

Chapter 11

Rolling, Torque, and Angular Momentum

First Edition

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Preface

This chapter develops the mechanics of bodies that both translate and rotate, and then extends the ideas of torque and momentum into a unified vector language. Rolling motion supplies the physical bridge: the centre of mass moves while the body simultaneously rotates, and the no-slip condition links those motions without making them identical. The same care with reference points then becomes essential for torque and angular momentum.

The chapter is written for calculus-based technical and engineering students. Its core requires vectors, cross products, Newton's laws, work–energy methods and the fixed-axis rotation ideas developed in the preceding chapter. The extension on gyroscopic precession adds vector-rate reasoning, while the advanced section derives the orbital-plus-spin decomposition and explains why the familiar scalar relation $L = I\omega$ is restricted to suitable axes.

Unless a dataset is explicitly identified, numerical values in examples are stipulated teaching inputs rather than measurements. Use $g = 9.81 \text{ m/s}^2$ throughout. The figures combine independently constructed physical models with explicitly credited TikZ construction methods. Synthetic teaching data are identified where they occur. The end-of-chapter source note identifies the supplied reference texts used comparatively for scientific checking and pedagogical coverage.

The Guideline

Treat every rotational problem as a modelling problem before treating it as an algebra problem. State the body or system, the reference frame, the origin or axis, the sign convention, the contact condition and the forces or impulses that are external to the chosen system. A correct equation with an unspecified origin can still represent the wrong physics.

Pedagogical boundary

The core chapter treats rigid-body rolling, vector torque, angular momentum, angular impulse and conservation. Full Euler-angle dynamics, detailed nutation, rotor stability, deformable-contact mechanics and three-dimensional rigid-body equations are outside the core. The advanced section introduces only the tensor ideas needed to understand the limits of scalar rotational formulas.

Conventions and notation

Vectors are bold. Cartesian unit vectors are $\hat{i}$, $\hat{j}$ and $\hat{k}$. Unless a different frame is stated, the laboratory frame is inertial and the horizontal ground is stationary. In planar diagrams, $+x$ points right, $+y$ points up and $+z$ points out of the page. Positive angular quantities are counterclockwise about $+z$.

Symbol	Meaning	SI unit
$\mathbf{R}_{\text{CM}}, \mathbf{V}_{\text{CM}}, \mathbf{a}_{\text{CM}}$	centre-of-mass position, velocity and acceleration	m, m/s, m/s ²
$\boldsymbol{\omega}, \boldsymbol{\alpha}$	angular velocity and angular acceleration	rad/s, rad/s ²
I_{CM}	rotational inertia about the stated CM axis	kg m ²
κ	dimensionless inertia ratio $I_{\text{CM}}/(MR^2)$	1
$\boldsymbol{\tau}_O$	torque about reference point O	N m
$\mathbf{L}_O$	angular momentum about reference point O	kg m ² /s
$\mathbf{J}$	linear impulse $\int \mathbf{F} dt$	N s
$\mathbf{J}_{\tau,O}$	angular impulse $\int \boldsymbol{\tau}_O dt$	N m s
f_x	signed horizontal friction component; $ f_x $ is its magnitude	N
Ω_p	precession angular speed	rad/s

A torque or angular momentum is never written without an origin or axis when that choice matters. For a particle at P ,

$$\boldsymbol{\tau}_O = \mathbf{r}_{P/O} \times \mathbf{F}, \quad \mathbf{L}_O = \mathbf{r}_{P/O} \times m\mathbf{v}.$$

Changing the origin generally changes both quantities. The notation I_{CM} is a scalar only after a specific axis or principal-axis model has been declared.

Caution

Radians are dimensionless in SI dimensional analysis, but writing “rad” is often useful when interpreting angular rates. In cross products, order matters: $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.

Chapter 11

Rolling, Torque, and Angular Momentum

A wheel rolling on a road seems familiar, but it hides several ideas that are easy to confuse. The point touching the road is instantaneously at rest in the road frame, yet it is accelerating. Static friction may be zero, uphill, or downhill depending on how the body is driven. A force may produce no torque about one point and a substantial torque about another. A particle can move in a straight line while carrying nonzero angular momentum about an off-line origin. These are not exceptions to mechanics; they are consequences of specifying geometry and reference choices carefully.

This chapter builds a common language for these situations. We begin with rigid-body velocity composition, use it to understand rolling constraints and energy, promote torque to a three-dimensional vector, define angular momentum for particles and systems, and derive the angular-momentum balance law. Conservation then becomes a statement about external torque impulse rather than a slogan about “spinning.” Gyroscopic precession provides an extension, and the advanced section exposes the orbital-plus-spin structure behind angular momentum of an extended body.

Learning goals

After studying the core and indicated extensions, you should be able to:

- compose translational and rotational velocities of points on a rolling body and state the frame in which a contact point is instantaneously at rest;
- derive rolling acceleration and required static friction, then test whether the no-slip assumption is feasible;
- calculate three-dimensional torque about a specified origin and use the origin-shift relation;
- calculate angular momentum of a particle or system about a stated origin and interpret straight-line motion correctly;
- apply $d\mathbf{L}_O/dt = \boldsymbol{\tau}_{O,\text{ext}}$ and its angular-impulse form under stated reference-origin conditions;
- use angular-momentum conservation when external torque impulse is negligible while keeping energy bookkeeping separate;
- estimate ideal slow gyroscopic precession and test the rapid-spin approximation;
- derive and apply the orbital-plus-spin decomposition of an extended body’s angular momentum;
- solve a pulled-spool problem in which the direction of static friction is an output, not an assumption.

Key message

The unifying theme is reference-aware vector mechanics. Rolling links translation and rotation; torque tells how external forces change angular momentum; conservation is the special case in which the appropriate external torque impulse vanishes.

11.1 Rolling kinematics**11.1.1 Translation and rotation describe different motions**

Rolling combines motion of the centre of mass with rotation about it. A wheel may spin rapidly while its centre remains nearly stationary, or slide rapidly with almost no spin. Consequently, translation and rotation must first be described independently. The condition of rolling without slipping then supplies a relation between them. That relation is a contact constraint; it is not Newton's second law and it does not determine the forces.

Let $\mathbf{r}_P = \mathbf{R}_{\text{CM}} + \boldsymbol{\rho}_P$ be the position of a material point P of a rigid body in an inertial laboratory frame. Here $\boldsymbol{\rho}_P = \mathbf{r}_{P/\text{CM}}$ joins the centre of mass to P . Rigid-body rotation gives $d\boldsymbol{\rho}_P/dt = \boldsymbol{\omega} \times \boldsymbol{\rho}_P$: the vector's length is fixed, but its direction changes. Differentiating the position therefore gives

$$\mathbf{v}_P = \mathbf{V}_{\text{CM}} + \boldsymbol{\omega} \times \boldsymbol{\rho}_P. \quad (11.1)$$

Every point receives the same translational term. Its rotational term is tangent to its circle about the CM, perpendicular to both the rotation axis and $\boldsymbol{\rho}_P$. Equation (11.1) applies instantaneously even when the centre accelerates or the rotation rate changes. For the planar wheels considered here, the axis retains its direction and $\boldsymbol{\omega} = \omega_z \hat{\mathbf{k}}$.

Use $+x$ right, $+y$ up, and $+z$ out of the page. The scalar ω_z is signed: counterclockwise is positive and clockwise is negative. If a circular wheel of radius R touches horizontal ground at its bottom, the material point currently at contact has $\boldsymbol{\rho}_C = -R\hat{\mathbf{j}}$. The cross product then gives

$$\mathbf{v}_C = (V_{\text{CM}} + R\omega_z)\hat{\mathbf{i}},$$

assuming the centre moves horizontally. On stationary ground, no slip means that the contacting surfaces have equal instantaneous velocities. Thus

$$V_{\text{CM}} + R\omega_z = 0. \quad (11.2)$$

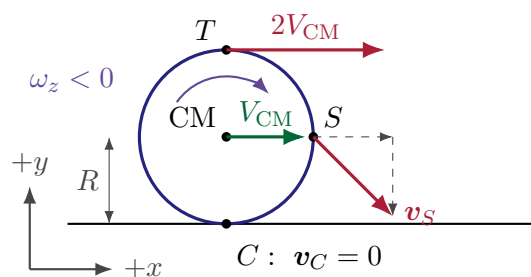
For motion right with clockwise angular-speed magnitude $\Omega = -\omega_z > 0$, this becomes $V_{\text{CM}} = R\Omega$. The signed form is safer when the direction can change. It also shows why $V_{\text{CM}} = R\omega_z$ would be wrong with the present axes.

Figure 11.1 shows how the terms combine. At the top, $\boldsymbol{\rho}_T = R\hat{\mathbf{j}}$, so $\omega_z \hat{\mathbf{k}} \times \boldsymbol{\rho}_T = -R\omega_z \hat{\mathbf{i}}$; translation and rotation add for rightward rolling. At the rightmost point S , $\boldsymbol{\rho}_S = R\hat{\mathbf{i}}$, and the rotational contribution is $R\omega_z \hat{\mathbf{j}}$, downward. The surface-speed magnitude relative to the CM is the same at every rim point, but the laboratory speeds are different.

Example 11.1: Three points on a rolling wheel

E11-01. A rigid wheel of radius $R = 0.30$ m rolls right on stationary ground with centre speed $V_{\text{CM}} = 1.80$ m/s and no slip. Find the angular velocity and the ground-frame velocities of its top, bottom and rightmost rim points. State their velocities relative to the CM.

Strategy and model. Use the same inertial axes as Fig. 11.1. The ground is stationary and



$$\mathbf{v}_P = \mathbf{V}_{\text{CM}} + \boldsymbol{\omega} \times \boldsymbol{\rho}_P$$

At the contact the terms cancel.

At the top the terms add.

At the side they are perpendicular.

Figure 11.1: Velocity field of a wheel rolling right without slipping on stationary ground. Red arrows represent ground-frame velocities and dashed lines resolve the side-point vector. The CM and top arrow lengths have the exact ratio 1 : 2; lengths use an arbitrary common velocity display scale.

the wheel is circular, so Eq. (11.2) determines the signed angular rate:

$$\omega_z = -\frac{1.80}{0.30} = -6.00 \text{ rad/s}, \quad \boldsymbol{\omega} = -6.00 \hat{\mathbf{k}} \text{ rad/s}.$$

The relative velocity is the rotational term, not the entire laboratory velocity. At the top that term is $+1.80 \hat{\mathbf{i}}$ m/s, giving $\mathbf{v}_T = 3.60 \hat{\mathbf{i}}$ m/s. At the bottom it is $-1.80 \hat{\mathbf{i}}$ m/s, giving $\mathbf{v}_C = \mathbf{0}$. At the rightmost point it is $-1.80 \hat{\mathbf{j}}$ m/s, giving

$$\mathbf{v}_S = (1.80 \hat{\mathbf{i}} - 1.80 \hat{\mathbf{j}}) \text{ m/s}, \quad |\mathbf{v}_S| = 1.80\sqrt{2} = 2.55 \text{ m/s}.$$

The last vector points 45° below $+x$. Relative to the CM the top, bottom and side velocities are, respectively, $+1.80 \hat{\mathbf{i}}$, $-1.80 \hat{\mathbf{i}}$ and $-1.80 \hat{\mathbf{j}}$, all in m/s.

Interpretation and check. Each relative speed is $R|\omega_z| = 1.80$ m/s, as required for rim points of a rigid wheel. The ground-frame values agree with a second construction: instantaneous rotation about C gives speed $|\omega_z|$ times distance from C . Those distances are $2R$, 0 and $\sqrt{2}R$ for T , C and S . This construction checks velocities at this instant; it does not make C a permanent pivot.

11.1.2 The contact point is instantaneously at rest, not fixed

Three different objects can be called “the contact point”: a material point of the wheel, a material point of the road, and the geometrical location where contact occurs. They coincide at one instant but have different subsequent histories. For a wheel rolling right, the road material stays where it is; the wheel material rises away from the road; and the geometrical contact location advances with the wheel. Confusing these histories produces the incorrect conclusion that a zero contact velocity implies zero contact acceleration.

Differentiate Eq. (11.1) for the *same material point*. Since $d\boldsymbol{\omega}/dt = \boldsymbol{\alpha}$ and $d\boldsymbol{\rho}_P/dt = \boldsymbol{\omega} \times \boldsymbol{\rho}_P$, its acceleration is

$$\mathbf{a}_P = \mathbf{a}_{\text{CM}} + \boldsymbol{\alpha} \times \boldsymbol{\rho}_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_P). \quad (11.3)$$

The second term is tangential acceleration relative to the CM. The third is inward normal acceleration: for planar motion it equals $-\omega_z^2 \boldsymbol{\rho}_P$. Both appear even though the body is rigid. Rigidity keeps relative distances constant; it does not make the points’ accelerations equal.

For a circular wheel remaining on a straight horizontal surface, $\mathbf{a}_{\text{CM}} = a \hat{\mathbf{i}}$. At the material point currently at the bottom,

$$\mathbf{a}_C = (a + R\alpha_z) \hat{\mathbf{i}} + R\omega_z^2 \hat{\mathbf{j}}.$$

The no-slip relation between the centre and angular rate holds throughout the motion. Differentiating

that *relation between generalized coordinates* gives $a + R\alpha_z = 0$, so

$$\boxed{\mathbf{a}_C = R\omega_z^2 \hat{\mathbf{j}}}$$
 for steady or accelerating no-slip rolling on fixed straight ground. (11.4)

Thus the upward acceleration survives when the wheel speeds up: its horizontal translational and tangential contributions cancel at the bottom. At the top they add, giving $\mathbf{a}_T = 2a\hat{\mathbf{i}} - R\omega_z^2 \hat{\mathbf{j}}$. Figure 11.2 displays these distinctly different results. The formulas assume maintained contact, a fixed radius and a straight surface; they are not a general result for a deforming tire or curved track.

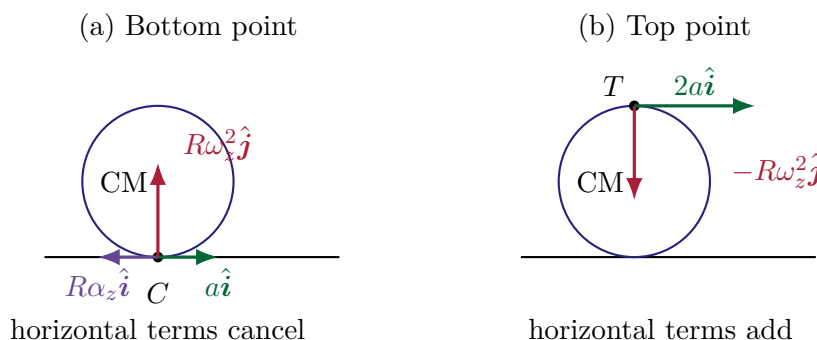


Figure 11.2: Acceleration of the material points presently at the bottom and top during accelerating rightward rolling without slip. For $a > 0$, $\alpha_z < 0$. Vector lengths are schematic; equal and opposite bottom horizontal components and twice the centre acceleration at the top are exact relations.

Example 11.2: Acceleration at a zero-velocity contact

At the instant a wheel has $R = 0.25$ m, $V_{CM} = 2.00$ m/s and $a = 1.20$ m/s² rightward, it rolls without slipping on fixed horizontal ground. Find ω_z , α_z and the accelerations of the bottom, top and rightmost material points.

Constraint first. The velocities give $\omega_z = -V_{CM}/R = -8.00$ rad/s, and its time derivative gives $\alpha_z = -a/R = -4.80$ rad/s². The inward acceleration magnitude of a rim point is $R\omega_z^2 = 16.0$ m/s². Hence

$$\mathbf{a}_C = 16.0\hat{\mathbf{j}} \text{ m/s}^2, \quad \mathbf{a}_T = (2.40\hat{\mathbf{i}} - 16.0\hat{\mathbf{j}}) \text{ m/s}^2.$$

For the rightmost point, $\rho_S = R\hat{\mathbf{i}}$. Its tangential contribution is $R\alpha_z \hat{\mathbf{j}} = -1.20\hat{\mathbf{j}}$ m/s² and its inward contribution is $-16.0\hat{\mathbf{i}}$ m/s². Adding the CM acceleration gives

$$\boxed{\mathbf{a}_S = (-14.8\hat{\mathbf{i}} - 1.20\hat{\mathbf{j}}) \text{ m/s}^2}.$$

Check and meaning. The wheel's centre accelerates right, while its rightmost rim point accelerates mainly left because inward normal acceleration dominates. At C, the horizontal components are $+1.20$ and -1.20 m/s² and cancel exactly. A zero instantaneous velocity therefore coexists with a substantial upward acceleration; the point immediately leaves the contact region.

11.1.3 Following one rim point: the cycloid

An instantaneous velocity field explains a snapshot; a trajectory follows one material point through time. Mark a rim point that starts at the bottom when the CM is at $(0, R)$. Let $\phi \geq 0$ be its clockwise rotation angle. No slip on fixed ground gives $x_{CM} = R\phi$. Relative to the CM, the marked point has coordinates $(-R \sin \phi, -R \cos \phi)$. Its laboratory path is therefore

$$\boxed{x_P = R(\phi - \sin \phi), \quad y_P = R(1 - \cos \phi).} \quad (11.5)$$

This curve is a cycloid. After one turn the CM has advanced $2\pi R$ and the marked point returns to the ground at $x_P = 2\pi R$. At half a turn, $\phi = \pi$, the point is at the top, $y_P = 2R$. The cusp at each ground contact is a consequence of momentarily vanishing velocity, not an abrupt impact in the ideal smooth rolling model.

For constant clockwise angular-speed magnitude $\Omega = d\phi/dt$, differentiating the parametric coordinates gives

$$v_{Px} = R\Omega(1 - \cos \phi), \quad v_{Py} = R\Omega \sin \phi, \quad (11.6)$$

$$a_{Px} = R\Omega^2 \sin \phi, \quad a_{Py} = R\Omega^2 \cos \phi. \quad (11.7)$$

The upward bottom acceleration and twice-CM top speed follow directly. Figure 11.3 uses dimensionless axes so that the path and its velocity history apply to any R and constant Ω . Its negative vertical-velocity interval is physically necessary: the marked point descends during the second half-turn even while the wheel continues right.

