

Chapter 10

Fixed-Axis Rotation

First Edition

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Preface

This chapter develops the physics of a rigid body rotating about a *fixed* axis. The phrase fixed axis is not decorative: it is the assumption that makes a single signed angular coordinate, a scalar moment of inertia about one axis, and the equation $\sum \tau_z = I_z \alpha$ sufficient. The chapter therefore builds a disciplined bridge from particle mechanics to extended bodies without prematurely importing rolling motion or the full vector theory of angular momentum.

The opening engineering question is practical: what do encoder readings actually tell us about a rotary stage? A display that repeats after one revolution hides how many turns have occurred; an angular speed alone does not tell us a rim speed until a radius is specified; and a torque does not predict angular acceleration until the mass distribution about the axis is known. These are not separate complications. They are the central structure of fixed-axis rotation.

The chapter uses rigid-body and fixed-axis models whenever they are appropriate, states their limits, and checks central results by more than one route. Worked examples use prescribed or explicitly synthetic parameters rather than invented measurements. A later chapter develops rolling motion and angular momentum for systems whose rotational and translational motions must be treated together.

The Guideline

Choose the rotation axis and the positive sense *before* writing signs. Keep angular position unwrapped when the number of turns matters. Label every moment of inertia with its axis. In dynamics, include all external torques about that axis, and do not promote the scalar fixed-axis relation $\sum \tau_z = I_z \alpha$ into a universal three-dimensional vector law.

Conventions and notation

A right-handed Cartesian system is used. Unless stated otherwise, the fixed rotation axis is the z axis and positive angular motion is counterclockwise when viewed from $+z$ toward the origin. Angular variables are signed scalars in the fixed-axis core:

$$\omega = \frac{d\theta}{dt}, \quad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}.$$

Angles in calculus and in relations such as $s = R\theta$ are in radians. The radian is dimensionless in SI, but writing “rad” is often useful for communicating what a numerical value represents.

Moments of inertia are written I_z , $I_{\text{CM},z}$, or with an equally explicit axis label. Torque components about the fixed axis are τ_z . Torque has unit N m; energy has unit J. They have the same base dimensions but are different physical quantities and should not be labelled interchangeably. The optional advanced tensor is written $\mathbf{I}$.

Pedagogical boundary

The core chapter treats rigid bodies about a fixed inertial axis with time-independent mass distribution. Rolling, moving rotation axes, angular momentum as a full vector, gyroscopic motion, Euler equations, and deliberately time-varying inertia are deferred. The advanced section previews why a scalar inertia is not enough for arbitrary three-dimensional rotation without making that material a prerequisite for the core.

Symbol	Meaning	Unit
θ, ω, α	signed angle, angular velocity, angular acceleration	rad, rad/s, rad/s ²
$r_{\perp}, R$	perpendicular distance to the rotation axis	m
$I_z, \mathbf{I}$	axis inertia; optional inertia tensor	kg m ²
$\tau_z, \boldsymbol{\tau}$	axial torque component; torque vector	N m
K, W, P	kinetic energy, mechanical work, power	J, J, W
λ, σ, ρ	line, surface, volume mass density	kg/m, kg/m ² , kg/m ³
g	specified local teaching acceleration, 9.81 m/s ²	m/s ²

Chapter 10

Fixed-Axis Rotation

A translational description asks where an object is and how its position changes. A fixed-axis rotational description asks for the orientation of the body about a specified axis and how that orientation changes. Every material point of a rigid body shares the same angular displacement, angular velocity and angular acceleration, but points at different radii generally have different linear speeds and accelerations. This single observation explains why rotation needs both angular variables and a new measure of mass distribution.

Learning goals

After studying the core of this chapter, you should be able to:

- use signed and unwrapped angular position, angular velocity and angular acceleration;
- apply constant-angular-acceleration kinematics and match piecewise motion segments;
- relate θ, ω, α to arc length, tangential speed, tangential acceleration and inward normal acceleration;
- derive and use $K_{\text{rot}} = \frac{1}{2}I_z\omega^2$ and calculate I_z for discrete and continuous mass distributions;
- derive and apply the parallel-axis theorem for parallel axes, one through the center of mass;
- compute signed torques about a fixed axis and solve $\sum \tau_z = I_z\alpha$ for rigid bodies with constant I_z ;
- calculate rotational work and power and use the work–kinetic-energy theorem as an independent check;
- evaluate the limits of point-mass, rigid-body, fixed-axis and lossless models in technical applications.

The core progression follows the fixed-axis treatments in Halliday–Resnick–Walker, Serway–Jewett, and OpenStax [1, 2, 3]. The model assumptions are stated explicitly because a relation valid for a fixed inertial axis need not extend to a freely rotating body.

10.1 Angular variables and radians

10.1.1 Orientation needs an axis and a reference

A rotating body does not have a useful scalar “angle” until an axis and a reference direction are chosen. Let a marked line fixed in the body make angle θ with a fixed laboratory direction. For fixed-axis motion, the angular displacement is

$$\Delta\theta = \theta_f - \theta_i. \quad (10.1)$$

The sign of $\Delta\theta$ records the sense of rotation. One complete positive revolution is

$$1 \text{ rev} = 2\pi \text{ rad} = 360^\circ. \quad (10.2)$$

The reason radians are natural is geometric. A point a perpendicular distance R from the axis moves through arc length

$$s = R\theta, \quad (10.3)$$

provided θ is in radians. This relation is independent of the object's angular speed and is the first bridge between angular and linear quantities.

Definition 10.1: Angular position, angular velocity, and angular acceleration

For fixed-axis rotation, angular position $\theta(t)$ specifies orientation relative to a reference. The signed angular velocity and angular acceleration are

$$\omega = \frac{d\theta}{dt}, \quad \alpha = \frac{d\omega}{dt}. \quad (10.4)$$

Their SI units are rad s^{-1} and rad s^{-2} , commonly written even though the radian is dimensionless.

10.1.2 Wrapped and unwrapped angle

An absolute encoder or graphical interface may report an angle only in $[0, 2\pi)$ or $(-\pi, \pi]$. Such a *wrapped* angle cannot by itself reveal the number of completed turns. An *unwrapped* angle carries the accumulated turn count and is the quantity needed for displacement, average angular velocity and trajectory reconstruction over multiple revolutions.

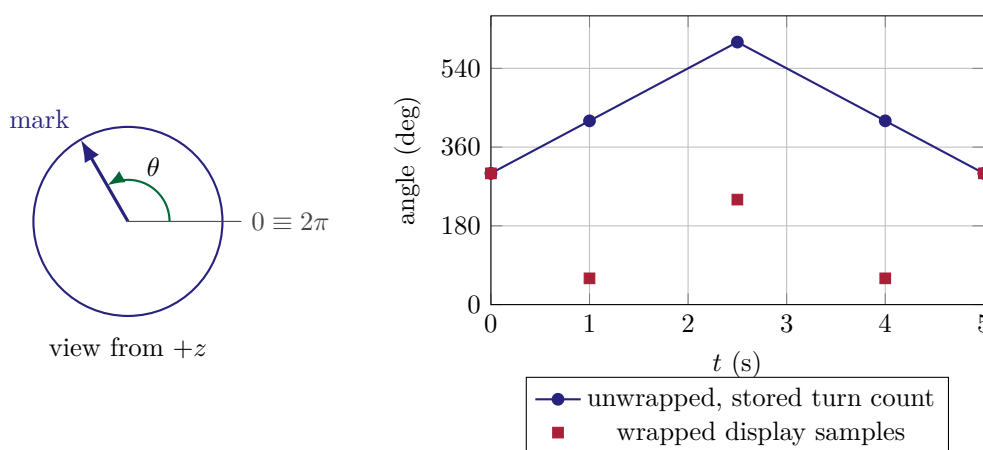


Figure 10.1: Encoder reference and recorded angular history. The blue record uses known turn counts; its straight segments describe the prescribed teaching motion. Red squares are discrete modulo-360° readings, with no connecting lines that could falsely imply reverse motion across a wrap.

Example 10.1: Encoder readings for a rotary stage

A controller stores the unwrapped angular positions $300^\circ, 420^\circ, 600^\circ, 420^\circ, 300^\circ$ at times 0, 1.0, 2.5, 4.0, 5.0 s. Its front panel displays only the values modulo 360° . The stage moves counterclockwise until $t = 2.5$ s and then reverses. Find the average angular velocity over 0–2.5 s and 2.5–5.0 s.

Model. The encoder axis is fixed and the stored turn count is trusted. Positive motion is counterclockwise.

Represent. Figure 10.1 distinguishes wrapped from unwrapped angle.

Relate and solve. The first displacement is $600^\circ - 300^\circ = 300^\circ = 5\pi/3$ rad, so

$$\bar{\omega}_1 = \frac{5\pi/3}{2.5 \text{ s}} = 2.09 \text{ rad s}^{-1}.$$

The second displacement is $300^\circ - 600^\circ = -300^\circ$, hence

$$\bar{\omega}_2 = -2.09 \text{ rad s}^{-1}.$$

Test. Equal-magnitude opposite displacements in equal time intervals must give average velocities of equal magnitude and opposite sign.

Interpret. A wrapped display alone would not reveal whether the stage moved by $+60^\circ$, $+420^\circ$, or another angle differing by an integer number of revolutions. Turn counting is physical information, not a plotting preference.

Guided checks

Check 1. Can a body have $\omega < 0$ and $\alpha > 0$? Yes. It is rotating in the negative sense while its signed angular velocity is increasing toward zero.

Check 2. Do two points on the same rigid rotor have the same ω ? Yes, for fixed-axis rigid rotation. Their tangential speeds differ if their radii differ.

Check 3. Is 5π rad the same orientation as π rad? Geometrically, yes modulo 2π ; as an unwrapped history, no: they differ by two full turns.

10.1.3 Why differentiation requires radians

The formula $s = R\theta$ contains a geometrical choice that is easy to overlook. An angle of one radian cuts an arc whose length equals its radius. Consequently, the numerical ratio s/R is the radian measure of the angle. A degree is equally useful for reporting an orientation, but it is a different numerical scale: if ϕ is the numerical angle in degrees, then $\theta = (\pi/180)\phi$. Differentiating $s = R\phi$ would therefore introduce a factor-of- $180/\pi$ error. The correct conversion must occur before the derivative is interpreted as a speed.

The same issue occurs in the derivative of a trigonometric function. Writing $x = R\cos\theta$ and using $d(\cos\theta)/d\theta = -\sin\theta$ assumes that θ is in radians. If a calculator or plotting system instead evaluates a degree argument ϕ , the derivative with respect to its numerical argument contains the conversion factor $\pi/180$. A unit label does not repair an equation after the wrong numerical scale has been differentiated.

Figure 10.2 connects the radian definition with the direction convention. The two arcs have equal angular displacement but different lengths. The reference line belongs to the laboratory, while the marked line belongs to the rotating body. Changing the initial reference changes every value of θ by the same constant; it does not change ω or α . Reversing the chosen positive sense changes the signs of all three signed angular variables, while leaving the speed, inward acceleration magnitude, and kinetic energy unchanged.

10.1.4 Angular displacement and total angular travel

An average angular velocity is an endpoint quantity:

$$\bar{\omega} = \frac{\theta(t_2) - \theta(t_1)}{t_2 - t_1} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \omega(t) dt. \quad (10.5)$$

In contrast, an average angular speed measures how much angular travel occurred, including travel that was later undone:

$$|\bar{\omega}| = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |\omega(t)| dt. \quad (10.6)$$

These are equal in magnitude only when the direction remains unchanged throughout the interval, apart from possible instants of rest. Positive and negative areas cancel in Eq. (10.5); they add in

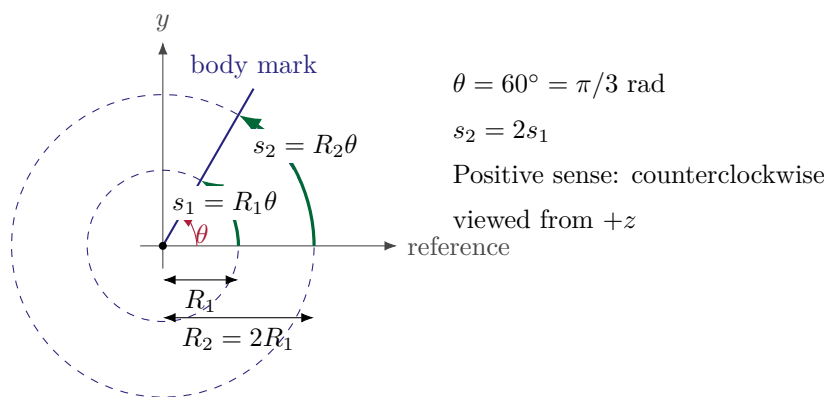


Figure 10.2: Radian geometry for two concentric paths. The angular displacement is shared; the traveled arc length is proportional to radius. The construction uses $\theta = \pi/3$ and $R_2 = 2R_1$, with arbitrary common length units.

Eq. (10.6). For a point at fixed radius R , the total traveled distance is $R \int |\omega| dt$, whereas $R\Delta\theta$ is a signed arc displacement. Neither quantity is generally the straight-line displacement between the point's initial and final Cartesian positions.

Example 10.2: A stage returns to its initial unwrapped angle

A prescribed fixed-axis motion has $\omega(t) = 3.0 - 1.5t$ in rad/s for $0 \leq t \leq 4.0$ s, with t in seconds. Find the reversal time, net angular displacement, total angular travel, and both angular averages. For a point 0.20 m from the axis, find the traveled distance.

Strategy. A direction reversal occurs at a zero of ω at which its sign changes. The angular displacement is the signed area under the graph; total travel requires separate integration on the two sides of the zero.

The stage first stops at $3.0 - 1.5t = 0$, giving $t = 2.0$ s. It rotates positively before this instant and negatively afterward. Integrating the velocity over the whole interval gives

$$\Delta\theta = \left[3.0t - 0.75t^2 \right]_0^4 = 0.$$

Each triangular area in Figure 10.3 has magnitude $\frac{1}{2}(2.0 \text{ s})(3.0 \text{ rad/s}) = 3.0 \text{ rad}$. Thus the total angular travel is $3.0 + 3.0 = 6.0 \text{ rad}$, and

$$\bar{\omega} = 0, \quad \overline{|\omega|} = \frac{6.0}{4.0} = 1.5 \text{ rad/s}.$$

The point travels $0.20(6.0) = 1.2 \text{ m}$ along its circular path.

Check and interpretation. The final angle equals the initial unwrapped angle, so the final orientation and the point's final position both equal their initial values. The traveled distance nevertheless remains positive. A speedometer responds to travel, while an endpoint displacement calculation responds to the net change.

10.1.5 When sampled encoder readings can be unwrapped

A wrapped encoder reports $\theta_{w,k}$ at sampling instants t_k . The actual increment between successive readings can be any value

$$\Delta\theta_k = \theta_{w,k+1} - \theta_{w,k} + 2\pi n_k, \quad n_k \text{ an integer.}$$

An algorithm cannot recover the unknown integer merely because one candidate looks smoother. Independent information must rule out the alternatives. A useful sufficient condition is a trusted

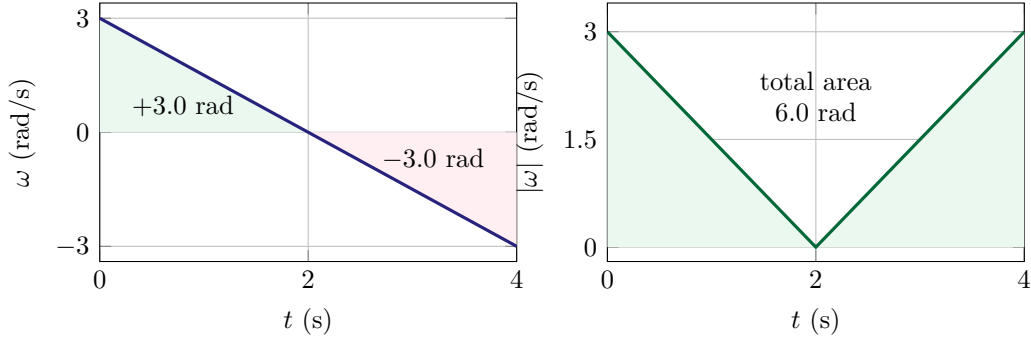


Figure 10.3: Signed and absolute areas for the return motion of Example 10.2. Identical horizontal scales make the reversal visible in both panels. Net displacement is zero; total angular travel is 6.0 rad.

bound $|\omega(t)| \leq \Omega_{\max}$ throughout every sampling interval together with

$$\Omega_{\max} \Delta t < \pi. \quad (10.7)$$

This ensures that the magnitude of the true angular increment is less than half a revolution. Exactly one candidate increment then lies in $(-\pi, \pi)$. The bound constrains motion between readings, rather than only speeds calculated afterward from the readings being decoded. The equality case is excluded because $+\pi$ and $-\pi$ cannot be distinguished.

With measurement errors bounded in magnitude by ε radians at each reading, a conservative margin is $\Omega_{\max} \Delta t + 2\varepsilon < \pi$. This allows for the largest possible error in a difference. A lost sample increases the interval and can destroy the guarantee. Also, unwrapping establishes the net increment; it does not exclude reversals hidden between samples or determine the exact total angular travel. The speed bound limits possible motion, but a sufficiently rapid back-and-forth motion can still have a small net increment.

Example 10.3: Unwrapping requires a sampling bound

An ideal encoder samples every 0.100 s. Its readings at $t = 0, 0.100, 0.200, 0.300, 0.400$ s are 5.800, 0.516815, 1.516815, 2.516815, and 3.516815 rad in $[0, 2\pi)$. An independently established limit is $|\omega| \leq 12.0$ rad/s. Recover the net displacement and average angular velocity. Explain whether the same readings determine the answer without the limit.

Model. The displayed values are synthetic, rounded representations of ideal samples. The bound applies continuously between the samples; it is not inferred from these five readings. Here $\Omega_{\max} \Delta t = 1.20$ rad, below π . The first raw difference is -5.283185 rad. Adding one revolution gives $+1.000000$ rad, the only admissible candidate of magnitude below 1.20 rad. The following three differences are already $+1.000000$ rad. The unwrapped record is consequently 5.800, 6.800, 7.800, 8.800, and 9.800 rad. It gives

$$\Delta\theta = 4.000 \text{ rad}, \quad \bar{\omega} = \frac{4.000}{0.400} = 10.00 \text{ rad/s}.$$

The record is consistent with constant positive motion, but the samples alone do not prove that motion was constant.

Without the speed bound, another constant velocity,

$$\omega' = 10.00 - \frac{2\pi}{0.100} = -52.8319 \text{ rad/s},$$

would produce exactly the same ideal wrapped readings. At each sample its unwrapped angle differs by an integer number of revolutions. Figure 10.4 displays the ambiguity. The trusted limit excludes this alternative because $52.8319 > 12.0$.

Check. For the permitted history, applying reduction modulo 2π to all five reconstructed angles reproduces the displayed values within their rounding. For the alternative, the accumulated difference at sample k is exactly $-2\pi k$. The strict maximum safe sampling interval under this sufficient condition is $\Delta t < \pi/12.0 = 0.262$ s; it is not a guarantee for an instrument whose actual speed bound is unknown.

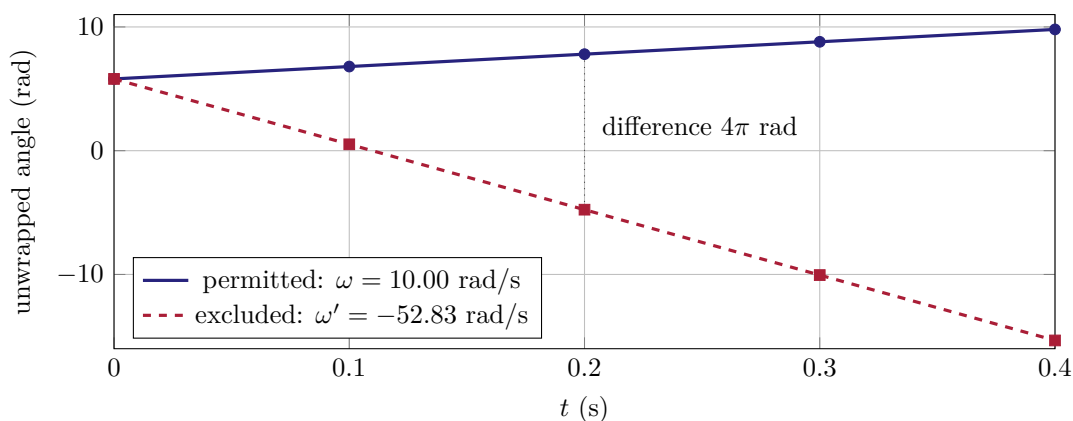


Figure 10.4: Two continuous angular histories with identical wrapped samples. Sample marks are shown on their unwrapped histories. The independent bound $|\omega| \leq 12.0$ rad/s selects the upper history; wrapped readings alone do not select it.

10.2 Rotation with constant angular acceleration

If α is constant over a time interval, integrate $d\omega/dt = \alpha$ from the initial state:

$$\omega = \omega_0 + \alpha t. \quad (10.8)$$

A second integration gives

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2. \quad (10.9)$$

Eliminating time yields

$$\omega^2 - \omega_0^2 = 2\alpha(\theta - \theta_0). \quad (10.10)$$

These are not new physical laws; they are consequences of the definitions of ω and α under the special assumption that α is constant.

Method

For piecewise rotational motion, do not use one constant- α equation across an interval in which the acceleration changes. Solve one segment at a time. The final θ and ω of one segment are the initial values for the next, and the area under an $\omega(t)$ graph must equal the angular displacement.

Example 10.4: Start-up, coast, and braking of a rotary table

A rotary table begins at $\theta_0 = 0$ with $\omega_0 = 2.0$ rad/s. It accelerates at $+3.0$ rad/s² for 2.0 s, coasts for 1.5 s, and then brakes at -4.0 rad/s² until it stops. Find the stopping time measured from the start and the total angular displacement.

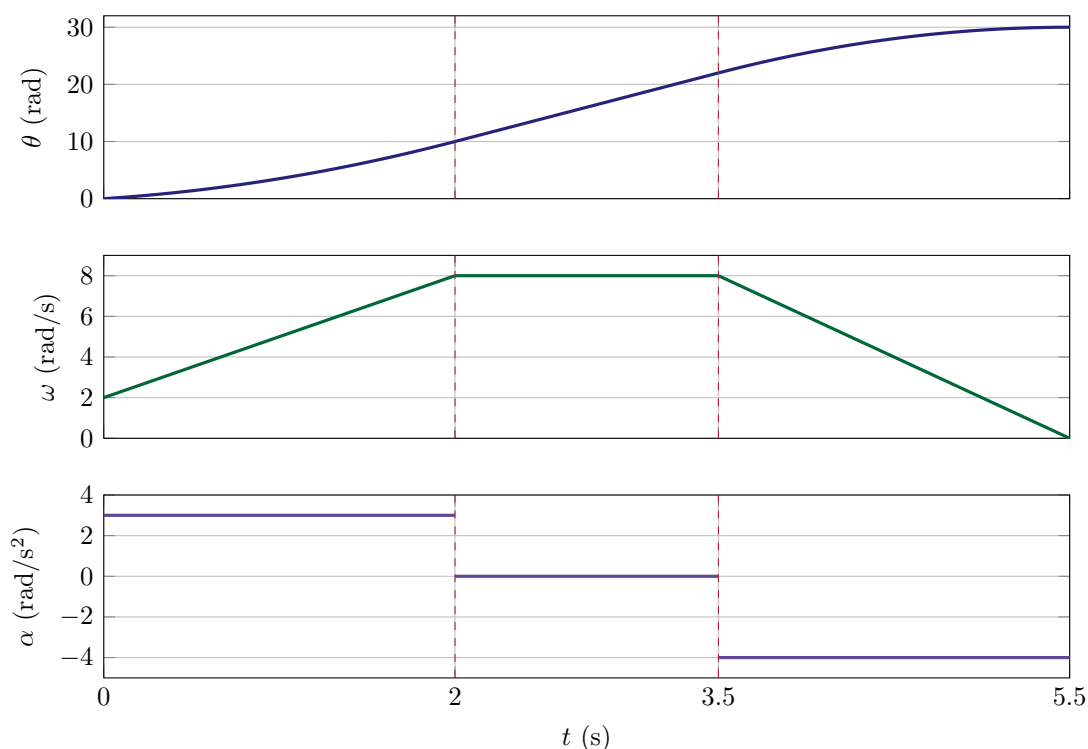


Figure 10.5: Constructed start-coast-brake motion used in Example 10.4. Event lines separate intervals with different angular acceleration. The signed area under $\omega(t)$ gives the change in angular position.

Segment 1. At $t = 2.0$ s,

$$\omega_1 = 2 + 3(2) = 8.0 \text{ rad/s}, \quad \Delta\theta_1 = 2(2) + \frac{1}{2}(3)(2)^2 = 10.0 \text{ rad}.$$

Segment 2. Coasting adds $\Delta\theta_2 = 8(1.5) = 12.0$ rad, giving $\theta = 22.0$ rad at $t = 3.5$ s.

Segment 3. The braking duration follows from $0 = 8 - 4t_b$, so $t_b = 2.0$ s. During braking,

$$\Delta\theta_3 = 8(2) + \frac{1}{2}(-4)(2)^2 = 8.0 \text{ rad}.$$

Thus the table stops at $t = 5.5$ s after turning through

$$\Delta\theta_{\text{total}} = 30.0 \text{ rad} = 4.77 \text{ rev}.$$

Independent check. The area under the $\omega(t)$ plot is a trapezoid of area 10 rad, a rectangle of area 12 rad, and a triangle of area 8 rad, again totaling 30 rad.

Caution

The equation $\omega^2 - \omega_0^2 = 2\alpha\Delta\theta$ contains no explicit sign for the direction of rotation. The signs are carried by α and $\Delta\theta$, while squaring removes the sign of ω . After solving, restore the physically consistent sign from the motion history.

10.2.1 General angular histories: integrate before choosing a formula

Constant- α equations are convenient because integration produces polynomials of low degree. When the acceleration varies, the definitions still apply, but the integrated functions must retain the actual

time dependence:

$$\omega(t) = \omega(t_0) + \int_{t_0}^t \alpha(u) du, \quad (10.11)$$

$$\theta(t) = \theta(t_0) + \int_{t_0}^t \omega(u) du. \quad (10.12)$$

The dummy variable u identifies the time being integrated; the upper limit t identifies the observation time. These integrals also make graph reading precise: slope in the θ panel is ω , slope in the ω panel is α , area under α is a change in velocity, and area under ω is an angular displacement. A position maximum or minimum requires $\omega = 0$ and a change of its sign. A zero of α instead locates a stationary value of ω , with a minimum or maximum only if the acceleration changes sign.

On a smooth branch with $\omega \neq 0$, angle can be used as the independent variable. If angular acceleration is supplied as a function of angle, the chain rule gives $\alpha = \omega d\omega/d\theta$. Integrating this relation gives

$$\frac{1}{2}(\omega_f^2 - \omega_i^2) = \int_{\theta_i}^{\theta_f} \alpha(\theta) d\theta.$$

This is a useful kinematic identity on a motion branch. It does not establish the sign of the final velocity after the square is taken, and it does not license replacing a general time-dependent acceleration by its arithmetic endpoint average. The appropriate integral depends on which variable the acceleration is actually given as a function of.

