.46 m s^{-1} . The curve uses the additionally stated 5 kg s^{-1} constant outflow; the endpoint solution itself needs only the given masses and burn duration.

9.12 Bridge to Chapter 10

This chapter reduced a translating many-particle system to collective quantities: center of mass, total momentum, and external impulse. The next chapter, *Rotation of Rigid Bodies about a Fixed Axis*, asks the corresponding question for rotational motion. Instead of describing every particle separately,

we will construct a mass-distribution quantity-the moment of inertia-that relates rotational kinetic energy and fixed-axis dynamics to angular speed and angular acceleration.

The bridge is conceptually important. Translational momentum depends on how mass moves; rotational inertia depends on how mass is distributed relative to an axis. Center-of-mass thinking remains useful because a rigid body's motion can often be separated into translation of the center of mass and rotation about it. Full angular momentum is reserved for the later rotational-dynamics chapter; Chapter 10 first builds the geometry and kinematics of rigid rotation carefully.

Bibliography

- [1] D. Halliday, R. Resnick, and J. Walker, *Fundamentals of Physics, Volume 1*, 12th ed., Wiley, 2022, Chapter 9.
- [2] R. A. Serway and J. W. Jewett, Jr., *Physics for Scientists and Engineers with Modern Physics*, 9th ed., Brooks/Cole, 2014, Chapter 9.
- [3] S. J. Ling, J. Sanny, and W. Moebs, *University Physics, Volume 1*, OpenStax, Rice University, 2018, Chapter 9.
- [4] I. Neutelings, *Center of mass*, TikZ.net, https://tikz.net/dynamics_center_of_mass/.
- [5] I. Neutelings, *Collision & conservation of momentum*, https://tikz.net/dynamics_collision/.
- [6] I. Neutelings, *Impulse*, https://tikz.net/dynamics_impulse/.