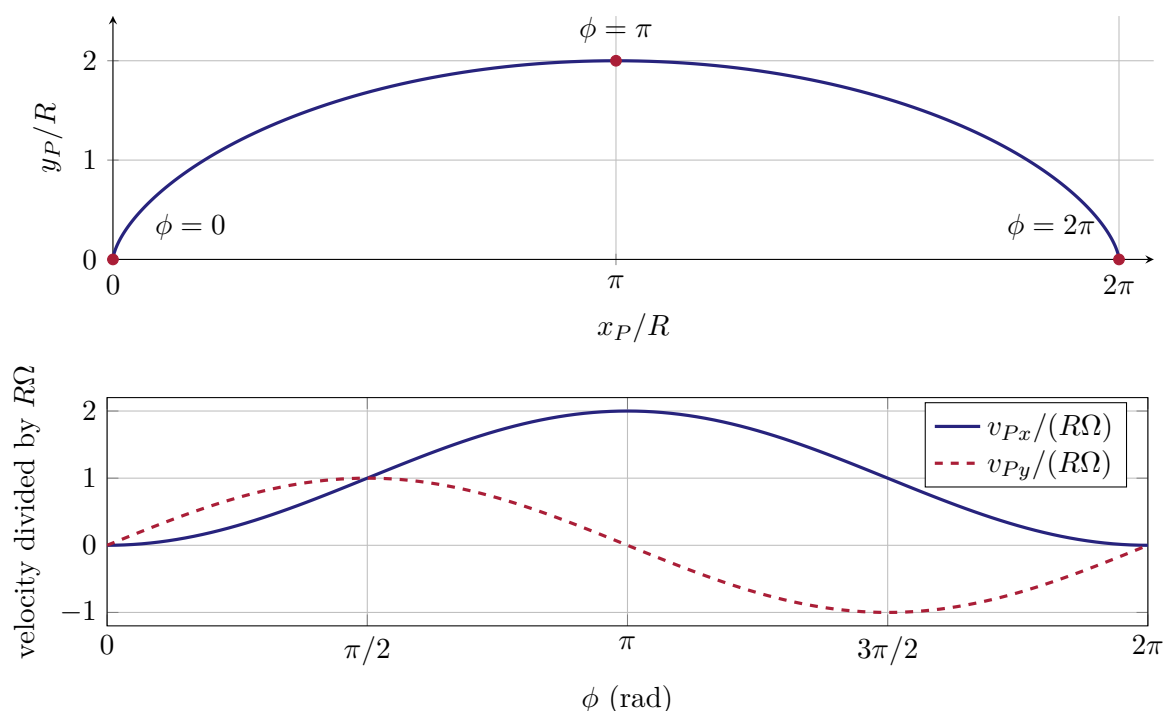


Figure 11.3: A marked rim point during one complete turn of uniform no-slip rolling. The upper panel is its cycloid in physical-coordinate ratios; the lower panel is its velocity history. Both components vanish at a cusp, while at the top the horizontal component is $2R\Omega$ and the vertical component is zero.

Example 11.3: Path length differs from wheel travel

A wheel of radius $R = 0.20$ m rolls uniformly at $V_{CM} = 1.00$ m/s. A marked rim point starts at the bottom. Find its position and velocity after a quarter-turn, the time for one full turn, and the distance that the marked point travels along its cycloid during that turn.

Geometry and time. Here $\Omega = V_{CM}/R = 5.00$ rad/s. At $\phi = \pi/2$,

$$(x_P, y_P) = (0.20(\pi/2 - 1), 0.20) \text{ m} = (0.114, 0.200) \text{ m},$$

and $\mathbf{v}_P = (1.00\hat{i} + 1.00\hat{j})$ m/s. A full turn takes $2\pi/\Omega = 1.257$ s; the CM advances $2\pi R = 1.257$ m.

Arc length. From the velocity components,

$$|\mathbf{v}_P| = R\Omega \sqrt{(1 - \cos \phi)^2 + \sin^2 \phi} = 2R\Omega \sin(\phi/2) \quad (0 \leq \phi \leq 2\pi).$$

The sine is nonnegative on this interval. Use $dt = d\phi/\Omega$ to obtain

$$s_P = \int_0^{2\pi} 2R \sin(\phi/2) d\phi = [-4R \cos(\phi/2)]_0^{2\pi} = 8R = \boxed{1.60 \text{ m}}.$$

Interpretation. The CM displacement, the wheel circumference and the marked point's path length answer different questions. The point rises and falls, so its path length $8R$ exceeds its displacement $2\pi R$. The derivation also checks that the point speed is zero at both contacts and reaches $2V_{\text{CM}}$ at the top.

11.1.4 Rolling on a moving belt and changing frames

Suppose a straight horizontal belt moves with laboratory velocity $U\hat{i}$. The surfaces do not slip when their contact velocities are equal, giving

$$\boxed{V_{\text{CM}} + R\omega_z = U.} \quad (11.8)$$

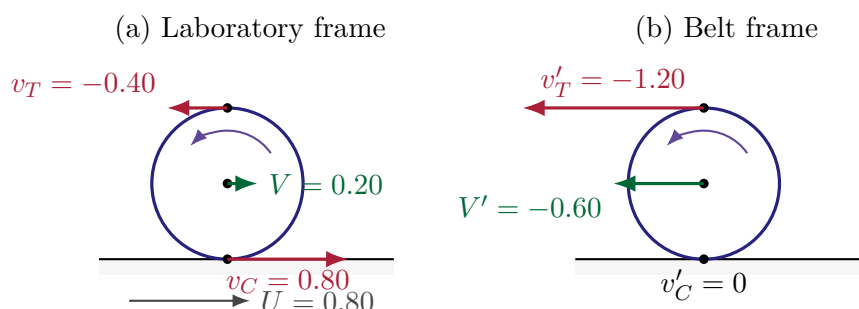
It is the *centre speed relative to the belt*, $V_{\text{CM}} - U$, that is tied to spin. A rightward-moving wheel can therefore rotate counterclockwise if the belt moves right faster than its centre. There is no contradiction: the bottom of a counterclockwise wheel moves right relative to its CM.

For constant U , a frame translating with the belt is inertial. Velocities transform as $\mathbf{v}'_P = \mathbf{v}_P - U\hat{i}$, while angular velocity is unchanged. In that frame the contact is at rest, the centre speed is $V_{\text{CM}} - U$, and the usual stationary-surface kinematics apply. If U changes, the instantaneous velocity subtraction remains valid, but the belt frame accelerates and Newton's laws require the corresponding inertial-force term. The laboratory point-acceleration equation remains valid directly.

For an accelerating belt, differentiating Eq. (11.8) gives $a + R\alpha_z = \dot{U}$. Substitution into the acceleration of the material bottom point yields

$$\mathbf{a}_C = \dot{U}\hat{i} + R\omega_z^2\hat{j}.$$

Thus it shares the belt's tangential acceleration but still has the upward component needed to leave the contact. Differentiating a velocity constraint is legitimate; concluding that the wheel material point and belt material point have identical full accelerations is not.



All indicated velocity components are in m/s; $\omega_z = +3.00$ rad/s in both frames.

Figure 11.4: The same no-slip state viewed from the laboratory and the belt. The belt moves faster rightward than the centre, so the wheel rotates counterclockwise. Velocity arrows use a common scale; the wheel's drawn radius is schematic. Subtracting $U\hat{i}$ makes the bottom velocity zero in the belt frame.

Example 11.4: A wheel turning against its laboratory travel

A wheel of radius $R = 0.20$ m moves right at $V_{\text{CM}} = 0.20$ m/s on a belt moving right at constant $U = 0.80$ m/s. It does not slip. Find its angular velocity, the laboratory top/contact velocities, and the corresponding belt-frame velocities.

Use the contact, not an assumed spin direction. Equation (11.8) gives $\omega_z = (U - V_{\text{CM}})/R = +3.00$ rad/s, counterclockwise. The bottom velocity is $V_{\text{CM}} + R\omega_z = 0.80$ m/s rightward, exactly equal to the belt. The top velocity is $V_{\text{CM}} - R\omega_z = -0.40$ m/s, leftward.

Frame transformation. Subtracting 0.80 m/s from each horizontal laboratory component gives $V'_{\text{CM}} = -0.60$ m/s, $v'_C = 0$ and $v'_T = -1.20$ m/s. The top speed is twice the centre speed in the belt frame, consistent with ordinary rolling left. Angular velocity is the same in both frames, since neither frame rotates.

Check. The equality $V'_{\text{CM}} + R\omega_z = -0.60 + 0.60 = 0$ independently confirms the constraint. If the rates remain constant, the bottom material acceleration is $R\omega_z^2 \hat{\mathbf{j}} = 1.80 \hat{\mathbf{j}}$ m/s², even though the belt material acceleration is zero. The contact-point velocity agreement is instantaneous.

Guided checks

Does no slip mean that every point of a wheel has zero velocity? No. It sets the relative velocity of the two contacting surfaces to zero; other wheel points generally move.

Can no-slip rolling continue with zero static friction? Yes. An ideal unforced wheel already rolling at constant speed on horizontal ground requires no horizontal force or torque.

Does instantaneous rotation about the contact justify treating it as a fixed pivot for acceleration? No. It reproduces the instantaneous velocity field, but the wheel material at contact has upward acceleration and is replaced by another material point.

11.2 Rolling energy and forces

11.2.1 Deriving the translation–rotation energy split

It is tempting to add a translational energy and a rotational energy simply because both motions are visible. The reason this addition is exact is the definition of the centre of mass. Divide a rigid body into mass elements m_i with positions $\boldsymbol{\rho}_i$ relative to its CM. In the laboratory frame,

$$K = \frac{1}{2} \sum_i m_i |\mathbf{V}_{\text{CM}} + \boldsymbol{\omega} \times \boldsymbol{\rho}_i|^2.$$

Expanding the square separates three terms:

$$K = \frac{1}{2} M V_{\text{CM}}^2 + \mathbf{V}_{\text{CM}} \cdot \left(\boldsymbol{\omega} \times \sum_i m_i \boldsymbol{\rho}_i \right) + \frac{1}{2} \sum_i m_i |\boldsymbol{\omega} \times \boldsymbol{\rho}_i|^2.$$

The middle term vanishes because $\sum_i m_i \boldsymbol{\rho}_i = 0$ at the CM. For the planar fixed-direction axis here, the final sum is $I_{\text{CM}} \omega_z^2$, where $I_{\text{CM}} = \sum_i m_i \rho_{i,\perp}^2$. Hence

$$K = \frac{1}{2} M V_{\text{CM}}^2 + \frac{1}{2} I_{\text{CM}} \omega_z^2. \quad (11.9)$$

The split holds for slipping as well as no-slip motion. Its two terms refer to the same body and frame; they must not be mistaken for energies of two independent bodies. Choosing an arbitrary reference point instead of the CM generally leaves a cross term.

For stationary-ground no-slip rolling, $\omega_z^2 = V_{\text{CM}}^2/R^2$. Define the dimensionless inertia ratio

$$\kappa = \frac{I_{\text{CM}}}{MR^2}, \quad K = \frac{1}{2} M (1 + \kappa) V_{\text{CM}}^2. \quad (11.10)$$

A uniform solid sphere has $\kappa = 2/5$, a uniform solid disk or cylinder has $\kappa = 1/2$, and a thin hoop has $\kappa = 1$. The radius here is the contact radius; substituting a hub radius into the rolling constraint would give an incorrect effective inertia. Bodies with more mass near the outer rim have a greater rotational energy for the same centre speed.

The translational and rotational fractions of total rolling kinetic energy are

$$\frac{K_{\text{trans}}}{K} = \frac{1}{1 + \kappa}, \quad \frac{K_{\text{rot}}}{K} = \frac{\kappa}{1 + \kappa}.$$

Figure 11.5 exposes this tradeoff without depending on M , R or speed. A hoop allocates half its energy to each form; a solid sphere allocates 5/7 to translation and 2/7 to rotation. These fractions describe a specified speed and mass distribution, not a mysterious preference of energy for one kind of motion.

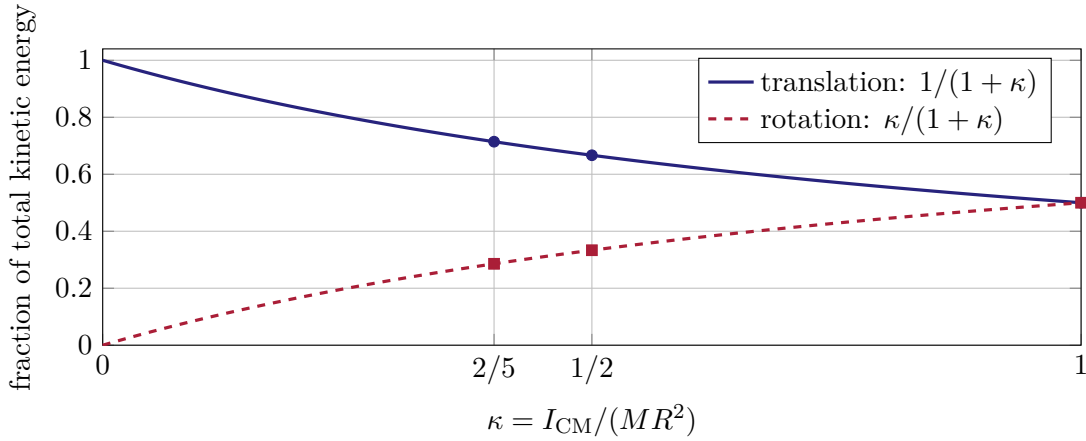


Figure 11.5: Energy partition for no-slip rolling on stationary ground. The two fractions sum to one at every inertia ratio. The marked solid-sphere, solid-disk and hoop values are quantitative; $\kappa = 0$ is an algebraic limiting case rather than a finite-mass rim-supported wheel.

11.2.2 Why ideal static friction can change spin without dissipating energy

A force's instantaneous power is $\mathbf{F} \cdot \mathbf{v}_{\text{application}}$, using the velocity of the material point on the body where it acts. For ideal point contact on stationary ground, $\mathbf{v}_C = 0$, so the contact friction has zero power on the wheel. This statement is stronger and more precise than saying “static friction never does work.” On a moving belt, its power can be nonzero because the application point moves in the laboratory. Deformation, rolling resistance and microscopic slip can also invalidate the ideal point-contact model.

There is no conflict between zero total frictional power and a friction torque changing the rotational energy. On horizontal ground, write the signed friction as $f_x \hat{\mathbf{i}}$. Its CM torque is $R f_x \hat{\mathbf{k}}$. Multiplying the translational balance by V_{CM} assigns a friction term $f_x V_{\text{CM}}$ to the translational-energy change; multiplying the rotational balance by ω_z assigns $R f_x \omega_z$ to the rotational-energy change. Their sum is

$$P_f = f_x (V_{\text{CM}} + R \omega_z) = 0 \quad (11.11)$$

for no slip on stationary ground. The two terms can be nonzero and opposite. The CM work term alone is sometimes called pseudowork: it belongs to the centre-of-mass balance and need not equal the actual work at the contact. Keeping both translation and rotation restores the physical energy accounting.

Consequently, a body released from rest through vertical drop h can satisfy

$$Mgh = \frac{1}{2} M (1 + \kappa) V_{\text{CM}}^2, \quad V_{\text{CM}} = \sqrt{\frac{2gh}{1 + \kappa}}, \quad (11.12)$$

if it remains in ideal no-slip rolling on a stationary track and loses negligible energy to air resistance, deformation or other mechanisms. The system includes the body and Earth for the gravitational potential energy. An energy equation determines the final speed but does not by itself establish that the surface can supply the necessary static friction; that requires force and torque balances.

11.2.3 A fixed incline: solve the force, then test the regime

Consider an unpowered round body on a straight incline descending rightward at angle θ to horizontal. Take $+s$ downhill and a normal coordinate $+n$ away from the surface. The pair $(\hat{s}, \hat{n})$ is right-handed with $+z$ out of the page. As the body accelerates downhill, it rotates clockwise, so $a = -R\alpha_z$. Let f_s denote the *positive uphill magnitude* of friction for this particular solution.

Figure 11.6 shows actual forces. Weight begins at the CM; the contact forces begin at the contact. The normal passes through the CM, and the weight acts at the CM, so neither produces CM torque. Static friction supplies the entire clockwise torque. Resolving Newton's second law gives

$$Mg \sin \theta - f_s = Ma, \quad N = Mg \cos \theta, \quad (11.13)$$

$$-Rf_s = I_{CM}\alpha_z, \quad \alpha_z = -\frac{a}{R}. \quad (11.14)$$

Substituting the constraint into the torque balance gives $f_s = I_{CM}a/R^2 = \kappa Ma$. The resulting acceleration and required friction are

$$a = \frac{g \sin \theta}{1 + \kappa}, \quad f_s = \frac{\kappa}{1 + \kappa} Mg \sin \theta. \quad (11.15)$$

Uphill friction slows the CM relative to frictionless sliding while supplying the torque needed to increase clockwise spin. It does not oppose rotation here. The sign follows from the coupled equations and the impending relative slip at the contact, rather than a general rule about the direction of centre motion.

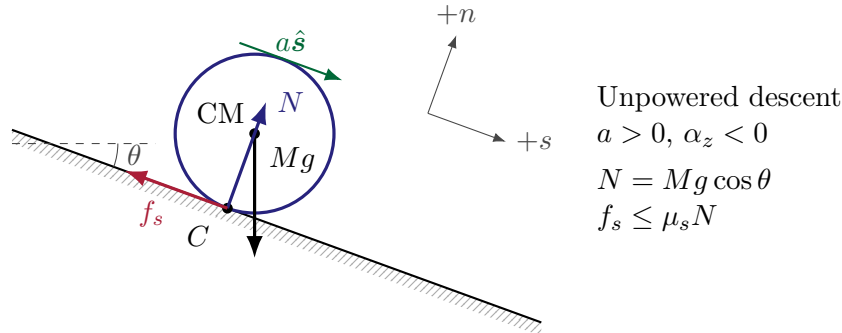


Figure 11.6: A round body descending a fixed incline to the right. Weight is vertical at the CM; the normal and uphill friction act at contact. The circle is tangent to the slope by construction. Force arrow lengths are schematic, and the green acceleration is a kinematic annotation rather than an additional force. Construction method adapted from Izaak Neutelings, TikZ.net.

Static friction can take whatever signed value is required *within its bound*. It is not generally equal to $\mu_s N$. The rolling solution is feasible only if

$$\frac{\kappa}{1 + \kappa} Mg \sin \theta \leq \mu_s Mg \cos \theta, \quad \mu_s \geq \mu_{s,\min} = \frac{\kappa}{1 + \kappa} \tan \theta. \quad (11.16)$$

The coefficient drops out of the rolling acceleration because it limits the contact force rather than prescribing its actual value. Raising μ_s above the threshold does not make ideal rolling faster. If the threshold is not met, using the rolling speed formula is inconsistent even if the algebra is correct.

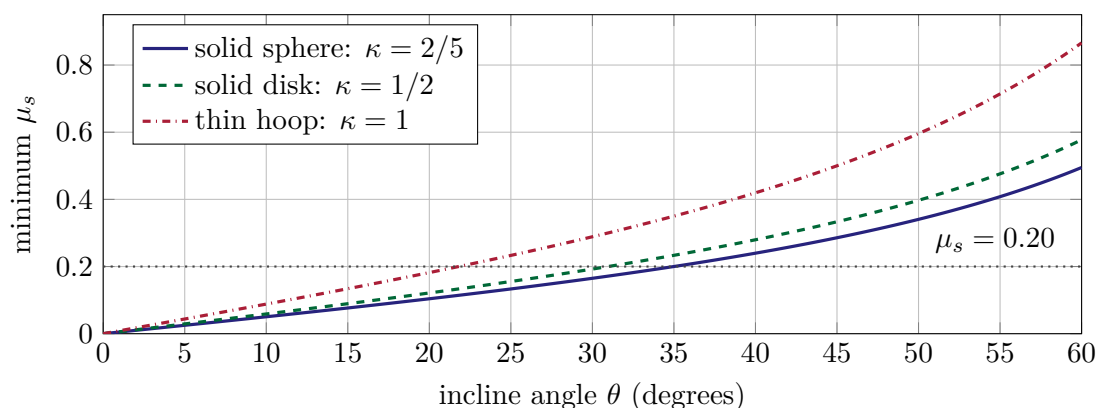


Figure 11.7: Required static-friction coefficient for unpowered no-slip descent. A specified material coefficient must lie on or above the relevant curve at the chosen angle. The horizontal line shows that $\mu_s = 0.20$ supports a larger angle for a solid sphere than for a hoop. The plot describes feasibility, not the actual friction force.