Example 10.5: A reversal under a changing angular acceleration

A stage starts with $\theta_0 = 0.400$ rad and $\omega_0 = -0.900$ rad/s. Its prescribed acceleration is $\alpha(t) = (1.20 \text{ rad/s}^3)t - 0.600 \text{ rad/s}^2$ for $0 \leq t \leq 3.00$ s. Find the velocity and angle histories, the reversal time, the final state, and total angular travel.

Relate. Two integrations with the stated initial conditions give

$$\begin{aligned} \omega(t) &= -0.900 - 0.600t + 0.600t^2, \\ \theta(t) &= 0.400 - 0.900t + 0.300t^2 + 0.200t^3, \end{aligned}$$

with t in seconds and the resulting units rad/s and rad. At $t = 3.00$ s, $\omega_f = +2.70$ rad/s and $\theta_f = 0.400$ rad. The net displacement is zero, although the body has moved throughout most of the interval.

Locate the reversal. Solving $0.600t^2 - 0.600t - 0.900 = 0$ gives

$$t = \frac{1 \pm \sqrt{7}}{2} \text{ s.}$$

Only $t_* = (1 + \sqrt{7})/2 = 1.823$ s belongs to the stated time interval. Before it, $\omega < 0$; afterward, $\omega > 0$. The minimum angle is $\theta(t_*) = -1.0260$ rad, so total angular travel is

$$[0.400 - (-1.0260)] + [0.400 - (-1.0260)] = 2.8520 \text{ rad.}$$

To the precision of the inputs, report 2.85 rad.

Check and interpret. Differentiating the angle recovers the velocity and differentiating once more recovers the prescribed acceleration. Independently, the integral of α over 0–3 s is +3.60 rad/s, equal to $\omega_f - \omega_0$. The acceleration vanishes at $t = 0.500$ s, when the velocity reaches its most negative value, -1.050 rad/s. This instant is different from the reversal at 1.823 s. Figure 10.6 aligns the two events so that their different meanings can be read directly.

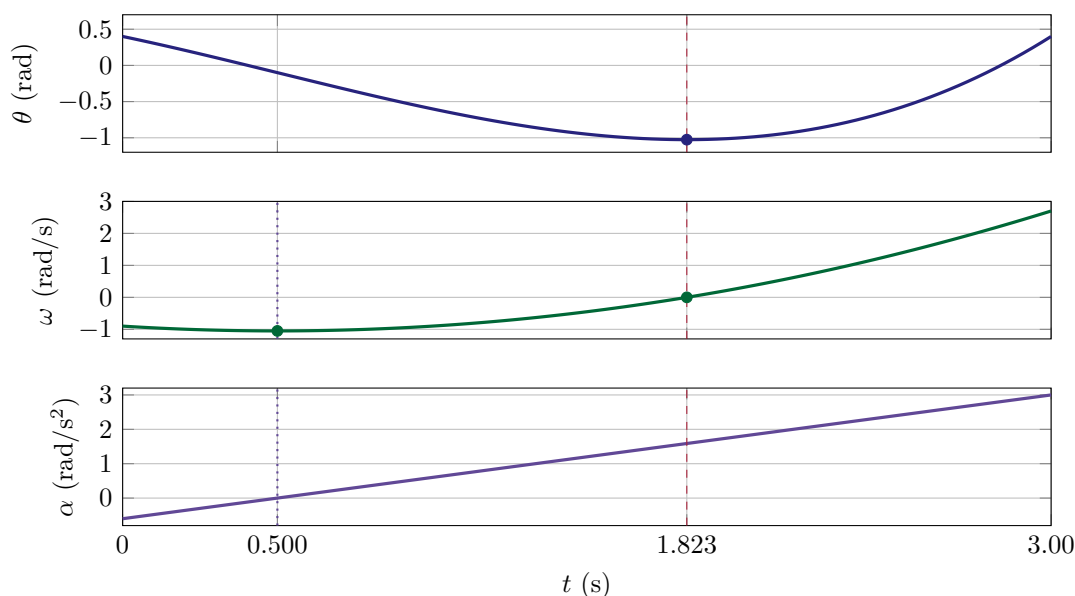


Figure 10.6: A common motion model generates all three histories in Example 10.5. The dotted event at 0.500 s is a velocity minimum, where $\alpha = 0$. The dashed event at 1.823 s is an angle minimum and direction reversal, where $\omega = 0$. The different vertical scales are explicitly labeled. Aligned derivative panels develop the construction method studied in Izaak Neutelings's *Kinematic xvat curves*, TikZ.net; the motion and plotted values are new.

10.3 Relating angular and linear motion

For a point at fixed perpendicular distance R from the axis, differentiate $s = R\theta$:

$$v_t = R\omega, \quad (10.13)$$

where v_t is a signed tangential component relative to a chosen positive tangential direction. Its speed is $|v_t| = R|\omega|$. Differentiating again gives the tangential acceleration

$$a_t = R\alpha. \quad (10.14)$$

Even when $\alpha = 0$, the velocity direction changes continuously. The inward normal acceleration has magnitude

$$a_n = \frac{v_t^2}{R} = R\omega^2. \quad (10.15)$$

The tangential and inward normal components are perpendicular, so

$$|\mathbf{a}| = \sqrt{a_t^2 + a_n^2}. \quad (10.16)$$

Example 10.6: Two markers on the same rotor

At an instant a rotor has $\omega = +4.0$ rad/s and $\alpha = -2.0$ rad/s². Compare markers at $R_A = 0.10$ m and $R_B = 0.30$ m.

For A ,

$$|v_A| = 0.400 \text{ m/s}, \quad a_{t,A} = -0.200 \text{ m/s}^2, \quad a_{n,A} = 1.60 \text{ m/s}^2,$$

so $|\mathbf{a}_A| = 1.61 \text{ m/s}^2$. For B ,

$$|v_B| = 1.20 \text{ m/s}, \quad a_{t,B} = -0.600 \text{ m/s}^2, \quad a_{n,B} = 4.80 \text{ m/s}^2,$$

and $|\mathbf{a}_B| = 4.84 \text{ m/s}^2$.

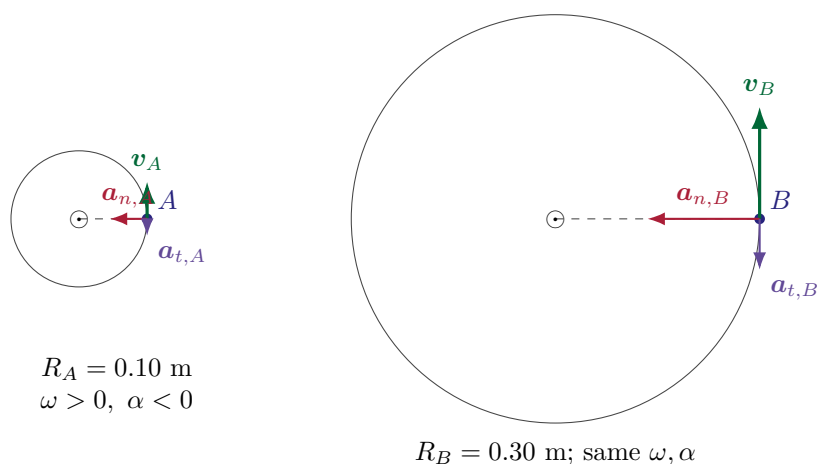


Figure 10.7: Two points of the same rotor, shown in separate views to keep their vectors distinct. The radii and the arrows within each vector family have the exact ratio 1 : 3. Velocity is upward, braking tangential acceleration downward, and normal acceleration inward. Different vector families use separate display scales. Axes point out of the page.

The ratio $|a_t|/|a_n| = |\alpha|/\omega^2 = 0.125$ is radius-independent, so both total-acceleration vectors are tilted by $\tan^{-1}(0.125) = 7.13^\circ$ from the inward direction toward the direction opposite the motion.

Key message

Equal angular speed does not mean equal linear speed. A rigid body can have a single ω while different points have different $v_t = R\omega$. This is why an extended body cannot generally be replaced by a point mass when its rotation matters.

10.3.1 Deriving the acceleration vector in a rotating basis

The inward acceleration is sometimes remembered as an extra term to attach to the tangential acceleration. A vector derivation shows why it is compulsory. For a point moving in the plane $z = \text{constant}$, define unit vectors

$$\hat{e}_r = (\cos \theta)\hat{i} + (\sin \theta)\hat{j}, \quad \hat{e}_\theta = -(\sin \theta)\hat{i} + (\cos \theta)\hat{j}.$$

The first points away from the axis; the second points in the direction of increasing angle. The vectors remain perpendicular, but they change direction in the laboratory as the body rotates. Differentiation using $d\theta/dt = \omega$ gives

$$\frac{d\hat{e}_r}{dt} = \omega\hat{e}_\theta, \quad \frac{d\hat{e}_\theta}{dt} = -\omega\hat{e}_r.$$

For a fixed radius R , the position relative to the axis is $\mathbf{r}_\perp = R\hat{e}_r$. Its first derivative is $\mathbf{v} = R\omega\hat{e}_\theta$. The product rule, including the derivative of the basis vector, now yields

$$\mathbf{a} = R\alpha\hat{e}_\theta - R\omega^2\hat{e}_r. \quad (10.17)$$

The minus sign of the radial term means inward acceleration regardless of whether ω is positive or negative. Its magnitude contains the square of the velocity because reversing the direction of traversal does not change the curvature of the circular path. The tangential term carries the sign of α relative to increasing angle. A negative tangential component does not automatically mean slowing down: it increases speed if the motion itself has negative angular velocity.

At the axis, $R = 0$, both terms vanish for a material point on a fixed rigid axis; a polar angle for that particular point is unnecessary. At a reversal, $\omega = 0$ and the instantaneous radial acceleration vanishes even though α may remain nonzero. For uniform rotation, $\alpha = 0$ but the radial acceleration remains. These limits provide independent interpretation checks on Eq. (10.17). The result assumes a fixed radius and a fixed inertial axis. A point sliding radially while the rotor turns has additional terms, which belong to a different model.

Example 10.7: Converting rotor acceleration to laboratory components

At a particular instant, a point on a rigid rotor is at $R = 0.180$ m and $\theta = 30.0^\circ$. The rotor has $\omega = -5.00$ rad/s and $\alpha = +4.00$ rad/s². Find the point's Cartesian velocity and acceleration, the acceleration magnitude, and whether its speed is increasing.

Model and geometry. The rotation axis is the fixed z axis. Figure 10.8 uses the positive $\hat{e}_\theta$ direction for components even though the current motion is opposite to that direction. At 30.0° ,

$$\hat{e}_r = 0.866025\hat{i} + 0.500000\hat{j}, \quad \hat{e}_\theta = -0.500000\hat{i} + 0.866025\hat{j}.$$

Hence $\mathbf{v} = (-0.900 \text{ m/s})\hat{e}_\theta$, or

$$\mathbf{v} = (0.450\hat{i} - 0.779\hat{j}) \text{ m/s}.$$

The tangential component is $R\alpha = +0.720$ m/s²; the outward radial component is $-R\omega^2 = -4.50$ m/s². Therefore

$$a_x = -4.50 \cos 30^\circ - 0.720 \sin 30^\circ = -4.26 \text{ m/s}^2,$$

$$a_y = -4.50 \sin 30^\circ + 0.720 \cos 30^\circ = -1.63 \text{ m/s}^2.$$

Perpendicular components give $|\mathbf{a}| = \sqrt{4.50^2 + 0.720^2} = 4.56 \text{ m/s}^2$.

Speed and check. Because $\omega < 0$, the derivative of speed is $d(R|\omega|)/dt = -R\alpha = -0.720$ m/s²: the point is slowing. Independently, $\mathbf{v} \cdot \mathbf{a} = R^2\omega\alpha = -0.648 \text{ m}^2/\text{s}^3$ is negative, so $d(v^2/2)/dt < 0$. Reconstructing the magnitude from the Cartesian components gives the same 4.56 m/s^2 because the basis transformation preserves lengths.

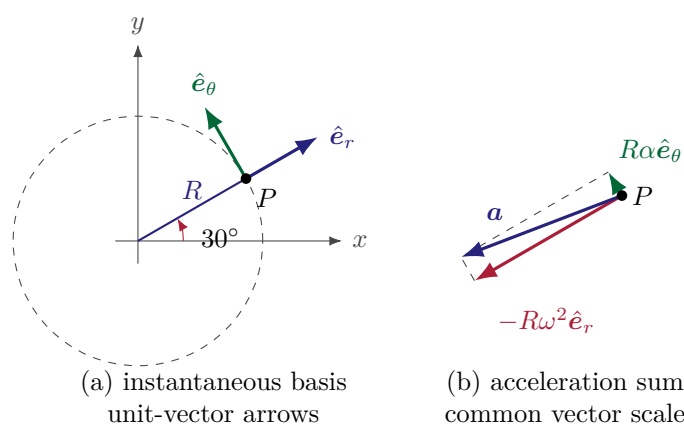


Figure 10.8: A rotating basis and the acceleration components for Example 10.7. The right panel uses 0.50 cm per m/s^2 for all acceleration arrows; the left panel has schematic length units. The inward component and positive tangential component combine to produce the laboratory acceleration. Polar point placement and angle-relative vector directions were studied in Izaak Neutelings's *Uniform circular motion*, TikZ.net; this basis/acceleration construction uses independently derived geometry.

10.4 Rotational kinetic energy and moment of inertia

Treat a rigid body as particles m_i at perpendicular distances $r_{\perp i}$ from the fixed axis. Each has speed $v_i = r_{\perp i}|\omega|$. The total kinetic energy associated with the rotation is

$$K_{\text{rot}} = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2} \left(\sum_i m_i r_{\perp i}^2 \right) \omega^2. \quad (10.18)$$

This motivates the moment of inertia about the specified axis,

$$I_z = \sum_i m_i r_{\perp i}^2, \quad (10.19)$$

and therefore

$$K_{\text{rot}} = \frac{1}{2} I_z \omega^2. \quad (10.20)$$

Unlike mass, I_z is not solely a property of the material object: it also depends on the chosen axis.

equal M , equal R_o ; both axes normal to the plane

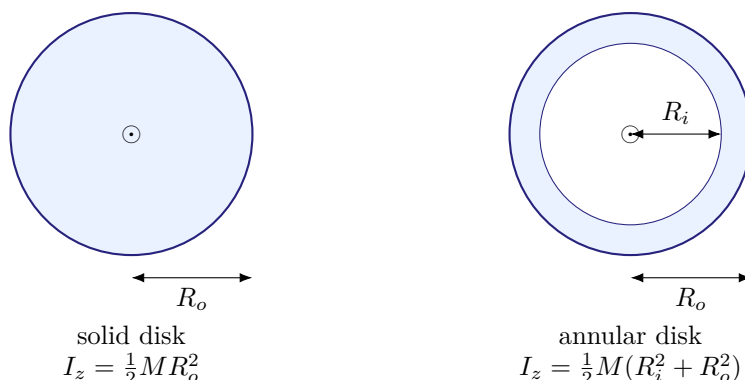


Figure 10.9: Solid and annular flywheels with equal mass and outer radius. Here $R_i/R_o = 0.75$, matching Example 10.8. Maintaining equal mass requires different surface densities. The annulus places a larger fraction of its mass farther from the normal symmetry axis.

Example 10.8: Solid and annular flywheels

Two ideal flywheels have mass $M = 20$ kg and outer radius $R_o = 0.40$ m. Flywheel A is a uniform solid disk. Flywheel B is a uniform annulus with inner radius $R_i = 0.30$ m. At $\omega = 30$ rad/s, compare their inertias and rotational kinetic energies.

Solid disk:

$$I_A = \frac{1}{2} M R_o^2 = 1.60 \text{ kg m}^2, \quad K_A = \frac{1}{2} I_A \omega^2 = 720 \text{ J}.$$

Annulus:

$$I_B = \frac{1}{2} M (R_i^2 + R_o^2) = 2.50 \text{ kg m}^2, \quad K_B = 1125 \text{ J}.$$

At the same angular speed the annulus stores 1.5625 times as much rotational kinetic energy. To store the same 720 J, it would need only

$$\omega_B = \sqrt{\frac{2K_A}{I_B}} = 24.0 \text{ rad/s}.$$

Interpretation. Mass farther from the axis raises I_z and therefore raises K_{rot} at fixed ω . This ideal comparison says nothing by itself about stress, fatigue, burst speed, bearings, containment, or safe flywheel design.

10.5 Calculating moments of inertia

For a continuous body the discrete sum becomes

$$I_z = \int r_{\perp}^2 dm. \quad (10.21)$$

The central modelling task is to express dm in terms of a coordinate and the appropriate mass density. For a line, surface and volume distribution, respectively,

$$dm = \lambda d\ell, \quad dm = \sigma dA, \quad dm = \rho dV.$$

10.5.1 Choosing a mass element and the distance to the axis

The integral $I_z = \int r_{\perp}^2 dm$ contains two different kinds of information. The mass element describes the material: where the mass is and how densely it is distributed. The squared distance describes the geometry of the selected rotation axis. These choices must be made together. A coordinate measured along a rod is the distance to a perpendicular end axis, but it is generally not the distance to an axis through the rod's center of mass. A cylindrical coordinate r is the distance to a cylinder's symmetry axis, but it is not the distance to a transverse diameter axis.

For a continuous distribution, think of the integral as the limit of a physical partition. Replace each small element by its mass located at a representative position, calculate $\sum r_{\perp}^2 \Delta m$, and refine the elements. The approximation improves when both the element mass and its axis distance are represented accurately. This viewpoint explains why the density, coordinate domain and chosen axis must all be specified before integrating. It also gives a useful independent numerical check on an analytic derivation.

When the body has uniform density, symmetry often suggests an element whose points all have the same distance from the axis. A complete narrow circular ring is suitable for a central-normal disk axis. A slender segment is suitable for a thin rod with a perpendicular axis. For a rectangular lamina, a narrow strip can simplify one coordinate integral, while a small area element makes the two-coordinate geometry explicit. There is no universal best element; its value lies in reducing the integral without concealing the model.

Method

Axis-first integration. State the axis and the domain. Express $r_{\perp}^2$ in the chosen coordinates. Write the density and dm with their units. Determine any density normalization from the specified total mass. Integrate, then test dimensions, symmetry, and a limiting geometry. The normalization step must precede a numerical comparison of mass distributions.

10.5.2 A uniform slender rod: center and end axes

A slender rod has length L , mass M and negligible transverse thickness. Choose coordinate x along it, measured from the left end, so $0 \leq x \leq L$. Its uniform linear density is $\lambda = M/L$, and a segment of length dx has mass $dm = (M/L)dx$. Let the fixed axis be perpendicular to the rod and pass through coordinate $x = a$. Its inertia is

$$\begin{aligned} I_{z,a} &= \frac{M}{L} \int_0^L (x-a)^2 dx \\ &= \frac{M}{3L} [(L-a)^3 + a^3] = M \left(\frac{L^2}{3} - La + a^2 \right). \end{aligned} \quad (10.22)$$

The sign of $x - a$ is irrelevant to its squared contribution, but keeping the signed coordinate makes the geometry transparent. For the end axis, $a = 0$ and

$$I_{z,0} = \frac{1}{3} ML^2. \quad (10.23)$$

For the center axis, $a = L/2$ and

$$I_{\text{CM},z} = \frac{1}{12}ML^2. \quad (10.24)$$

Completing the square in Eq. (10.22) gives

$$I_{z,a} = \frac{1}{12}ML^2 + M\left(a - \frac{L}{2}\right)^2.$$

Thus the center axis has the smallest inertia among this family of parallel axes. The end-axis value is four times the center-axis value because the distances have changed, although neither the mass nor the material has changed. This is an axis comparison, not a comparison between two rods.

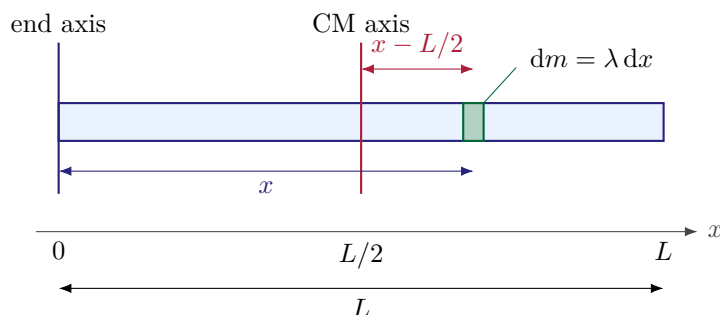


Figure 10.10: Two perpendicular rotation axes for the same slender rod. The mass element has distance x from the end axis and $|x - L/2|$ from the center axis. The axes lie in the page, while the rod rotates out of the page about either chosen axis. Rod thickness and element width are schematic.

The slender-rod qualification matters. A rod of appreciable radius also contains mass away from its longitudinal line, so its exact transverse inertia includes a thickness contribution. The formulas above are accurate when that contribution is negligible on the scale ML^2 . They do not describe rotation about the rod's own long axis; in the ideal line model that axial inertia would vanish.

10.5.3 Example derivation: a uniform annular disk

Let a thin annulus have inner radius R_i , outer radius R_o , uniform surface density $\sigma = M/[\pi(R_o^2 - R_i^2)]$, and axis through its center perpendicular to the plane. A thin ring of radius r and width dr has

$$dm = \sigma(2\pi r \, dr).$$

Hence

$$I_z = \int_{R_i}^{R_o} r^2 \sigma(2\pi r) \, dr = 2\pi\sigma \left[\frac{r^4}{4} \right]_{R_i}^{R_o} \quad (10.25)$$

$$= \frac{1}{2}M(R_i^2 + R_o^2). \quad (10.26)$$

Setting $R_i = 0$ recovers the solid-disk result $I_z = \frac{1}{2}MR_o^2$. Taking a thin-ring limit in which the mass is concentrated near one radius with fixed total mass gives $I_z \rightarrow MR^2$.

10.5.4 The parallel-axis theorem

Suppose a known axis through the center of mass is parallel to a second axis displaced by vector $\mathbf{d}$ in the plane perpendicular to the axes. For a mass element with center-of-mass coordinates $\mathbf{r}'$ in the perpendicular plane (or their perpendicular projection for a three-dimensional body), its squared distance from the shifted axis contains

$$|\mathbf{r}' + \mathbf{d}|^2 = r'^2 + d^2 + 2\mathbf{r}' \cdot \mathbf{d}.$$

Integrating over the body,

$$I_z = \int r'^2 dm + d^2 \int dm + 2\mathbf{d} \cdot \int \mathbf{r}' dm.$$

The final integral vanishes because the origin of $\mathbf{r}'$ is the center of mass. Therefore

$$I_z = I_{\text{CM},z} + Md^2. \quad (10.27)$$

The axes must be parallel and one of the two axes in Eq. (10.27) must pass through the center of mass.

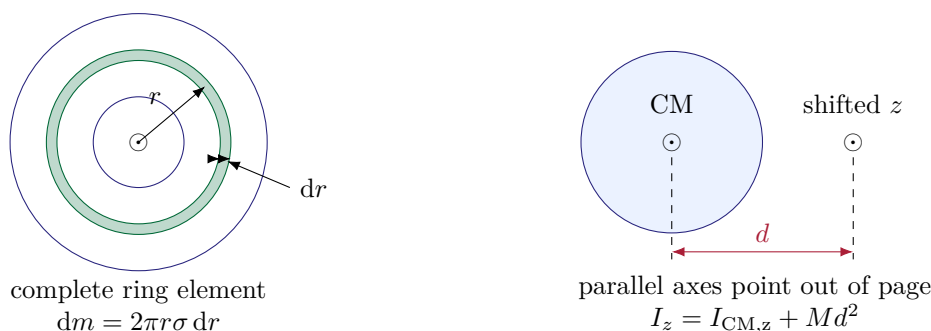


Figure 10.11: The inertia integral uses a full thin ring, not a rectangular mass patch. The right panel is a plan view of two parallel axes normal to the disk. The separation d lies in the plane perpendicular to both axes. Ring thickness is enlarged for clarity.

Example 10.9: A composite rotor with an offset component

A uniform disk of mass 4.0 kg and radius 0.20 m rotates about its central z axis. A smaller uniform disk of mass 1.0 kg and radius 0.050 m is rigidly attached with its center 0.25 m from that axis; its own symmetry axis is parallel to z . Find the composite I_z .

The central disk contributes

$$I_1 = \frac{1}{2}(4.0)(0.20)^2 = 0.0800 \text{ kg m}^2.$$

For the smaller disk,

$$I_{2,\text{CM}} = \frac{1}{2}(1.0)(0.050)^2 = 1.25 \times 10^{-3} \text{ kg m}^2,$$

and the parallel-axis theorem gives

$$I_2 = I_{2,\text{CM}} + M_2 d^2 = 0.06375 \text{ kg m}^2.$$

Therefore

$$I_z = 0.14375 \text{ kg m}^2 \approx 0.144 \text{ kg m}^2.$$

The shift term $M_2 d^2$ is much larger than the small disk's own central inertia; treating the attached disk as if it remained on the main axis would be a major modelling error.

Caution

A table value for moment of inertia is incomplete without its axis. The same rod, disk, or plate can have many different moments of inertia. The parallel-axis theorem moves between *parallel* axes; it does not rotate one axis into another.

10.5.5 A graded rod: the center of mass is not the midpoint

Consider a thin rod with a prescribed density increasing linearly toward its right end:

$$\lambda(x) = \lambda_0 \left(1 + \frac{x}{L}\right), \quad 0 \leq x \leq L. \quad (10.28)$$

The constant λ_0 has units kg/m. It is not M/L : the rod is denser than λ_0 except at its left end. Normalizing the distribution gives

$$M = \int_0^L \lambda(x) dx = \frac{3}{2} \lambda_0 L, \quad \lambda_0 = \frac{2M}{3L}.$$

The first mass moment locates the center of mass:

$$x_{\text{CM}} = \frac{1}{M} \int_0^L x \lambda(x) dx = \frac{\lambda_0}{M} \left(\frac{L^2}{2} + \frac{L^2}{3} \right) = \frac{5}{9} L. \quad (10.29)$$

It lies beyond $L/2$ toward the denser end, as it must. The inertia about the perpendicular left-end axis follows from the second mass moment:

$$I_{z,0} = \lambda_0 \int_0^L x^2 \left(1 + \frac{x}{L}\right) dx = \frac{7}{12} \lambda_0 L^3 = \frac{7}{18} M L^2. \quad (10.30)$$

The uniform rod gives $M L^2/3$, so grading in this direction increases the left-end inertia. More of the same total mass lies at large x , where its contribution is amplified by x^2 .

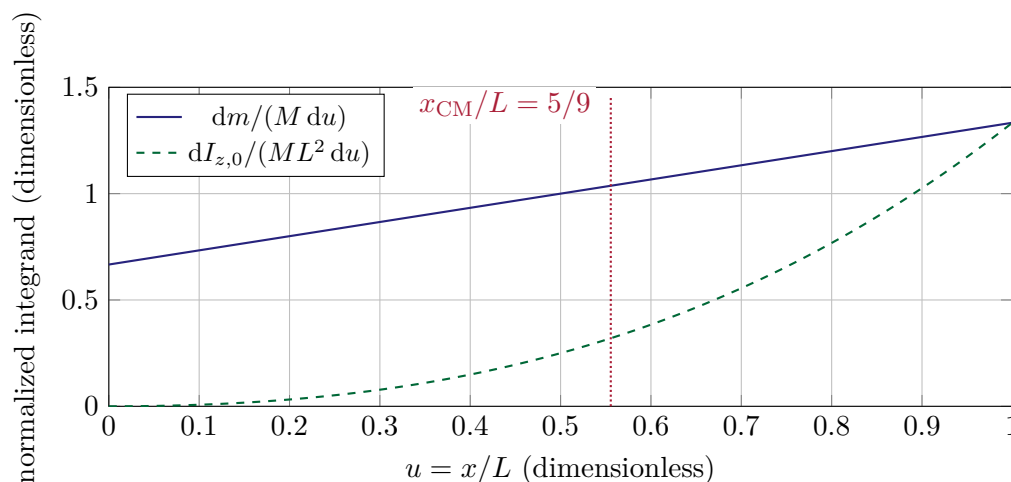


Figure 10.12: Mass and left-end inertia are different weighted integrals for the graded rod. The blue curve integrates to 1; the dashed green curve integrates to $7/18$. Their different shapes show why a small amount of material near the far end can contribute strongly to inertia. Both curves are analytic, with $u = x/L$.