Example 11.5: Two rolling bodies from the same height

E11-02. A uniform solid disk and a thin hoop have the same mass $M = 2.50$ kg and radius. Each is released from rest on the same 20° incline through vertical drop $h = 1.20$ m. The static coefficient is $\mu_s = 0.20$. Compare accelerations, friction, travel times and bottom speeds, and check no slip.

Model and geometry. Assume rigid bodies, a straight stationary incline, maintained contact and negligible rolling resistance. Use $g = 9.81$ m/s². The incline length is $s = h / \sin 20^\circ = 3.509$ m, and both normal forces are $Mg \cos 20^\circ = 23.05$ N. The two bodies differ in κ , not in their prescribed mass or contact radius.

Disk, $\kappa = 1/2$. Force and torque balances give

$$a_d = \frac{9.81 \sin 20^\circ}{1.5} = 2.237 \text{ m/s}^2, \quad f_{s,d} = \frac{1}{3} Mg \sin 20^\circ = 2.80 \text{ N}$$

uphill. Its threshold is $\mu_{s,\min,d} = \tan 20^\circ / 3 = 0.121$, below 0.20. Constant acceleration from rest gives

$$t_d = \sqrt{2s/a_d} = 1.77 \text{ s}, \quad v_d = \sqrt{2a_d s} = 3.96 \text{ m/s}.$$

Hoop, $\kappa = 1$. Similarly,

$$a_h = \frac{9.81 \sin 20^\circ}{2} = 1.678 \text{ m/s}^2, \quad f_{s,h} = \frac{1}{2} Mg \sin 20^\circ = 4.19 \text{ N}$$

uphill. Its threshold is $\mu_{s,\min,h} = \tan 20^\circ / 2 = 0.182$, also below 0.20. Therefore

$$t_h = \sqrt{2s/a_h} = 2.05 \text{ s}, \quad v_h = \sqrt{2a_h s} = 3.43 \text{ m/s}.$$

Both required forces are below $\mu_s N = 4.61$ N, although the hoop is closer to the limit.

Independent energy check. The available gravitational energy is $Mgh = 29.43$ J. Substitution into $v = \sqrt{2gh/(1 + \kappa)}$ gives the same two speeds without using the force-derived acceleration. The disk's translational fraction is $2/3$; the hoop's is $1/2$, explaining their different speeds at equal total energy.

Interpretation. The disk reaches the bottom first, and the hoop requires greater friction. If the coefficient were 0.15, the disk's rolling model would remain feasible but the hoop's would fail. The threshold checks establish which branch of the dynamics can be used; the final-speed energy equation alone cannot make that decision.

11.2.4 When the incline threshold fails

If a body released from rest lacks sufficient static friction, the bottom initially tends to slip downhill. Kinetic friction then acts uphill with magnitude $f_k = \mu_k Mg \cos \theta$. Translation and rotation evolve independently:

$$a = g(\sin \theta - \mu_k \cos \theta), \quad \alpha_z = -\frac{\mu_k g \cos \theta}{\kappa R}. \quad (11.17)$$

For this downhill-slipping branch, define the contact slip speed relative to the plane by $u = V_{\text{CM}} + R\omega_z$. Its rate is

$$\dot{u} = g \left[\sin \theta - \mu_k \cos \theta \left(1 + \frac{1}{\kappa} \right) \right]. \quad (11.18)$$

If $\mu_k \leq \mu_s < \kappa \tan \theta / (1 + \kappa)$, this derivative is positive: the stipulated kinetic-friction law is consistent with continuing downhill slip from rest. One must not retain $a = -R\alpha_z$ after abandoning no slip. For other initial spins, u can initially have the opposite sign, requiring the opposite friction direction and a piecewise solution until a transition occurs.

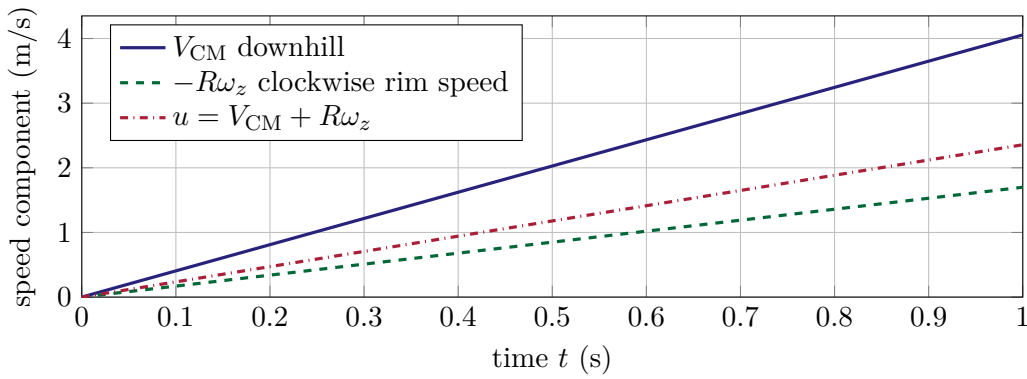


Figure 11.8: Slipping descent in the failed-friction regime of Example 11.6. The centre accelerates faster than the clockwise rim speed grows, leaving a positive and increasing downhill contact slip. The gap between the blue and dashed green curves equals the red curve. All curves are analytical constant-acceleration solutions.

Example 11.6: Diagnosing an impossible rolling assumption

A uniform solid cylinder with $R = 0.10$ m is released from rest on a 30° slope with $\mu_s = 0.12$ and $\mu_k = 0.10$. Determine whether it rolls without slip; if it slips, find its centre speed, signed spin and slip speed after 1.00 s. Assume the slope is sufficiently long and use $g = 9.81$ m/s².

Regime decision. For $\kappa = 1/2$, the required coefficient is $\tan 30^\circ / 3 = 0.192$. Since $0.12 < 0.192$, a no-slip solution is unavailable. Starting from rest, translation initially outruns spin, so kinetic friction is uphill.

Independent dynamics. Equation (11.17) gives

$$a = 9.81(0.500 - 0.10 \cos 30^\circ) = 4.055 \text{ m/s}^2, \quad \alpha_z = -\frac{0.10(9.81) \cos 30^\circ}{0.50(0.10)} = -16.99 \text{ rad/s}^2.$$

After one second, $V_{\text{CM}} = 4.06$ m/s and $\omega_z = -17.0$ rad/s. Consequently

$$u = V_{\text{CM}} + R\omega_z = 4.055 - 1.699 = \boxed{2.36 \text{ m/s}}$$

downhill. The positive $\dot{u} = 2.356$ m/s² confirms the chosen friction direction throughout the interval.

Contrast and check. The rejected rolling acceleration would be $g \sin 30^\circ / 1.5 = 3.27$ m/s². The actual CM acceleration is greater, while the actual spin grows more slowly. During slip, mechanical energy is converted to internal energy at rate $f_k|u|$; imposing conservation of body–Earth mechanical energy would therefore introduce a second inconsistent assumption.

11.2.5 Driven, braked and pulled wheels

The incline result does not establish a universal friction direction. A motor can turn a wheel clockwise, making its bottom tend to slip left and requiring forward friction. A brake can reduce clockwise spin while the centre still moves right, making the bottom tend to slip right and requiring backward friction. An axle force can pull the centre forward without directly spinning the wheel, also requiring backward friction. These different causes are displayed in Fig. 11.9.

To cover all three with one model, consider a single rigid wheel of mass M , radius R and inertia $I_{\text{CM}} = \kappa MR^2$ on fixed horizontal ground. Let F_a be a signed horizontal force through its axle, T_z an externally applied signed couple torque, and f_x signed contact friction. The wheel is the system; the actuator producing T_z is external. Force, torque and no-slip equations are

$$Ma = F_a + f_x, \quad I_{\text{CM}}\alpha_z = T_z + Rf_x, \quad a = -R\alpha_z.$$

Eliminating α_z and f_x yields

$$a = \frac{F_a - T_z/R}{M(1 + \kappa)}, \quad f_x = -\frac{\kappa F_a + T_z/R}{1 + \kappa}. \quad (11.19)$$

For $F_a = 0$ and a clockwise motor torque $T_z < 0$, both a and f_x are positive. The ground supplies the horizontal force; the motor supplies energy. For $F_a = 0$ and a counterclockwise braking torque $T_z > 0$, both are negative, so the rightward-moving wheel slows. For an axle pull $F_a > 0$ with $T_z = 0$, acceleration is forward but friction is backward, providing clockwise torque.

These signs are outputs of a stated actuator model. A car wheel additionally experiences axle forces from the vehicle, and its normal load need not be Mg for the wheel alone. Such systems require those forces and loads rather than automatic substitution of the single-wheel formulas. In this isolated-wheel model the feasibility test is $|f_x| \leq \mu_s Mg$. During pure torque drive, it imposes $|T_z| \leq \mu_s MgR(1 + \kappa)$.

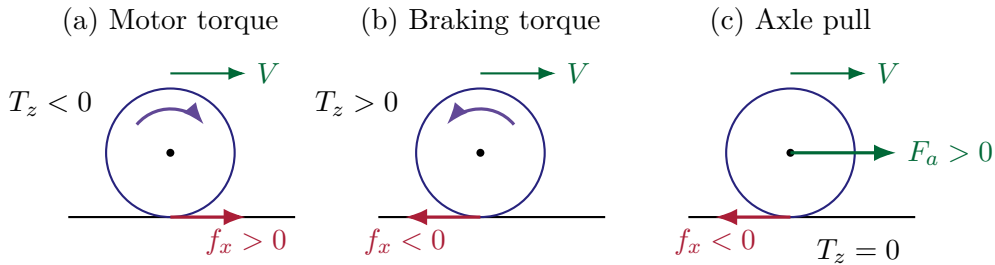


Figure 11.9: Horizontal force and torque causes can require different static-friction directions for wheels moving right without slip. Weight and the balancing normal force are omitted from this sign-comparison schematic. Panels (a) and (b) have $F_a = 0$; panel (c) has $T_z = 0$. Arrow lengths do not specify force magnitudes.

Example 11.7: Traction and braking of an ideal wheel

A wheel has $M = 4.0 \text{ kg}$, $R = 0.20 \text{ m}$, $\kappa = 0.50$ and $\mu_s = 0.30$. There is no horizontal axle force. First apply a motor torque $T_z = -3.0 \text{ N m}$. Then consider the same wheel rolling right at 2.0 m/s while a brake applies $T_z = +2.0 \text{ N m}$. Find acceleration, friction and angular acceleration in each case. For braking, find stopping time and distance, and check energy accounting.

Motor case. The inertia is $I_{\text{CM}} = 0.50(4.0)(0.20)^2 = 0.080 \text{ kg m}^2$. Equation (11.19) gives $a = 2.50 \text{ m/s}^2$, $f_x = +10.0 \text{ N}$ and $\alpha_z = -12.5 \text{ rad/s}^2$. Torque substitution checks the result: $T_z + Rf_x = -3.0 + 0.20(10.0) = -1.0 \text{ N m} = I_{\text{CM}}\alpha_z$. The available friction is $0.30(4.0)(9.81) =$

11.77 N, so rolling is feasible.

Brake case. Changing the applied torque gives

$$a = -1.667 \text{ m/s}^2, \quad f_x = -6.67 \text{ N}, \quad \alpha_z = +8.33 \text{ rad/s}^2.$$

The initially clockwise spin $\omega_{z0} = -2.0/0.20 = -10.0 \text{ rad/s}$ approaches zero. Required friction is again within the bound. The constant deceleration gives

$$t_{\text{stop}} = \frac{2.0}{1.667} = 1.20 \text{ s}, \quad s_{\text{stop}} = \frac{(2.0)^2}{2(1.667)} = 1.20 \text{ m}.$$

Energy check. Initially $K_0 = \frac{1}{2}M(1 + \kappa)V_0^2 = 12.0 \text{ J}$. While stopping, the signed angular displacement is $\Delta\theta_z = -s_{\text{stop}}/R = -6.00 \text{ rad}$. The brake's work is $T_z\Delta\theta_z = -12.0 \text{ J}$, equal to $K_f - K_0$. Ideal static contact friction does no total work on the wheel; the brake removes energy. Once the wheel stops, the stated braking process ends rather than allowing the constant braking torque to accelerate it backward.

11.2.6 Slipping on level ground can evolve into rolling

A wheel put down with the wrong combination of speed and spin initially slips. Friction then changes both translation and rotation in a coordinated way. Unlike the insufficient-friction incline example, a free wheel on level ground can reach a no-slip state because there is no continuing downhill force to outrun its spin.

For a wheel with initial V_0 and signed ω_{z0} , define

$$u = V_{\text{CM}} + R\omega_z, \quad u_0 = V_0 + R\omega_{z0}.$$

This is the horizontal velocity of the material bottom point relative to fixed ground. Its sign specifies kinetic-friction direction. With constant $\mu_k > 0$, no other horizontal force or applied torque, and maintained contact,

$$f_x = -\mu_k Mg \operatorname{sgn}(u), \quad \dot{V}_{\text{CM}} = \frac{f_x}{M}, \quad \dot{\omega}_z = \frac{Rf_x}{I_{\text{CM}}}.$$

Until u reaches zero, friction has a constant sign. The combination obeys

$$\dot{u} = -\mu_k g \left(1 + \frac{1}{\kappa}\right) \operatorname{sgn}(u_0), \quad t_r = \frac{|u_0|}{\mu_k g(1 + 1/\kappa)}. \quad (11.20)$$

The solution must stop using kinetic friction at t_r . Continuing its constant-friction equations past that time would reverse u while keeping the wrong friction sign. On ideal horizontal ground, after matching speed and spin, the free wheel requires zero friction and continues in uniform rolling.

Integrating the two balances over t_r gives

$$V_r = V_0 - \frac{\kappa}{1 + \kappa} u_0 = \frac{V_0 - \kappa R\omega_{z0}}{1 + \kappa}, \quad \omega_{zr} = -\frac{V_r}{R}. \quad (11.21)$$

If the initial spin is too slow for rightward travel, $u_0 > 0$, friction acts left, reducing V and increasing clockwise spin. If the wheel is over-spinning clockwise, $u_0 < 0$, friction acts right, increasing V while reducing the clockwise spin magnitude. Either way $|u|$ decreases until the two motions match.

The dissipative power at the slipping contact is $f_x u = -\mu_k Mg|u|$. Since $|u|$ falls linearly from $|u_0|$ to zero, its time integral is a triangular area. Thus the decrease in wheel kinetic energy is

$$K_0 - K_r = \frac{1}{2}\mu_k Mg|u_0|t_r = \frac{1}{2}M\frac{\kappa}{1 + \kappa}u_0^2. \quad (11.22)$$

Substitution of V_r and ω_{zr} into the translation–rotation energy formula gives the same difference, providing an independent check. Within this model the final state and total energy loss are independent of nonzero μ_k ; the coefficient controls how long capture takes. The limit $\mu_k \rightarrow 0$ sends t_r to infinity. At exactly zero friction, an initially slipping wheel never reaches rolling in finite time.

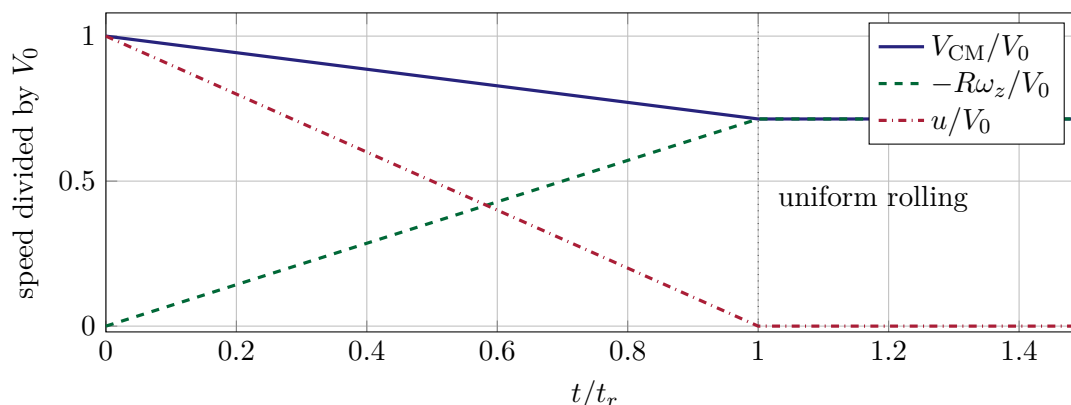


Figure 11.10: Capture into rolling for an initially nonspinning solid sphere with $\kappa = 2/5$. Kinetic friction decreases centre speed and increases clockwise spin until their associated speeds agree. The change of regime at $t = t_r$ keeps all velocities continuous; friction becomes zero after capture. The common final speed is $5V_0/7$.

Example 11.8: A sliding sphere begins to roll

A uniform solid sphere with $M = 2.0$ kg and $R = 0.15$ m is launched along horizontal ground at $V_0 = 3.0$ m/s with no initial spin. Take $\mu_k = 0.20$ and $g = 9.81$ m/s², neglect other resistance, and assume maintained contact. Find the time and centre displacement before rolling, the final speed and spin, and the kinetic-energy loss.

Friction and transient. With $\kappa = 2/5$ and $u_0 = 3.0$ m/s > 0 , friction acts left with magnitude $\mu_k Mg = 3.924$ N. Hence $a = -1.962$ m/s² and $R\alpha_z = -\mu_k g/\kappa = -4.905$ m/s². Their sum makes the slip speed decrease at 6.867 m/s², so

$$t_r = \frac{3.0}{6.867} = 0.437 \text{ s}, \quad x_r = V_0 t_r + \frac{1}{2} a t_r^2 = 1.12 \text{ m}.$$

Captured rolling state. Equation (11.21) gives

$$V_r = \frac{3.0}{1.4} = 2.143 \text{ m/s}, \quad \omega_{zr} = -\frac{2.143}{0.15} = -14.29 \text{ rad/s}.$$

The sphere is still moving right; the lost centre speed has helped establish clockwise rotation while some mechanical energy has also been dissipated.

Energy and two checks. Initially $K_0 = \frac{1}{2}(2.0)(3.0)^2 = 9.00$ J. Finally $K_r = \frac{1}{2}(2.0)(1.4)(2.143)^2 = 6.43$ J, so the loss is 2.57 J. Independently, Eq. (11.22) gives $\frac{1}{2}(2.0)(0.4/1.4)(3.0)^2 = 2.57$ J.

The integrated relative slip distance is $\int_0^{t_r} u \, dt = \frac{1}{2} u_0 t_r = 0.655$ m. Multiplying by 3.924 N gives the same 2.57 J loss. This distance differs from the CM displacement 1.12 m because the contact's relative motion includes the increasing spin. At capture, $V_r + R\omega_{zr} = 0$ within rounding; thereafter no contact friction is required by the ideal free-rolling model.

11.2.7 A pulled spool: friction reversal and application-point work

Consider a spool with outer contact radius R and inner hub radius $r < R$. A light string leaves the *top* of the hub horizontally and is pulled right with force $F > 0$. The chosen geometry matters:

a string leaving the bottom or at an angle produces different torques and a different solution. Figure 11.11 fixes the upper-hub geometry, with positive α_z explicitly counterclockwise.