For the actual center axis, the parallel-axis theorem gives

$$I_{\text{CM},z} = I_{z,0} - M x_{\text{CM}}^2 = \left(\frac{7}{18} - \frac{25}{81} \right) M L^2 = \frac{13}{162} M L^2. \quad (10.31)$$

The same result follows by directly integrating $\int_0^L (x - 5L/9)^2 \lambda(x) dx$. Notice that this center-axis inertia is slightly smaller than the uniform value $M L^2/12$. This does not contradict the increased left-end inertia: shifting the mass changes both its spread about the center of mass and the location of that center. Comparisons must state which axis is held fixed.

Example 10.10: Inertia of a graded instrument boom

Statement. A rigid slender boom has $L = 0.900$ m, $M = 3.00$ kg and the density in Eq. (10.28). Find its density normalization, center of mass, inertias about perpendicular axes through the two ends and the center of mass, and its kinetic energy at $\omega = 4.00$ rad/s about the left-end axis. Compare the latter with a uniform boom of the same length and mass.

Model and strategy. The thickness is neglected and each selected axis is fixed. Determine the mass distribution before applying either inertia integrals or an axis shift. The drawing and weighting plot in Figs. 10.10 and 10.12 specify the distances.

Solution. The normalization and center location are

$$\lambda_0 = \frac{2(3.00)}{3(0.900)} = 2.22222 \text{ kg/m}, \quad x_{\text{CM}} = \frac{5}{9}(0.900) = 0.500 \text{ m}.$$

The left-end and center-axis inertias are

$$I_{z,0} = \frac{7}{18}(3.00)(0.900)^2 = 0.945 \text{ kg m}^2, \quad I_{\text{CM},z} = \frac{13}{162}(3.00)(0.900)^2 = 0.195 \text{ kg m}^2.$$

The right-end axis is $L - x_{\text{CM}} = 0.400$ m from the center axis. Hence

$$I_{z,L} = I_{\text{CM},z} + M(L - x_{\text{CM}})^2 = 0.195 + 3.00(0.400)^2 = 0.675 \text{ kg m}^2.$$

Equivalently, direct integration of $(L - x)^2 \lambda(x)$ gives $I_{z,L} = 5ML^2/18$. At the prescribed speed the left-end rotational energy is

$$K_{z,0} = \frac{1}{2}(0.945)(4.00)^2 = 7.56 \text{ J}.$$

The uniform boom has $I_{z,0}^{\text{uniform}} = ML^2/3 = 0.810 \text{ kg m}^2$ and $K^{\text{uniform}} = 6.48 \text{ J}$. Grading raises this energy by 1.08 J, or 16.7%, at the same speed about the same left-end axis.

Check and interpretation. The right-end inertia is smaller because the denser material lies near that axis. Direct integration gives the same 0.675 kg m^2 without using an axis shift. Also $I_{z,0} - Mx_{\text{CM}}^2 = 0.945 - 0.750 = 0.195 \text{ kg m}^2 > 0$, as required. Using the midpoint as the center of mass would give a wrong axis shift even though the units would remain correct.

10.5.6 A cylinder: symmetry axis and transverse axis

A uniform cylinder of radius R , axial length H and mass M makes the importance of axis naming especially clear. About its central symmetry axis z , every thin cylindrical shell has the same radius at all points. Its volume is $dV = 2\pi r H dr$, so

$$I_z = \int_0^R r^2 \rho(2\pi r H) dr = \frac{\pi \rho H R^4}{2} = \frac{1}{2} M R^2,$$

where $M = \rho \pi R^2 H$. The length H cancels at fixed M and R : extending the body along the rotation axis changes no perpendicular distances if the radial mass distribution is held the same.

For a transverse center axis x , the answer is different. First consider a thin uniform disk in the xy plane. Since each mass element has $r^2 = x^2 + y^2$,

$$I_z = \int (x^2 + y^2) dm = I_x + I_y.$$

Rotational symmetry gives $I_x = I_y$, so $I_x = MR^2/4$. The relation $I_z = I_x + I_y$ is the perpendicular-axis identity for a planar lamina; it is not a general relation for a thick three-dimensional body.

Now slice the cylinder into thin disks along z , with $-H/2 \leq z \leq H/2$ and $dm = (M/H)dz$. A disk's own center-diameter inertia is $R^2 dm/4$. Its diameter axis is parallel to the cylinder's center x axis

and displaced from it by $|z|$, so the disk contribution is $(R^2/4 + z^2)dm$. Summing the slices gives

$$I_x = \frac{M}{H} \int_{-H/2}^{H/2} \left(\frac{R^2}{4} + z^2 \right) dz = \frac{M}{12} (3R^2 + H^2). \quad (10.32)$$

This expression tends to the thin-disk diameter value when $H/R \rightarrow 0$ and to the slender-rod center value $MH^2/12$ when $R/H \rightarrow 0$. The limits identify which geometric contribution matters.

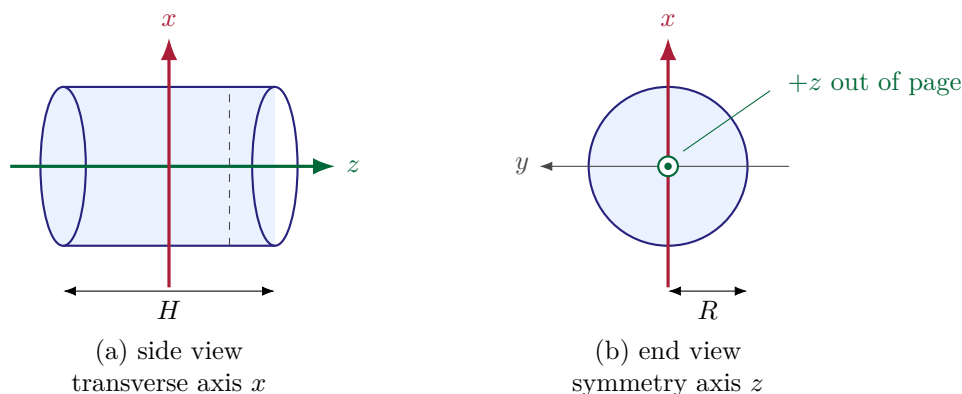


Figure 10.13: Two distinct fixed axes through the center of a uniform cylinder. The side view identifies the axial length H ; the end view identifies radius R and the normal z axis. The perspective is schematic. The perpendicular-axis identity applies to each thin disk slice, while the whole cylinder's transverse inertia also includes the slices' axial offsets.

Example 10.11: Which cylinder axis gives the larger inertia?

Statement. A uniform test cylinder has $M = 5.00$ kg, $R = 0.120$ m and $H = 0.500$ m. Compare its inertias and rotational energies at $\omega = 6.00$ rad/s for fixed rotation about the central symmetry axis and about a transverse center axis. Determine the aspect ratio at which these inertias would be equal.

Strategy. Figure 10.13 identifies two different experiments with the same body. Use the appropriate axis formula in each experiment; no rolling or simultaneous rotation about both axes is assumed.

Solution. The symmetry-axis inertia is

$$I_z = \frac{1}{2}(5.00)(0.120)^2 = 0.0360 \text{ kg m}^2.$$

The transverse-axis inertia is

$$I_x = \frac{5.00}{12} [3(0.120)^2 + (0.500)^2] = 0.122167 \text{ kg m}^2.$$

Thus $K_z = \frac{1}{2}I_z\omega^2 = 0.648$ J and $K_x = 2.199$ J. Rotation about x requires about 3.39 times the kinetic energy at this angular speed because the cylinder's length puts mass far from that transverse axis.

Setting the two inertia expressions equal gives $6R^2 = 3R^2 + H^2$, hence $H/R = \sqrt{3}$. A shorter cylinder has $I_x < I_z$; a longer cylinder has $I_x > I_z$.

Check. Decomposing I_x gives $MR^2/4 = 0.0180$ kg m² and $MH^2/12 = 0.104167$ kg m², which add to the result. The latter dominates for this geometry. Using the thin-disk diameter value alone would miss most of the inertia.

10.5.7 A rectangular lamina and the role of two coordinates

Let a uniform thin rectangular plate have sides a along x and b along y , centered at the origin in the xy plane. Its surface density is $\sigma = M/(ab)$. For the normal z axis, a small rectangle $dx dy$ has $dm = \sigma dx dy$ and squared axis distance $x^2 + y^2$. Therefore

$$\begin{aligned} I_z &= \sigma \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} (x^2 + y^2) dy dx \\ &= \sigma \left[b \frac{a^3}{12} + a \frac{b^3}{12} \right] = \frac{M}{12} (a^2 + b^2). \end{aligned} \quad (10.33)$$

For the in-plane x axis, the distance is $|y|$, so $I_x = Mb^2/12$. Similarly $I_y = Ma^2/12$. These values also confirm $I_z = I_x + I_y$. The double integral is simply the continuous sum over rows and columns; doing the y integral first accounts for every mass element within one column, then the x integral sums all the columns.

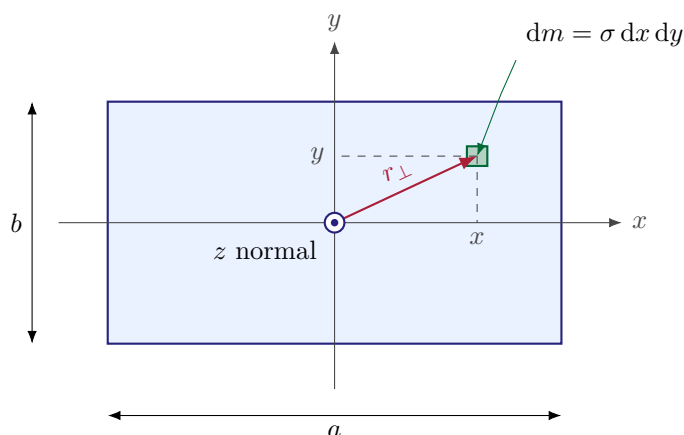


Figure 10.14: Inertia integral for a thin rectangular plate. The normal-axis distance satisfies $r_{\perp}^2 = x^2 + y^2$, while the distance to the in-plane x axis is $|y|$. The small element is schematic; the plate is uniform and its axes pass through its center.

Example 10.12: A rectangular panel on two different hinges

Statement. A uniform thin panel has $M = 2.00$ kg, $a = 0.600$ m and $b = 0.400$ m, as in Fig. 10.14. Find its inertia about the central normal axis and about a hinge along the edge $y = -b/2$, parallel to x . Find the angular acceleration about each axis under a net torque of magnitude 0.240 N m directed along that axis.

Model. The panel is rigid, each hinge is a fixed inertial axis, and the quoted torque is the net torque about that axis. Any gravitational or bearing contribution has already been included in that specified net torque.

Solution. The normal-axis inertia is

$$I_z = \frac{2.00}{12} [(0.600)^2 + (0.400)^2] = 0.0866667 \text{ kg m}^2.$$

The center x axis has $I_{\text{CM},x} = Mb^2/12 = 0.0266667 \text{ kg m}^2$. The edge hinge is parallel to it and displaced by $b/2 = 0.200$ m. Thus

$$I_{\text{edge},x} = I_{\text{CM},x} + M(b/2)^2 = 0.0266667 + 2.00(0.200)^2 = 0.106667 \text{ kg m}^2.$$

The angular accelerations in the applied-torque sense are $\alpha_z = 0.240/0.0866667 = 2.77 \text{ rad/s}^2$ and $\alpha_{\text{edge},x} = 0.240/0.106667 = 2.25 \text{ rad/s}^2$.

Independent check. About the edge hinge, strips at distance $s = y + b/2$ have $0 \leq s \leq b$ and mass $(M/b)ds$. Direct integration gives $I_{\text{edge},x} = (M/b) \int_0^b s^2 ds = Mb^2/3 = 0.106667$

kg m^2 . The side a does not appear because moving mass along the hinge direction leaves its perpendicular distance unchanged.

10.5.8 Composite bodies and material removed as a hole

Inertia is additive for masses rotating about the *same specified axis*. For attached parts, calculate each part's inertia about that axis and add. For a hole, imagine the complete uniform body and subtract the mass and inertia of the material that would occupy the hole. This bookkeeping does not assign negative mass to physical matter. It subtracts an integral over a region from the integral over a larger region, leaving the integral over the material that remains.

The virtual removed part must have the original body's density. If a hole is off center, its own central inertia must be shifted to the chosen common axis before subtraction. Removing an off-center hole also shifts the remaining body's center of mass. Consequently the inertia about the original geometric center is generally not the inertia about the new center of mass. These are separate questions, both answerable from the same mass distribution.

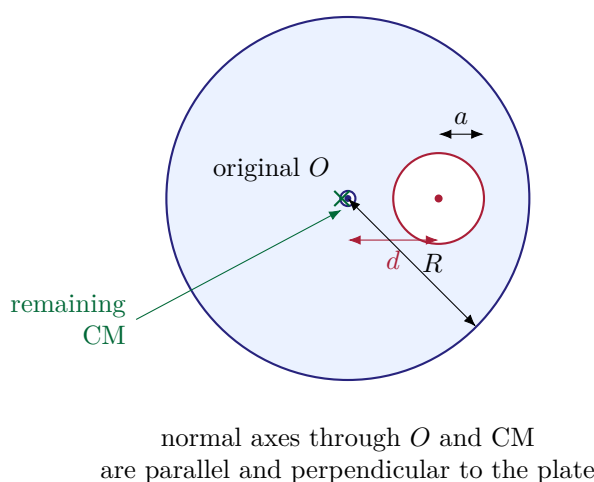


Figure 10.15: An off-center circular hole in a uniform disk. The remaining center of mass shifts away from the removed material. The geometry shown has $a/R = 1/4$ and $d/R = 1/2$; the central-axis markers are enlarged for clarity. The subtraction calculation uses one common normal axis through O .

Example 10.13: A perforated rotor and its shifted center of mass

Statement. A uniform full disk would have mass $M_0 = 8.00 \text{ kg}$ and radius $R = 0.300 \text{ m}$. A circular hole of radius $a = 0.0750 \text{ m}$ is cut with its center at $d = 0.150 \text{ m}$ on the $+x$ side. Find the remaining mass, the inertia about the normal axis through the original center O , the remaining center of mass, and the normal-axis inertia through that center of mass.

Strategy and geometry. Figure 10.15 has the same radius and offset ratios. The hole is wholly inside the disk because $d + a = 0.225 \text{ m} < R$. Use the full disk's uniform surface density for the virtual removed disk.

Solution. The removed mass is $m_h = M_0 a^2 / R^2 = 0.500 \text{ kg}$, and the remaining mass is $M = 7.50 \text{ kg}$. The full-disk inertia is $I_{0,z} = M_0 R^2 / 2 = 0.360 \text{ kg m}^2$. About O , the removed material contributes

$$I_{h,O} = \frac{1}{2} m_h a^2 + m_h d^2 = 0.00140625 + 0.0112500 = 0.01265625 \text{ kg m}^2.$$

Consequently $I_{O,z} = 0.34734375 \text{ kg m}^2$. The full disk's first moment about O vanishes.

Removing a first moment $m_h d$ leaves

$$x_{\text{CM}} = \frac{0 - m_h d}{M} = -0.0100 \text{ m}, \quad y_{\text{CM}} = 0.$$

The CM axis is parallel to the original normal axis, so

$$I_{\text{CM},z} = I_{O,z} - Mx_{\text{CM}}^2 = 0.34734375 - 0.00075000 = 0.34659375 \text{ kg m}^2.$$

Checks and interpretation. Both inertias are positive and less than the full-disk value; the CM-axis value is the smaller of the two. The first-moment balance $Mx_{\text{CM}} + m_h d = 0$ is satisfied. Subtracting only $m_h a^2/2$ would omit the removed material's offset contribution. The result describes ideal rigid-body inertia and does not assess vibration or the bearing load of a rotor operated about O rather than its center of mass.

10.5.9 Radius of gyration and bounds that catch modelling errors

The radius of gyration about a specified axis is

$$k_z = \sqrt{\frac{I_z}{M}}, \quad I_z = Mk_z^2. \quad (10.34)$$

It is a root-mean-square distance of the mass from the axis. A hypothetical thin ring containing the same mass at radius k_z would have the same scalar inertia. This equivalence concerns inertia about that axis; it does not assert that the body has become a ring, has the same stresses, or has the same inertia about another axis. For a uniform disk $k_z = R/\sqrt{2}$; for a uniform slender rod about its center $k_z = L/\sqrt{12}$.

Simple bounds provide a useful check before any elaborate integration. If all the material lies between perpendicular distances $r_{\min}$ and $r_{\max}$, nonnegative density implies

$$Mr_{\min}^2 \leq I_z \leq Mr_{\max}^2, \quad r_{\min} \leq k_z \leq r_{\max}. \quad (10.35)$$

The proof is to multiply $r_{\min}^2 \leq r_{\perp}^2 \leq r_{\max}^2$ by the positive mass element and integrate. Equality requires essentially all mass to lie at the relevant limiting radius. A filled disk has $0 < I_z < MR^2$, while a thin ring at radius R reaches MR^2 . These statements assume $M > 0$; a body with all its mass exactly on the axis has zero inertia in the ideal line model.

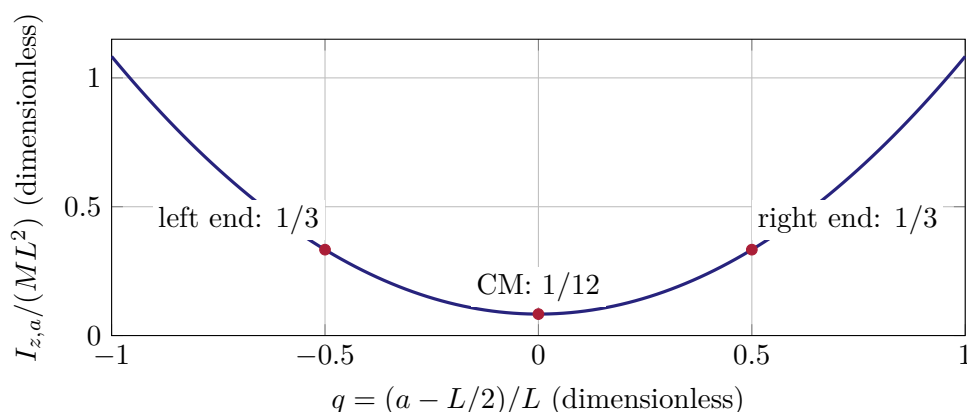


Figure 10.16: Axis dependence for one uniform slender rod. The coordinate a locates a perpendicular axis along the rod or its extension. The analytic parabola $I_{z,a}/(ML^2) = 1/12 + q^2$ has its minimum at the center of mass. Shifts in either direction increase inertia equally for this symmetric body.

The parallel-axis theorem can also be written $k_{z,a}^2 = k_{\text{CM},z}^2 + d^2$. Figure 10.16 shows the corresponding squared-distance dependence for a rod. This geometrical check catches a common sign error: shifting away from a center-of-mass axis increases inertia. To infer the CM value from an offset-axis value, subtract Md^2 , but a negative answer would reveal an inconsistent mass, distance, measured inertia or axis identification.

10.6 Torque about a fixed axis

For a force $\mathbf{F}$ applied at position $\mathbf{r}$ relative to a chosen origin on the rotation axis, the torque vector is

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}. \quad (10.36)$$

The fixed-axis equation uses the component along the axis,

$$\tau_z = (\mathbf{r} \times \mathbf{F})_z. \quad (10.37)$$

For forces and position vectors in the plane perpendicular to z , $|\tau_z| = rF \sin \phi = \ell F = r|F_t|$, where ℓ is the perpendicular distance from the axis to the force line of action. In three dimensions $|\boldsymbol{\tau}| = rF \sin \phi$ is the total torque magnitude; it need not equal $|\tau_z|$. A force can be large yet exert zero torque about a selected axis if its line of action passes through that axis.

10.6.1 The axis torque component and the force's lever arm

For the fixed z axis, Cartesian components give the particularly useful expression

$$\tau_z = xF_y - yF_x. \quad (10.38)$$

Here x, y locate the point of force application relative to an origin on the axis. The component F_z produces no torque about z , although it may produce a transverse torque or an axial load. Moving the reference origin along the same z axis leaves x, y unchanged and therefore leaves τ_z unchanged. Moving to a different axis generally changes the torque component.

In a strictly planar force geometry, with $\mathbf{r}$ and $\mathbf{F}$ in the plane perpendicular to z , $|\tau_z| = rF \sin \phi = F\ell$, where ϕ is the angle between the two vectors and ℓ is the shortest distance from the axis point to the force's line of action. This ℓ is the force lever arm. It is not generally the same as the application point's radius r from the axis: equality occurs only for a tangential force. For a general three-dimensional force, use the projection $(\mathbf{r} \times \mathbf{F})_z$ rather than equating the full vector magnitude $|\mathbf{r} \times \mathbf{F}|$ with $|\tau_z|$.

The lever-arm construction explains why a radial force has zero axis torque: its line of action passes through the axis. Such a force can still be essential for keeping material on a circular path. Zero torque does not imply zero force, and zero axis torque does not imply that every force component is dynamically irrelevant. Likewise, a force applied at a point far from the axis can have small torque if it points nearly radially.

Example 10.14: A force whose large component gives no axis torque

Statement. A force is applied at $(x, y, z) = (0.300, 0.100, 0)$ m on a rigid body with $I_z = 0.0800$ kg m². Its components are $(F_x, F_y, F_z) = (12.0, 20.0, 40.0)$ N. A separate bearing-resistance torque is -0.800 N m. Find the applied z torque, net z torque and angular acceleration, and explain why the 40.0 N component does not enter these results.

Model and solution. The body rotates about a fixed z axis, with positive sense given by the right-hand rule. Equation (10.38) gives

$$\tau_z = (0.300)(20.0) - (0.100)(12.0) = 4.80 \text{ N m}.$$

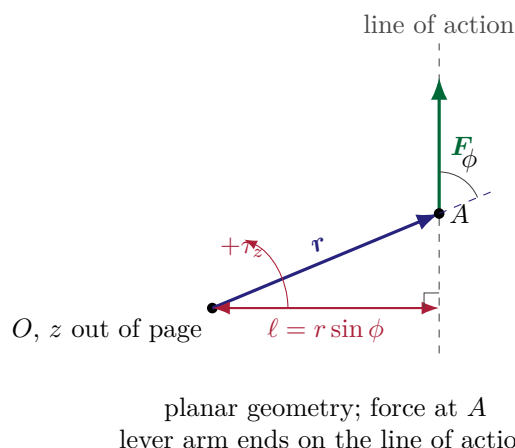


Figure 10.17: The force lever arm is a perpendicular distance to its line of action. It need not end at the force application point. Here the upward force at the right-hand point produces a positive z torque, and its lever arm is the horizontal distance to the vertical force line. The force-arrow length is schematic.

The net axis torque is $4.80 - 0.800 = 4.00 \text{ N m}$, so $\alpha = 4.00/0.0800 = 50.0 \text{ rad/s}^2$ in the positive sense. The axial force has no tangential component around z , so it cannot change this rotational degree of freedom.

Independent geometrical check. The in-plane force magnitude is $F_{xy} = \sqrt{12.0^2 + 20.0^2} = 23.3238 \text{ N}$. Its line of action passes through $(0.300, 0.100)$ in direction $(12.0, 20.0)$. Its perpendicular distance from the origin is

$$\ell = \frac{|xF_y - yF_x|}{F_{xy}} = 0.205798 \text{ m},$$

so $F_{xy}\ell = 4.80 \text{ N m}$. This checks the lever-arm interpretation in the planar projection. The full cross product is $(4.00, -12.0, 4.80) \text{ N m}$; its magnitude is about 13.5 N m and is not the axis torque. Ideal supports provide whatever transverse constraint torques the fixed-axis model requires; they do not turn that full magnitude into τ_z .

10.6.2 Why the scalar equation follows from particle dynamics

The equation $\sum \tau_z = I_z \alpha$ can be derived without guessing an analogy with translation. Partition a rigid body into material particles. For particle i , let r_i be its perpendicular distance from the fixed axis and let $\hat{e}_{t,i}$ be the local unit vector in the positive tangential direction. Its tangential acceleration is $a_{t,i} = r_i \alpha$. Its inward radial acceleration is $r_i \omega^2$, but that acceleration has zero tangential projection.

Newton's second law, projected on the tangential direction for each particle, is

$$F_{t,i}^{\text{ext}} + F_{t,i}^{\text{int}} = m_i r_i \alpha.$$

Multiplying by r_i converts each force projection to its torque about the axis, then summing gives

$$\sum_i r_i F_{t,i}^{\text{ext}} + \sum_i r_i F_{t,i}^{\text{int}} = \left(\sum_i m_i r_i^2 \right) \alpha. \quad (10.39)$$

For the ordinary rigid-body model, internal interaction torques cancel in the total. In the simplest particle description, an interaction pair obeys equal and opposite central forces: the pair's total torque is $(\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij} = 0$ because the separation and force are parallel. A more general continuum description gives the same net cancellation when internal stresses transmit forces and couples

consistently and no unmodelled internal angular-momentum source is present. The cancellation is a property of the complete chosen body, not of an arbitrary subset of its particles.

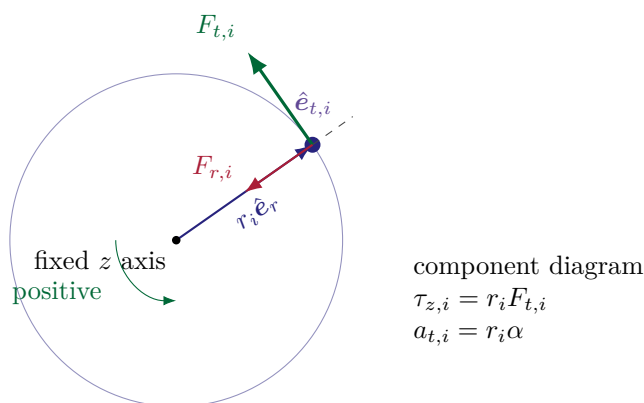


Figure 10.18: Local tangential and radial components at one material particle. Only the tangential component enters its torque about the fixed axis. The two arrows are force components, not a force plus two additional forces. Arrow lengths are schematic; the positive tangential direction is counterclockwise.

External support forces must still be treated honestly. An ideal axle force acting on the axis has zero z torque, but a bearing friction couple generally does not. Supports can supply transverse torques that prevent the axis direction from changing. Those transverse torques need not vanish and do not enter the scalar equation for motion around z . With all external axis torques included, Eq. (10.39) reduces to

$$\sum \tau_z^{\text{ext}} = \left(\sum_i m_i r_i^2 \right) \alpha = I_z \alpha.$$

The same result holds for a continuum by replacing the mass sum with the inertia integral. Rigidity keeps the perpendicular distances fixed, so I_z is constant, and every particle shares the same α . If the axis moves, the body deforms, or material redistributes, this derivation's steps must be reconsidered; the broader angular-momentum treatment is developed in the next chapter.

Key message

The inertia integral, torque projection and scalar equation describe the same chosen axis. Mass at larger perpendicular distance contributes more strongly to inertia. Force contributes to axis torque only through its local tangential projection. Radial forces provide inward acceleration while ideal fixed-axis constraint forces and transverse torques maintain the assumed motion.

For a rigid body constrained to rotate about a fixed inertial axis with time-independent mass distribution,

$$\sum \tau_z = I_z \alpha. \quad (10.40)$$

This scalar relation is the rotational counterpart of $\sum F_x = M a_x$ for the specified one-dimensional rotational degree of freedom.

Caution

Equation (10.40) is not the universal vector equation $\boldsymbol{\tau} = I\boldsymbol{\alpha}$. Away from fixed-axis motion or when the inertia distribution changes, angular momentum is the safer starting point. That generalization belongs to the next chapter.

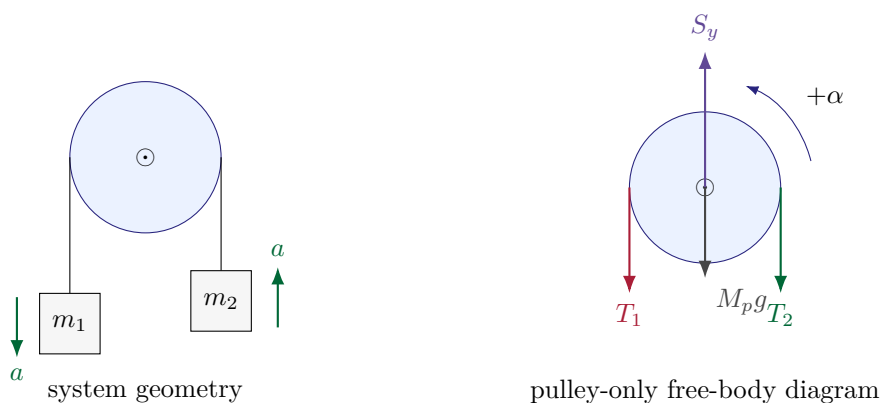


Figure 10.19: Atwood geometry and the isolated pulley. Left-side downward motion corresponds to counterclockwise rotation, so $\tau_z = (T_1 - T_2)R$. The axle support S_y and pulley weight pass through the axis and produce no z torque. Force lengths are schematic; the no-slip constraint is $a = R\alpha$ with the shown signs.

Example 10.15: Atwood machine with a massive pulley

Masses $m_1 = 4.0$ kg and $m_2 = 2.0$ kg are connected by a light inextensible cord over a uniform solid pulley of mass $M_p = 2.0$ kg and radius $R = 0.10$ m. The cord does not slip and axle friction is negligible. The system is released from rest. Find the linear acceleration, the two tensions and the pulley angular acceleration. Then independently check the speed after m_1 descends $h = 0.80$ m.

Model and signs. Take m_1 downward, m_2 upward and the corresponding pulley rotation as positive. The pulley inertia is $I_z = \frac{1}{2}M_p R^2 = 0.0100$ kg m².

Translation:

$$\begin{aligned} m_1 g - T_1 &= m_1 a, \\ T_2 - m_2 g &= m_2 a. \end{aligned}$$

Rotation:

$$(T_1 - T_2)R = I_z \alpha, \quad \alpha = \frac{a}{R}.$$

Eliminating the tensions gives

$$a = \frac{(m_1 - m_2)g}{m_1 + m_2 + I_z/R^2} = 2.803 \text{ m/s}^2. \quad (10.41)$$

Thus

$$T_1 = 28.03 \text{ N}, \quad T_2 = 25.23 \text{ N}, \quad \alpha = 28.03 \text{ rad/s}^2.$$

The tension difference is physical: if $T_1 = T_2$, the pulley would have no net torque despite its nonzero inertia.

Independent energy check. Kinematics gives $v = \sqrt{2ah} = 2.118$ m/s. The gravitational energy decrease is

$$(m_1 - m_2)gh = 15.696 \text{ J}.$$

The total kinetic energy is

$$\frac{1}{2}(m_1 + m_2)v^2 + \frac{1}{2}I_z \left(\frac{v}{R}\right)^2 = 15.696 \text{ J},$$

which matches to the shown precision.

10.6.3 Rotational equilibrium and uniform rotation

If $\sum \tau_z = 0$, then $\alpha = 0$ under the fixed-axis assumptions. The body may be at rest, or it may already be rotating with constant angular velocity. Rotational equilibrium means no angular acceleration, not necessarily zero angular velocity. Static equilibrium additionally requires the body to be at rest in the chosen frame.

10.6.4 A driven pulley: distinguish the force from the tension

An applied force and a cord tension are equal only when the model makes them equal. A light cord transmits one tension along a freely moving straight segment, but a massive rotating pulley can sustain different tensions on its two sides. A useful test of force bookkeeping is a pulley driven by a prescribed downward force on the left while a hanging load rises on the right. Figure 10.20 separates the apparatus from the two bodies whose equations are required. The arrows on the isolated pulley are forces *on the pulley*. The tension arrow on the isolated load is a force *on the load*. It has the same magnitude as the right-side cord force on the pulley under the light-cord model, but these two forces are not a third-law pair: each partner acts on the cord.

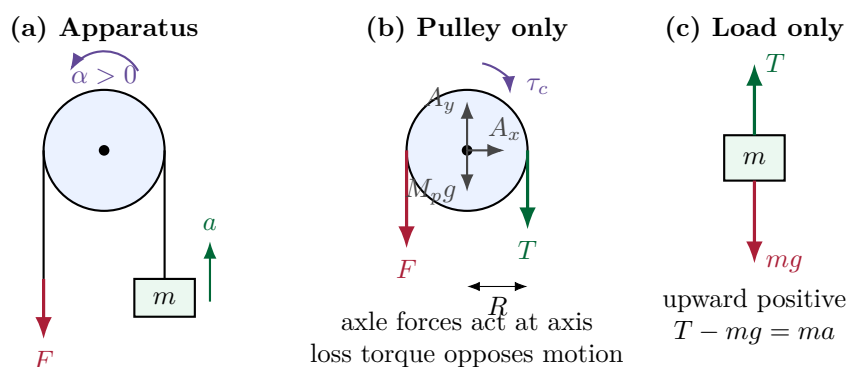


Figure 10.20: A force-driven pulley raises a load. Counterclockwise rotation is positive, so the left force produces positive torque and the right tension produces negative torque. The axle reaction and pulley weight have zero moment about the fixed axis. Arrow lengths and apparatus dimensions are schematic.

Assume an inextensible light cord, no slip, a fixed pulley axis, and a constant pulley inertia I_z . During positive rotation let a bearing loss have constant magnitude τ_c . Taking upward load acceleration and counterclockwise angular acceleration as positive gives

$$\begin{aligned} T - mg &= ma, & a &= R\alpha, \\ (F - T)R - \tau_c &= I_z\alpha. \end{aligned} \quad (10.42)$$

Eliminating T shows how the load contributes an effective rotational inertia:

$$\alpha = \frac{FR - mgR - \tau_c}{I_z + mR^2}. \quad (10.43)$$

The term mR^2 is not an extra physical component attached to the pulley. It is the load's translational kinetic-energy contribution expressed using $v = R\omega$. The axis remains fixed and the pulley's own inertia remains constant.

Example 10.16: A load lifted by a force-driven pulley

A pulley has $I_z = 0.0400 \text{ kg m}^2$ and $R = 0.0800 \text{ m}$. A constant force $F = 20.0 \text{ N}$ pulls the left cord downward and lifts $m = 1.20 \text{ kg}$ on the right. The loss torque has magnitude 0.0800 N m during positive rotation. Starting from rest, find the angular acceleration, load acceleration, tension, and speed after the load rises 0.400 m . Check the energy balance.

Strategy and solution. The applied force is prescribed, whereas the tension must follow from the load's acceleration. Equation (10.43) gives

$$\alpha = \frac{20(0.08) - 1.2(9.81)(0.08) - 0.08}{0.04 + 1.2(0.08)^2} = 12.13 \text{ rad/s}^2.$$

Consequently $a = 0.9702 \text{ m/s}^2$ upward and

$$T = m(g + a) = 12.94 \text{ N}.$$

Here $T > mg$ because the load accelerates upward, and $T < F$ because the pulley also accelerates and dissipates energy. The no-slip displacement is $\Delta\theta = h/R = 5.00 \text{ rad}$, so

$$\omega_f = \sqrt{2\alpha\Delta\theta} = 11.01 \text{ rad/s}.$$

The input work $Fh = 8.000 \text{ J}$ divides into gravitational energy $mgh = 4.7088 \text{ J}$, bearing loss $\tau_c\Delta\theta = 0.4000 \text{ J}$, and kinetic energies:

$$K_{\text{pulley}} = 2.4255 \text{ J}, \quad K_{\text{load}} = \frac{1}{2}m(R\omega_f)^2 = 0.4657 \text{ J}.$$

Their sum is 8.0000 J . This check would fail if one set $T = F$ or assigned all input work to the pulley.

Validity. The numerator is positive, as assumed. Starting upward requires $F > mg + \tau_c/R = 12.77 \text{ N}$ within this loss model. A negative numerator requires reconsidering the direction of motion and the friction sign; the positive-motion expression must not be used unchanged for a falling load.

10.6.5 Gravity changes torque throughout a rotation

For a rod swinging in a vertical plane, the force mg is constant but its lever arm changes. Torque determines the *instantaneous* angular acceleration; energy relates two orientations without finding the elapsed time. These methods answer different questions and together reveal more than either alone.

Let a uniform rod of mass M and length L be pivoted at one end and released from rest horizontally. Define ϕ as the downward angle from the initial horizontal, so increasing ϕ is clockwise. This local coordinate has $\phi = -\theta$ relative to a counterclockwise angular coordinate. Write $\omega_\phi = \dot{\phi}$ and $\alpha_\phi = \ddot{\phi}$. With $I_{\text{pivot}} = ML^2/3$, gravity gives

$$I_{\text{pivot}}\alpha_\phi = \frac{MgL}{2} \cos \phi, \quad \alpha_\phi = \frac{3g}{2L} \cos \phi. \quad (10.44)$$

The pivot reaction has zero moment about the pivot. It need not be zero as a force: the pivot prevents translation of one endpoint and generally exchanges momentum with the rod.

Taking the initial CM height as potential zero, the CM drops by $(L/2) \sin \phi$. The pivot point does not move, so the pivot does no mechanical work in the ideal model. Energy conservation yields

$$\frac{1}{2}I_{\text{pivot}}\omega_\phi^2 = \frac{MgL}{2} \sin \phi, \quad \omega_\phi^2 = \frac{3g}{L} \sin \phi. \quad (10.45)$$

Differentiating ω_ϕ^2 with respect to ϕ gives $2\alpha_\phi = 3g \cos \phi/L$, using $\alpha_\phi = \omega_\phi d\omega_\phi/d\phi$ away from the initial rest point. Its continuous limit agrees with Equation (10.44). Thus the energy and torque predictions are consistent, including the limiting initial acceleration.

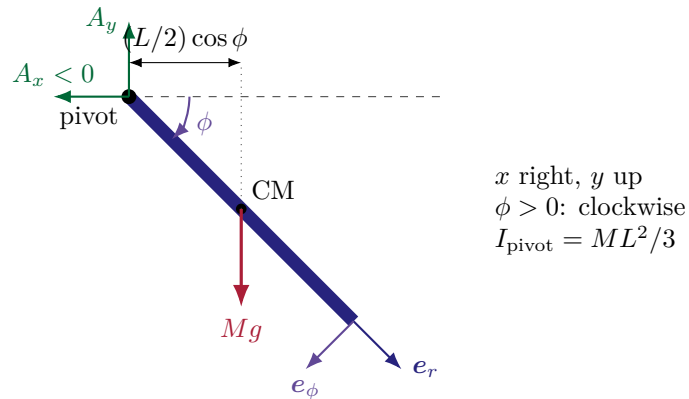


Figure 10.21: A falling rod with a fixed end pivot. The horizontal distance from pivot to gravity's line of action is the moment arm. The local unit vector e_ϕ points toward increasing downward angle. Reaction arrows show the actual directions for the pictured falling motion; A_x is the signed Cartesian component. The drawing uses $\phi = 45^\circ$ to illustrate the geometry.

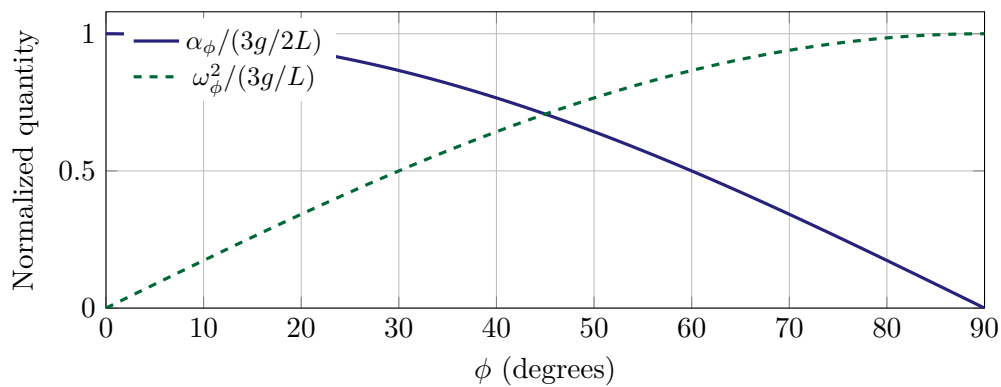


Figure 10.22: From horizontal release to the first downward vertical crossing, torque and angular acceleration decrease while kinetic energy and squared speed increase. At $\phi = 90^\circ$ the torque vanishes but the speed is maximal on this interval. The axes are explicitly normalized; degrees serve only as the plotted orientation coordinate.

Example 10.17: Speed, acceleration, and support force of a falling rod

For $M = 2.00$ kg and $L = 1.20$ m, calculate the rod's angular speed, angular acceleration, and pivot reaction at $\phi = 60^\circ$. Explain why zero torque at the vertical does not imply zero support force.

Rotational motion. Equations (10.44) and (10.45) give

$$\omega_\phi = 4.609 \text{ rad/s}, \quad \alpha_\phi = 6.131 \text{ rad/s}^2.$$

Neither constant-acceleration kinematics nor the initial gravitational torque can determine this finite-angle speed, because α_ϕ changes with orientation.

Translation of the CM. With x right and y up, $\mathbf{e}_r = (\cos \phi, -\sin \phi)$ and $\mathbf{e}_\phi = (-\sin \phi, -\cos \phi)$. The CM acceleration is

$$\mathbf{a}_{\text{CM}} = \frac{L}{2} (\alpha_\phi \mathbf{e}_\phi - \omega_\phi^2 \mathbf{e}_r).$$

Its components at 60° are $a_x = -9.558 \text{ m/s}^2$ and $a_y = 9.197 \text{ m/s}^2$. Newton's translational equation for the entire rod therefore gives

$$A_x = Ma_x = -19.12 \text{ N}, \quad A_y - Mg = Ma_y, \quad A_y = 38.01 \text{ N}.$$

The reaction is leftward and upward, with magnitude 42.55 N. This force calculation requires both tangential and inward normal acceleration.

Independent limits. At release $\omega_\phi = 0$, and the CM initially accelerates downward at $3g/4$, giving $A_y = Mg/4$. At the first vertical crossing $\alpha_\phi = 0$ but $\omega_\phi^2 = 3g/L$. The CM then accelerates upward at $3g/2$, so $A_y = 5Mg/2 = 49.05 \text{ N}$. A large support force and zero torque coexist because the support acts at the pivot and the moving CM requires inward acceleration.

10.7 Rotational work and power

A tangential force F_t applied at radius r through an angular displacement $d\theta$ moves its point of application through $ds = r d\theta$. The incremental work is

$$dW = F_t ds = (rF_t) d\theta = \tau_z d\theta.$$

Therefore

$$W = \int_{\theta_i}^{\theta_f} \tau_z(\theta) d\theta. \quad (10.46)$$

For net torque on a fixed-axis rigid body,

$$W_{\text{net}} = \Delta K_{\text{rot}} = \frac{1}{2} I_z \omega_f^2 - \frac{1}{2} I_z \omega_i^2. \quad (10.47)$$

The instantaneous mechanical power is

$$P = \frac{dW}{dt} = \tau_z \omega. \quad (10.48)$$

Signs matter: positive P means the torque is adding rotational kinetic energy at that instant; negative P means it is removing it.

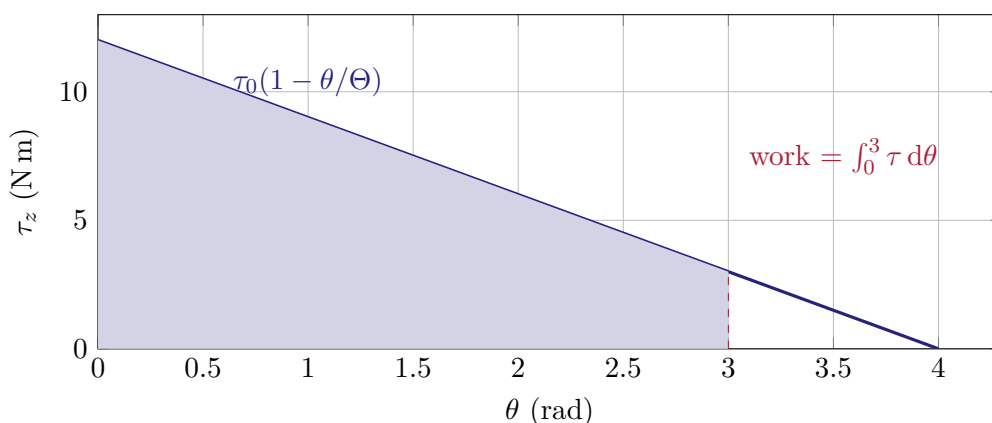


Figure 10.23: Prescribed variable torque for Example 10.18. The shaded area is rotational work, with units $\text{N m} = \text{J}$. The torque itself is still labelled in N m , not J .

Example 10.18: A positioning drive with angle-dependent torque

A rotor with $I_z = 3.0 \text{ kg m}^2$ begins at $\theta = 0$ with $\omega_0 = 1.0 \text{ rad/s}$. Over $0 \leq \theta \leq 4.0 \text{ rad}$, a drive applies

$$\tau_z(\theta) = \tau_0 \left(1 - \frac{\theta}{\Theta}\right), \quad \tau_0 = 12 \text{ N m}, \quad \Theta = 4.0 \text{ rad}.$$

Neglect losses. Find the speed at $\theta = 3.0 \text{ rad}$ and the instantaneous power there.

The work is

$$W = \tau_0 \left[\theta - \frac{\theta^2}{2\Theta} \right]_0^3 = 12 \left(3 - \frac{9}{8} \right) = 22.5 \text{ J}.$$

Initially $K_i = \frac{1}{2}(3)(1)^2 = 1.5 \text{ J}$, so $K_f = 24.0 \text{ J}$ and

$$\omega_f = \sqrt{\frac{2K_f}{I_z}} = 4.00 \text{ rad/s}.$$

At $\theta = 3 \text{ rad}$, $\tau_z = 3.0 \text{ N m}$, hence

$$P = \tau_z \omega_f = 12.0 \text{ W}.$$

Check. The torque remains positive over the interval, so the speed must increase. The area in Fig. 10.23 is a trapezoid with parallel sides 12 and 3 N m and width 3 rad , giving 22.5 J independently.

10.7.1 Braking models predict different stopping histories

The work–energy theorem constrains the total energy removed, but a braking law determines how quickly it is removed. Two models can begin with the same torque and still predict different stopping times and angles. During positive rotation, constant-magnitude braking gives $I_z \dot{\omega} = -\tau_c$, whereas viscous braking gives $I_z \dot{\omega} = -b\omega$. Here $b > 0$ has unit N m s and the viscous torque automatically reverses sign when the rotation reverses.

For a constant brake and $\omega_0 > 0$,

$$\omega(t) = \omega_0 - \frac{\tau_c}{I_z} t, \quad t_s = \frac{I_z \omega_0}{\tau_c}, \quad \Delta\theta_s = \frac{I_z \omega_0^2}{2\tau_c}. \quad (10.49)$$

These equations describe $0 \leq t \leq t_s$. A passive brake cannot continue the same negative torque beyond rest: that would make $\omega < 0$, then give $P = (-\tau_c)\omega > 0$, so the device would add energy.

At rest the appropriate static-contact or controller model takes over. In the simple idealization with no driving torque the rotor remains at rest.

For viscous braking, separation of variables gives

$$\omega(t) = \omega_0 e^{-t/t_b}, \quad t_b = \frac{I_z}{b}, \quad \Delta\theta(t) = \omega_0 t_b (1 - e^{-t/t_b}). \quad (10.50)$$

The angular speed approaches zero asymptotically, and the total angle approaches $\omega_0 t_b$. Since $P = -b\omega^2 \leq 0$, the model is passive for either sense of rotation. A real detection threshold can define an operational stopping time; the exact exponential model has no finite zero-speed event.

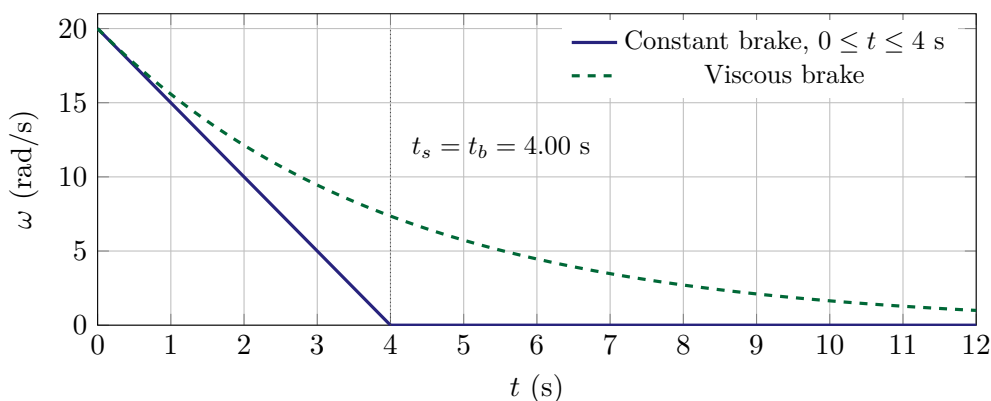


Figure 10.24: Brakes with equal initial torque can have different histories. Here $I_z = 0.0800 \text{ kg m}^2$, $\omega_0 = 20.0 \text{ rad/s}$, $\tau_c = 0.400 \text{ N m}$, and $b = 0.0200 \text{ N m s}$. The constant brake switches to a rest regime at 4 s; the dashed exponential still has positive speed.

Example 10.19: Equal initial braking torque, unequal stopping angle

Use the parameters in Figure 10.24. Find the constant-brake stopping angle and the viscous-brake speed and angle after 4.00 s. Check energy removal and find the viscous time to reach 1% of the initial speed.

Constant brake. The initial energy is $K_0 = \frac{1}{2}(0.08)(20)^2 = 16.0 \text{ J}$. The constant angular deceleration is -5.00 rad/s^2 , the stopping time is 4.00 s, and $\Delta\theta_s = 40.0 \text{ rad}$. Independently, the brake work is $-\tau_c \Delta\theta_s = -16.0 \text{ J}$, exactly the kinetic-energy change.

Viscous brake. Its time scale is also $t_b = 4.00 \text{ s}$, but

$$\omega(4) = 20e^{-1} = 7.358 \text{ rad/s}, \quad \Delta\theta(4) = 80(1 - e^{-1}) = 50.57 \text{ rad}.$$

The remaining energy is $K(4) = 16e^{-2} = 2.165 \text{ J}$. Integrating its dissipated power independently gives

$$\int_0^4 b\omega^2 dt = b\omega_0^2 \frac{t_b}{2} (1 - e^{-2}) = 13.835 \text{ J} = K_0 - K(4).$$

To reach $\omega/\omega_0 = 0.01$, solve $e^{-t/t_b} = 0.01$: $t = 4 \ln 100 = 18.42 \text{ s}$. The limiting total angle is 80.0 rad. Thus describing both cases as a four-second brake would conceal a physically important difference.

10.7.2 A motor's torque–speed curve sets its steady speed

A prescribed constant torque is often useful over a limited interval. A different model is needed when the drive torque decreases as the shaft speed rises. Consider the idealized positive-speed law

$$\tau_m(\omega) = \tau_s \left(1 - \frac{\omega}{\omega_f}\right), \quad \tau_{\text{load}}(\omega) = \tau_c + b\omega.$$

The parameter τ_s is the drive's zero-speed torque and ω_f its unloaded zero-torque speed in this linear model. These are teaching parameters, not specifications for a named motor. The law is used only on its declared positive-speed interval. The opposing constant load term describes positive motion; a static threshold must be considered at startup.

Subtracting load torque from motor torque gives a first-order equation,

$$I_z \dot{\omega} = (\tau_s - \tau_c) - k\omega, \quad k = \frac{\tau_s}{\omega_f} + b. \quad (10.51)$$

If $\tau_s > \tau_c$, the steady speed and response time are $\omega_\infty = (\tau_s - \tau_c)/k$ and $t_m = I_z/k$. For a start from rest,

$$\omega(t) = \omega_\infty(1 - e^{-t/t_m}). \quad (10.52)$$

Substitution checks both the initial condition and the differential equation. The intersection of the torque curves is stable: a speed slightly below it has positive net torque, and a speed slightly above it has negative net torque. At steady speed the motor still delivers positive power, all of which the load receives or dissipates. Zero angular acceleration does not mean zero power.

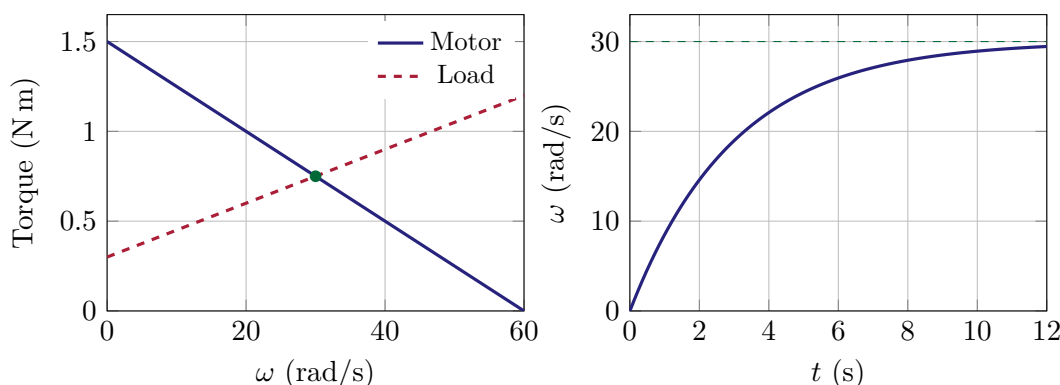


Figure 10.25: The motor/load intersection determines the steady speed; inertia determines the time needed to approach it. Parameters are $\tau_s = 1.50$ N m, $\omega_f = 60.0$ rad/s, $\tau_c = 0.300$ N m, $b = 0.0150$ N m s, and $I_z = 0.120$ kg m². Torque balance occurs at 30.0 rad/s and 0.750 N m.

Example 10.20: Acceleration and power of a loaded rotary drive

For Figure 10.25, find the steady speed, time to reach 90% of it, and shaft power at 6.00 s. Identify where the power goes.