Write $q = r/R$ and $I_{\text{CM}} = \kappa MR^2$. The string force has CM torque $-rF$, because its application point is above the CM and the force is rightward. Signed ground friction f_x has torque $+Rf_x$. Force, torque and no-slip balances are

$$F + f_x = Ma, \quad (11.23)$$

$$-rF + Rf_x = I_{\text{CM}}\alpha_z, \quad (11.24)$$

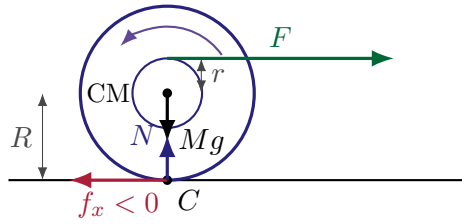
$$a = -R\alpha_z. \quad (11.25)$$

Substituting the last equation into the torque balance and dividing by R gives $-qF + f_x = -\kappa Ma$. Combine this with $f_x = Ma - F$ to find

$$a = \frac{F(1+q)}{M(1+\kappa)}, \quad f_x = \frac{F(q-\kappa)}{1+\kappa}. \quad (11.26)$$

The spool accelerates right for all $0 < q < 1$ in this no-slip model, but friction changes sign at $q = \kappa$. It is leftward if $q < \kappa$, rightward if $q > \kappa$, and zero at equality. In the frictionless instantaneous tendency, the contact acceleration would be $(F/M)(1 - q/\kappa)$: translation dominates for $q < \kappa$, so friction must act left; string-induced clockwise spin dominates for $q > \kappa$, so friction must act right. This independently explains the algebraic sign.

positive α_z : counterclockwise



String leaves the upper hub.
Friction shown for $q < \kappa$.
Actual rolling right has $\alpha_z < 0$.

Figure 11.11: The upper-hub pulled spool. The positive rotation arrow denotes the counterclockwise sign convention, while the actual angular acceleration for rightward rolling is clockwise. Leftward friction is shown for $q = r/R < \kappa$; it reverses across the threshold. Force arrow lengths are schematic.

Since the string is horizontal, $N = Mg$. The required signed friction must satisfy

$$\frac{F|q - \kappa|}{1 + \kappa} \leq \mu_s Mg. \quad (11.27)$$

For $q \neq \kappa$, this gives $F \leq \mu_s Mg(1 + \kappa)/|q - \kappa|$. At $q = \kappa$, the ideal force and torque supplied by the string alone already match the rolling constraint, so no friction is required. The disappearance of this particular traction bound does not remove limits due to strength, deformation, loss of the ideal string geometry or the available pulling apparatus.

Figure 11.12 plots the force ratio and makes the sign change visible. The torque itself also deserves attention: rightward friction, when present, supplies a counterclockwise torque and partially counters the clockwise string torque. The net torque still produces the negative angular acceleration required for rolling right. Friction direction cannot be inferred from the direction of net torque alone.

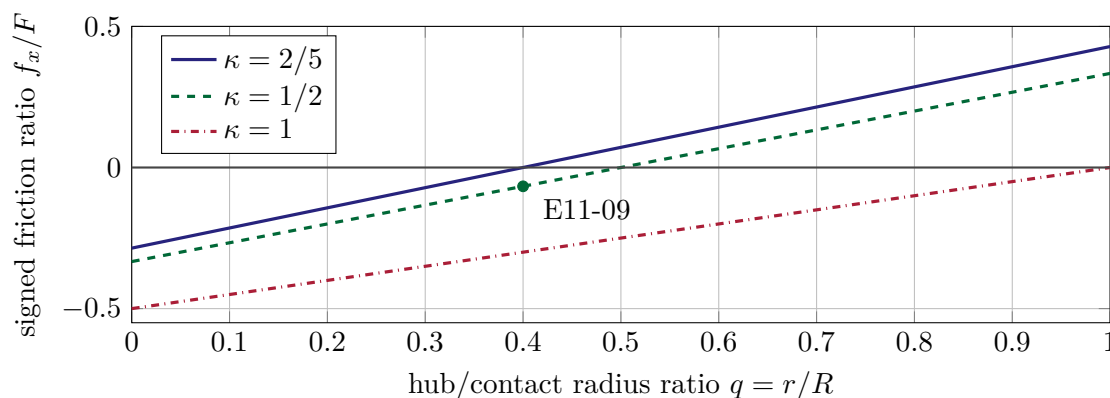


Figure 11.12: Signed friction for a spool pulled right from the top of its hub. Negative values mean leftward friction. Each line crosses zero at $q = \kappa$; a hoop-like $\kappa = 1$ remains on the negative side for the permitted $q < 1$. The marker represents E11-09. A plotted force ratio becomes physically attainable only after the static-friction bound is checked.

Example 11.9: A spool driven by a horizontal string

E11-09. A spool has $M = 3.0$ kg, $R = 0.20$ m, $r = 0.080$ m and $I_{\text{CM}} = 0.50MR^2$. A horizontal force $F = 12$ N pulls the upper tangent of the hub; $\mu_s = 0.25$. Determine centre acceleration, angular acceleration and friction. Verify the rolling assumption. As an energy check, suppose it starts at rest and its centre moves 0.50 m.

Geometry and balance. Here $q = 0.40$, $\kappa = 0.50$ and $I_{\text{CM}} = 0.060$ kg m². With signed counterclockwise-positive rotation,

$$a = \frac{12(1 + 0.40)}{3.0(1 + 0.50)} = 3.733 \text{ m/s}^2, \quad f_x = \frac{12(0.40 - 0.50)}{1.50} = -0.800 \text{ N},$$

and $\alpha_z = -a/R = -18.67$ rad/s². Thus the centre accelerates right, the spin accelerates clockwise, and friction points left. Direct substitution checks both balances:

$$F + f_x = 12 - 0.8 = 11.2 \text{ N} = Ma,$$

$$-rF + Rf_x = -0.080(12) + 0.20(-0.8) = -1.12 \text{ N m} = I_{\text{CM}}\alpha_z.$$

The available static friction is $\mu_s Mg = 0.25(3.0)(9.81) = 7.36$ N, comfortably above 0.800 N. The solution is feasible. Its minimum required coefficient is $|f_x|/(Mg) = 0.0272$.

Work acts through the string displacement. During centre displacement $s = 0.50$ m, clockwise rotation magnitude is $s/R = 2.50$ rad. The string unwinds through an additional length $r(s/R) = 0.20$ m, so its pulled end travels $s(1 + q) = 0.70$ m. For an inextensible massless string, that is the force application displacement. Hence $W_F = 12(0.70) = 8.40$ J.

Constant-acceleration kinematics gives $V = \sqrt{2as} = 1.932$ m/s. The total kinetic energy is $\frac{1}{2}M(1 + \kappa)V^2 = 8.40$ J, exactly matching the string work. Using $Fs = 6.00$ J alone would miss the extra unwinding displacement. This independently checks both the $1 + q$ factor and the zero actual work of ideal ground friction.

Caution

Determine contact regime and friction direction from relative slip or the coupled balances. The condition $|f_s| \leq \mu_s N$ permits a range of static forces; equality identifies a threshold. Ideal static contact on fixed ground can transmit torque without dissipating wheel energy, while sliding contact dissipates mechanical energy and moving contact can transfer energy from the support.

11.3 Vector torque about a stated origin

The incline and spool calculations used the component of torque along a known rotation axis. A bracket in three dimensions, a freely translating platform, or a particle passing a detector requires a more general description. The first decision is the reference point. A force may have a large moment about one point and no moment about another; changing the reference point changes the quantity being calculated, rather than changing the force itself.

11.3.1 Cross product, orientation and perpendicular lever arm

Let a force $\mathbf{F}$ act at point P , and let $\mathbf{r}_{P/O}$ be the vector from the selected point O to P . The torque of this force about O is

$$\boxed{\boldsymbol{\tau}_O = \mathbf{r}_{P/O} \times \mathbf{F}.} \quad (11.28)$$

This definition can be used at any instant, whether or not O is moving. A separate question is whether its torque can be equated directly to the time derivative of a particular angular momentum. We derive the conditions for that dynamical statement below.

The cross product is perpendicular to both of its factors. To determine its direction, place the position and force vectors tail to tail without rotating either vector, then curl the fingers of the right hand from $\mathbf{r}$ toward $\mathbf{F}$ through the smaller angle. The thumb gives the direction of $\boldsymbol{\tau}$. In a right-handed Cartesian basis, $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$, $\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}$ and $\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$; reversing either order reverses the result. For forces in the xy plane, positive τ_z corresponds to a counterclockwise turning tendency. The torque vector points along the associated axis, rather than along the curved motion drawn on a page.

The magnitude follows from the cross-product definition:

$$\tau_O = rF \sin \phi = rF_{\perp} = F\ell_{\perp}, \quad (11.29)$$

where ϕ is the angle between the vectors placed tail to tail, $F_{\perp}$ is the component perpendicular to $\mathbf{r}$, and $\ell_{\perp}$ is the shortest distance from O to the force's line of action. A force directed toward or away from O has zero moment about O , even if its magnitude is large. Conversely, the distance OP is not generally the lever arm: the perpendicular distance to the line of action is what enters $F\ell_{\perp}$.

Figure 11.13 separates direction from magnitude. In its first panel, the positive torque is perpendicular to the plane of $\mathbf{r}$ and $\mathbf{F}$. In the second panel, the horizontal force has negative z torque because its line of action lies above O . These signs can be checked directly with $\tau_z = xF_y - yF_x$.

Torque has SI units N m. Work has the same dimensions, but the quantities play different roles: torque is an axial vector, whereas work is a scalar energy transfer obtained from force dotted with displacement. Keeping N m for torque and J for energy helps preserve that distinction.

For an arbitrary three-dimensional force, calculate all components rather than inferring a direction from a planar sketch:

$$\boldsymbol{\tau}_O = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} = (yF_z - zF_y)\hat{\mathbf{i}} + (zF_x - xF_z)\hat{\mathbf{j}} + (xF_y - yF_x)\hat{\mathbf{k}}. \quad (11.30)$$

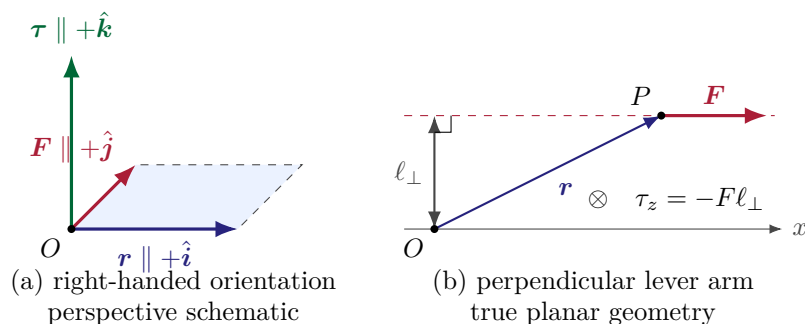


Figure 11.13: Two uses of the torque cross product. The perspective panel identifies its orientation; the planar panel identifies the perpendicular lever arm and the into-page sign. Arrow lengths are schematic. The symbols $\odot$ and $\otimes$ denote out of and into the page, respectively. Panel (b) adapts the perpendicular-projection method of I. Neutelings, TikZ.net; its geometry is independently constructed.

The determinant is a compact component mnemonic; the minus sign in its middle cofactor must be retained. Two useful checks follow from the geometry: $\boldsymbol{\tau} \cdot \mathbf{r} = 0$ and $\boldsymbol{\tau} \cdot \mathbf{F} = 0$. They cannot prove that every component is correct, but failure of either test exposes a mistake.

11.3.2 Changing the origin and recognizing a couple

Let O' be displaced from O by $\mathbf{a} = \mathbf{r}_{O'/O}$. The position of the application point from O' is $\mathbf{r}_{P/O'} = \mathbf{r}_{P/O} - \mathbf{a}$. Substitution into the definition gives

$$\boxed{\boldsymbol{\tau}_{O'} = \boldsymbol{\tau}_O - \mathbf{a} \times \mathbf{F}.} \quad (11.31)$$

For several forces, sum each force moment about the same origin:

$$\boldsymbol{\tau}_{O',\text{net}} = \boldsymbol{\tau}_{O,\text{net}} - \mathbf{a} \times \mathbf{F}_{\text{net}}. \quad (11.32)$$

This relation is especially useful when a support force is inconvenient about one point but has zero moment about another. It does not remove that support force from the body's force balance. Moving the origin along the direction of a single force leaves its torque unchanged because $\mathbf{a} \times \mathbf{F} = 0$; moving the origin onto the line of action makes its torque zero.

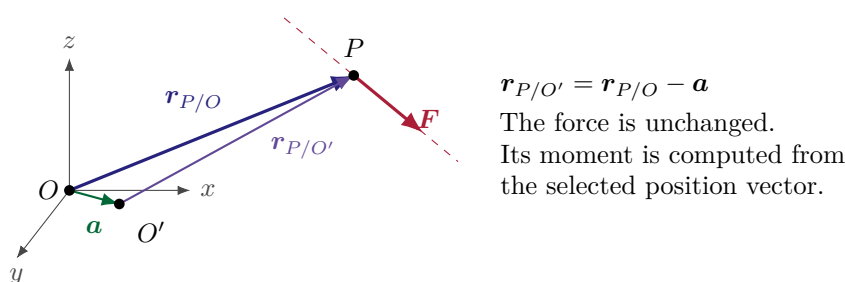


Figure 11.14: Origin shift for the bracket in Example 11.10. The positions are a common linear projection of the stated three-dimensional coordinates; the force uses a separate display scale. The dashed line is its line of action. Projected angles need not equal physical three-dimensional angles.

Example 11.10: A force acting on an offset bracket

A force $\mathbf{F} = (30\hat{\mathbf{i}} + 20\hat{\mathbf{j}} - 10\hat{\mathbf{k}})$ N acts at $\mathbf{r}_{P/O} = (0.40\hat{\mathbf{i}} - 0.20\hat{\mathbf{j}} + 0.10\hat{\mathbf{k}})$ m. A second origin is displaced by $\mathbf{a} = (0.10\hat{\mathbf{i}} + 0.05\hat{\mathbf{j}})$ m. Find the torque about both origins and verify the shift identity.

Model and strategy. We calculate the moment of the specified force alone. No assumption about static equilibrium or a bracket support reaction is needed. Figure 11.14 shows the two position vectors; the full component calculation retains information lost in the projection.

Using Eq. (11.30), the torque components about O are

$$\begin{aligned}\tau_x &= (-0.20)(-10) - (0.10)(20) = 0, \\ \tau_y &= (0.10)(30) - (0.40)(-10) = 7.0, \\ \tau_z &= (0.40)(20) - (-0.20)(30) = 14.0,\end{aligned}$$

all in N m. Therefore

$$\boldsymbol{\tau}_O = (0\hat{\mathbf{i}} + 7.0\hat{\mathbf{j}} + 14.0\hat{\mathbf{k}}) \text{ N m.}$$

From O' , the position is $(0.30\hat{\mathbf{i}} - 0.25\hat{\mathbf{j}} + 0.10\hat{\mathbf{k}}) \text{ m}$, giving

$$\boldsymbol{\tau}_{O'} = (0.50\hat{\mathbf{i}} + 6.0\hat{\mathbf{j}} + 13.5\hat{\mathbf{k}}) \text{ N m.}$$

For an independent geometric check, $\mathbf{a} \times \mathbf{F} = (-0.50\hat{\mathbf{i}} + 1.0\hat{\mathbf{j}} + 0.50\hat{\mathbf{k}}) \text{ N m}$. Subtracting this from $\boldsymbol{\tau}_O$ reproduces $\boldsymbol{\tau}_{O'}$. Also $\boldsymbol{\tau}_O \cdot \mathbf{r} = 7(-0.20) + 14(0.10) = 0$ and $\boldsymbol{\tau}_O \cdot \mathbf{F} = 7(20) + 14(-10) = 0$. The moment is perpendicular to both factors, as required.

Interpretation. The x component vanishes because two contributions cancel, rather than because the complete torque vanishes. An equal opposite force applied at a different point would create a couple with this force; the torque of that pair would then be independent of origin.

A couple consists of equal opposite forces with different parallel lines of action. If $\mathbf{F}$ acts at A and $-\mathbf{F}$ at B , its net torque is

$$\boldsymbol{\tau}_{\text{couple}} = (\mathbf{r}_A - \mathbf{r}_B) \times \mathbf{F}, \quad (11.33)$$

and its net force is zero. Consequently the origin correction in Eq. (11.32) vanishes. A couple can change spin while providing no net translational force; a single off-centre force generally does both.

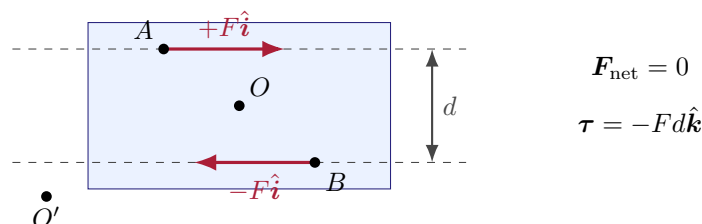


Figure 11.15: A clockwise couple. The two forces have zero sum but their moments add. Its torque has the same value about O , O' and every other origin. The displayed application points are schematic; d is the perpendicular line separation.

Example 11.11: A couple and a misleading lever arm

Two equal opposite horizontal forces, each of magnitude 4.0 N, act as in Fig. 11.15. Their lines are separated by $d = 1.50 \text{ m}$. Find the net force and torque, and explain whether the horizontal separation of their application points matters.

The net force is zero. About the midpoint O , each line has lever arm $d/2$ and each force has a clockwise moment, so

$$\tau_z = -F \frac{d}{2} - F \frac{d}{2} = -Fd = -6.0 \text{ N m.}$$

Equivalently, choose representative application points with $\mathbf{r}_A - \mathbf{r}_B = (-2\hat{\mathbf{i}} + 1.5\hat{\mathbf{j}}) \text{ m}$; crossing this with $4\hat{\mathbf{i}} \text{ N}$ gives the same $-6.0\hat{\mathbf{k}} \text{ N m}$. The x separation contributes zero because it is parallel to the forces. The origin-shift correction is also zero because $\mathbf{F}_{\text{net}} = 0$. Thus neither moving the origin nor sliding either force along its own line changes the couple moment.

11.4 Angular momentum of particles and systems

Linear momentum describes translational motion without reference to a particular spatial origin. Its moment about an origin is angular momentum. This distinction makes angular momentum useful for a projectile passing a pivot, a captured object at the edge of a disk, and a freely spinning body, even though these motions look different.

11.4.1 Definition and a straight-line counterexample

For a particle of constant mass m , use its velocity in the stated inertial frame to define $\mathbf{p} = m\mathbf{v}$. Its angular momentum about O is

$$\mathbf{L}_O = \mathbf{r}_{P/O} \times \mathbf{p}. \quad (11.34)$$

The magnitude is $L_O = rp \sin \phi$. It may also be written pb , with b the perpendicular distance from O to the line through the particle along its instantaneous momentum. For a curved path this is an instantaneous tangent-line construction, rather than the minimum distance from O to the entire path. Units are $\text{kg m}^2/\text{s}$, or equivalently N m s .

Circular motion is not required. Suppose a free particle moves along $y = b$ with constant velocity $v\hat{\mathbf{i}}$. At an event its position is $x\hat{\mathbf{i}} + b\hat{\mathbf{j}}$, and

$$\mathbf{L}_O = (x\hat{\mathbf{i}} + b\hat{\mathbf{j}}) \times mv\hat{\mathbf{i}} = -bmv\hat{\mathbf{k}}. \quad (11.35)$$

The changing x coordinate lies parallel to the momentum and therefore contributes nothing. Although the particle's distance from O changes, the product $r \sin \phi = b$ remains constant. Angular momentum is constant about this fixed origin, including before and after the closest approach. Placing an origin on the path gives zero angular momentum about that origin.

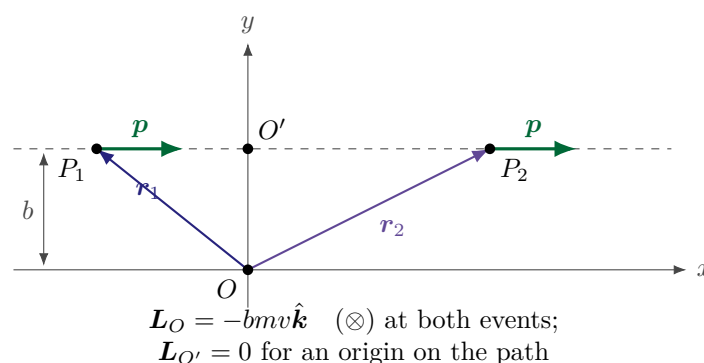


Figure 11.16: The exact two event positions of Example 11.12, drawn with a common position scale. Both have the same perpendicular offset from O and therefore the same angular momentum, despite different position vectors.

Example 11.12: A particle passing a detector offset

A particle with $m = 0.150 \text{ kg}$ moves freely with $\mathbf{v} = 12.0\hat{\mathbf{i}} \text{ m/s}$ along $y = 0.40 \text{ m}$. Find its angular momentum about the laboratory origin at $x = -0.50 \text{ m}$ and $x = +0.80 \text{ m}$. Compare

an origin on its path.

Strategy. Figure 11.16 identifies the perpendicular offset. The inertial momentum is $\mathbf{p} = 1.80\hat{\mathbf{i}} \text{ kg m/s}$ at both events. Consequently

$$\begin{aligned} \mathbf{L}_{O,1} &= (-0.50\hat{\mathbf{i}} + 0.40\hat{\mathbf{j}}) \times 1.80\hat{\mathbf{i}} = -0.720\hat{\mathbf{k}} \text{ kg m}^2/\text{s}, \\ \mathbf{L}_{O,2} &= (+0.80\hat{\mathbf{i}} + 0.40\hat{\mathbf{j}}) \times 1.80\hat{\mathbf{i}} = -0.720\hat{\mathbf{k}} \text{ kg m}^2/\text{s}. \end{aligned}$$

The cross-product orientation is into the page. As a geometric check, the respective distances are $r_1 = 0.640 \text{ m}$ and $r_2 = 0.894 \text{ m}$, but each has $r \sin \phi = 0.40 \text{ m}$. Thus $rp \sin \phi = 0.40(1.80) = 0.720 \text{ kg m}^2/\text{s}$.

For O' on $y = 0.40 \text{ m}$, the relative position is parallel to the momentum, so $\mathbf{L}_{O'} = 0$. The descriptions refer to different origins. Neither changes the particle's actual motion, and both are consistent with zero torque about their respective fixed origins.

11.4.2 Circular motion and radial forces

For a particle in a circle of radius R about its centre, the velocity is tangent, so $\mathbf{r} \cdot \mathbf{p} = 0$ and $L = mRv$. Counterclockwise motion in the xy plane gives $\mathbf{L} = mRv\hat{\mathbf{k}}$. If the motion is uniform, the momentum vector continually changes direction, but the angular momentum vector is constant. The required centripetal force is radial and has zero moment about the circle centre. A zero torque therefore permits a nonzero force.

More generally, resolve the velocity at an instant into a radial component and a transverse component in the plane of motion. The radial part contributes no angular momentum; the transverse part gives $L_z = mrv_{\text{transverse}}$. The later gravitation chapter will use this result to connect a central force with conserved angular momentum along a noncircular path. Here it supplies a check on the statement that force and torque answer different dynamical questions.

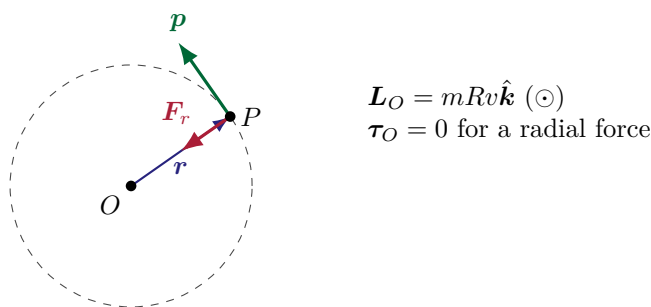


Figure 11.17: A radial force changes linear momentum without producing torque about its centre. Uniform counterclockwise circular motion has constant out-of-page angular momentum. The tangent and radial directions are calculated from the same polar angle. The polar-point and tangent-arrow method adapts I. Neutelings, TikZ.net, with a planar physical model.

11.4.3 System sum and origin dependence

For a fixed set of particles, angular momentum is the vector sum

$$\mathbf{L}_O = \sum_i \mathbf{r}_{i/O} \times m_i \mathbf{v}_i. \quad (11.36)$$

All positions use the same origin and all velocities use the same inertial frame. The magnitudes must not be added when the vectors have different directions. The same origin-shift algebra used

for torque now gives

$$\boxed{\mathbf{L}_{O'} = \mathbf{L}_O - \mathbf{a} \times \mathbf{P}, \quad \mathbf{P} = \sum_i m_i \mathbf{v}_i.} \quad (11.37)$$

If the total linear momentum is zero, angular momentum is independent of origin at that instant. A rotating pair illustrates why zero linear momentum does not mean zero angular momentum.

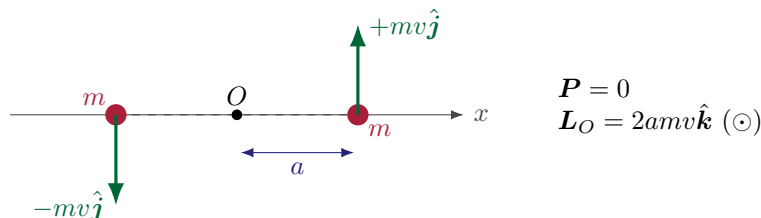


Figure 11.18: Two opposite linear momenta can add to zero while their angular momenta add. For this configuration the angular momentum is independent of origin because the total linear momentum is zero. The drawing is schematic.

Example 11.13: A rotating pair with zero total momentum

Two particles of equal mass $m = 0.20 \text{ kg}$ are at $\mathbf{r}_1 = +0.30\hat{i} \text{ m}$ and $\mathbf{r}_2 = -0.30\hat{i} \text{ m}$. Their velocities are $\mathbf{v}_1 = +2.0\hat{j} \text{ m/s}$ and $\mathbf{v}_2 = -2.0\hat{j} \text{ m/s}$. Find $\mathbf{P}$ and $\mathbf{L}_O$, and decide whether an origin shift changes the result.

The momenta are $+0.40\hat{j}$ and $-0.40\hat{j} \text{ kg m/s}$, so $\mathbf{P} = 0$. Each cross product is $+0.120\hat{k} \text{ kg m}^2/\text{s}$; hence

$$\boxed{\mathbf{L}_O = +0.240\hat{k} \text{ kg m}^2/\text{s}.$$

The shift term $-\mathbf{a} \times \mathbf{P}$ is zero for every $\mathbf{a}$. As a separate rigid-pair check, the inertia about the midpoint is $I = 2ma^2 = 0.0360 \text{ kg m}^2$ and the instantaneous angular speed is $v/a = 6.67 \text{ rad/s}$, giving $I\omega = 0.240 \text{ kg m}^2/\text{s}$. The total linear momentum carries no information about this internal rotational motion.

An extended body can carry both an orbital contribution from the motion of its CM and a spin contribution from motion relative to the CM. The advanced section derives this decomposition without assuming rigidity. Even for a rigid body, $\mathbf{L}_{\text{CM}} = I\boldsymbol{\omega}$ with a single scalar I requires an appropriate principal-axis or symmetry model. The full vector sum is the definition; the scalar expression is a useful consequence under stated conditions. In particular, a body constrained to rotate about an arbitrary fixed axis can have $L_z = I_z\omega_z$ while also carrying transverse angular-momentum components.

11.5 Torque and angular-momentum balance

11.5.1 A fixed inertial origin

For a particle about an origin fixed in an inertial frame, $d\mathbf{r}/dt = \mathbf{v}$. Differentiate the definition using the product rule for a cross product:

$$\frac{d\mathbf{L}_O}{dt} = \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times \frac{d\mathbf{p}}{dt} = \mathbf{r} \times \mathbf{F} = \boldsymbol{\tau}_O. \quad (11.38)$$

The first term vanishes because parallel vectors have zero cross product; the second uses Newton's law in the inertial frame. Thus the torque determines how the vector angular momentum changes. For nonzero angular momentum, a parallel torque changes its magnitude, a perpendicular torque changes its direction instantaneously, and a general torque can do both.

For a fixed system of particles, split each net force into external forces and internal interactions. For one internal pair, with $\mathbf{F}_{ij}$ the force on i from j , Newton's third law gives $\mathbf{F}_{ji} = -\mathbf{F}_{ij}$. The pair's combined torque is

$$\mathbf{r}_i \times \mathbf{F}_{ij} + \mathbf{r}_j \times \mathbf{F}_{ji} = (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij}. \quad (11.39)$$

It vanishes when the forces are central, directed along the line joining the particles. Equal opposite forces by themselves are insufficient: an equal opposite noncentral pair has the structure of a couple. For the mechanical particle models used here, the internal forces are central or internal torques are balanced by angular momentum in included internal degrees of freedom. Under that condition,

$$\boxed{\frac{d\mathbf{L}_O}{dt} = \boldsymbol{\tau}_{O,\text{ext}}}. \quad (11.40)$$

The system is closed to particle transfer, the frame is inertial, and the origin is fixed. These are the hypotheses behind the formula. A model involving omitted fields, material flowing through a boundary, or an omitted rotor needs additional bookkeeping rather than an automatic cancellation claim.

11.5.2 A moving origin and the centre-of-mass exception

The most common ambiguity arises from combining positions measured from a moving point with momenta measured in the laboratory. Let $A(t)$ have position $\mathbf{a}(t)$ and velocity $\mathbf{V}_A$ in the inertial frame, and define

$$\mathbf{L}_A^{\text{abs}} = \sum_i (\mathbf{r}_i - \mathbf{a}) \times m_i \mathbf{v}_i.$$

The superscript reminds us that the velocities are absolute inertial velocities, despite the relative position vectors. Differentiation gives

$$\begin{aligned} \frac{d\mathbf{L}_A^{\text{abs}}}{dt} &= \sum_i (\mathbf{v}_i - \mathbf{V}_A) \times m_i \mathbf{v}_i + \sum_i (\mathbf{r}_i - \mathbf{a}) \times \mathbf{F}_i \\ &= \boldsymbol{\tau}_{A,\text{ext}} - \mathbf{V}_A \times \mathbf{P}. \end{aligned} \quad (11.41)$$

The correction is caused by origin velocity. It remains present for a moving origin with constant velocity if that velocity is not parallel to total momentum. The torque itself still means the sum of force moments about the current position of A .

There is an important exact exception. For $A = \text{CM}$, $\mathbf{V}_A = \mathbf{V}_{\text{CM}}$ and $\mathbf{P} = M\mathbf{V}_{\text{CM}}$, so their cross product vanishes. Furthermore,

$$\sum_i \boldsymbol{\rho}_i \times m_i \mathbf{v}_i = \sum_i \boldsymbol{\rho}_i \times m_i (\mathbf{v}_i - \mathbf{V}_{\text{CM}}), \quad \boldsymbol{\rho}_i = \mathbf{r}_i - \mathbf{R}_{\text{CM}},$$

because $\sum_i m_i \boldsymbol{\rho}_i = 0$. Angular momentum about the CM can therefore be computed with absolute velocities or velocities relative to the CM; both sums agree. It obeys

$$\boxed{\frac{d\mathbf{L}_{\text{CM}}}{dt} = \boldsymbol{\tau}_{\text{CM},\text{ext}}}. \quad (11.42)$$

This result remains valid when the CM accelerates, provided the system and internal-torque conditions remain the same and vector derivatives are evaluated in inertial, nonrotating axes. Accelerating a CM origin does not by itself invalidate this special result.

If relative velocities are instead used about a general moving point, define $\mathbf{L}_A^{\text{rel}} = \sum_i (\mathbf{r}_i - \mathbf{a}) \times m_i (\mathbf{v}_i - \mathbf{V}_A)$. Direct differentiation gives

$$\frac{d\mathbf{L}_A^{\text{rel}}}{dt} = \boldsymbol{\tau}_{A,\text{ext}} - M(\mathbf{R}_{\text{CM}} - \mathbf{a}) \times \mathbf{A}_A, \quad (11.43)$$

where $\mathbf{A}_A = d\mathbf{V}_A/dt$. The correction now involves origin acceleration. It vanishes at the CM because $\mathbf{R}_{CM} - \mathbf{a} = 0$. Distinguishing the two definitions prevents a velocity correction from being confused with an acceleration correction. A basis rotating with the body would require a further distinction between inertial derivatives and rotating-basis component derivatives; that is outside this core treatment.

Example 11.14: A moving detector origin

A free particle has $\mathbf{p} = 3.0\hat{\mathbf{i}} \text{ kg m/s}$ and moves on $y = 1.0 \text{ m}$. A detector point has $\mathbf{a}(t) = Ut\hat{\mathbf{j}}$ with $U = 0.50 \text{ m/s}$; it starts at the laboratory origin. Determine the particle's angular momentum about the detector using inertial momentum, and its derivative.

At time t , the relative position is $x(t)\hat{\mathbf{i}} + (1.0 - Ut)\hat{\mathbf{j}}$ in metres. Consequently

$$\mathbf{L}_A^{\text{abs}} = -3.0(1.0 - Ut)\hat{\mathbf{k}} \text{ kg m}^2/\text{s}, \quad \frac{d\mathbf{L}_A^{\text{abs}}}{dt} = +1.50\hat{\mathbf{k}} \text{ N m}.$$

No force acts, so the actual torque about the detector is zero. The moving-origin identity gives the same derivative: $-\mathbf{V}_A \times \mathbf{p} = -(0.50\hat{\mathbf{j}}) \times (3.0\hat{\mathbf{i}}) = +1.50\hat{\mathbf{k}} \text{ N m}$. At $t = 2.0 \text{ s}$ the detector lies on the particle's line of motion and the angular momentum about it is zero. Its change arose from the moving reference point, rather than from a physical torque on the particle. This example uses absolute momentum; switching to detector-relative momentum would define a different quantity.

11.5.3 Angular impulse: the time integral of torque

For a fixed inertial origin, integrating Eq. (11.40) over an interaction gives

$$\Delta \mathbf{L}_O = \int_{t_i}^{t_f} \boldsymbol{\tau}_{O,\text{ext}} dt \equiv \mathbf{J}_{\tau,O}. \quad (11.44)$$

The signed area beneath one torque component's time graph gives the change in the corresponding angular-momentum component. Different torque components require separate integrals. A large torque acting briefly may have the same angular impulse as a smaller torque acting longer; the final change depends on the integral.

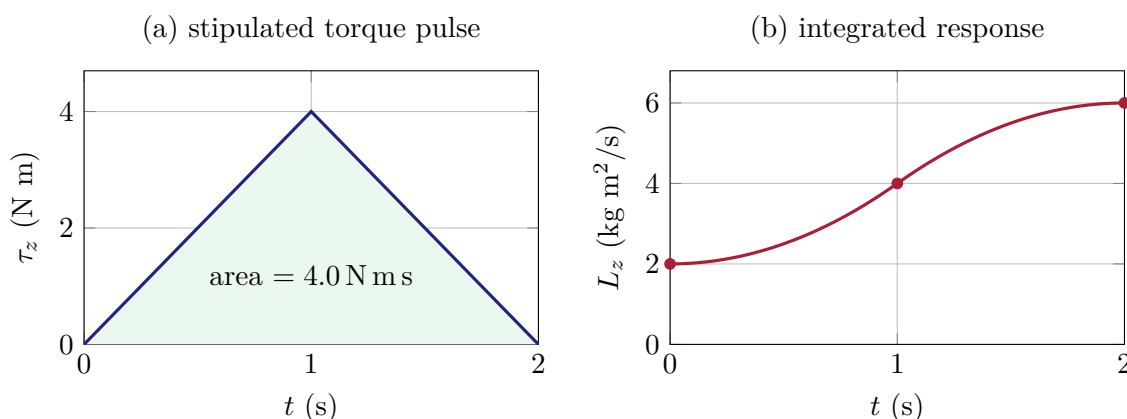


Figure 11.19: Angular impulse as torque area. The angular momentum changes from $2.0 \text{ kg m}^2/\text{s}$ to $6.0 \text{ kg m}^2/\text{s}$. Its slope equals the torque, so it steepens during the rising pulse and flattens during the falling pulse. The curves represent an analytical teaching model.

Example 11.15: Reading a torque pulse and checking its integral

A rotor on a fixed axis has $I_z = 2.0 \text{ kg m}^2$ and initial $L_z = 2.0 \text{ kg m}^2/\text{s}$. The only external z torque rises linearly from zero to 4.0 N m during the first second and then falls to zero during the next second, as in Fig. 11.19. Find the final angular speed and the work done.

The area of the triangle is $\Delta L_z = \frac{1}{2}(2.0 \text{ s})(4.0 \text{ N m}) = 4.0 \text{ N m s}$. Thus $L_{z,f} = 6.0 \text{ kg m}^2/\text{s}$, $\omega_{z,i} = L_{z,i}/I_z = 1.0 \text{ rad/s}$ and $\omega_{z,f} = 3.0 \text{ rad/s}$. Integration on each interval gives

$$L_z(t) = \begin{cases} 2 + 2t^2, & 0 \leq t \leq 1, \\ 6 - 2(2-t)^2, & 1 \leq t \leq 2, \end{cases}$$

with t in seconds and L_z in $\text{kg m}^2/\text{s}$. The values and slopes agree at $t = 1$. Differentiation recovers the stated piecewise torque.

For fixed I_z , the work is $\Delta K = \frac{1}{2}I_z(\omega_f^2 - \omega_i^2) = 8.0 \text{ J}$. An independent power calculation uses $P = \tau_z \omega_z = (L_z/I_z)dL_z/dt$, giving $W = (L_f^2 - L_i^2)/(2I_z) = 8.0 \text{ J}$. Torque area gives an angular-momentum change; it does not itself have energy units.

If a short linear impulse $\mathbf{J} = \int \mathbf{F} dt$ acts at a location whose position relative to the selected origin changes negligibly during impact, then

$$\Delta \mathbf{L}_O \simeq \mathbf{r}_{P/O} \times \mathbf{J}. \quad (11.45)$$

The exact expression is $\int \mathbf{r}_{P/O}(t) \times \mathbf{F}(t) dt$. Pulling $\mathbf{r}$ out of the integral is an approximation justified by a short interaction and small geometric change. For a free rigid platform, linear impulse determines the CM translation, while angular impulse about the CM determines its spin.