The coefficient $k = 1.5/60 + 0.015 = 0.0400$ N m s gives $\omega_\infty = 30.0$ rad/s and $t_m = 3.00$ s. The fractional remaining speed difference is $e^{-t/3}$, so 90% is reached at $t = 3 \ln 10 = 6.908$ s.

At 6.00 s, $\omega = 25.94$ rad/s, $\tau_m = 0.8515$ N m, and $\tau_{\text{load}} = 0.6891$ N m. The motor shaft power is

$$P_m = \tau_m \omega = 22.09 \text{ W}.$$

The load receives $P_{\text{load}} = 17.88$ W, leaving $P_{\text{net}} = 4.213$ W to increase kinetic energy. Independently, $I_z \omega \dot{\omega} = 4.213$ W follows by differentiating $K = \frac{1}{2} I_z \omega^2$. At steady speed both torque curves give 0.750 N m; motor and load powers are each 22.5 W and the energy-growth rate is zero.

10.7.3 Constant power requires a startup torque limit

The identity $P = \tau \omega$ does not specify a drive law. In a constant-power operating regime, $\tau = P_0/\omega$ for $\omega > 0$, so the torque decreases with speed. Extending that law to rest would require an infinite torque. A simple lossless drive model instead supplies its maximum torque τ_{max} until the speed

reaches $\omega_c = P_0/\tau_{\max}$, then supplies constant power:

$$\tau_m(\omega) = \begin{cases} \tau_{\max}, & 0 \leq \omega \leq \omega_c, \\ P_0/\omega, & \omega \geq \omega_c. \end{cases} \quad (10.53)$$

Starting from rest, the transition time is $t_c = I_z P_0 / \tau_{\max}^2$. In the first regime $\omega = \tau_{\max} t / I_z$. After the transition, energy increases at rate P_0 , giving

$$\omega(t) = \begin{cases} \tau_{\max} t / I_z, & 0 \leq t \leq t_c, \\ \sqrt{\omega_c^2 + (2P_0/I_z)(t - t_c)}, & t \geq t_c. \end{cases} \quad (10.54)$$

Speed, torque, power, and angular acceleration match at the transition. The startup power rises from zero to P_0 instead of being constant from the first instant.

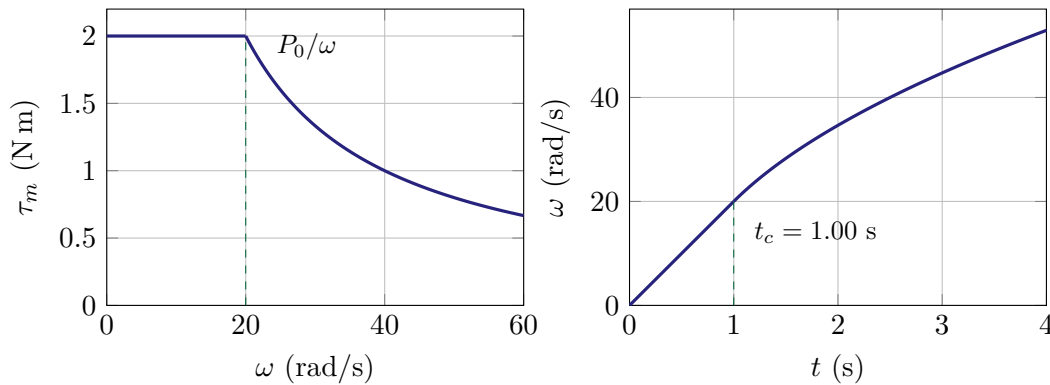


Figure 10.26: A torque-limited startup joins smoothly to constant-power operation. The constructed ideal model uses $I_z = 0.100 \text{ kg m}^2$, $P_0 = 40.0 \text{ W}$, and $\tau_{\max} = 2.00 \text{ N m}$. The torque plateau prevents a zero-speed singularity; the second speed segment grows as a square root.

Example 10.21: Energy and angle for a torque-limited constant-power drive

For Figure 10.26, calculate the speed, energy, and angular displacement at 4.00 s. Compare with a proposed calculation $W = P_0 t$ from startup.

Here $\omega_c = 20.0 \text{ rad/s}$ and $t_c = 1.00 \text{ s}$. At 4.00 s,

$$\omega = \sqrt{400 + 800(4 - 1)} = \sqrt{2800} = 52.92 \text{ rad/s}.$$

The startup energy is $\frac{1}{2} I_z \omega_c^2 = 20.0 \text{ J}$. The constant-power interval supplies another $40(3) = 120 \text{ J}$, so $K(4) = 140 \text{ J}$, in agreement with $\frac{1}{2} I_z \omega(4)^2$. The proposed $40(4) = 160 \text{ J}$ overestimates energy because power is smaller than 40 W during startup.

The first interval contributes $\theta_c = \frac{1}{2} (\tau_{\max}/I_z) t_c^2 = 10.0 \text{ rad}$. In the second interval use $I_z \omega d\omega = P_0 dt$, hence

$$\Delta\theta_{\text{second}} = \frac{I_z}{3P_0} [\omega(4)^3 - \omega_c^3] = 116.80 \text{ rad}.$$

The total is 126.80 rad, or 20.18 revolutions. Differentiating this integrated expression recovers $\dot{\theta} = \omega$, while $\tau_m(4) = 40/52.92 = 0.756 \text{ N m}$ still gives $P = 40.0 \text{ W}$.

10.8 Engineering, computation, and measurement

10.8.1 Engineering connection: flywheel energy and power

A first-pass flywheel calculation often starts with

$$E = \frac{1}{2} I_z \omega^2, \quad P = \tau_z \omega.$$

These equations answer energy and instantaneous shaft-power questions within the rigid fixed-axis model. They do *not* certify a rotor structurally. A real design additionally requires stress and fatigue analysis, bearing and shaft loads, material defects, thermal limits, containment, overspeed margins, vibration modes, and applicable engineering standards. The simple mechanics is useful because it isolates the energetic scale before those additional models are introduced.

For example, the annular flywheel in Example 10.8 stores 1125 J at 30 rad/s. If a motor delivers a constant shaft power of 1.50 kW at that instant, the corresponding ideal torque is $P/\omega = 50.0$ N m. A constant-power model cannot be extended to $\omega = 0$ by writing $\tau = P/\omega$; that singularity signals that the motor/control model must change near startup.

10.8.2 Computational connection: inertia and prescribed torque histories

For an arbitrary discrete mass model,

$$I_z = \sum_i m_i (x_i^2 + y_i^2)$$

if the axis is the z axis through the coordinate origin. For a prescribed torque history and constant I_z , one can integrate

$$\alpha(t) = \frac{\tau_z(t)}{I_z}, \quad \omega(t) = \omega_0 + \int_0^t \alpha(t') dt', \quad \theta(t) = \theta_0 + \int_0^t \omega(t') dt'.$$

A minimal implementation is shown below. The code is not required to compile this chapter; it is printed here as reproducible teaching content.

Listing 10.1: Direct inertia calculation and time stepping for a prescribed torque.

```
def inertia_z(cells):
    # cell = (mass, x, y), SI units
    return sum(m*(x*x + y*y) for (m, x, y) in cells)

def advance(theta, omega, torque, Iz, dt):
    alpha = torque/Iz
    # constant-alpha update over one small interval
    theta_new = theta + omega*dt + 0.5*alpha*dt*dt
    omega_new = omega + alpha*dt
    return theta_new, omega_new
```

A verification sequence should include: (1) point masses with hand-calculated I_z ; (2) a discretized uniform disk approaching $MR^2/2$ under refinement; (3) zero torque giving constant ω ; and (4) constant torque reproducing Eqs. (10.8)–(10.9). A numerical trajectory that “looks smooth” is not by itself evidence that the physical model or code is correct.

10.8.3 Time stepping: test an actual torque history

The printed update treats torque as constant during each small interval. That is exact for truly constant torque, but a varying torque introduces an approximation. A time-step study asks a physical question: how accurately does the calculated motion respond to the specified input torque?

A smooth-looking trajectory is insufficient evidence. Compare the numerical final speed and angle with independent analytic integrations, then halve the interval.

Consider a fixed-axis rotor with constant inertia I_z and prescribed net torque $\tau(t) = \tau_0 + ct$. Integrating the dynamical equation gives

$$\omega(t) = \omega_0 + \frac{\tau_0}{I_z}t + \frac{c}{2I_z}t^2, \quad \theta(t) = \theta_0 + \omega_0 t + \frac{\tau_0}{2I_z}t^2 + \frac{c}{6I_z}t^3.$$

These functions satisfy the initial conditions and recover the specified torque when $I_z d\omega/dt$ is evaluated. They are a benchmark for the algorithm, independent of its finite-step recurrence.

Example 10.22: A refinement test for a ramped net torque

Take $I_z = 0.500 \text{ kg m}^2$, $\tau_0 = 0.200 \text{ N m}$, $c = 0.100 \text{ N m/s}$, $\omega_0 = 1.00 \text{ rad/s}$, $\theta_0 = 0$, and final time $T = 2.00 \text{ s}$. Determine the exact final state and assess the earlier left-endpoint torque update at step sizes $h = 0.500, 0.250, 0.125, 0.0625 \text{ s}$. Give a better update for this prescribed ramp and explain how to check it.

Exact state. The angular acceleration is $0.400 + 0.200t \text{ rad/s}^2$. The final values are $\omega(T) = 2.200000 \text{ rad/s}$ and $\theta(T) = 3.066666667 \text{ rad}$. Substitution into the continuous equations confirms both values without invoking a numerical mesh.

Computed refinement. On each interval the algorithm uses $\alpha_k = (\tau_0 + ct_k)/I_z$, updates angle using the old speed and α_k , and then updates speed. Its calculated endpoint values are

$h \text{ (s)}$	$\omega_N \text{ (rad/s)}$	$\theta_N \text{ (rad)}$	$ \theta_N - \theta(T) \text{ (rad)}$
0.5000	2.100000	2.975000000	0.091666667
0.2500	2.150000	3.018750000	0.047916667
0.1250	2.175000	3.042187500	0.024479167
0.0625	2.187500	3.054296875	0.012369792

The leading errors shrink by approximately two when h is halved. The method is first-order for this variable-torque history, even though the angle formula contains an h^2 term. A polynomial term in the update does not by itself determine the global accuracy.

Independent error derivation. Sum the arithmetic series of the sampled ramp torques and the resulting speed increments. For $N = T/h$ this gives

$$\omega_N - \omega(T) = -\frac{cTh}{2I_z}, \quad \theta_N - \theta(T) = -\frac{cT^2h}{4I_z} + \frac{cTh^2}{12I_z}.$$

These formulas reproduce the table and predict the leading linear dependence on h in Figure 10.27.

Improved model integration. Because $\alpha(t)$ is linear on this particular interval, the exact local updates are

$$\omega_{k+1} = \omega_k + \alpha_k h + \frac{c}{2I_z}h^2, \quad \theta_{k+1} = \theta_k + \omega_k h + \frac{1}{2}\alpha_k h^2 + \frac{c}{6I_z}h^3.$$

They reproduce the continuous polynomial at every node up to floating-point roundoff. This exactness is specific to the prescribed ramp. For a nonlinear or speed-dependent torque, evaluate a suitable integration method and repeat a convergence or exact-limit test; do not assume the ramp update remains exact.

The inertia mesh and time step are independent sources of error. First verify a density integral against a continuum benchmark, then hold that verified inertia fixed while testing the time integrator. Finally refine both when both are approximations. Replacing I_z by a discretized I_N changes the predicted acceleration even if the time integration is exact. This separation helps diagnose whether an incorrect final speed arises from the body model, the torque model, or the integration algorithm.

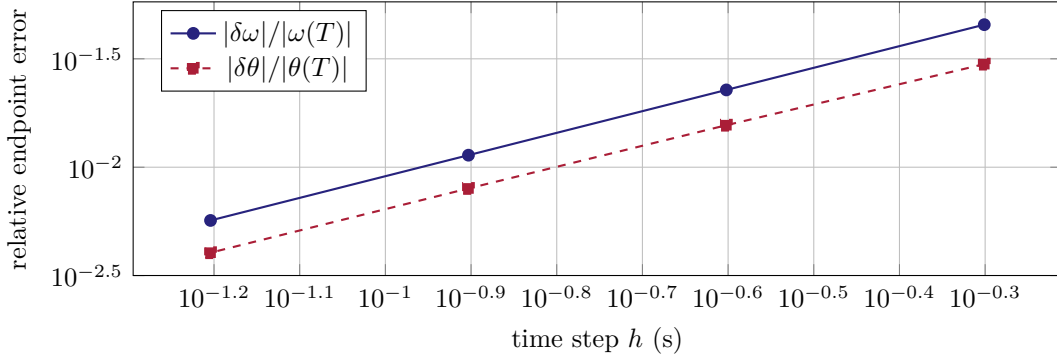


Figure 10.27: Relative endpoint errors for the left-endpoint ramped-torque update, normalized by the nonzero exact endpoint speed and angle. Both approach first-order behavior as the time step decreases; dimensionless normalization permits a common error axis.

10.8.4 Measurement connection: encoder resolution and inferred rim speed

An encoder measures angle rather than speed directly. A finite difference over a time interval Δt estimates $\bar{\omega} = \Delta\theta/\Delta t$, an interval average. It equals the instantaneous velocity throughout the interval only for constant-speed motion. For varying motion, an average must be assigned to an interval or a specified approximation to a time within it; merely increasing the number of digits on the display does not change its physical meaning.

For an ideal absolute encoder with N equally spaced positions per revolution, the angular increment is $q = 2\pi/N$. Suppose each reported angle is rounded to the nearest position. Each individual quantization error then lies in $[-q/2, q/2]$, so the worst possible error in a difference is q . If, for a statistical uncertainty model, the two rounding errors are independent and uniform within their bins, their individual standard uncertainties are $q/\sqrt{12}$ and the difference has standard uncertainty $q/\sqrt{6}$. This statistical assumption should be stated: correlated errors, calibration offsets, shaft compliance, eccentricity, or deterministic phase locking can behave differently.

For independent angular-difference and time-interval uncertainties, first-order propagation gives

$$\sigma_{\bar{\omega}}^2 \simeq \left(\frac{q}{\sqrt{6}\Delta t} \right)^2 + \left(\frac{\bar{\omega} \sigma_{\Delta t}}{\Delta t} \right)^2. \quad (10.55)$$

Both terms decrease with a longer interval if the underlying speed is approximately constant and $\sigma_{\Delta t}$ is fixed. A longer interval, however, reduces time resolution and may violate the sampling condition needed for unwrapping. Figure 10.28 shows this tradeoff for a constructed measurement model. The speed bound and the quantization resolution address different problems: one chooses a turn count, while the other estimates how accurately a permitted angle difference is known.

Example 10.23: An uncertainty budget for a rim-speed estimate

An ideal 4096-position absolute encoder records a change of 391 positions over $\Delta t = 0.02000$ s. Assume approximately constant positive motion during this interval. A continuously valid independent speed limit is 40.0 rad/s. The interval's standard uncertainty is $20 \mu\text{s}$, and the independent uniform quantization model above is adopted. A rim radius is measured as $R = 0.1500$ m with standard uncertainty 0.0005 m. Estimate the mean angular velocity, mean rim speed, and their standard uncertainties.

Sampling and central values. Since $40.0(0.02000) = 0.800$ rad is below π by a wide margin, the prescribed interval satisfies the sufficient unwrapping condition; even the maximum difference quantization error is small compared with that margin. With $q = 2\pi/4096 = 0.00153398$ rad,

$$\Delta\theta = 391q = 0.599786 \text{ rad}, \quad \bar{\omega} = 29.9893 \text{ rad/s}.$$

The inferred mean rim speed is $R\bar{\omega} = 4.49840$ m/s, assuming the motion remains positive during the interval. The two angular-velocity uncertainty contributions are

$$\sigma_{\omega,q} = 0.03131 \text{ rad/s}, \quad \sigma_{\omega,t} = \frac{29.9893(20 \times 10^{-6})}{0.02000} = 0.02999 \text{ rad/s}.$$

Combining independent contributions in quadrature gives $\sigma_{\omega} = 0.04336$ rad/s. Radius uncertainty enters the linear speed through $v = R\omega$, so

$$\sigma_v = \sqrt{(\bar{\omega}\sigma_R)^2 + (R\sigma_{\omega})^2} = 0.01634 \text{ m/s}.$$

The results may be reported as $\bar{\omega} = (29.989 \pm 0.043)$ rad/s and $\bar{v} = (4.498 \pm 0.016)$ m/s, where the $\pm$ values denote standard uncertainties under the stated model.

Interpret and check. The radius contribution to σ_v is 0.01499 m/s, larger than the encoder/time contribution 0.00650 m/s. Increasing encoder resolution alone therefore has limited benefit for the final rim speed. In a worst-case bound analysis, the angle-difference error would instead be bounded by q , not $q/\sqrt{6}$; a standard uncertainty must not be presented as a guaranteed maximum error. All quoted inputs are synthetic teaching specifications, not the performance of a named instrument.

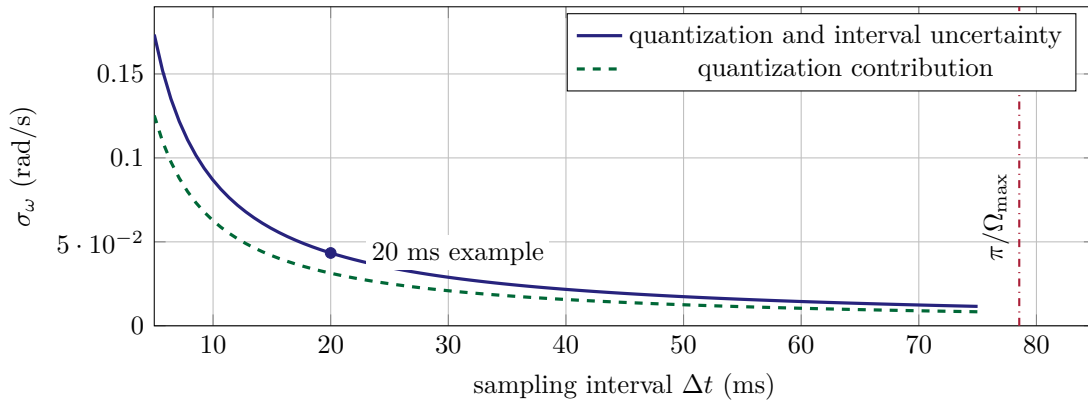


Figure 10.28: Uncertainty versus sampling interval for a synthetic constant-speed model with $\omega = 30$ rad/s, $N = 4096$, and $\sigma_{\Delta t} = 20 \mu\text{s}$. The independently trusted bound $\Omega_{\max} = 40$ rad/s requires $\Delta t < 78.54$ ms for the stated unwrapping guarantee. Curves stop below that limit. The example's speed differs slightly from 30 rad/s because its count change is an integer.

10.8.5 An inverse inertia experiment: accelerated tension matters

A hanging mass can drive a low-speed platform through a cord wrapped around a hub. The observed quantity is an unwrapped angular history; a quadratic fit $\theta(t) = \theta_0 + \omega_0 t + \alpha t^2/2$ estimates acceleration when it is approximately constant. The inferred quantity is the platform inertia, and possibly a loss torque. Figure 10.29 shows the apparatus and the graph that the model requires. The radius is the *effective winding radius*; cord layers changing that radius violate the constant-radius model.

Take descending load motion as positive. For a mass m and hub radius R_h , the separate body equations are

$$mg - T = mR_h\alpha, \quad TR_h - \tau_c = I_z\alpha.$$

The correct available torque is therefore

$$\tau_{\text{corr}} = m(g - R_h\alpha)R_h = I_z\alpha + \tau_c. \quad (10.56)$$

Plotting τ_{corr} against α separates the slope I_z from the intercept τ_c under a constant-loss model. Plotting mgR_h instead gives $mgR_h = (I_z + mR_h^2)\alpha + \tau_c$. Because m changes between runs, that expression does not describe one common straight line with slope I_z .

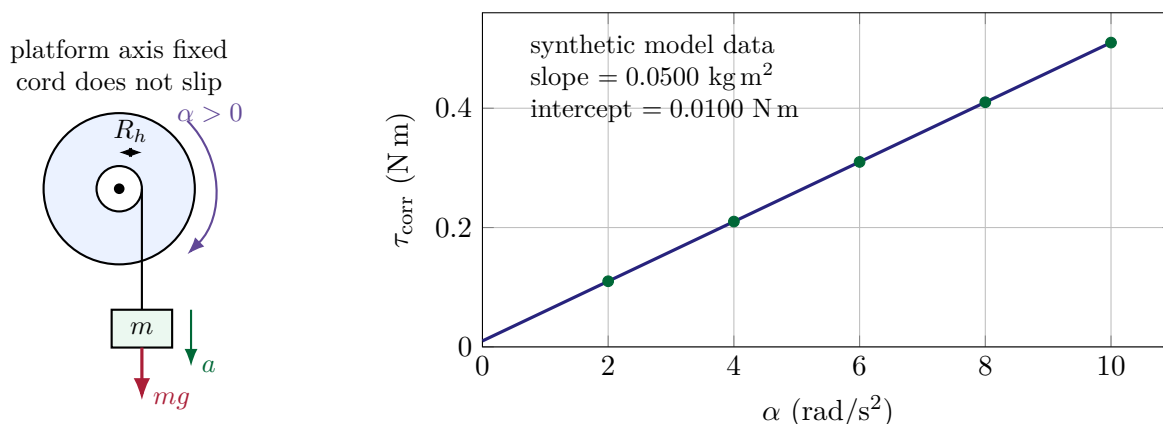


Figure 10.29: A proposed hanging-mass inertia experiment and its corrected-torque graph. The points are explicitly synthetic teaching data generated from $I_z = 0.0500 \text{ kg m}^2$, $R_h = 0.0500 \text{ m}$, and constant $\tau_c = 0.0100 \text{ N m}$. Apparatus dimensions and arrow lengths are schematic. The regression variables are linked by the accelerated-tension model.

The following values are constructed, not measured. For each chosen acceleration the hanging mass was generated from $m = (I_z\alpha + \tau_c)/(gR_h - R_h^2\alpha)$, retaining the same platform inertia and loss torque. Displayed masses are rounded; guard digits were used for the torque column.

$\alpha \text{ (rad/s}^2\text{)}$	$m \text{ (kg)}$	$T \text{ (N)}$	$\tau_{\text{corr}} \text{ (N m)}$
2.00	0.226571	2.2000	0.1100
4.00	0.437045	4.2000	0.2100
6.00	0.651945	6.2000	0.3100
8.00	0.871413	8.2000	0.4100
10.00	1.095596	10.2000	0.5100

Example 10.24: Recovering inertia and estimating uncertainty

Use the synthetic table to infer I_z and τ_c . For the $\alpha = 6.00 \text{ rad/s}^2$ row, estimate the standard uncertainty of a one-run inertia estimate if independent input standard uncertainties are $u_m = 0.0002 \text{ kg}$, $u_R = 0.0002 \text{ m}$, $u_\alpha = 0.04 \text{ rad/s}^2$, and $u_\tau = 0.002 \text{ N m}$. Treat $g = 9.81 \text{ m/s}^2$ as the stated teaching constant.

Inverse graph. The endpoint slope is $I_z = (0.5100 - 0.1100)/(10 - 2) = 0.0500 \text{ kg m}^2$. The intercept is $0.1100 - I_z(2) = 0.0100 \text{ N m}$. All five rows lie on the line by construction. Their agreement demonstrates the calculation, not experimental confirmation of the apparatus model.

Why the correction is consequential. In the middle row, $mgR_h = 0.319779 \text{ N m}$, while $TR_h = 0.310000 \text{ N m}$. Neglecting the falling mass's acceleration and otherwise using the correct loss would give $(mgR_h - \tau_c)/\alpha = 0.051630 \text{ kg m}^2$, about 3.26% too large.

Uncertainty calculation. With a separately estimated loss torque,

$$I_z = \frac{mgR_h - \tau_c}{\alpha} - mR_h^2.$$

For small independent input uncertainties, a first-order standard uncertainty is the square root

of the sum of squared sensitivity contributions. The required derivatives are

$$\begin{aligned}\frac{\partial I_z}{\partial m} &= \frac{gR_h}{\alpha} - R_h^2, & \frac{\partial I_z}{\partial R_h} &= \frac{mg}{\alpha} - 2mR_h, \\ \frac{\partial I_z}{\partial \alpha} &= -\frac{mgR_h - \tau_c}{\alpha^2}, & \frac{\partial I_z}{\partial \tau_c} &= -\frac{1}{\alpha}.\end{aligned}$$

The absolute contributions are, in kg m^2 , 0.0000159, 0.000200, 0.000344, and 0.000333. Their quadrature gives $u_I = 0.000520 \text{ kg m}^2$, or about 1.04% of the nominal inertia.

Interpretation. This is an illustrative propagated uncertainty, not a confidence interval derived from actual repeats. A radius calibration error is common to all runs and does not disappear when more loads are used. Moreover, α appears in both graph coordinates, so an experimental regression should account for uncertainty in both coordinates and their correlation. A small ordinary least-squares slope error alone would not describe the full uncertainty.

10.8.6 Residual structure tests the loss model

An excellent straight line can hide an inadequate experiment. If every higher acceleration is observed at a higher speed, viscous torque can be absorbed into an apparent inertia slope. Varying load and sampling speed separately helps identify that confounding. A residual is the observed corrected torque minus the torque predicted by the fitted model, evaluated at the same acceleration and speed.

As a transparent synthetic diagnostic, suppose the true loss is $\tau_{\text{loss}} = 0.0100 + 0.00100\omega$ in N m for positive rotation, with ω in rad/s. Hold the comparison acceleration at 6.00 rad/s^2 by choosing the required loads, and sample speeds 5, 10, 15, 20, and 25 rad/s. Equation (10.56), now including viscous loss, gives corrected torques 0.315, 0.320, 0.325, 0.330, and 0.335 N m. A constant-loss fit at this one acceleration predicts their mean, 0.325 N m. Its residuals have zero mean but systematically increase with speed.

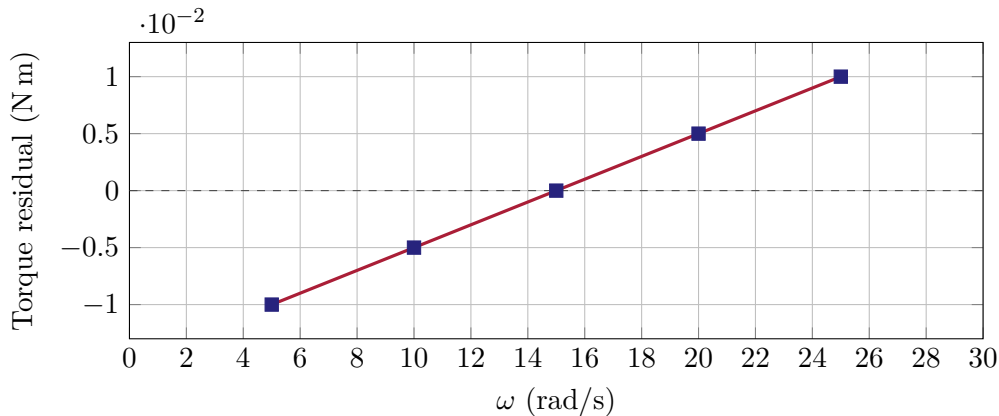


Figure 10.30: Synthetic residuals from fitting a constant loss to speed-dependent loss. The zero mean is guaranteed by the fitted mean torque and does not validate the constant-loss model. The systematic slope identifies an omitted viscous contribution $b = 0.00100 \text{ N m s}$. These are model-generated local observations, not a constant-acceleration time history.