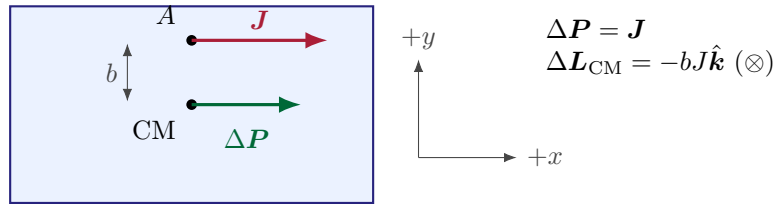


Figure 11.20: Off-centre impulse on a free platform, viewed from above. The positive y offset and positive x impulse give a negative z spin impulse. The momentum-change relations are placed outside the body to distinguish them from its CM and application point; body dimensions and arrow lengths are schematic.

Example 11.16: Off-centre impulse on a platform

A free rigid platform has $M = 20 \text{ kg}$ and $I_{\text{CM}} = 8.0 \text{ kg m}^2$ about its vertical principal axis. It starts at rest. A short impulse $\mathbf{J} = 30\hat{\mathbf{i}} \text{ N s}$ acts at $\mathbf{r}_{A/\text{CM}} = 0.25\hat{\mathbf{j}} \text{ m}$. Neglect all other impulses and geometric change during impact. Find the CM velocity and spin angular velocity.

System and strategy. Use the entire platform. Its CM is free to translate; it is not a fixed pivot. Linear impulse gives

$$M\mathbf{V}_{\text{CM},f} = \mathbf{J}, \quad \boxed{\mathbf{V}_{\text{CM},f} = 1.50\hat{\mathbf{i}} \text{ m/s.}}$$

The CM balance applies despite that translation. From the top-view geometry in Fig. 11.20,

$$\Delta \mathbf{L}_{\text{CM}} = (0.25\hat{\mathbf{j}}) \times (30\hat{\mathbf{i}}) = -7.50\hat{\mathbf{k}} \text{ N m s.}$$

Using the stated principal-axis inertia,

$$\boxed{\omega_{z,f} = \frac{-7.50}{8.0} = -0.9375 \text{ rad/s} \simeq -0.938 \text{ rad/s.}}$$

The spin is clockwise, as the cross-product direction predicts. The resulting kinetic energy is $K_{\text{trans}} = \frac{1}{2}(20)(1.50)^2 = 22.5 \text{ J}$ and $K_{\text{spin}} = (-7.50)^2/(2 \times 8.0) = 3.52 \text{ J}$, for a total of 26.0 J. These are consequences of the final motion; the force history and detailed work transfer were not specified.

Central-impulse comparison. Applying the same impulse through the CM gives the same $\mathbf{V}_{\text{CM}}$ but zero spin impulse. The point of application controls the angular response, not the linear impulse. A displacement parallel to $\mathbf{J}$ would likewise leave the spin impulse unchanged because it leaves the perpendicular lever arm unchanged.

11.6 Conservation, capture and changing inertia

11.6.1 Conservation is a statement about external torque

For the chosen fixed inertial origin, the integral balance reads

$$\mathbf{L}_{O,f} - \mathbf{L}_{O,i} = \int_{t_i}^{t_f} \boldsymbol{\tau}_{O,\text{ext}} dt. \quad (11.46)$$

Angular momentum is conserved throughout an interval when the external torque vanishes throughout that interval. Equal initial and final angular momentum requires only zero net torque impulse; nonzero positive and negative torques could cancel in the integral. Approximate conservation requires that the omitted external torque impulse be small compared with the angular-momentum scale relevant to the question. Merely calling an axle frictionless does not eliminate its force or every component of its moment.

Conservation can also apply to one component while other components change. For a fixed vertical axle with negligible external torque about that axis, L_z is conserved even if support forces supply transverse torques. In a planar capture experiment, the axle force acts through the selected origin and has no z moment. Horizontal linear momentum of the captured particle and disk, however, need not be conserved: the axle can exert a substantial horizontal impulse.

11.6.2 A sticking collision about a pivot

During a brief collision, choose an origin at the pivot so that the unknown pivot impulse has zero moment. If gravitational or other torque impulses are negligible on that timescale, conserve angular momentum about the pivot. Sticking is an inelastic process; it imposes a common final motion without conserving kinetic energy. The energy difference appears in deformation, internal motion and thermal energy within the enlarged system.

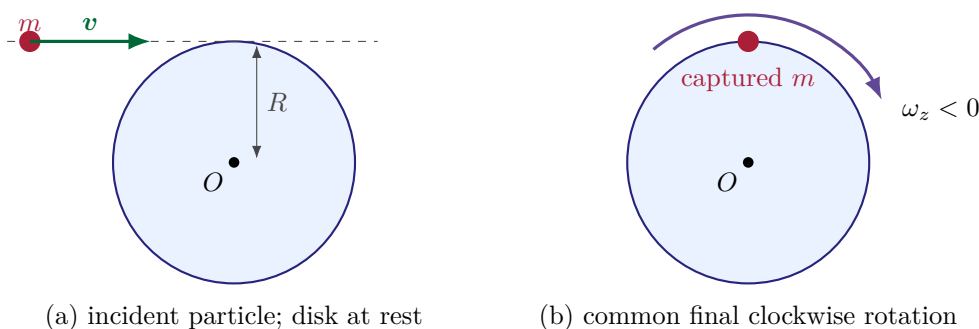


Figure 11.21: An inelastic capture with a fixed frictionless vertical axle at O . The tangential incoming motion carries negative z angular momentum; the final motion has the same sign. The axle impulse can change total linear momentum while giving zero torque impulse about its own axis.

Example 11.17: Tangential capture by a stationary disk

A particle with $m = 0.20$ kg and speed $v = 6.0$ m/s strikes tangentially and sticks at the top rim of a stationary uniform disk with $M = 3.0$ kg and $R = 0.40$ m. The disk has a fixed frictionless vertical axle. Use Fig. 11.21 to find the final angular speed and the kinetic-energy loss. Take $+z$ out of the top-view page. The particle moves along $+x$ at $y = R$, so $L_{z,i} = -mvR = -0.480$ kg m²/s. The disk inertia is $I_d = MR^2/2 = 0.240$ kg m², and after sticking

$$I_f = I_d + mR^2 = 0.240 + 0.0320 = 0.272 \text{ kg m}^2.$$

The axle impulse has zero moment about O ; neglected torques have negligible impulse during capture. Thus

$$\omega_{z,f} = \frac{-0.480}{0.272} = -1.76 \text{ rad/s.}$$

Initially $K_i = mv^2/2 = 3.60$ J; finally

$$K_f = \frac{L_i^2}{2I_f} = \frac{0.480^2}{2(0.272)} = 0.424 \text{ J.}$$

The decrease is 3.18 J, or 88.2% of the initial kinetic energy. As an algebraic check,

$$\frac{K_f}{K_i} = \frac{mR^2}{I_d + mR^2} = \frac{m}{M/2 + m} = \frac{0.20}{1.70} = 0.1176.$$

This fraction is between zero and one and tends toward zero for a very massive disk, as expected. The support force excludes linear-momentum conservation for the particle–disk system; inelastic sticking excludes kinetic-energy conservation. Neither prevents the valid angular-momentum balance about the axle.

11.6.3 Changing inertia on a fixed principal axis

Suppose a symmetric system rotates about a fixed principal axis and its mass distribution changes while remaining symmetric about that axis. Its axial angular momentum is $L_z = I(t)\omega_z$. Radial motions contribute no axial moment directly. For negligible external axial torque,

$$I_i\omega_i = I_f\omega_f, \quad I\alpha_z + \dot{I}\omega_z = 0. \quad (11.47)$$

The differential form follows by applying the product rule. In general, $\tau_z = I\alpha_z + \dot{I}\omega_z$, rather than $I\alpha_z$ alone. When I falls, a nonzero positive ω_z grows. This is a change of mass distribution; the complete system need not be rigid during the rearrangement.

At endpoints where radial motion has stopped, its rotational kinetic energy can be written

$$K_{\text{rot}} = \frac{1}{2}I\omega^2 = \frac{L_z^2}{2I}. \quad (11.48)$$

Conserved nonzero angular momentum therefore implies greater rotational kinetic energy when inertia decreases. The actuator or person moving the masses supplies work. Conservation of angular momentum constrains an axial vector; it neither supplies energy nor requires constant kinetic energy. During rapid radial motion the masses also carry radial kinetic energy, so the endpoint formula must not be misidentified as the complete kinetic energy at every intermediate instant.

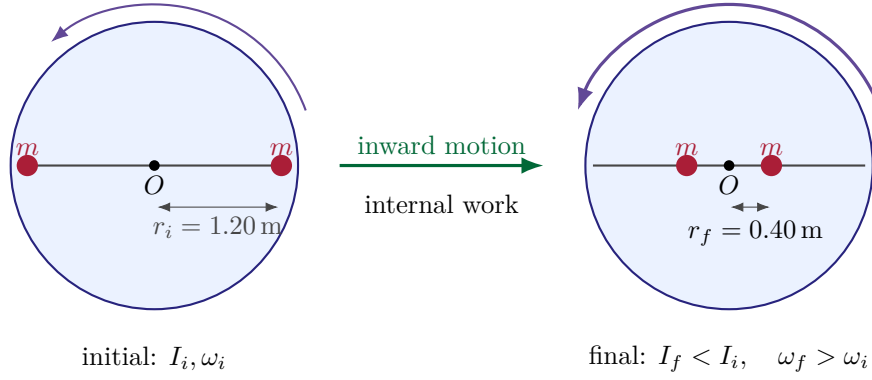


Figure 11.22: Symmetric inward rearrangement for Example 11.18. The particle radii use the same scale, displaying the actual 3 : 1 initial-to-final ratio; the platform footprint is schematic. Curved arrows indicate the common counterclockwise sign. They are not quantitative angular-speed scales.

Example 11.18: A rotating platform with movable masses

A platform has fixed inertia $I_0 = 12.0 \text{ kg m}^2$. Two masses, each with $m = 2.0 \text{ kg}$, lie symmetrically at $r_i = 1.20 \text{ m}$. The system initially rotates at $\omega_i = 1.50 \text{ rad/s}$ and the masses are pulled inward to $r_f = 0.40 \text{ m}$. External torque about the vertical axis is negligible. Find the final angular speed and the change in rotational kinetic energy.

Model. The masses are represented as point particles, move radially in a symmetric arrangement, and have zero radial speed at both endpoints. The platform's stated I_0 includes all fixed parts. Choose counterclockwise rotation as positive.

The two endpoint inertias are

$$I_i = I_0 + 2mr_i^2 = 12.0 + 4(1.20)^2 = 17.76 \text{ kg m}^2,$$

$$I_f = I_0 + 2mr_f^2 = 12.0 + 4(0.40)^2 = 12.64 \text{ kg m}^2.$$

The conserved axial angular momentum is $L_z = I_i\omega_i = 26.64 \text{ kg m}^2/\text{s}$. Thus

$$\omega_f = \frac{26.64}{12.64} = 2.108 \text{ rad/s}.$$

Keeping guard digits, the endpoint rotational energies are

$$K_i = \frac{1}{2}(17.76)(1.50)^2 = 19.98 \text{ J}, \quad K_f = \frac{26.64^2}{2(12.64)} = 28.07 \text{ J}.$$

Therefore $\Delta K_{\text{rot}} = +8.09 \text{ J}$. An independent ratio check gives $K_f/K_i = I_i/I_f = 1.4051$, the same ratio as ω_f/ω_i . The increased kinetic energy is supplied by internal work during rearrangement; if there is additional internal dissipation, the actuator must supply still more energy than this net increase.

Limit check. If the final radius equalled the initial radius, the inertias, angular speeds and endpoint energies would agree. Moving the masses all the way to the axis would leave inertia I_0 , so the attainable speed in this ideal model remains finite rather than diverging.

The continuous energy accounting reinforces this conclusion. At fixed L_z ,

$$\frac{dK_{\text{rot}}}{dt} = -\frac{L_z^2}{2I^2}\dot{I} = -\frac{1}{2}\omega_z^2\dot{I}. \quad (11.49)$$

For two equal point masses, $I = I_0 + 2mr^2$ and $\dot{I} = 4mr\dot{r}$; consequently $dK_{\text{rot}}/dt = -2mr\omega_z^2\dot{r}$. In a

slow radial rearrangement, the total inward actuator force magnitude is approximately $2mr\omega_z^2$ and its power is $(-2mr\omega_z^2)\dot{r}$. With $\dot{r} < 0$, this power is positive and agrees with the derivative above. If radial accelerations are significant, their kinetic energy must be included as well.

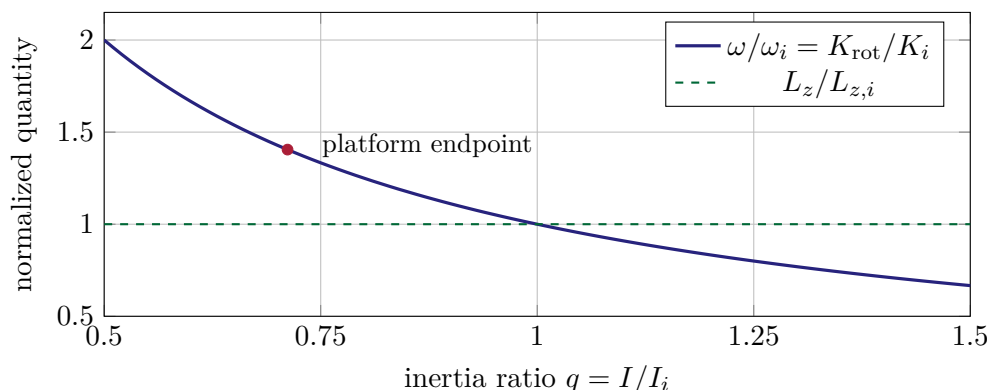


Figure 11.23: At fixed nonzero axial angular momentum, angular speed and rotational kinetic energy have the same reciprocal dependence on inertia. The marked endpoint is Example 11.18. Only rotational energy is plotted; possible radial kinetic energy during rearrangement is excluded explicitly.

Guided checks

Origin choice. If a force passes through O , must its torque vanish about every other origin? No: only origins on its line of action have zero moment. Equation (11.31) supplies the correction.

Internal forces. Do equal opposite internal forces always have zero combined torque? No: their combined moment is $(\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij}$, which vanishes for a central pair but can form a couple otherwise.

Momentum and energy. Does pulling masses inward without external torque leave kinetic energy unchanged? No: at fixed nonzero L_z , $K_{\text{rot}} = L_z^2/(2I)$ increases when I decreases. Internal work supplies the increase; radial motion and dissipation require further energy accounting.

11.7 Engineering, computation, and measurement

The same rolling model can answer three different questions: what acceleration is predicted, whether the required contact force is available, and whether observations support the assumed motion. These questions should be answered in that order. A numerical value for acceleration does not establish the validity of a no-slip constraint, and a plausible arrival time does not directly measure the velocity of the contact point.

11.7.1 A regime check before a numerical prediction

Write the CM moment of inertia as $I_{\text{CM}} = \kappa MR^2$. For a body released on a fixed incline, with no applied drive torque and negligible rolling resistance, the ideal rolling solution requires

$$a_{\text{roll}} = \frac{g \sin \theta}{1 + \kappa}, \quad \mu_{\text{req}} = \frac{\kappa}{1 + \kappa} \tan \theta. \quad (11.50)$$

The friction coefficient is a condition on this solution, not a factor that should be inserted into its acceleration formula. Provided $\mu_s \geq \mu_{\text{req}}$, static friction supplies the required torque. Below that boundary, a slipping model with additional contact information is needed. Replacing the required friction by $\mu_s N$ while retaining the rolling constraint would combine incompatible models.

Figure 11.7 separates the feasible and infeasible regions. A hoop needs more friction than a solid disk on the same incline because a larger fraction of its energy and torque budget belongs to rotation.

The increasing curves also explain why a shallow-ramp trial can succeed while the same body slips on a steeper ramp.

For an illustrative surface with $\mu_s = 0.15$, the feasibility boundary can also be read inversely. Solving Eq. (11.50) for the maximum angle gives $\theta_{\max} = \arctan[\mu_s(1 + \kappa)/\kappa]$. The disk permits 24.2° and the hoop 16.7° in this ideal model. These are predictions for a stipulated friction coefficient, not measured contact limits.

A minimal computational check evaluates a_{roll} and μ_{req} from the input κ , angle in radians, and stipulated μ_s , then returns both numbers together with the logical test $\mu_s \geq \mu_{\text{req}}$. Inputs should first satisfy $\kappa \geq 0$, $0 \leq \theta < \pi/2$, and $\mu_s \geq 0$. At equality the model is marginal; an experimental uncertainty that crosses the boundary prevents a confident regime assignment. Returning an acceleration without the feasibility flag hides the most important physical decision.

11.7.2 What a rolling experiment measures

In a proposed ramp investigation, distance, elapsed time, incline angle, and calibrated video positions are observations. Acceleration and the inertia factor are inferred quantities. Markers on both the body and the ramp permit simultaneous tracking of $s(t)$ and the rotation angle. Agreement of V_{CM} with $R|\omega|$ is a stronger test of rolling than arrival time alone.

For release from rest with approximately constant acceleration,

$$a_{\text{inferred}} = \frac{2s}{t^2}, \quad \kappa_{\text{inferred}} = \frac{g \sin \theta}{a_{\text{inferred}}} - 1. \quad (11.51)$$

A nonzero release speed, angle-dependent rolling resistance, slipping, or a biased clock can change this inference. A mismatch with the geometrical value of κ therefore identifies a model or measurement issue; it does not by itself identify the cause.

Example 11.19: Inferring inertia from synthetic rolling times

A solid disk is released from rest down a ramp. For this teaching investigation, the path length is $s = 1.200$ m with standard uncertainty $u_s = 0.003$ m; the angle is $\theta = 10.0^\circ$ with $u_\theta = 0.2^\circ$. Five synthetic arrival times are 1.453, 1.460, 1.449, 1.457, and 1.455 s. The independent timing-calibration standard uncertainty is 0.010 s. Use $g = 9.81$ m/s² as a stipulated model input. Estimate a and κ , propagate the stated uncertainties, and interpret the result.

The mean time is $\bar{t} = 1.4548$ s. The sample standard deviation is $s_t = 0.00415$ s; the standard uncertainty of the mean from repeatability is $s_t/\sqrt{5} = 0.00185$ s. Calibration uncertainty does not shrink when a reading is repeated. Treating these contributions as independent gives

$$u_t = \sqrt{(0.00185)^2 + (0.010)^2} = 0.01017 \text{ s}.$$

The inferred acceleration and inertia factor are

$$a = \frac{2(1.200)}{1.4548^2} = 1.13398 \text{ m/s}^2, \quad \kappa = \frac{9.81 \sin 10^\circ}{1.13398} - 1 = 0.50222.$$

For independent small uncertainties, differentiate Eq. (11.51). The relative acceleration uncertainty is

$$\frac{u_a}{a} = \sqrt{\left(\frac{u_s}{s}\right)^2 + \left(2\frac{u_t}{\bar{t}}\right)^2}, \quad u_a = 0.0161 \text{ m/s}^2.$$

The uncertainty in κ follows from its angle and acceleration dependence:

$$u_\kappa = (\kappa + 1)\sqrt{(\cot \theta u_\theta)^2 + (u_a/a)^2} = 0.0366,$$

where $u_\theta = 0.00349$ radians must be used in differentiation. Thus the appropriately rounded inferences are

$$a = (1.134 \pm 0.016) \text{ m/s}^2, \quad \kappa = 0.502 \pm 0.037.$$

These are standard uncertainties, not guaranteed bounds. The result is consistent with a solid disk's $\kappa = 0.500$ under the assumptions. The angle contribution dominates the uncertainty in κ ; improving only repeat timing would have limited benefit. The model requires $\mu_s \geq \tan 10^\circ/3 = 0.0588$. Without independent contact or angular tracking information, these times do not prove that requirement was met or that the body never slipped.

Repeated times quantify scatter but can leave a systematic model error invisible. Position data add a test of the predicted curve. For example, fitting the rest-release form $s = \frac{1}{2}at^2$ to the separate synthetic data in Table 11.1 gives

$$a_{\text{fit}} = \frac{2 \sum_j t_j^2 s_j}{\sum_j t_j^4} = 1.13723 \text{ m/s}^2.$$

This expression minimizes $\sum_j [s_j - \frac{1}{2}at_j^2]^2$ when the times are treated as exact and all position errors have equal variance. If time uncertainty is consequential, that statistical model must be revised. The zero-time point fixes the release origin here and is not an additional nonzero-leverage acceleration observation.

Table 11.1: Synthetic video-style positions and residuals about the fitted rest-release model. Times are stipulated; displayed positions are rounded from the embedded data.

t (s)	s (m)	$s - \frac{1}{2}a_{\text{fit}}t^2$ (mm)
0.00	0.00000	0.00
0.30	0.05310	1.93
0.60	0.20142	-3.28
0.90	0.46394	3.37
1.20	0.81567	-3.13
1.50	1.28062	1.24

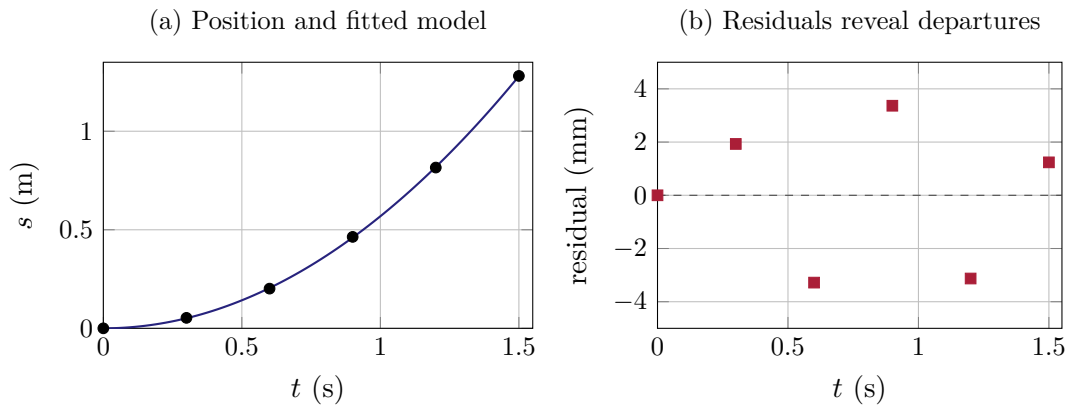


Figure 11.24: A synthetic rolling-data example. The near-parabolic position plot alone hides millimetre-scale differences. The residual panel uses exactly the same fitted acceleration and embedded positions. Six points are too few to establish a detailed noise distribution or to exclude every model defect.

The alternating residuals in Fig. 11.24 contain no obvious sustained curvature at this resolution. A linear trend could indicate an unmodeled release velocity; systematic curvature could indicate a wrong acceleration or time-varying forces. Those patterns are diagnostic possibilities, not unique

diagnoses. A defensible investigation reports calibration, release controls, the fitted model, residual units, and simultaneous angular observations, then explains which alternative mechanisms remain unresolved.

11.7.3 A numerical benchmark with a known answer

Numerical integration is useful when forces change with time or the contact regime changes. Its simplest check should be a case whose answer is known independently. Consider the feasible constant-slope rolling model

$$\dot{s} = v, \quad \dot{v} = a, \quad s(0) = v(0) = 0, \quad a = \frac{g \sin \theta}{1 + \kappa}.$$

Forward Euler with time step h updates $s_{n+1} = s_n + hv_n$ and $v_{n+1} = v_n + ha$. Both right-hand sides use values at the old time. Updating position with the newly calculated velocity is a different algorithm and has the opposite leading position error for this example.

Example 11.20: Testing step convergence against exact rolling motion

For a disk with $\kappa = 1/2$ on a 10° ramp, stipulate $\mu_s = 0.10$ and use $g = 9.81 \text{ m/s}^2$. Integrate from rest to $T = 2.0 \text{ s}$ using forward Euler. Find the exact answer, predict the step-size error, and compare $h = 0.20, 0.10, 0.05$, and 0.025 s .

First, $\mu_{\text{req}} = 0.0588 < 0.10$, so the assumed regime is feasible within the ideal model. The acceleration is $a = 1.13565908 \text{ m/s}^2$. Direct integration gives $v(T) = aT = 2.27131816 \text{ m/s}$ and $s(T) = aT^2/2 = 2.27131816 \text{ m}$; the equality of the numerical magnitudes here is incidental because their units differ.

Euler gives $v_n = nah$ exactly at the sample times. Summing the position increments for $N = T/h$ gives

$$s_N = ah^2 \sum_{n=0}^{N-1} n = \frac{1}{2}aT(T-h), \quad s(T) - s_N = \frac{1}{2}aTh.$$

Therefore halving h halves the position error: this is first-order convergence. The results are

h (s)	Euler s_N (m)	position error (m)
0.200	2.044186	0.227132
0.100	2.157752	0.113566
0.050	2.214535	0.056783
0.025	2.242927	0.028391

The relative position error is h/T , so $h = 0.10 \text{ s}$ gives a 5% underestimate even though the computed velocity is exact. Using the trapezoidal position update $s_{n+1} = s_n + h(v_n + v_{n+1})/2$ reproduces the exact quadratic trajectory here. This exactness relies on constant acceleration; it does not establish exactness for a general torque or friction model.

Numerical refinement tests the integration method, whereas a friction check tests the physical regime. Neither substitutes for the other. A precisely converged computation of an infeasible rolling constraint is still an invalid physical prediction. In a more general computation, also check energy and angular-momentum balances with the same system boundary, and locate contact transitions rather than stepping past them unnoticed.

11.7.4 Angular-momentum exchange in a reaction wheel

A spacecraft body and an internal wheel can exchange angular momentum through an internal motor torque. In the ideal one-axis model, let I_b be the body's moment of inertia excluding the

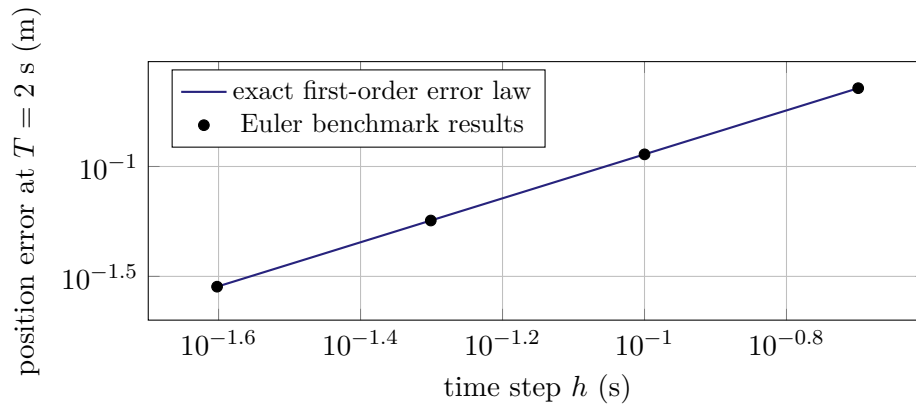


Figure 11.25: An independently derived error law and the numerical constant-acceleration benchmark agree. Unit slope on logarithmic axes expresses first-order convergence. Agreement at one time step would provide much weaker evidence.

wheel and I_w the wheel's moment of inertia. Both axes are coaxial and fixed in the idealized geometry. Angular velocities ω_b and ω_w are measured in the same inertial frame. If the combined system initially has zero angular momentum and external torque impulse is negligible,

$$I_b\omega_b + I_w\omega_w = 0.$$

An encoder normally supplies the wheel's speed relative to the body, $\Omega_{\text{rel}} = \omega_w - \omega_b$. Substitution gives

$$\omega_b = -\frac{I_w}{I_b + I_w}\Omega_{\text{rel}}, \quad \omega_w = \frac{I_b}{I_b + I_w}\Omega_{\text{rel}}. \quad (11.52)$$

The denominator is $I_b + I_w$ because the relative wheel reading already contains body rotation. Treating that reading as an inertial angular velocity loses this distinction.

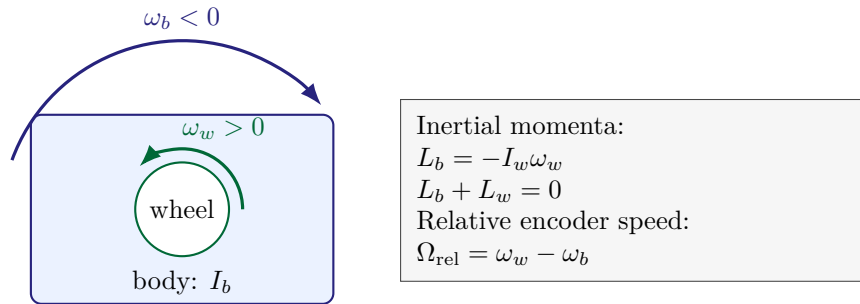


Figure 11.26: An ideal coaxial reaction-wheel model viewed along the axis. The wheel rotates counterclockwise and the body clockwise; arrow lengths are schematic. Opposite angular momenta preserve the total while motor work can raise the combined kinetic energy.

Example 11.21: Relative speed, body response, and motor work

An ideal body has $I_b = 18.0 \text{ kg m}^2$ and its wheel has $I_w = 0.120 \text{ kg m}^2$. Initially both are at rest. The motor brings the relative wheel speed to $\Omega_{\text{rel}} = +60.0 \text{ rad/s}$. Find both inertial speeds, verify angular-momentum conservation, and calculate the minimum mechanical work for this ideal final state.

Equation (11.52) gives

$$\omega_b = -0.397351 \text{ rad/s}, \quad \omega_w = +59.602649 \text{ rad/s}.$$

The separate angular momenta are $I_b\omega_b = -7.152318 \text{ kg m}^2/\text{s}$ and $I_w\omega_w = +7.152318 \text{ kg m}^2/\text{s}$, so they cancel. The final mechanical energy is

$$\begin{aligned} K_f &= \frac{1}{2}I_b\omega_b^2 + \frac{1}{2}I_w\omega_w^2 \\ &= \frac{1}{2} \frac{I_b I_w}{I_b + I_w} \Omega_{\text{rel}}^2 = 214.570 \text{ J}. \end{aligned}$$

With no other final kinetic energy or dissipation, the motor supplies $W = \Delta K = 214.570 \text{ J}$. An actual electrical energy input can exceed this value. If τ denotes motor torque on the wheel, the opposite body torque is $-\tau$ and the total mechanical power is

$$P = \tau\omega_w - \tau\omega_b = \tau\Omega_{\text{rel}}.$$

Internal torques cancel in the angular-momentum balance but their powers need not cancel because the two angular velocities differ. The result therefore illustrates conservation of angular momentum without conservation of mechanical energy during motor operation.

This ideal exchange model describes a principle rather than a mission's hardware. It does not supply motor limits, saturation speed, structural flexibility, or three-axis attitude control. A nonzero external torque changes the total angular momentum according to its time integral; an internal wheel cannot erase that accumulated total merely by changing its own speed.

11.8 Advanced Topics for Technical and Engineering Students

These extensions require the vector cross product, the CM definition, and elementary differentiation. The tensor discussion adds a matrix acting on a vector, with its components explained explicitly. The purpose is to identify the assumptions behind familiar scalar formulas. Full Euler-angle dynamics, rotor stability, and detailed spacecraft control lie beyond this chapter.

11.8.1 Orbital-plus-spin decomposition for an extended system

Let a fixed set of particles have total mass M , CM position $\mathbf{R}_{\text{CM}}$, and CM velocity $\mathbf{V}_{\text{CM}}$ in an inertial frame with fixed origin O . Write each particle's position and velocity as

$$\mathbf{r}_i = \mathbf{R}_{\text{CM}} + \boldsymbol{\rho}_i, \quad \mathbf{v}_i = \mathbf{V}_{\text{CM}} + \dot{\boldsymbol{\rho}}_i.$$

The defining CM sums are $\sum_i m_i \boldsymbol{\rho}_i = 0$ and, after differentiation for fixed masses, $\sum_i m_i \dot{\boldsymbol{\rho}}_i = 0$. Expanding the particle definition of angular momentum gives

$$\begin{aligned} \mathbf{L}_O &= \sum_i (\mathbf{R}_{\text{CM}} + \boldsymbol{\rho}_i) \times m_i (\mathbf{V}_{\text{CM}} + \dot{\boldsymbol{\rho}}_i) \\ &= \mathbf{R}_{\text{CM}} \times M \mathbf{V}_{\text{CM}} + \mathbf{R}_{\text{CM}} \times \sum_i m_i \dot{\boldsymbol{\rho}}_i + \left(\sum_i m_i \boldsymbol{\rho}_i \right) \times \mathbf{V}_{\text{CM}} + \sum_i \boldsymbol{\rho}_i \times m_i \dot{\boldsymbol{\rho}}_i. \end{aligned}$$

The two mixed terms vanish because of the CM identities. Hence

$$\boxed{\mathbf{L}_O = \mathbf{R}_{\text{CM}} \times M \mathbf{V}_{\text{CM}} + \mathbf{L}_{\text{CM}}.} \quad (11.53)$$

The first term is orbital angular momentum of the CM about O . The second is angular momentum of motion relative to the CM. This is an exact decomposition for the fixed particle system, including a deforming body; rigidity is only needed for the subsequent replacement of relative particle velocities by $\boldsymbol{\omega} \times \boldsymbol{\rho}_i$.

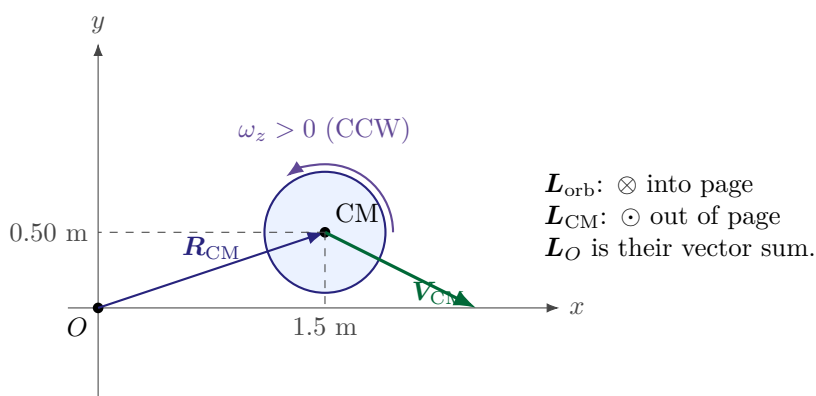


Figure 11.27: The translating disk of the next example. CM coordinates are drawn on a common scale; disk size and velocity-arrow length are schematic. Positive counterclockwise spin gives $\mathbf{L}_{\text{CM}}$ out of the page, whereas the CM position and momentum produce orbital angular momentum into it.

Example 11.22: A translating and spinning disk

A symmetric disk has mass $M = 4.0 \text{ kg}$, CM position $\mathbf{R}_{\text{CM}} = (1.5\hat{\mathbf{i}} + 0.50\hat{\mathbf{j}}) \text{ m}$, and CM velocity $\mathbf{V}_{\text{CM}} = (2.0\hat{\mathbf{i}} - 1.0\hat{\mathbf{j}}) \text{ m/s}$. Its spin-axis inertia is $I_{\text{CM}} = 0.32 \text{ kg m}^2$ and $\boldsymbol{\omega} = 5.0\hat{\mathbf{k}} \text{ rad/s}$. Find its angular momentum about O and compare it with the CM-relative value.

The momentum is $\mathbf{P} = (8\hat{\mathbf{i}} - 4\hat{\mathbf{j}}) \text{ kg m/s}$. The cross product has only a z component:

$$\mathbf{L}_{\text{orb}} = [(1.5)(-4) - (0.50)(8)]\hat{\mathbf{k}} = -10.0\hat{\mathbf{k}} \text{ kg m}^2/\text{s}.$$

The disk spins about a principal symmetry axis, so its CM-relative contribution is

$$\mathbf{L}_{\text{CM}} = I_{\text{CM}}\boldsymbol{\omega} = +1.60\hat{\mathbf{k}} \text{ kg m}^2/\text{s}.$$

Adding the vectors, rather than their magnitudes, gives

$$\mathbf{L}_O = -8.40\hat{\mathbf{k}} \text{ kg m}^2/\text{s}.$$

The clockwise orbital contribution dominates the counterclockwise spin contribution about O . About the CM, only $+1.60\hat{\mathbf{k}} \text{ kg m}^2/\text{s}$ remains. As an origin-shift check, set $\mathbf{a} = \mathbf{R}_{\text{CM}}$ in $\mathbf{L}_{O'} = \mathbf{L}_O - \mathbf{a} \times \mathbf{P}$: the result is $-8.40\hat{\mathbf{k}} - (-10.0\hat{\mathbf{k}}) = +1.60\hat{\mathbf{k}}$. The same physical state therefore carries different angular momentum about these different origins.

11.8.2 An origin that moves: which velocities are being used?

For two fixed origins separated by $\mathbf{a}$, the instantaneous shift law is

$$\mathbf{L}_{O'} = \mathbf{L}_O - \mathbf{a} \times \mathbf{P}. \quad (11.54)$$

Angular momentum is unchanged under a shift parallel to $\mathbf{P}$, and under every origin shift if $\mathbf{P} = 0$. This statement concerns the same particle velocities in the same frame.

If the reference point Q moves with velocity $\mathbf{u}_Q$, define first the angular momentum using inertial particle velocities:

$$\mathbf{L}_Q^{\text{lab}} = \sum_i (\mathbf{r}_i - \mathbf{q}) \times m_i \mathbf{v}_i.$$

Differentiation contributes an extra term from $\dot{\mathbf{q}}$:

$$\frac{d\mathbf{L}_Q^{\text{lab}}}{dt} = \boldsymbol{\tau}_{Q,\text{ext}} - \mathbf{u}_Q \times \mathbf{P}. \quad (11.55)$$

The term $\mathbf{v}_i \times m_i \mathbf{v}_i$ vanishes individually, but the moving-origin term does not generally vanish. A moving point is therefore not automatically entitled to the fixed-origin torque law.

A second definition uses velocities relative to Q :

$$\mathbf{L}_Q^{\text{rel}} = \sum_i (\mathbf{r}_i - \mathbf{q}) \times m_i (\mathbf{v}_i - \mathbf{u}_Q) = \mathbf{L}_Q^{\text{lab}} - M(\mathbf{R}_{\text{CM}} - \mathbf{q}) \times \mathbf{u}_Q.$$

For fixed masses and ordinary cancelling internal torques, its balance is

$$\frac{d\mathbf{L}_Q^{\text{rel}}}{dt} = \boldsymbol{\tau}_{Q,\text{ext}} - M(\mathbf{R}_{\text{CM}} - \mathbf{q}) \times \mathbf{a}_Q. \quad (11.56)$$

When Q is the CM, the displacement sum vanishes and $\mathbf{u}_Q \times \mathbf{P} = \mathbf{V}_{\text{CM}} \times M\mathbf{V}_{\text{CM}} = 0$. Both definitions then coincide and give the usual CM torque balance, even if the CM accelerates. For another moving point, the velocity convention must be stated before applying a conservation argument.