The model-consistent comparison loads are obtained from $m = (I_z\alpha + \tau_c + b\omega)/(gR_h - R_h^2\alpha)$, so no instantaneous observation violates the coupled mass–platform equations. This construction also explains what a real protocol would require: local acceleration estimates, recorded speed, an unchanged hub radius, and loads chosen to explore more than one independent condition. If acceleration changes appreciably during a run, fit the evolving torque model rather than one quadratic over the whole motion.

The residual slope in Figure 10.30 is $(0.010 - (-0.010))/(25 - 5) = 0.00100 \text{ N m s}$. Adding $b\omega$ removes these residuals in the constructed case. In actual data, remaining scatter must be compared with observational uncertainty, while drift, curvature, or dependence on winding angle can indicate a changing radius, speed-dependent losses, or slipping cord. An assessment of the proposed method should reward correct body equations, appropriate angle unwrapping, separate calibration uncertainty, and an interpretation of residuals; a precise-looking inertia number alone is insufficient.

10.9 Advanced Topics for Technical and Engineering Students

The following material extends the fixed-axis core without becoming a prerequisite for the chapter exercises marked “core.” The additional mathematical tools are numerical quadrature, elementary series/error reasoning, and matrix–vector multiplication. The first module quantifies a nonuniform mass distribution and numerical convergence; the second previews why a scalar moment of inertia is insufficient for arbitrary three-dimensional rotation.

10.9.1 Nonuniform bodies and numerical inertia

For a thin axisymmetric disk whose surface density varies with radius,

$$\sigma(r) = \sigma_0 \left(1 + \beta \frac{r^2}{R^2} \right), \quad 0 \leq r \leq R, \quad (10.57)$$

the mass and fixed-axis inertia are

$$M = 2\pi \int_0^R \sigma(r) r \, dr = \pi \sigma_0 R^2 \left(1 + \frac{\beta}{2} \right), \quad (10.58)$$

$$I_z = 2\pi \int_0^R \sigma(r) r^3 \, dr = \pi \sigma_0 R^4 \left(\frac{1}{2} + \frac{\beta}{3} \right). \quad (10.59)$$

Therefore

$$\frac{I_z}{MR^2} = \frac{\frac{1}{2} + \frac{\beta}{3}}{1 + \frac{\beta}{2}}. \quad (10.60)$$

For $\beta = 0$ the uniform-disk result $1/2$ is recovered. For positive β , more mass lies toward the rim and the normalized inertia increases.

10.9.2 Density constraints and what is held fixed

For the model in Eq. (10.57), take $\sigma_0 > 0$. A nonnegative surface density throughout the disk requires $\beta \geq -1$: its smallest value is at the edge when $\beta < 0$. If strictly positive density is required even at the edge, use $\beta > -1$. Algebraically evaluating the ratio for a parameter outside this physical domain would produce a formal number for a body with negative mass density.

Two comparisons must be distinguished. If σ_0 and R are held fixed while β changes, both the mass and inertia change. If the total mass M and radius R are held fixed, the central density must change according to

$$\sigma_0(\beta) = \frac{M}{\pi R^2 (1 + \beta/2)}.$$

This is a deliberate change in the physical model between disks. It is not permission to alter the density normalization as a numerical grid is refined for one disk.

Introduce the dimensionless ratio $q = I_z/(MR^2)$. Its derivative is

$$\frac{dq}{d\beta} = \frac{1}{12(1 + \beta/2)^2} > 0 \quad (\beta \geq -1).$$

Thus the ratio increases monotonically with outward grading. At the physical endpoint $\beta = -1$, $q = 1/3$, while the uniform disk has $q = 1/2$. As $\beta \rightarrow \infty$, $q \rightarrow 2/3$, rather than unity: even strongly graded disks in this particular family retain material inside the rim. A thin ring is a different density family, not the infinite- β limit of this polynomial model.

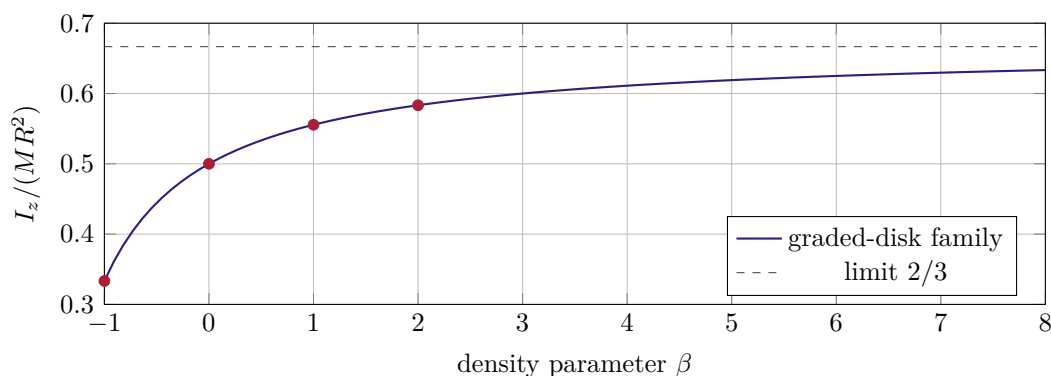


Figure 10.31: Normalized inertia for the nonnegative polynomial-density family. Total mass and radius are fixed between curves' parameter values, with the central density adjusted explicitly. The asymptote is $2/3$, because this family does not concentrate all mass at the rim.

Example 10.25: A radially graded rotor and a convergence check

Let $R = 0.30$ m, $M = 6.0$ kg and $\beta = 1$. From Eq. (10.60),

$$I_z = \frac{5}{9}MR^2 = 0.300000 \text{ kg m}^2,$$

and the required central density is $\sigma_0 = 14.1471 \text{ kg/m}^2$.

Discretize the disk into N equal-width radial cells. At midpoint $r_j = (j + \frac{1}{2})R/N$, approximate

$$\Delta m_j \approx 2\pi r_j \sigma(r_j) \Delta r, \quad I_z^{(N)} = \sum_j r_j^2 \Delta m_j.$$

The resulting calculated values are

N	$M^{(N)}$ (kg)	$I_z^{(N)}$ (kg m ²)	relative inertia error
4	5.937500	0.285205	4.932×10^{-2}
8	5.984375	0.296263	1.246×10^{-2}
16	5.996094	0.299063	3.122×10^{-3}
32	5.999023	0.299766	7.811×10^{-4}
64	5.999756	0.299941	1.953×10^{-4}

Each doubling beyond the coarse grid reduces the inertia error by approximately a factor of four, consistent with second-order midpoint quadrature for this smooth integrand. Checking total mass at the same time is important: a grid can produce an apparently converged inertia for the wrong normalization if the cell masses do not sum toward the prescribed M .

10.9.3 A second quadrature model and its verification

The midpoint rule approximates both the mass of a radial cell and the inertia contribution using the midpoint density. A useful alternative is to integrate each cell's mass exactly, then place that entire mass at its radial midpoint. For cell boundaries $a_j = jR/N$, $b_j = (j + 1)R/N$, and $r_j = (a_j + b_j)/2$,

$$\Delta m_j = \pi \sigma_0 (b_j^2 - a_j^2) + \frac{\pi \sigma_0 \beta}{2R^2} (b_j^4 - a_j^4), \quad \tilde{I}_N = \sum_{j=0}^{N-1} r_j^2 \Delta m_j.$$

The total mass now equals the analytic M on every grid, because summing these exact cell masses telescopes. The inertia still has a quadrature error: replacing a cell's range of r^2 values by one representative value is an approximation. Correct total mass is a valuable test, but it does not prove

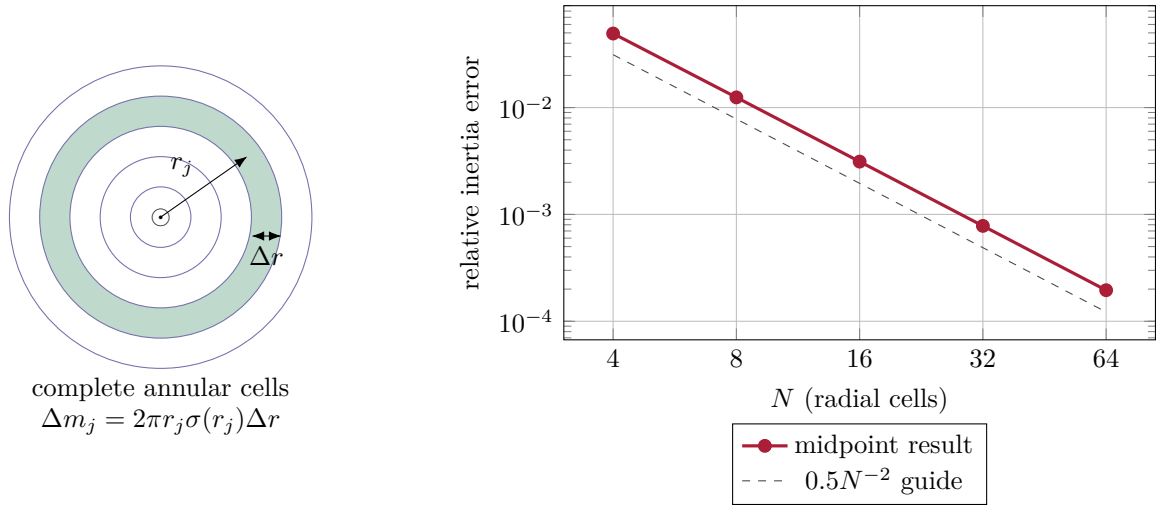


Figure 10.32: Radial midpoint approximation to the graded disk and its convergence. Each lump represents a complete annulus. Density is held fixed as the grid is refined; neither mass nor inertia is silently renormalized. Embedded errors come from the same midpoint sums as the table.

correct inertia. Conversely, normalizing approximate cell masses to enforce the correct total mass can hide the mass error without eliminating the inertia error.

The two discretizations therefore answer a precise computational question: how well does a chosen representation recover a known continuum benchmark? Hold R , β , and σ_0 fixed through refinement. Record M_N , I_N , and the errors against independently integrated values. Halve the grid spacing and compare errors. For a smooth integrand the leading midpoint error is proportional to $(\Delta r)^2$, so the asymptotic error ratio is four. At coarse resolution, higher-order terms may make this ratio differ from four without indicating a programming failure.

An additional check uses geometric bounds. Every cell lies between a_j and b_j , hence for nonnegative density

$$\sum_j a_j^2 \Delta m_j \leq I_z \leq \sum_j b_j^2 \Delta m_j.$$

These bounds contract as the grid is refined and require no assumed convergence order. They are particularly useful for density profiles whose derivatives are not smooth. A numerical result outside them has failed a physical check even if its successive digits appear stable.

Reference calculation for the refinement investigation. Use the preceding graded-disk parameters $M = 6.00$ kg, $R = 0.300$ m, and $\beta = 1$. Keep $\sigma_0 = M/[\pi R^2(1 + \beta/2)]$ fixed. The independently integrated continuum inertia is $I_z = 0.300000$ kg m². Evaluating the exact cell masses and then their midpoint inertias gives

N	$\sum_j \Delta m_j$ (kg)	$\tilde{I}_N$ (kg m ²)	relative inertia error
4	6.000000	0.2879296875	0.0402343750
8	6.000000	0.2969604492	0.0101318359
16	6.000000	0.2992387390	0.0025375366
32	6.000000	0.2998095989	0.0006346703

For this particular linear-in- r^2 density, summing the powers of the cell boundaries and midpoints gives

$$\frac{\tilde{I}_N}{I_z} = 1 - \frac{13}{20N^2} + \frac{1}{10N^4}.$$

The preceding midpoint-density rule instead gives $I_N/I_z = 1 - 4/(5N^2) + 7/(40N^4)$ and total mass $M_N = M[1 - 1/(6N^2)]$. Integrating each cell mass exactly reduces the leading inertia error coefficient

from 0.800 to 0.650 for this profile; it does not remove the representative-radius approximation. The displayed inertia errors decrease by factors 3.97, 3.99, and 4.00, approaching second-order convergence. The geometric bounds contain the continuum inertia on every listed grid; for example, at $N = 4$ they give $0.20224 \leq I_z \leq 0.39050 \text{ kg m}^2$. A satisfactory investigation reports both mass and inertia, distinguishes the two rules, and tests refinement and bounds rather than merely printing a plausible final value.

10.9.4 Inertia tensor preview: why $\mathbf{L}$ need not follow $\boldsymbol{\omega}$

The scalar I_z answers a fixed-axis question: how much rotational inertia does a body have about one specified axis? For arbitrary three-dimensional rotation, mass distribution relative to several directions matters simultaneously. A compact description is the inertia tensor

$$I_{ij} = \int (r^2 \delta_{ij} - r_i r_j) dm. \quad (10.61)$$

At a principal-axis frame through the center of mass, $\mathbf{I}$ is diagonal. For a rigid body rotating about its center of mass,

$$\mathbf{L} = \mathbf{I} \boldsymbol{\omega} \quad (10.62)$$

is the relation that reveals a key geometric point: unless the active components encounter equal principal moments or $\boldsymbol{\omega}$ lies along one principal axis, $\mathbf{L}$ and $\boldsymbol{\omega}$ need not be parallel.

Example 10.26: A rectangular plate rotating about a non-principal direction

A thin rectangular plate of mass $M = 3.0 \text{ kg}$ has side lengths $a = 0.60 \text{ m}$ along x and $b = 0.20 \text{ m}$ along y . About its center,

$$I_x = \frac{Mb^2}{12} = 0.0100, \quad I_y = \frac{Ma^2}{12} = 0.0900, \quad I_z = \frac{M(a^2 + b^2)}{12} = 0.100 \text{ kg m}^2.$$

If at an instant $\boldsymbol{\omega} = (4, 2, 0) \text{ rad/s}$, the principal-axis relation gives

$$\mathbf{L} = (I_x \omega_x, I_y \omega_y, I_z \omega_z) = (0.040, 0.180, 0) \text{ kg m}^2/\text{s}.$$

The angle between $\mathbf{L}$ and $\boldsymbol{\omega}$ is about 50.9° , so the vectors are plainly not parallel. No contradiction exists with the core chapter: the core never claimed that a single scalar inertia applies to arbitrary rotation axes.

10.9.5 Recovering the scalar result from the tensor

Let $\mathbf{n}$ be a unit vector along a prescribed axis through the center of mass. Fixed-axis angular velocity is $\boldsymbol{\omega} = \omega \mathbf{n}$. The inertia about this direction and the kinetic energy are

$$I_n = \mathbf{n}^T \mathbf{I} \mathbf{n}, \quad K = \frac{1}{2} \boldsymbol{\omega}^T \mathbf{I} \boldsymbol{\omega} = \frac{1}{2} I_n \omega^2.$$

The perpendicular distance formula gives the same result directly: $r_\perp^2 = r^2 - (\mathbf{r} \cdot \mathbf{n})^2$. Substituting the tensor definition yields $I_n = \int r_\perp^2 dm$. The scalar inertia used throughout the chapter is therefore an axis projection of the tensor, rather than an unrelated approximation to it.

For the rectangular plate example, the angular-velocity direction is $\mathbf{n} = (2, 1, 0)/\sqrt{5}$. It gives $I_n = 0.0260 \text{ kg m}^2$ and

$$K = \frac{1}{2} (0.0260) (4^2 + 2^2) = 0.260 \text{ J}.$$

The independent component calculation is $K = \frac{1}{2} [0.0100(4)^2 + 0.0900(2)^2] = 0.260 \text{ J}$. The two methods agree even though $\mathbf{L}$ is not parallel to $\boldsymbol{\omega}$.

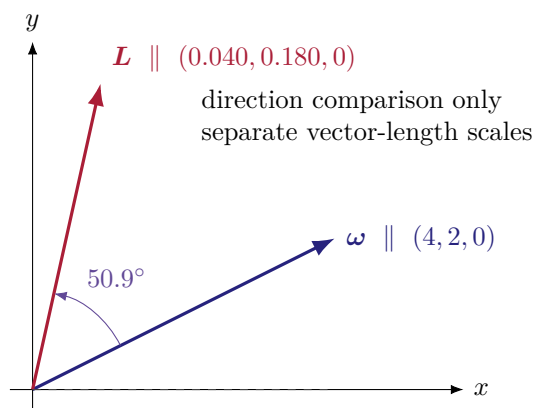


Figure 10.33: Directions of angular velocity and angular momentum for the plate example. Separate positive scale factors are used for their lengths because the vectors have different units. Their directions, and the angle between them, follow the calculated components.

If a rigid body is constrained to rotate about a fixed inertial axis that is not principal, bearings can supply transverse torques. Those torques govern the direction of angular momentum but do not contribute to the axial power $P = \tau_z \omega$. For ideal constraints the support points do not move, so their forces do no work. This is why the axial scalar equation remains useful while the unrestricted vector equation $\boldsymbol{\tau} = I_z \boldsymbol{\alpha}$ remains invalid. Following the full three-dimensional motion of a free body requires the later angular-momentum treatment.

Pedagogical boundary

Equation (10.62) is only a preview of the geometry. The evolution of a freely rotating body, body-frame derivatives, Euler equations, gyroscopic effects and general torque–angular-momentum dynamics are outside this chapter. Likewise, if a body’s mass distribution changes with time, the core equation $\sum \tau_z = I_z \alpha$ cannot simply be reused unchanged; the next chapter’s angular-momentum framework is the correct extension.

10.9.6 Advanced investigation: sensitivity of energy to radial mass redistribution

At fixed total mass and fixed angular speed, a small mass δm moved from radius r_1 to r_2 changes the inertia by

$$\delta I_z = \delta m(r_2^2 - r_1^2),$$

and therefore changes the rotational kinetic energy by

$$\delta K_{\text{rot}} = \frac{1}{2} \omega^2 \delta m(r_2^2 - r_1^2).$$

This linear sensitivity in δm but quadratic sensitivity in radius explains why rim mass can dominate rotational response. The expression also provides a quick numerical check on a discretized model: moving a known cell radially should change the computed inertia by exactly the corresponding $m(r_2^2 - r_1^2)$ if all other cells remain fixed.

10.10 Chapter summary and a working formula sheet

Summary

Fixed-axis rotation is a one-coordinate description of a rigid body's orientation about a specified axis. Radians connect angle to arc length. Angular velocity and acceleration are shared by every point of a rigid body, whereas tangential speed and linear accelerations scale with radius. The moment of inertia I_z measures mass distribution relative to the selected axis and enters both rotational kinetic energy and fixed-axis dynamics. Torque is the axis-dependent turning action of a force. Under a fixed inertial axis and constant mass distribution, $\sum \tau_z = I_z \alpha$. Rotational work is the signed area under a torque–angle curve and changes $\frac{1}{2} I_z \omega^2$; instantaneous power is $\tau_z \omega$.

Formula summary

Kinematics

$$\omega = \frac{d\theta}{dt}, \quad \alpha = \frac{d\omega}{dt}, \quad s = R\theta,$$

$$\omega = \omega_0 + \alpha t, \quad \theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2, \quad \omega^2 - \omega_0^2 = 2\alpha(\theta - \theta_0) \quad (\alpha = \text{constant}).$$

Point on a rigid rotor

$$|v_t| = R|\omega|, \quad a_t = R\alpha, \quad a_n = R\omega^2 \quad (\text{inward}).$$

Inertia and energy

$$I_z = \sum_i m_i r_{\perp i}^2 = \int r_{\perp}^2 dm, \quad I_z = I_{\text{CM},z} + Md^2, \quad K_{\text{rot}} = \frac{1}{2} I_z \omega^2.$$

Fixed-axis dynamics, work, power

$$\tau_z = (\mathbf{r} \times \mathbf{F})_z, \quad \sum \tau_z = I_z \alpha, \quad W = \int \tau_z d\theta, \quad P = \tau_z \omega.$$

The equation $\sum \tau_z = I_z \alpha$ is a fixed-axis scalar relation under the stated assumptions, not a universal vector identity.

10.11 Exercises and problems

Use $g = 9.81 \text{ m/s}^2$ unless stated otherwise. Treat the rotor as rigid, the axis as fixed in an inertial frame, and its inertia as constant unless a task explicitly asks you to diagnose a failed assumption. Choose the positive angular sense before writing equations; angles used in calculus are in radians. Problems 1–25 develop the core. Problems 26–31 retain the advanced investigations, while Problems 32–45 connect the representations and models. Every item has a complete solution in the following section. Constructed numerical data are teaching data, not experimental observations.

Problem 1: crossing an encoder boundary

An encoder display changes from 350° to 20° in 0.20 s . The controller records continuous counter-clockwise motion with no full extra turn. Find (a) the angular displacement and (b) the average angular velocity. Explain which additional information makes the displayed values interpretable.

Problem 2: stopping during negative rotation

A shaft starts at $\omega_0 = -6.0$ rad/s and has constant $\alpha = +2.0$ rad/s². Find (a) when it stops momentarily and (b) the signed angular displacement by then. Is positive acceleration evidence of positive rotation?

Problem 3: differentiating an angular history

A rotor obeys $\theta(t) = 0.40t^3 - 1.20t^2 + 2.0t$ in SI units. Find (a) $\omega(2.0$ s) and (b) $\alpha(2.0$ s). State the SI dimensions of the three coefficients.

Problem 4: acceleration without a stated time

A wheel accelerates uniformly from 5.0 to 25.0 rad/s while turning through 60 rad. Find (a) α and (b) elapsed time. Check the displacement using an independent kinematic relation.

Problem 5: revolutions and radians

A process runs at 1200 rpm for 18 s. Find (a) the number of revolutions and (b) angular speed in rad/s. Identify the conversion that a calculation using rpm inside $v = R\omega$ would omit.

Problem 6: acceleration of a rotor point

At $R = 0.18$ m, a point has $\omega = 12$ rad/s and $\alpha = +5.0$ rad/s². Find (a) speed, (b) signed tangential acceleration, (c) inward normal acceleration and (d) total acceleration magnitude. Explain why the last two components are combined quadratically; compare with the geometry of Fig. 10.7.

Problem 7: two radii, one rigid body

Sensors are at radii 0.08 m and 0.32 m on the same rigid rotor. At a nonzero angular speed, compare (a) their tangential speeds and (b) their normal acceleration magnitudes. Do the sensors have different angular velocities?

Problem 8: grinding-wheel rim speed

A grinding wheel of radius 0.125 m spins at 3000 rpm. Find its rim speed and explain why this kinematic result is not a safety certification.

Problem 9: what normal acceleration reveals

A point has $a_n = 18$ m/s² at radius 0.20 m. Find $|\omega|$. Can the sign of ω be inferred from this measurement alone? Explain the ambiguity physically.

Problem 10: four point masses

Four point masses of 0.50 kg lie at $(\pm 0.20, 0)$ m and $(0, \pm 0.30)$ m. Find I_z about the origin. Check whether reversing either coordinate sign changes the answer.

Problem 11: deriving a rod inertia

Derive the moment of inertia of a uniform thin rod of mass M and length L about its central axis perpendicular to the rod. State the density and integration limits explicitly. Check dimensions and derive the corresponding end-axis inertia.

Problem 12: a rotating solid disk

A uniform solid disk has $M = 8.0$ kg and $R = 0.25$ m. About its central perpendicular axis find (a) I_z and (b) K_{rot} at 20 rad/s.

Problem 13: the same disk about a tangent axis

The disk in Problem 12 rotates about an axis parallel to its central perpendicular axis and tangent to its rim. Find its new inertia. Use the parallel-axis geometry of Fig. 10.11. Explain why the parallel-axis theorem applies and why its energy at the same ω would increase.

Problem 14: ring versus disk

A thin ring and a uniform solid disk have the same M , R , and nonzero ω . Find $K_{\text{ring}}/K_{\text{disk}}$ about their respective central perpendicular axes. Explain the role of radial mass distribution.

Problem 15: moving a component radially

A 2.0 kg point mass is moved from radius 0.10 m to 0.40 m, with all other rotor components unchanged. Find the difference between the initial and final inertias. This compares two assembled configurations; no assumption of constant angular speed during the physical movement is required.

Problem 16: an oblique force

A 40 N force acts at radius 0.30 m at angle 25° to the radius vector, both vectors lying in the plane perpendicular to the axis. Find the torque magnitude. Is its sign determined by the magnitude of the angle alone?

Problem 17: force along a lever

A 60 N force acts at the end of a 0.50 m lever, but its line of action passes through the pivot. Find its torque about the pivot and explain why a large force can have zero turning effect there.

Problem 18: adding signed torques

A rotor with $I_z = 0.80$ kg m² experiences torques +5.0, -1.5, and -0.5 N m. Find α . Does its sign alone determine whether the rotor's speed is increasing?

Problem 19: a massive-pulley Atwood machine

In a massive-pulley Atwood machine, $m_1 = 3.0$ kg, $m_2 = 1.0$ kg, $R = 0.15$ m, and $I_z = 0.045$ kg m². The cord is light, inextensible, and does not slip; losses are negligible. Use the opposing tension directions in Fig. 10.19. Find the acceleration and compare it with the massless-pulley result. Include the two tensions in your derivation.

Problem 20: initial fall of a rod

A uniform rod of mass 4.0 kg and length 1.2 m is pivoted frictionlessly about one end and initially horizontal. Find its initial angular acceleration due to gravity. Choose the downward rotational sense as positive, and explain why the mass cancels.

Problem 21: equilibrium during rotation

A rotor spins at constant -10 rad/s while its net axial torque is zero. Is it in rotational equilibrium? Is it at rest? Distinguish these two claims without changing the fixed-axis model.

Problem 22: constant-torque work

A constant torque of 8.0 N m turns a rotor through 15 rad in the positive sense. Find (a) work and (b) final angular speed if $I_z = 2.0 \text{ kg m}^2$ and the rotor starts from rest. Check with dynamics.

Problem 23: shaft power

A motor delivers 4.0 kW of mechanical power to a shaft rotating at +80 rad/s. Find the applied torque at that instant. Explain why the formula cannot represent finite constant-power operation from exactly zero speed.

Problem 24: stopping with a constant brake torque

A brake applies constant torque -12 N m to a rotor with $I_z = 3.0 \text{ kg m}^2$ and initial $\omega = 20 \text{ rad/s}$. Find (a) stopping time and (b) angular displacement before stopping. Check the energy removed. The specified braking model ends when the rotor reaches rest.

Problem 25: verifying inertia software

Design a verification plan for code evaluating $I_z = \sum_i m_i(x_i^2 + y_i^2)$. Include (a) an exact discrete benchmark and a symmetry test, (b) a limiting case and a dimensional/scaling test, and (c) a convergent uniform-disk construction. Give concrete expected outputs and a criterion for judging the plan.

Problem 26: a graded disk

For the graded disk $\sigma(r) = \sigma_0[1 + \beta(r/R)^2]$ with the prescribed total mass kept fixed, evaluate $I_z/(MR^2)$ for (a) $\beta = -0.5$ and (b) $\beta = 2$. Explain the radial redistribution and check that both density profiles remain nonnegative.

Problem 27: measuring quadrature order

For the graded-disk benchmark $M = 6.0 \text{ kg}$, $R = 0.30 \text{ m}$, $\beta = 1$, the exact inertia is $I = 0.300000000000 \text{ kg m}^2$. Unrounded midpoint sums give $I_{16} = 0.2990633010864258$ and $I_{32} = 0.2997656750679016 \text{ kg m}^2$. Define $e_N = |I_N - I|/I$. Find (a) e_{16}/e_{32} and (b) $p = \log_2(e_{16}/e_{32})$. Compare with second-order behavior. Compare with the error trend in Fig. 10.32. Do not form ratios from the six-decimal rounded display values in the earlier table.

Problem 28: angular momentum in principal axes

A principal-axis body has $(I_x, I_y, I_z) = (2, 3, 5) \text{ kg m}^2$ and $\boldsymbol{\omega} = (1, 2, 0) \text{ rad/s}$. Compute $\mathbf{L}$ and determine whether it is parallel to $\boldsymbol{\omega}$. This is an instantaneous tensor calculation, not a request for the subsequent free-body trajectory.

Problem 29: doubling a component radius

Show that moving a point mass m from radius R to $2R$ multiplies its inertia contribution by four. Show the same factor for its rotational-energy contribution when the two configurations are compared at the same ω . Distinguish a multiplication factor from an increment.

Problem 30: the wrong disk weights

A numerical disk model puts N equal masses M/N at equally spaced midpoint radii $r_j = (j + \frac{1}{2})R/N$, $j = 0, \dots, N - 1$, instead of assigning masses proportional to annular area. (a) Explain why this is not a uniform physical disk. (b) Find the limiting value of $I_N/(MR^2)$ as $N \rightarrow \infty$. (c) State a correct area-weighted alternative.

Problem 31: an invalid universal torque equation

A student writes $\tau = I_z \alpha$ for a freely rotating asymmetric body. Identify the fixed-axis assumption that has been lost. Name the more general dynamical quantity and explain why a scalar inertia cannot generally describe the situation.

Problem 32: encoder sampling and aliasing

A wrapped encoder reports 350° at $t = 0$ and 20° at $t = 0.020$ s. Direction is not recorded, but an independent bound gives $|\omega(t)| \leq 100$ rad/s throughout the interval. (a) Find every net displacement compatible with the two readings and the bound. (b) Could an additional complete revolution have occurred? (c) If only the bound were used, what maximum sampling interval guarantees a unique shortest-angle reconstruction for arbitrary adjacent readings? Distinguish unique net displacement from knowledge of any reversals within an interval.

Problem 33: reading a reversal graph

Figure 10.34 specifies an angular-velocity history by straight segments. The unwrapped initial angle is zero. Find (a) net angular displacement, (b) total angular distance, (c) average angular velocity and average angular speed over $0 \leq t \leq 6$ s, and (d) the times at which the direction reverses. Describe the curvature of $\theta(t)$ on each open segment.

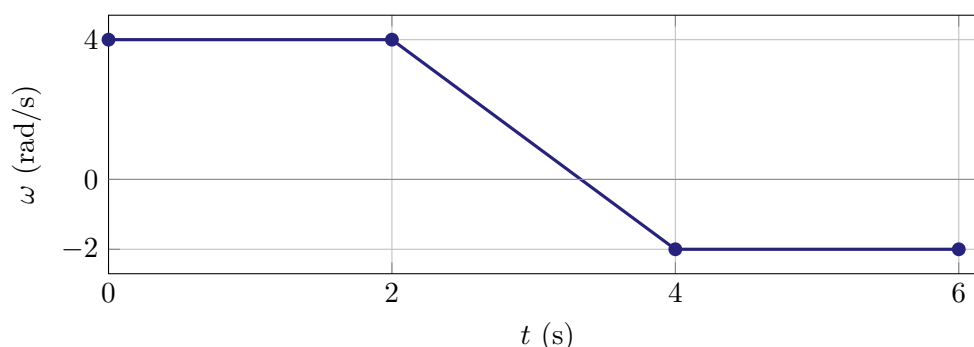


Figure 10.34: Constructed angular-velocity graph for Problem 33. Straight joining segments specify the history; slopes are angular accelerations.

Problem 34: a torque-time history

A rotor with $I_z = 2.0$ kg m² begins at $\omega_0 = -1.0$ rad/s. Its net torque is prescribed by Fig. 10.35: $+4$ N m for $0 < t < 2$ s and -2 N m for $2 < t < 5$ s. Find (a) $\omega(2)$ and $\omega(5)$, (b) angular displacement in each segment, and (c) total net work. Check the work by integrating $\tau\omega$ in time. Values at the instantaneous torque switches do not affect the integrals.

Problem 35: signed work from a torque-angle graph

A rotor with $I_z = 0.50$ kg m² starts at $\theta = 0$ with $\omega = +2.0$ rad/s. Over $0 \leq \theta \leq 4$ rad, the net torque follows the straight line from $+6$ to -2 N m in Fig. 10.36. Find (a) work to $\theta = 4$, (b) the final angular velocity, and (c) the maximum angular speed and its angular position. Check that the assumed continued positive motion is consistent with the energy throughout this interval.

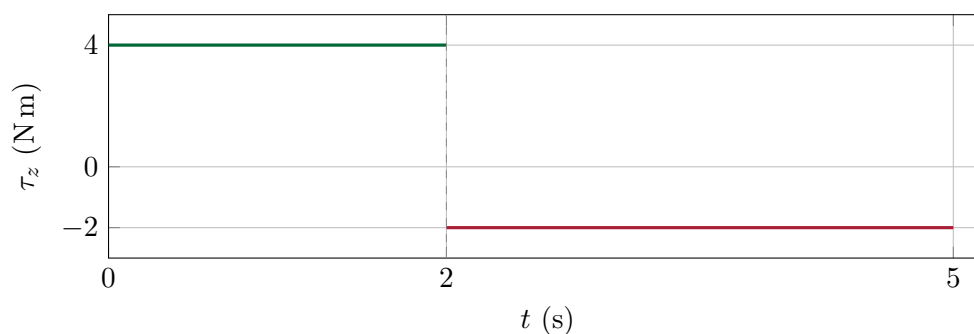


Figure 10.35: Piecewise constant net torque for Problem 34. The jump specifies a change in the drive, while angular velocity remains continuous.

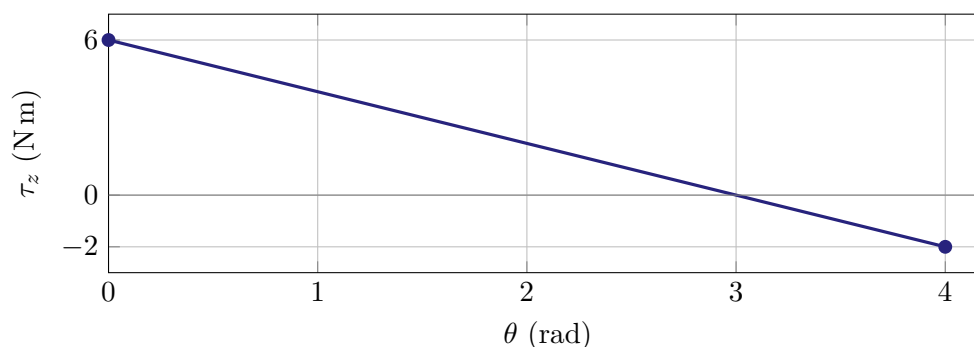


Figure 10.36: Constructed net torque-angle graph for Problem 35. Areas above and below the axis have opposite work signs.

Problem 36: acceleration vectors in Cartesian form

A marker is at $\theta = 30^\circ$, $R = 0.20$ m, with $\omega = -3.0$ rad/s and $\alpha = +4.0$ rad/s². The geometric basis is shown in Fig. 10.37. Find (a) Cartesian velocity, (b) Cartesian acceleration, and (c) whether the speed is increasing or decreasing. Draw the velocity and both acceleration contributions on a copy of the figure.

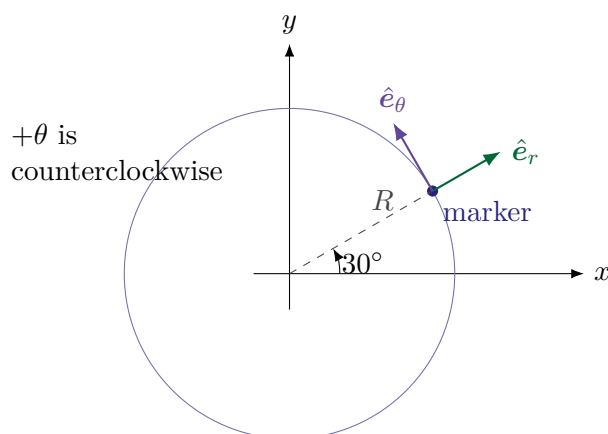


Figure 10.37: Radial and positive tangential unit vectors for Problem 36. The arrows define directions, not the unknown physical velocity or acceleration.

Problem 37: recovering angular motion from acceleration

At radius $R = 0.25$ m a sensor resolves an inward normal component $a_n = 9.0$ m/s² and a signed positive tangential component $a_t = 1.5$ m/s². (a) Find $|\omega|$ and α . (b) Find the total acceleration and

its angle from inward radial toward positive tangential. (c) State what extra observation determines whether the rotor is gaining or losing speed. Use the orthogonal directions of Fig. 10.37, with the applicable sensor radius.

Problem 38: an offset instrument on a rotor

A uniform main disk of mass 5.0 kg and radius 0.20 m has its central perpendicular axis fixed. An attached rectangular instrument of mass 1.0 kg has sides $a = 0.10$ m and $b = 0.06$ m in the disk plane; its center is $d = 0.18$ m from the main axis. Assume the instrument has uniform areal density. (a) Find the combined inertia. (b) Find the constant net torque needed to change ω from 2 to 12 rad/s in 4.0 s. (c) Compare with the point-mass approximation for the instrument and state the relative inertia error.

Problem 39: separating inertia and bearing loss

A cord on a hub of radius $R_h = 0.050$ m drives a platform as in Fig. 10.38. Constructed runs with hanging masses 0.20 kg and 0.40 kg give constant positive angular accelerations 3.0 and 7.0 rad/s², respectively. Model bearing resistance as a common constant torque $\tau_{\text{loss}} > 0$ opposing positive rotation. (a) Derive two equations for platform inertia and loss torque, including acceleration of the hanging mass. (b) Determine both quantities. (c) Explain which measurement checks would test the model.

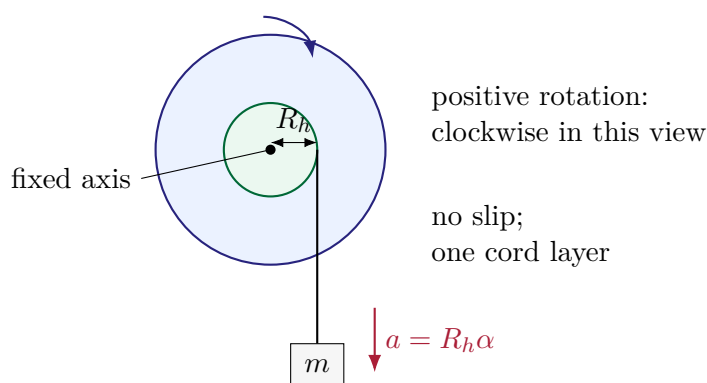


Figure 10.38: Idealized hanging-mass inertia experiment for Problem 39. The cord pulls tangentially on the hub; the hanging mass and platform must be modeled together.

Problem 40: sensitivity of an inferred inertia

A calibrated net torque $\tau = 0.600 \pm 0.012$ N m produces $\alpha = 2.00 \pm 0.08$ rad/s² on a fixed-axis platform. The quoted uncertainties are independent standard uncertainties and losses have already been subtracted. (a) Find I_z . (b) Using first-order propagation, estimate its standard uncertainty. (c) Compare with the conservative interval obtained by treating each quoted plus/minus as a hard bound. Explain the different meanings.

Problem 41: power during negative rotation

At an instant a rotor has $I_z = 0.50$ kg m², $\omega = -8.0$ rad/s, and net torque +3.0 N m. (a) Find α and net power. (b) Decide whether speed and rotational energy are increasing. (c) If this torque is held constant until the rotor stops, find stopping time and signed displacement and check the work. State what happens if the torque continues afterward.

Problem 42: a linearly graded rod

A thin rod occupies $0 \leq x \leq L$ and has $\lambda(x) = \lambda_0(1 + x/L)$. Its total mass is M . Find (a) λ_0 , (b) center-of-mass coordinate x_{CM} , (c) inertia about the perpendicular axis at $x = 0$, and (d) inertia about the parallel center-of-mass axis. Check positivity and compare the end-axis value with a uniform rod of the same M, L .

Problem 43: an area-weighted disk refinement

Model a uniform disk by concentric annular cells with edges $r_j = jR/N$. Assign each cell its exact mass and place that mass at radial midpoint $\bar{r}_j = (j + \frac{1}{2})R/N$. (a) Derive Δm_j . (b) Prove $I_N = MR^2[\frac{1}{2} - 1/(4N^2)]$. (c) Calculate relative inertia errors for $N = 10, 20$ and contrast this convergent construction with Problem 30. Exact mass in each cell does not imply exact inertia in each cell.

Problem 44: constant power above zero speed

An ideal fixed-axis rotor with $I_z = 2.0 \text{ kg m}^2$ starts at $\omega_0 = +5.0 \text{ rad/s}$. A drive supplies constant positive mechanical power $P = 100 \text{ W}$ for 3.0 s. Losses are absent. Find (a) $\omega(t)$, (b) final torque and (c) angular displacement over the interval. Explain why constant-angular-acceleration equations do not apply and why this model is restricted to positive nonzero speed.

Problem 45: scalar inertia along a chosen tensor direction

At the illustrated instant, a body's principal moments are $(I_x, I_y, I_z) = (1, 3, 4) \text{ kg m}^2$. Bearings constrain its rotation to the fixed unit direction $\mathbf{n} = (\hat{\mathbf{i}} + \hat{\mathbf{j}})/\sqrt{2}$, with signed speed $\omega = 2.0 \text{ rad/s}$. Figure 10.39 gives the principal-plane geometry. (a) Find $\mathbf{L} = \mathbf{I}(\omega\mathbf{n})$ and $L_n = \mathbf{n} \cdot \mathbf{L}$. (b) Find $I_n = \mathbf{n}^T \mathbf{I} \mathbf{n}$ and rotational energy. (c) Explain how $\tau_n = I_n \alpha$ can be valid while $\mathbf{L}$ is not parallel to ω . Do not assume that this is free rotation without bearing reactions.

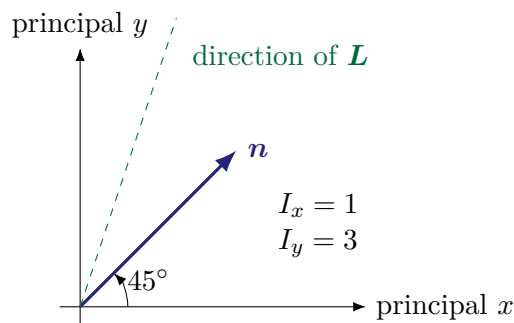


Figure 10.39: Principal-plane directions for Problem 45. The dashed direction indicates the effect of unequal principal moments; its length is not a scale for angular momentum units.

10.12 Complete solutions

The solutions keep the axis, signs and model visible. Intermediate guard digits are retained for checking; reported numerical precision reflects the pedagogical input rather than a claim of laboratory accuracy. A check tests the stated model and arithmetic, whereas agreement between two formulas derived from the same assumption cannot by itself establish that a real apparatus satisfies that assumption.

Solution 1: crossing an encoder boundary

Counterclockwise is positive. With no extra full turn, the final unwrapped angle is 380° , giving (a)

$$\Delta\theta = 380^\circ - 350^\circ = 30^\circ = \pi/6 = 0.523599 \text{ rad.}$$

Then (b) $\bar{\omega} = \Delta\theta/\Delta t = 2.61799 \text{ rad/s}$, about 2.62 rad/s . Subtracting the displayed readings directly would produce -330° , the wrong history. Direction and turn-count information select the correct branch; the wrapped readings alone allow displacements $30^\circ + 360^\circ k$ for integer k .

Solution 2: stopping during negative rotation

For constant acceleration, (a) $0 = \omega_0 + \alpha t = -6 + 2t$, hence $t = 3.00 \text{ s}$. Before this instant $\omega < 0$, so the displacement must be negative. Indeed, (b)

$$\Delta\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = -6(3) + \frac{1}{2}(2)(3)^2 = -9.00 \text{ rad.}$$

The graph-area check is a triangle below the time axis, $\frac{1}{2}(-6)(3) = -9 \text{ rad}$. Positive acceleration means increasing signed velocity; here the magnitude of velocity decreases. At the stopping instant the speed is zero, and continued positive acceleration would cause positive rotation.

Solution 3: differentiating an angular history

Differentiate the entire polynomial before substituting the time:

$$\omega(t) = 1.20t^2 - 2.40t + 2.0, \quad \alpha(t) = 2.40t - 2.40.$$

Consequently (a) $\omega(2) = 4.8 - 4.8 + 2 = 2.00 \text{ rad/s}$ and (b) $\alpha(2) = 4.8 - 2.4 = 2.40 \text{ rad/s}^2$. The coefficients of t^3, t^2, t have dimensions $\text{s}^{-3}, \text{s}^{-2}, \text{s}^{-1}$, respectively, since angle is dimensionless; writing $\text{rad/s}^3, \text{rad/s}^2, \text{rad/s}$ makes their angular roles explicit. Acceleration is time dependent, so constant- α formulas are inappropriate for a finite interval of this history.

Solution 4: acceleration without a stated time

Eliminate time using the constant- α relation:

$$(a) \quad \alpha = \frac{\omega_f^2 - \omega_0^2}{2\Delta\theta} = \frac{25^2 - 5^2}{2(60)} = 5.00 \text{ rad/s}^2.$$

Then (b) $t = (\omega_f - \omega_0)/\alpha = 4.00 \text{ s}$. Both endpoint velocities are positive and the acceleration is positive, consistent with increasing speed throughout. An independent representation gives the trapezoidal area under $\omega(t)$: $[(5 + 25)/2](4) = 60 \text{ rad}$, reproducing the prescribed displacement.

Solution 5: revolutions and radians

The rotational frequency is $1200/60 = 20.0 \text{ rev/s}$. Thus (a) $N = (20)(18) = 360 \text{ rev}$ and (b) $\omega = 2\pi(20) = 40\pi = 125.664 \text{ rad/s}$. Using 1200 directly in $R\omega$ omits both division by 60 s/min and multiplication by $2\pi \text{ rad/rev}$. The displacement check is $\omega t = 720\pi \text{ rad}$, which divided by 2π again gives 360 revolutions.

Solution 6: acceleration of a rotor point

At fixed radius, (a) $v = R|\omega| = 0.18(12) = 2.16 \text{ m/s}$, (b) $a_t = R\alpha = +0.900 \text{ m/s}^2$, and (c) $a_n = R\omega^2 = 0.18(144) = 25.92 \text{ m/s}^2$, inward. Radial and tangential directions are perpendicular, so (d)

$$|\mathbf{a}| = \sqrt{a_n^2 + a_t^2} = \sqrt{25.92^2 + 0.900^2} = 25.9356 \text{ m/s}^2.$$

Adding magnitudes directly would disregard vector geometry. The result is slightly larger than a_n , as required. The tilt from the inward direction is only $\tan^{-1}(0.900/25.92) = 1.99^\circ$ toward the positive tangent.

Solution 7: two radii, one rigid body

Rigid fixed-axis rotation imposes one common signed ω . Since $v = R|\omega|$, (a) $v_{\text{outer}}/v_{\text{inner}} = 0.32/0.08 = 4$. Since $a_n = R\omega^2$, (b) the normal-acceleration ratio is also 4. Their accelerations differ even though their angular velocities are equal. Normal acceleration scales linearly with radius at fixed ω , rather than quadratically: the square applies to angular speed. At zero speed both normal accelerations vanish and their ratio is undefined, which is why the statement specifies nonzero speed.

Solution 8: grinding-wheel rim speed

Convert first: $\omega = 3000(2\pi)/60 = 100\pi$ rad/s. The rim speed is therefore $v = 0.125(100\pi) = 39.2699$ m/s, about 39.3 m/s. The dimensional check is m times s^{-1} , giving m/s. This relation contains no stress, material strength, defect, temperature, bearing, guard, or permitted operating-speed information. It describes motion under the rigid model; it does not establish whether a particular wheel can safely sustain that motion.

Solution 9: what normal acceleration reveals

From $a_n = R\omega^2$, $|\omega| = \sqrt{a_n/R} = \sqrt{18/0.20} = \sqrt{90} = 9.48683$ rad/s. The quotient has units s^{-2} before taking the square root. Squaring removes the sign, so both $+9.49$ and -9.49 rad/s yield the same inward acceleration. A tangential velocity observation or direction-resolving encoder is needed to distinguish them. Inward acceleration is not evidence that the marker travels inward: its radius remains constant while its velocity direction changes.

Solution 10: four point masses

For the z axis through the origin, $r_{\perp}^2 = x^2 + y^2$. Pair the equal radii:

$$I_z = 2(0.50)(0.20)^2 + 2(0.50)(0.30)^2 = 0.0400 + 0.0900 = 0.130 \text{ kg m}^2.$$

Coordinate sign reversals leave every square unchanged. As a bound, all masses lie within 0.30 m and total mass is 2 kg, so $I_z \leq 2(0.30)^2 = 0.180 \text{ kg m}^2$; the result satisfies that check. The outer pair contributes more than twice as much as the inner pair despite identical pair masses.

Solution 11: deriving a rod inertia

Put the uniform rod along x , with center at zero. Its linear density is $\lambda = M/L$ and $dm = \lambda dx$. The chosen perpendicular axis gives $r_{\perp}^2 = x^2$, hence

$$I_{\text{CM},z} = \frac{M}{L} \int_{-L/2}^{L/2} x^2 dx = \frac{M}{L} \left[\frac{x^3}{3} \right]_{-L/2}^{L/2} = \frac{ML^2}{12}.$$

The two symmetric halves contribute equally. Dimensions are kg m^2 . Moving to the parallel perpendicular axis at an end uses $d = L/2$:

$$I_{\text{end},z} = I_{\text{CM},z} + M(L/2)^2 = \frac{ML^2}{3}.$$

A direct integral $(M/L) \int_0^L x^2 dx$ gives the same end-axis result. The parallel-axis theorem connects these axes because one passes through the center of mass and their directions are equal.

Solution 12: a rotating solid disk

The disk is uniform and the axis is its central perpendicular axis. Therefore (a)

$$I_z = \frac{1}{2}MR^2 = \frac{1}{2}(8.0)(0.25)^2 = 0.250 \text{ kg m}^2.$$

Then (b) $K_{\text{rot}} = \frac{1}{2}I_z\omega^2 = \frac{1}{2}(0.250)(20)^2 = 50.0 \text{ J}$. The disk inertia is half that of the same mass concentrated at its rim, consistent with its distributed radii. Squaring the angular speed makes the energy identical for either sense of rotation.

Solution 13: the same disk about a tangent axis

The central axis passes through the center of mass, the tangent axis is parallel, and their separation is R . Thus

$$I_{\text{tangent}} = I_{\text{CM}} + MR^2 = 0.250 + 8.0(0.25)^2 = 0.750 \text{ kg m}^2.$$

The shift increases inertia by 0.500 kg m^2 and triples it. At the same nonzero ω , $K = I\omega^2/2$ likewise triples. This energy statement compares rotation about two fixed axes; it does not infer that relocating a pivot would occur without work.

Solution 14: ring versus disk

A thin ring places essentially all its mass at radius R , so $I_{\text{ring}} = MR^2$. A solid uniform disk has $I_{\text{disk}} = MR^2/2$. The equal angular speeds and factor $1/2$ cancel in the energy ratio:

$$\frac{K_{\text{ring}}}{K_{\text{disk}}} = \frac{I_{\text{ring}}}{I_{\text{disk}}} = 2.$$

The disk's interior mass has smaller r^2 weights. The larger ring energy at equal speed follows from that distribution, not from a difference in total mass.

Solution 15: moving a component radially

Subtract the two point-mass contributions; every other component cancels:

$$\Delta I_z = m(r_f^2 - r_i^2) = 2.0(0.40^2 - 0.10^2) = 0.300 \text{ kg m}^2.$$

The positive result agrees with outward redistribution. The contributions themselves are 0.0200 and 0.320 kg m^2 . Any prediction for angular speed or work while the component actually moves would need an additional dynamical description; this static comparison alone determines the two inertias.

Solution 16: an oblique force

Only the component perpendicular to the radius gives axial torque. Its magnitude is $F_{\perp} = F \sin 25^\circ$, so

$$|\tau_z| = rF \sin 25^\circ = (0.30)(40) \sin 25^\circ = 5.07142 \text{ N m}.$$

Equivalently, the perpendicular distance from the pivot to the force line is $r \sin 25^\circ$. This is smaller than the maximal $rF = 12 \text{ N m}$, a useful check. The unsigned angle leaves two mirror-image force directions possible; one turns clockwise and the other counterclockwise. Geometry and a positive-sense convention are needed for a signed answer.

Solution 17: force along a lever

The perpendicular lever arm from the pivot to the force line is zero. Therefore $\tau_z = F\ell_{\perp} = 0$, equivalently $rF \sin \phi = 0$ because the vectors are collinear. Multiplying 60 N by the lever length would incorrectly treat the force as perpendicular. The force may produce a support reaction or axial stress, but it produces no turning moment about this pivot in the stated geometry.

Solution 18: adding signed torques

The net torque is $\tau_{\text{net}} = 5.0 - 1.5 - 0.5 = +3.0$ N m. Constant-inertia fixed-axis dynamics gives

$$\alpha = \frac{\tau_{\text{net}}}{I_z} = \frac{3.0}{0.80} = +3.75 \text{ rad/s}^2.$$

This is the rate of increase of signed angular velocity. It raises speed if $\omega > 0$ and lowers speed if $\omega < 0$. Since no initial velocity is supplied, the acceleration's sign alone cannot decide the speed trend. The same distinction appears through power: $dK/dt = \tau_{\text{net}}\omega$.

Solution 19: a massive-pulley Atwood machine

Take m_1 downward, m_2 upward, and the corresponding pulley rotation as positive. The free-body equations are $m_1g - T_1 = m_1a$ and $T_2 - m_2g = m_2a$. No slip gives $\alpha = a/R$. The axial torque equation becomes $(T_1 - T_2)R = I_z a/R$. Elimination yields

$$a = \frac{(m_1 - m_2)g}{m_1 + m_2 + I_z/R^2} = \frac{2(9.81)}{4 + 0.045/0.15^2} = 3.270 \text{ m/s}^2.$$

The tensions are $T_1 = 3(9.81 - 3.27) = 19.62$ N and $T_2 = 1(9.81 + 3.27) = 13.08$ N. Their difference gives $6.54(0.15) = 0.981$ N m, equal to $I_z\alpha = 0.045(3.27/0.15)$. The massless-pulley limit gives 4.905 m/s², larger because it assigns no energy to pulley rotation. Unequal tensions are necessary to accelerate this massive pulley; replacing them by one common tension would erase the required torque.

Solution 20: initial fall of a rod

Choose downward rotation positive. The weight acts at the center of mass, $L/2$ from the pivot; initially the rod is horizontal so the force is perpendicular to the radius. Thus $\tau_z = MgL/2$ and $I_z = ML^2/3$. The initial acceleration is

$$\alpha = \frac{MgL/2}{ML^2/3} = \frac{3g}{2L} = \frac{3(9.81)}{2(1.2)} = 12.2625 \text{ rad/s}^2.$$

Mass cancels because torque and inertia are both proportional to M . This is only the initial value: as the rod turns, gravity's lever arm changes. The center's initial tangential acceleration is $(L/2)\alpha = 3g/4$, compatible with the pivot constraint rather than unrestricted free fall.

Solution 21: equilibrium during rotation

Under the fixed-axis constant-inertia assumptions, zero net axial torque gives $\alpha = 0$. Therefore the rotor is in rotational equilibrium: its angular velocity does not change. It is not at rest because $\omega = -10$ rad/s is nonzero. The negative sign specifies rotational sense and does not alter this equilibrium criterion. If a bearing exerts resistance in a real device, another torque would have to balance it to preserve this motion.

Solution 22: constant-torque work

Since torque and displacement have the same sign, (a) $W = \tau\Delta\theta = 8.0(15) = 120$ J. With no initial kinetic energy, (b)

$$\frac{1}{2}I_z\omega_f^2 = W \quad \Rightarrow \quad \omega_f = +\sqrt{\frac{2(120)}{2.0}} = 10.9545 \text{ rad/s}.$$

The positive root follows from the positive accelerating torque applied from rest. Dynamics gives $\alpha = 8/2 = 4$ rad/s² and $\omega_f^2 = 2\alpha\Delta\theta = 120 \text{ s}^{-2}$, confirming the result. N m times a dimensionless angle has energy units, while torque retains its N m label.

Solution 23: shaft power

Delivered mechanical power is positive, and angular velocity is positive, so torque is positive:

$$\tau_z = \frac{P}{\omega} = \frac{4000}{80} = +50.0 \text{ N m.}$$

This relation describes an instantaneous operating point. A finite nonzero constant power would require $\tau = P/\omega$ to diverge as $\omega \rightarrow 0$. A real drive starts with a bounded-torque control or another regime; the ideal constant-power formula cannot model its entire startup from rest.

Solution 24: stopping with a constant brake torque

The acceleration is $\alpha = \tau/I_z = -12/3 = -4.0 \text{ rad/s}^2$. Thus (a) $0 = 20 - 4t$ gives $t = 5.00 \text{ s}$. During this interval the rotor continues in the positive sense, giving (b)

$$\Delta\theta = 20(5) + \frac{1}{2}(-4)(5)^2 = 50.0 \text{ rad.}$$

Initially $K_i = \frac{1}{2}(3)(20)^2 = 600 \text{ J}$. Brake work is $(-12)(50) = -600 \text{ J}$, so the final energy is zero. The $\omega(t)$ graph has triangular area $\frac{1}{2}(20)(5) = 50 \text{ rad}$, a second check. The model ends at rest; a brake need not continue to exert the same negative torque and drive the rotor backward.

Solution 25: verifying inertia software

A defensible exemplar specifies the axis and inputs in SI, calculates exact expectations, and checks that the implementation returns them within a chosen floating-point tolerance. For (a), a 2 kg mass at (3, 4) m gives $I_z = 2(9 + 16) = 50 \text{ kg m}^2$. Reflecting to $(-3, 4)$ or rotating to $(5, 0)$ must preserve 50. The four masses in Problem 10 must return 0.130 kg m^2 , testing summation as well as geometry.

For (b), every mass at (0, 0) must return zero and multiplying every mass by c must multiply I by c . Multiplying all coordinates by b must multiply I by b^2 . With the first benchmark and $b = 0.01$, the answer is 0.00500 kg m^2 . This catches an unnoticed centimeter-to-meter mistake. Translating coordinates while keeping the physical axis fixed generally changes inertia; translation is a valid invariance test only if the axis is translated with the object.

For (c), subdivide a uniform disk into annuli with exact masses $M(r_{j+1}^2 - r_j^2)/R^2$ and midpoint radii. The mass sum must equal M , while inertia tends to $MR^2/2$; Problem 43 derives its finite-grid error. For $M = 2 \text{ kg}$, $R = 1 \text{ m}$, $N = 10, 20$, the expected inertias are 0.995 and 0.99875 kg m^2 . An explicit reference test is:

```
def inertia_z(cells):
    return sum(m*(x*x+y*y) for m,x,y in cells)
assert abs(inertia_z([(2.0,3.0,4.0)])-50.0) < 1e-12
assert abs(inertia_z([(2.0,-3.0,4.0)])-50.0) < 1e-12
assert inertia_z([(2.0,0.0,0.0)]) == 0.0
assert abs(inertia_z([(2.0,0.03,0.04)])-0.005) < 1e-12
```

A useful assessment rubric awards credit for correct axis/units, a nontrivial exact expected value, a valid symmetry/scaling test, and refinement with both mass and inertia checks. A smooth-looking result or a single positive output earns no verification credit by itself. Other benchmarks are acceptable if their independent expected results and acceptance criteria are justified.

Solution 26: a graded disk

Normalization and inertia must use the same density. Integrating annuli gives

$$M = \pi\sigma_0 R^2(1 + \beta/2), \quad I_z = \pi\sigma_0 R^4(1/2 + \beta/3),$$

so $I_z/(MR^2) = (1/2 + \beta/3)/(1 + \beta/2)$. For (a) $\beta = -0.5$ the result is $(1/3)/(3/4) = 4/9 = 0.444444$. For (b) $\beta = 2$ it is $(7/6)/2 = 7/12 = 0.583333$. Both bracket the uniform value $1/2$. With fixed total mass, the first density gives relatively less rim mass and the second relatively more. Over $0 \leq r \leq R$, their smallest density factors are respectively 0.5 and 1, so neither assigns negative mass. Holding σ_0 fixed instead would change the total mass and would not be the same comparison.

Solution 27: measuring quadrature order

Use the prescribed exact benchmark rather than differences between rounded table entries. The relative errors are

$$e_{16} = 0.00312232971191, \quad e_{32} = 0.000781083106995.$$

Therefore (a) $q = e_{16}/e_{32} = 3.99743593$ and (b) $p = \ln q / \ln 2 = 1.99907490$. A second-order method predicts error proportional to h^2 , so halving $h = R/N$ predicts $q \rightarrow 4$ and $p \rightarrow 2$. The small difference is consistent with higher-order terms at finite N . This benchmark supports the quadrature implementation for this smooth density; it does not establish convergence for every density or validate an incorrect mass model.

Solution 28: angular momentum in principal axes

In principal coordinates the tensor is diagonal. Multiply componentwise:

$$\mathbf{L} = (I_x\omega_x, I_y\omega_y, I_z\omega_z) = (2, 6, 0) \text{ kg m}^2/\text{s}.$$

Parallelism would require one proportionality constant for every nonzero component. Here $L_x/\omega_x = 2$ and $L_y/\omega_y = 3$, so the vectors are not parallel. Alternatively their z cross-product component is $\omega_x L_y - \omega_y L_x = 6 - 4 = 2$, nonzero. Their angle is $\cos^{-1}[14/(\sqrt{5}\sqrt{40})] = 8.13^\circ$. This is geometry at one instant, consistent with the tensor preview's limited scope.

Solution 29: doubling a component radius

The two contributions are $I_i = mR^2$ and $I_f = m(2R)^2 = 4mR^2$, giving multiplication factor $I_f/I_i = 4$. The increment is $\Delta I = 3mR^2$, not $4mR^2$. At equal angular speed, $K_i = \frac{1}{2}mR^2\omega^2$ and $K_f = 4K_i$, so $\Delta K = 3K_i$. Equal speeds are an imposed comparison of configurations, not a conservation-law prediction for a mass moved on a freely evolving rotor.

Solution 30: the wrong disk weights

For (a), annular area grows as $2\pi r dr$. Equal-width annuli near the rim contain more mass than those near the center in a uniform disk. Equal masses per radial interval instead approximate constant dm/dr , producing an areal density proportional to $1/r$ away from the origin. This is a different distribution even when its total mass is exactly correct.

For (b), the specified midpoint construction gives

$$\frac{I_N}{MR^2} = \frac{1}{N^3} \sum_{j=0}^{N-1} (j + \frac{1}{2})^2 = \frac{1}{3} - \frac{1}{12N^2} \rightarrow \frac{1}{3}.$$

The identity follows by expanding the square and using the sums of j and j^2 ; equivalently its limit is $\int_0^1 u^2 du = 1/3$. It therefore misses the uniform-disk value $1/2$ by a finite amount even as N grows. For (c), use exact cell masses $\Delta m_j = M[(j+1)^2 - j^2]/N^2$ at the midpoint radii, or a properly normalized annular quadrature with weights proportional to $r_j \Delta r$. Refining geometry cannot repair the wrong mass measure.

Solution 31: an invalid universal torque equation

The core restricts motion to one fixed inertial axis, with a time-independent mass distribution, so its one angular coordinate and axial torque suffice. A freely rotating asymmetric body does not retain that single fixed-axis constraint. Its angular momentum obeys the general dynamical relation $\boldsymbol{\tau} = d\mathbf{L}/dt$ about a fixed inertial origin (or for spin about the center of mass), and its center-of-mass spin relation uses the inertia tensor, $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$. Unequal principal moments can make $\mathbf{L}$ and $\boldsymbol{\omega}$ nonparallel. Thus replacing the tensor and vector evolution by one scalar I_z loses physical information. Subsequent free-body dynamics is deferred to Chapter 11.

Solution 32: encoder sampling and aliasing

Every compatible net displacement has form $\Delta\theta = \pi/6 + 2\pi k$. The speed bound gives

$$|\Delta\theta| = \left| \int_0^{0.020} \omega(t) dt \right| \leq \int_0^{0.020} |\omega(t)| dt \leq 2.00 \text{ rad.}$$

For (a), only $k = 0$ lies within $[-2, 2]$, so the unique net displacement is $+\pi/6$ rad and average velocity is 26.1799 rad/s. For (b), even total angular distance is at most 2 rad, less than one full revolution; a complete extra turn is excluded. Short reversals are still possible because a bound on speed does not specify its time history.

For (c), shortest-angle reconstruction for arbitrary neighboring readings is unique if $\omega_{\max}\Delta t < \pi$: a displacement equal to π could represent either sense. Thus $\Delta t < \pi/100 = 0.0314159$ s is a sufficient strict condition. Its reciprocal is a sampling rate exceeding 31.83 samples/s. This is a bound for recovering net angular changes under the model, not a guarantee of resolving all within-sample acceleration or reversal features.

Solution 33: reading a reversal graph

The graph areas in the three segments are 8, $[(4 - 2)/2](2) = 2$, and -4 rad. Thus (a) net displacement is $8 + 2 - 4 = 6.00$ rad. During the middle segment $\omega = 4 - 3(t - 2)$, so (d) the direction reverses at $t = 2 + 4/3 = 10/3$ s. Its positive triangular area is $\frac{1}{2}(4)(4/3) = 8/3$ rad and its negative area has magnitude $\frac{1}{2}(2)(2/3) = 2/3$ rad. Therefore (b)

$$\text{angular distance} = 8 + \frac{8}{3} + \frac{2}{3} + 4 = \frac{46}{3} = 15.3333 \text{ rad.}$$

For (c), average velocity is $6/6 = 1.00$ rad/s, whereas average angular speed is $(46/3)/6 = 23/9 = 2.55556$ rad/s. Cancellation affects velocity, not distance. The angle graph is linear and rising on $(0, 2)$, concave downward on $(2, 4)$ with a maximum at $10/3$ s, and linear and falling on $(4, 6)$. Its first derivative remains continuous at 2 and 4 s, although the acceleration changes there.

Solution 34: a torque-time history

The accelerations are $+4/2 = +2$ rad/s² and $-2/2 = -1$ rad/s². Thus (a) $\omega(2) = -1 + 2(2) = 3$ rad/s and $\omega(5) = 3 - 1(3) = 0$. For (b), the first displacement is $-1(2) + \frac{1}{2}(2)(2)^2 = 2$ rad; the second is $3(3) - \frac{1}{2}(3)^2 = 4.5$ rad. The first segment includes a reversal at 0.5 s, but the signed-displacement formula already handles it.

For (c), $W = 4(2) - 2(4.5) = -1$ J. Initially $K_i = \frac{1}{2}(2)(-1)^2 = 1$ J and finally $K_f = 0$, confirming $\Delta K = -1$ J. Explicit power integration gives $\int_0^2 4(-1 + 2t)dt = 8$ J and $\int_0^3 (-2)(3 - u)du = -9$ J. At early negative ω the positive torque removes energy; it adds energy after the first reversal. Using unsigned area or assuming positive torque always adds energy would miss this behavior.

Solution 35: signed work from a torque-angle graph

The line is $\tau_z(\theta) = 6 - 2\theta$ N m when θ is inserted in radians. Hence (a)

$$W(0 \rightarrow 4) = \int_0^4 (6 - 2\theta) d\theta = [6\theta - \theta^2]_0^4 = 8.00 \text{ J.}$$

Graphically, the positive triangle has area 9 J and the negative triangle has area -1 J. Initial energy is $\frac{1}{2}(0.50)(2)^2 = 1$ J, so (b) $K_f = 9$ J and $\omega_f = +\sqrt{2(9)/0.50} = +6.00$ rad/s.

For (c), kinetic energy is $K(\theta) = 1 + 6\theta - \theta^2$. Its derivative is the torque, which vanishes at $\theta = 3$ rad. There $K_{\max} = 10$ J and $\omega_{\max} = \sqrt{40} = 6.32456$ rad/s. The energy is positive throughout $[0, 4]$, so no stopping point invalidates the positive-motion branch. The final speed is smaller than the maximum because negative torque beyond 3 rad removes one joule. Figure 10.40 shows the resulting energy curve and maximum.

Solution 36: acceleration vectors in Cartesian form

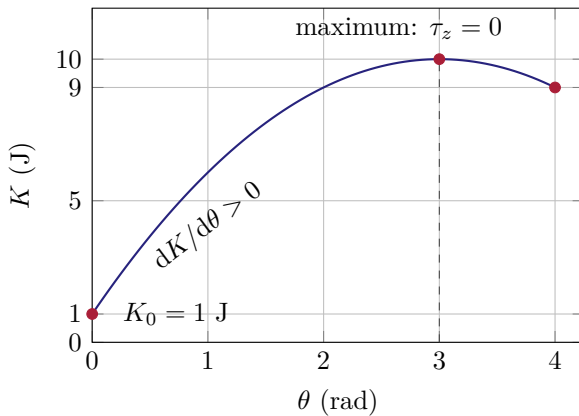
At 30° , $\hat{e}_r = (\sqrt{3}/2, 1/2)$ and $\hat{e}_\theta = (-1/2, \sqrt{3}/2)$. These directions are set geometrically even though the body rotates clockwise. For (a),

$$\mathbf{v} = R\omega\hat{e}_\theta = -0.600\hat{e}_\theta = (+0.300, -0.519615) \text{ m/s.}$$

For (b), the inward and tangential contributions are $-R\omega^2\hat{e}_r = -1.80\hat{e}_r$ and $R\alpha\hat{e}_\theta = +0.800\hat{e}_\theta$. Thus

$$\mathbf{a} = (-1.95885, -0.207180) \text{ m/s}^2, \quad |\mathbf{a}| = \sqrt{1.80^2 + 0.800^2} = 1.96977 \text{ m/s}^2.$$

The annotated construction in Figure 10.40 shows the requested drawing. The vector drawing must put velocity opposite the positive tangent, normal acceleration toward the center, and tangential acceleration along the positive tangent. For (c), $\omega\alpha < 0$, so speed decreases: $d|\omega|/dt = -4$ rad/s² at this instant. The scalar product check is $\mathbf{v} \cdot \mathbf{a} = R^2\omega\alpha = -0.480 \text{ m}^2/\text{s}^3$, confirming a decrease in speed squared.



Solution 36: common tail at marker
axes parallel to laboratory x, y

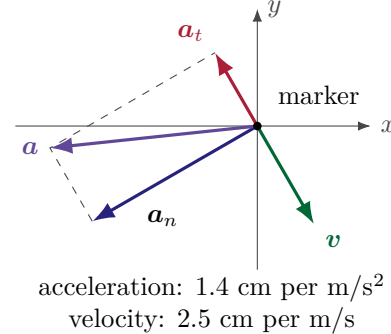


Figure 10.40: Annotated solution models for Problems 35 and 36. Left: integrating the prescribed torque gives $K = 1 + 6\theta - \theta^2$ J, so the maximum speed occurs at $\theta = 3$ rad and the final energy is 9 J. Right: velocity is opposite the positive tangent, normal acceleration points inward, and tangential acceleration opposes velocity; the dashed parallelogram constructs $\mathbf{a} = \mathbf{a}_n + \mathbf{a}_t$. All acceleration arrows share one scale. Velocity uses a separate disclosed scale because its units differ.

Solution 37: recovering angular motion from acceleration

For (a), $|\omega| = \sqrt{a_n/R} = \sqrt{9/0.25} = 6.00$ rad/s and $\alpha = a_t/R = 1.5/0.25 = +6.00$ rad/s². For (b), $|\mathbf{a}| = \sqrt{9^2 + 1.5^2} = 9.12414 \text{ m/s}^2$ and $\psi = \tan^{-1}(1.5/9) = 9.46232^\circ$. The direction is inward tilted toward the positive tangent, regardless of rotational sense.

For (c), a velocity-direction observation is needed. If $\omega = +6$, then α increases speed; if $\omega = -6$, it decreases speed. Normal acceleration is invariant under $\omega \mapsto -\omega$, so the two measured acceleration components do not remove this ambiguity. Repeated time-resolved positions, with adequate sampling and turn counting, would supply the missing direction.

Solution 38: an offset instrument on a rotor

For (a), the main disk contributes $\frac{1}{2}(5)(0.20)^2 = 0.100 \text{ kg m}^2$. The rectangular plate contributes its central perpendicular inertia plus the parallel-axis shift:

$$I_{\text{inst}} = \frac{m(a^2 + b^2)}{12} + md^2 = \frac{0.10^2 + 0.06^2}{12} + 0.18^2 = 0.0335333 \text{ kg m}^2.$$

Thus $I_z = 0.1335333 \text{ kg m}^2$. For (b), uniform change over 4 s requires $\alpha = (12 - 2)/4 = 2.50 \text{ rad/s}^2$, so $\tau_{\text{net}} = I_z \alpha = 0.333833 \text{ N m}$. Work-energy predicts a change $\frac{1}{2}I_z(12^2 - 2^2) = 9.34733 \text{ J}$, equal to $\tau \Delta\theta$ with $\Delta\theta = [(2 + 12)/2]4 = 28 \text{ rad}$.

For (c), collapsing the instrument to its center gives $I_{\text{approx}} = 0.100 + 0.0324 = 0.1324 \text{ kg m}^2$. It omits $0.00113333 \text{ kg m}^2$, or 0.848727% of the exact combined inertia. A small error here follows from the small instrument dimensions relative to its offset; it is not a universal justification for treating every component as a point mass.

Solution 39: separating inertia and bearing loss

Take the hanging mass downward and the driven platform rotation as positive. For (a), $mg - T = mR_h \alpha$, while $TR_h - \tau_{\text{loss}} = I_z \alpha$. Eliminate T :

$$mgR_h - mR_h^2 \alpha = I_z \alpha + \tau_{\text{loss}}.$$

For the two runs the numerical equations are $0.09660 = 3I_z + \tau_{\text{loss}}$ and $0.18920 = 7I_z + \tau_{\text{loss}}$, with torques in N m. For (b), subtracting gives $I_z = (0.18920 - 0.09660)/4 = 0.02315 \text{ kg m}^2$; substitution gives $\tau_{\text{loss}} = 0.02715 \text{ N m}$. Both are positive.

The inferred tensions are $0.20(9.81 - 0.05 \times 3) = 1.932 \text{ N}$ and $0.40(9.81 - 0.05 \times 7) = 3.784 \text{ N}$. Multiplying by R_h reproduces the left sides. For (c), check unwrapped angle histories for approximately quadratic growth, no slip, single-layer effective radius, stable axis, repeated runs and loss dependence on speed. With more masses, fit the same two-parameter equation and inspect residuals: curvature or systematic residual trends can expose changing friction or winding radius. The two constructed runs determine the ideal model exactly; they cannot alone certify its adequacy for an apparatus.

Solution 40: sensitivity of an inferred inertia

For (a), $I_z = \tau/\alpha = 0.600/2.00 = 0.300 \text{ kg m}^2$. For independent small standard uncertainties, differentiate $I(\tau, \alpha)$ and add variances:

$$(b) \quad u_I^2 = \left(\frac{u_\tau}{\alpha}\right)^2 + \left(\frac{\tau u_\alpha}{\alpha^2}\right)^2, \quad \frac{u_I}{I} = \sqrt{(0.012/0.600)^2 + (0.08/2.00)^2}.$$

Thus $u_I = 0.0134164 \text{ kg m}^2$, conventionally about 0.013 kg m^2 . The acceleration uncertainty contributes more variance here.

For (c), treating the quoted ranges as hard bounds gives $I_{\text{min}} = 0.588/2.08 = 0.282692$ and $I_{\text{max}} = 0.612/1.92 = 0.318750 \text{ kg m}^2$. This interval combines the extremal numerator and denominator values; it is wider than one standard uncertainty and is asymmetric about 0.300 . Independent standard uncertainties describe dispersion and are not strict limits. First-order propagation also assumes small relative uncertainties and no omitted model bias; subtracting loss torque incorrectly would violate that premise.

Solution 41: power during negative rotation

For (a), $\alpha = 3/0.50 = +6.00 \text{ rad/s}^2$ and $P = \tau\omega = 3(-8) = -24.0 \text{ W}$. For (b), the torque opposes the negative motion: signed velocity increases toward zero while speed and energy decrease. Initially $K = \frac{1}{2}(0.50)(8)^2 = 16 \text{ J}$.

For (c), $0 = -8 + 6t$ gives $t_s = 4/3 \text{ s}$. Displacement is $-8(4/3) + \frac{1}{2}(6)(4/3)^2 = -16/3 \text{ rad}$. Work is $3(-16/3) = -16 \text{ J}$, exactly removing the initial kinetic energy. If the positive torque continues after stopping, it drives positive rotation and subsequently adds energy. Thus the same torque has negative power before rest and positive power afterward; the sign of torque alone does not determine whether energy is supplied.

Solution 42: a linearly graded rod

For (a), integrate the actual density rather than replacing it by a mean too early:

$$M = \lambda_0 \int_0^L (1 + x/L) dx = \frac{3}{2}\lambda_0 L, \quad \lambda_0 = \frac{2M}{3L}.$$

For (b), $Mx_{\text{CM}} = \lambda_0 \int_0^L x(1 + x/L) dx = \lambda_0 L^2(1/2 + 1/3)$, so $x_{\text{CM}} = 5L/9$. The center shifts toward the denser end, beyond $L/2$. For (c),

$$I_0 = \lambda_0 \int_0^L x^2(1 + x/L) dx = \lambda_0 L^3(1/3 + 1/4) = \frac{7}{18}ML^2.$$

For (d), the axes are parallel and the center-of-mass axis is identified, so $I_{\text{CM}} = I_0 - Mx_{\text{CM}}^2 = (7/18 - 25/81)ML^2 = 13ML^2/162$. It is positive. The density factor lies between 1 and 2, with $\lambda_0 > 0$, and the end-axis inertia exceeds the uniform value $ML^2/3$ by $ML^2/18$, consistent with more mass farther from that end. Subtracting $M(L/2)^2$ would use the wrong center of mass.

Solution 43: an area-weighted disk refinement

For (a), the area fraction of cell j gives $\Delta m_j = M[(j+1)^2 - j^2]/N^2 = M(2j+1)/N^2$. The mass sum telescopes to M . For (b), put $q_j = j + 1/2$; then $2j+1 = 2q_j$ and

$$\frac{I_N}{MR^2} = \frac{2}{N^4} \sum_{j=0}^{N-1} q_j^3 = \frac{2}{N^4} \left(\frac{N^4}{4} - \frac{N^2}{8} \right) = \frac{1}{2} - \frac{1}{4N^2}.$$

The finite sum follows by expanding $(j + 1/2)^3$ and using the standard sums of j, j^2, j^3 ; it can also be verified by induction from the displayed expression. For (c), relative error against $I = MR^2/2$ is $e_N = 1/(2N^2)$. Thus $e_{10} = 0.00500$ and $e_{20} = 0.00125$, or 0.500% and 0.125%. Doubling N reduces error by four.

The area weights repair Problem 30's distribution error. The remaining error comes from replacing every cell's range of r^2 by its midpoint value. The limiting value is now correct; exact total cell mass and increasing resolution address different checks and should both be monitored.

Solution 44: constant power above zero speed

For (a), constant power is an energy-rate statement:

$$\frac{d}{dt} \left(\frac{1}{2} I_z \omega^2 \right) = P \quad \Rightarrow \quad \omega(t) = + \sqrt{\omega_0^2 + \frac{2P}{I_z} t} = \sqrt{25 + 100t} \text{ rad/s}.$$

The positive root follows the stated positive initial motion and energy input. At $t = 3 \text{ s}$, $\omega_f = \sqrt{325} = 18.0278 \text{ rad/s}$. For (b), $\tau_f = P/\omega_f = 5.54700 \text{ N m}$; torque begins at 20 N m and decreases as speed grows.

For (c), integrate the varying velocity:

$$\Delta\theta = \int_0^3 \sqrt{25 + 100t} dt = \frac{(325)^{3/2} - (25)^{3/2}}{150} = 38.2269 \text{ rad.}$$

The energy check gives $\Delta K = \frac{1}{2}(2)(325 - 25) = 300 \text{ J}$, equal to Pt . Angular acceleration is $P/(I_z\omega)$ and is not constant, so a mean of the endpoint velocities is not exact. Starting at a positive finite speed avoids the singular torque demand of finite constant power at rest; the model must be replaced if startup or direction reversal reaches zero speed.

Solution 45: scalar inertia along a chosen tensor direction

For (a), $\boldsymbol{\omega} = \omega\mathbf{n} = (\sqrt{2}, \sqrt{2}, 0) \text{ rad/s}$, so

$$\mathbf{L} = (\sqrt{2}, 3\sqrt{2}, 0) \text{ kg m}^2/\text{s}, \quad L_n = \frac{\sqrt{2} + 3\sqrt{2}}{\sqrt{2}} = 4.00 \text{ kg m}^2/\text{s}.$$

The component ratios differ, so the vectors are not parallel. For (b), $I_n = I_x n_x^2 + I_y n_y^2 + I_z n_z^2 = (1 + 3)/2 = 2.00 \text{ kg m}^2$ and $K = \frac{1}{2}I_n\omega^2 = 4.00 \text{ J}$. Equivalently, $K = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L} = \frac{1}{2}(2 + 6) = 4 \text{ J}$.

For (c), the chosen direction is fixed in inertial space and fixed in the constrained body; rotation about that direction leaves I_n constant. The axial projection is $L_n = I_n\omega$, so projecting $\boldsymbol{\tau} = d\mathbf{L}/dt$ along this fixed $\mathbf{n}$ gives $\tau_n = I_n\alpha$. Transverse bearing torques can maintain the imposed axis and account for the changing transverse angular-momentum direction. Thus a valid axial scalar equation does not assert a universal vector identity. Removing the bearings changes the physical constraint and requires the more general dynamics of Chapter 11.

10.13 Bridge to Chapter 11

This chapter has deliberately kept the rotation axis fixed. The next step is to allow a rigid body to translate while it rotates, as in rolling, and to describe rotational dynamics through angular momentum. That language handles systems whose axis is not fixed in the laboratory and clarifies what changes when the mass distribution or orientation evolves.

The core ideas survive the transition. Moment of inertia still measures mass distribution relative to an axis; torque still depends on a reference point or axis; energy methods remain valuable; and careful sign and geometry choices remain essential. What changes is the level of generality: a single scalar equation $\sum \tau_z = I_z\alpha$ is replaced by angular-momentum reasoning appropriate to the actual motion.

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