Example 11.23: A changing moment without an external torque

A free particle of mass 2.0 kg has $\mathbf{r}(t) = [(3 + 2t)\hat{\mathbf{i}} + \hat{\mathbf{j}}] \text{ m}$, where numerical t is in seconds. The reference point follows $\mathbf{q}(t) = t\hat{\mathbf{j}} \text{ m}$. Find both moving-origin angular momenta and explain their different balances.

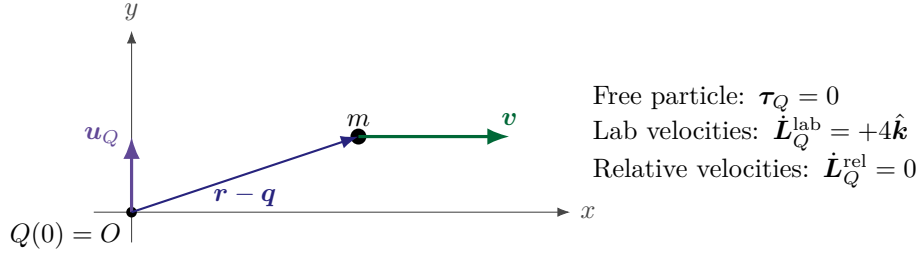


Figure 11.28: A free particle and a reference point moving upward at constant speed. This is a change of reference point, not an added physical torque. The next example gives the units and exact values of the displayed vector rates.

The inertial momentum is $\mathbf{P} = 4\hat{\mathbf{i}}$ kg m/s and the relative position is $[(3 + 2t)\hat{\mathbf{i}} + (1 - t)\hat{\mathbf{j}}]$ m. Directly,

$$\mathbf{L}_Q^{\text{lab}} = -4(1 - t)\hat{\mathbf{k}} \text{ kg m}^2/\text{s}, \quad \dot{\mathbf{L}}_Q^{\text{lab}} = +4\hat{\mathbf{k}} \text{ N m}.$$

Equation (11.55) independently predicts

$$-\mathbf{u}_Q \times \mathbf{P} = -\hat{\mathbf{j}} \times 4\hat{\mathbf{i}} = +4\hat{\mathbf{k}} \text{ N m},$$

although the physical torque is zero. Relative velocity is $(2\hat{\mathbf{i}} - \hat{\mathbf{j}})$ m/s, so

$$\mathbf{L}_Q^{\text{rel}} = 2[-(3 + 2t) - 2(1 - t)]\hat{\mathbf{k}} = -10\hat{\mathbf{k}} \text{ kg m}^2/\text{s}.$$

It is constant because $\mathbf{a}_Q = 0$ and the external torque is zero. These different results follow from different definitions; neither contradicts the particle's force-free motion. The numerical expressions apply with t entered in seconds, not with a dimensionless replacement of the physical time variable.

11.8.3 Why angular momentum need not follow angular velocity

For a rigid body relative to its CM, $\dot{\boldsymbol{\rho}} = \boldsymbol{\omega} \times \boldsymbol{\rho}$. Insert this into the angular-momentum integral and use the vector identity

$$\boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) = \rho^2 \boldsymbol{\omega} - (\boldsymbol{\rho} \cdot \boldsymbol{\omega}) \boldsymbol{\rho}.$$

Each component of angular momentum is therefore a linear combination of angular-velocity components:

$$\mathbf{L}_{\text{CM}} = \mathbf{I}_{\text{CM}} \boldsymbol{\omega}, \quad I_{ab} = \int (\rho^2 \delta_{ab} - \rho_a \rho_b) dm. \quad (11.57)$$

Here δ_{ab} is one when the component indices match and zero otherwise. The diagonal element $I_{xx} = \int (y^2 + z^2) dm$ is the ordinary inertia about x ; an off-diagonal element such as $I_{xy} = - \int xy dm$ couples two directions. The discrete version replaces the integral by a mass-weighted sum.

The tensor is real and symmetric, so mutually perpendicular principal axes can be chosen that make it diagonal. In those coordinates,

$$\mathbf{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}, \quad \mathbf{L} = (I_1 \omega_1, I_2 \omega_2, I_3 \omega_3).$$

Different inertia factors multiply different components. Parallel vectors result for rotation along one principal axis, or when all occupied components have equal principal inertia. They are not the general outcome of three-dimensional rotation. The rotational energy is the scalar

$$K_{\text{rot}} = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{I} \boldsymbol{\omega} = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{L}. \quad (11.58)$$

It remains positive for a nonzero ordinary body's rotation, regardless of whether the two vectors are parallel.

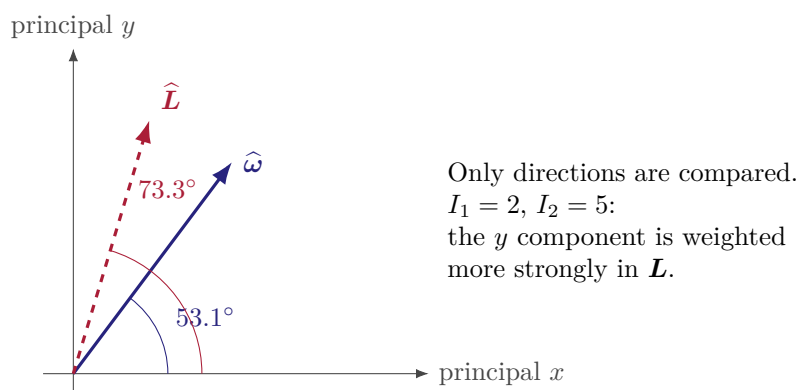


Figure 11.29: Principal-axis inertia changes the direction as well as the magnitude of an angular-velocity vector. Both vectors have been normalized separately to show their directions on the same geometric scale; their different physical units prohibit a direct magnitude comparison.

Example 11.24: An oblique spin and its angular momentum

In principal-axis coordinates, take $(I_1, I_2, I_3) = (2.0, 5.0, 6.0) \text{ kg m}^2$ and $\boldsymbol{\omega} = (3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}) \text{ rad/s}$. Find $\mathbf{L}$, its angular separation from $\boldsymbol{\omega}$, and the rotational energy.

Matrix multiplication gives

$$\mathbf{L} = (6\hat{\mathbf{i}} + 20\hat{\mathbf{j}}) \text{ kg m}^2/\text{s}.$$

Both vectors lie in the first quadrant. Their directions are $\arctan(4/3) = 53.1301^\circ$ and $\arctan(20/6) = 73.3008^\circ$, giving separation 20.1707° . An independent check uses

$$\cos \phi = \frac{\boldsymbol{\omega} \cdot \mathbf{L}}{|\boldsymbol{\omega}| |\mathbf{L}|} = \frac{98}{5\sqrt{436}},$$

which gives the same angle. The energy is

$$K_{\text{rot}} = \frac{1}{2}[2(3^2) + 5(4^2)] = \boxed{49 \text{ J}}.$$

This agrees with $\frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L} = 98/2$. These are instantaneous statements; they do not determine the subsequent torque-free motion merely by treating the principal-axis components as constant in an inertial frame.

11.8.4 Gyroscopic precession as a qualified extension

Consider a rapidly spinning symmetric rotor supported at a fixed point O , with its CM a distance d along the symmetry direction $\hat{\mathbf{s}}$. Let that direction make angle θ with upward vertical $\hat{\mathbf{k}}$. Gravitational torque about the support is

$$\boldsymbol{\tau}_O = d\hat{\mathbf{s}} \times (-Mg\hat{\mathbf{k}}), \quad |\boldsymbol{\tau}_O| = Mgd \sin \theta. \quad (11.59)$$

The support force has zero moment about its point of application. That does not imply force equilibrium at the CM: the CM follows a precessing path and the support force also supplies the required translational acceleration.

In the spin-dominated approximation, $\mathbf{L}_O \simeq L_{\text{spin}}\hat{\mathbf{s}}$ with $L_{\text{spin}} = I_s\omega_s$. Steady precession about $+z$ has

$$\frac{d\mathbf{L}_O}{dt} \simeq \Omega_p \hat{\mathbf{k}} \times \mathbf{L}_{\text{spin}}, \quad |\dot{\mathbf{L}}_O| \simeq \Omega_p L_{\text{spin}} \sin \theta.$$

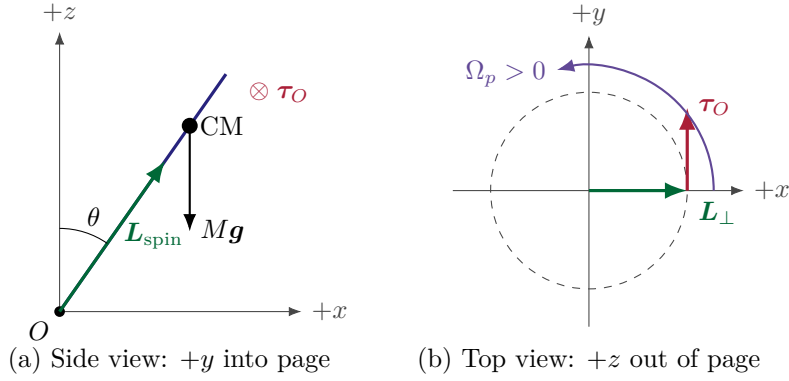


Figure 11.30: Consistent side and top views of slow precession at the displayed instant. In the side view, torque is perpendicular to the x - z plane, into the page. In the top view, it is tangent to the circle swept by the horizontal angular-momentum component. Positive precession is counterclockwise about $+z$. Arrow lengths are schematic.

Equating torque with this vector rate, keeping the same angle factor on both sides, gives

$$\Omega_p \simeq \frac{Mgd}{I_s \omega_s}. \quad (11.60)$$

Torque turns the dominant angular-momentum vector rather than merely reducing its magnitude. The frequently attempted formula $\Omega_p = \tau/L$ omits the projection factor for nonhorizontal tilt. At exactly $\theta = 0$, both sides before cancellation vanish; the orientation does not sweep a cone, so the divided expression is not an independently determined observable precession rate in that degenerate configuration.

The small rate ratio $\eta = |\Omega_p/\omega_s|$ is a useful timescale check. Spin dominance also needs transverse angular momentum to be small. If $I_\perp$ denotes the transverse inertia about the support, a sufficient scale comparison away from pathological geometries is

$$\epsilon \equiv \frac{I_\perp |\Omega_p|}{I_s |\omega_s|} \ll 1. \quad (11.61)$$

The precession angular momentum is at most of this scale; the detailed tilt projections determine its components. A small η alone is not sufficient if $I_\perp/I_s$ is very large. Nor do either of these checks rule out nutation, the oscillation of the tilt angle that can follow an unsuitable release. Approximately steady precession is an additional initial-state and dynamical assumption.

Example 11.25: Ideal slow-precession rate and its limits

A rotor has $M = 2.0 \text{ kg}$, CM offset $d = 0.080 \text{ m}$, spin-axis inertia $I_s = 0.015 \text{ kg m}^2$, and $\omega_s = 200 \text{ rad/s}$. Its axis makes $\theta = 35^\circ$ with the vertical during approximately steady precession. Estimate the rate and state exactly what can be checked from these inputs. Use $g = 9.81 \text{ m/s}^2$.

The spin angular momentum is $L_{\text{spin}} = 3.00 \text{ kg m}^2/\text{s}$, and the torque magnitude is

$$\tau = (2.0)(9.81)(0.080) \sin 35^\circ = 0.900286 \text{ N m}.$$

The ideal rate is

$$\Omega_p = \frac{(2.0)(9.81)(0.080)}{(0.015)(200)} = \boxed{0.5232 \text{ rad/s}}.$$

Thus $f_p = \Omega_p/(2\pi) = 0.08327 \text{ Hz}$ and $T_p = 2\pi/\Omega_p = 12.01 \text{ s}$. The vector-rate check gives $\Omega_p L_{\text{spin}} \sin 35^\circ = 0.900286 \text{ N m}$, matching the torque, whereas $\tau/L_{\text{spin}} = 0.300095 \text{ s}^{-1}$ would be the incorrect precession rate because it omits $\sin \theta$.

The timescale ratio is $\eta = 0.002616 \ll 1$. This supports slow precession relative to spin. However, $I_{\perp}$ is not supplied, so $\epsilon = (I_{\perp}/I_s)(0.002616)$ cannot be evaluated completely. Spin dominance is plausible for comparable inertias but is conditional on their ratio. The supplied numbers also cannot verify absence of nutation or describe startup. The calculation is therefore an internally consistent estimate under the stated steady, spin-dominated assumptions, not a full solution of the rotor's motion.

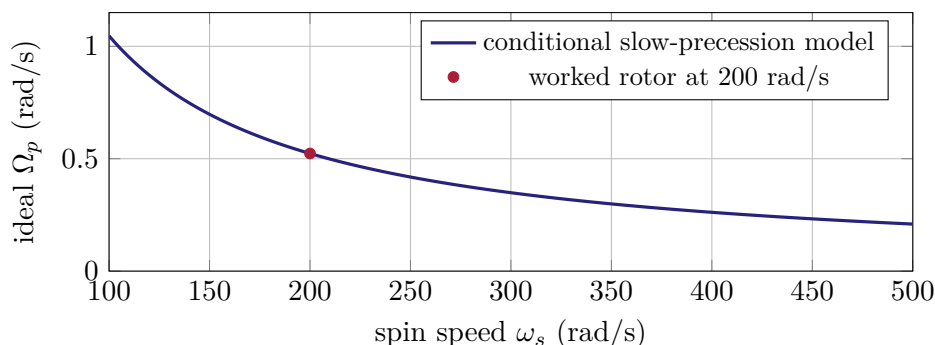


Figure 11.31: For fixed M , d , and I_s , the spin-dominated estimate predicts an inverse relation between precession and spin speed. The displayed curve does not extend toward zero spin, where this approximation fails. Each point still requires the spin-dominance and approximately steady-precession assumptions.

The ideal curve shows why a slower rotor precesses faster within the model, but it cannot be extrapolated to a nonspinning supported body. Frictional spin-down changes the assumptions; startup can excite nutation; and a strongly nonprincipal angular velocity requires the tensor description. These limits connect the extension back to the chapter's central discipline: specify the origin, the system, and the approximation before replacing a vector balance by a scalar relation.

11.9 Chapter summary and working formula sheet

A rolling body has both translation and rotation. The two motions are tied by a contact constraint only while the surfaces do not slide relative to one another. The contact is a succession of material points: a point can have zero instantaneous ground-frame velocity and a nonzero upward acceleration. Solve the translational and rotational balances first, and then compare the required static friction with the available bound. A failure of that bound is a change of physical regime, not a small correction to a rolling formula.

Torque and angular momentum are vectors about specified origins. A force through the origin has no moment about that origin even when it has a moment about another point. A particle moving on a straight line can have angular momentum about an origin off that line. Conservation follows from the external angular impulse for the chosen system and origin; it does not follow merely because something rotates.

The centre of mass separates translational motion from internal spin. For an extended body, the orbital contribution depends on the chosen origin, while the spin about the CM describes motion relative to the CM. A tensor is needed when angular velocity is not along a principal axis. Changing inertia can increase the spin rate at constant angular momentum, with the extra kinetic energy supplied through internal work. Slow gyroscopic precession is a controlled approximation in which torque mainly changes the direction of the spin angular momentum.

Formula summary**Rigid-body kinematics (inertial frame)**

$$\begin{aligned}\mathbf{v}_P &= \mathbf{V}_{\text{CM}} + \boldsymbol{\omega} \times \mathbf{r}_{P/\text{CM}}, \\ \mathbf{a}_P &= \mathbf{a}_{\text{CM}} + \boldsymbol{\alpha} \times \mathbf{r}_{P/\text{CM}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{P/\text{CM}}).\end{aligned}$$

On a horizontal belt, with $+x$ right and ω_z positive anticlockwise,

$$V_x + R\omega_z = U, \quad a_x + R\alpha_z = \dot{U}.$$

For stationary ground and rightward rolling, $\omega_z = -V_x/R$. A material rim point currently at the bottom has $\mathbf{a}_C = (V_x^2/R)\hat{\mathbf{j}}$ when the ground is horizontal and stationary.

Planar energy and unpowered incline (stationary surface)

$$K = \frac{1}{2}MV_{\text{CM}}^2 + \frac{1}{2}I_{\text{CM}}\omega^2, \quad \kappa = \frac{I_{\text{CM}}}{MR^2}, \quad a = \frac{g \sin \theta}{1 + \kappa}, \quad \mu_{s,\min} = \frac{\kappa \tan \theta}{1 + \kappa}.$$

The required friction is uphill, $f_s = \kappa Ma$, and must obey $f_s \leq \mu_s Mg \cos \theta$.

Upper-hub spool, horizontal rightward pull

$$a_x = \frac{F(1 + r/R)}{M(1 + \kappa)}, \quad f_x = \frac{F(r/R - \kappa)}{1 + \kappa}, \quad \alpha_z = -a_x/R.$$

Here f_x is signed. Its feasibility test is $|f_x| \leq \mu_s Mg$; the reversal is at $r/R = \kappa$.

Formula summary**Origins, angular impulse and spin**

$$\begin{aligned}\boldsymbol{\tau}_O &= \sum_i \mathbf{r}_{i/O} \times \mathbf{F}_{i,\text{ext}}, & \mathbf{L}_O &= \sum_i \mathbf{r}_{i/O} \times m_i \mathbf{v}_i, \\ \mathbf{L}_O &= \mathbf{R}_{\text{CM}/O} \times \mathbf{P} + \mathbf{L}_{\text{CM}}, & \mathbf{P} &= M\mathbf{V}_{\text{CM}}, \\ \mathbf{L}_{O'} &= \mathbf{L}_O - \mathbf{a} \times \mathbf{P}, & \boldsymbol{\tau}_{O'} &= \boldsymbol{\tau}_O - \mathbf{a} \times \mathbf{F}_{\text{ext}}.\end{aligned}$$

For the shifts above, $\mathbf{a}$ points from O to O' . For a fixed inertial O ,

$$\dot{\mathbf{L}}_O = \boldsymbol{\tau}_O, \quad \Delta \mathbf{L}_O = \int \boldsymbol{\tau}_O dt.$$

For an arbitrary moving origin using *inertial particle momenta*,

$$\dot{\mathbf{L}}_O = \boldsymbol{\tau}_O - \mathbf{V}_O \times \mathbf{P}.$$

At the CM, that extra term is zero because $\mathbf{P} = M\mathbf{V}_{\text{CM}}$. Internal moments cancel in a closed particle model with central action–reaction pairs; otherwise internal spin and couple transfers must be included explicitly.

Inertia and approximation limits

$$\mathbf{L}_{\text{CM}} = \mathbf{I}_{\text{CM}}\boldsymbol{\omega}, \quad K_{\text{rot}} = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I}_{\text{CM}}\boldsymbol{\omega}.$$

On a torque-free fixed principal axis, $I_i\omega_i = I_f\omega_f$ and $K = L^2/(2I)$. Ideal slow precession gives

$$\Omega_p \simeq \frac{Mgd}{I_s\omega_s}, \quad \frac{I_\perp\Omega_p}{I_s\omega_s} \ll 1.$$

The first relation presumes steady-cone geometry, negligible nutation and spin dominance; a small Ω_p/ω_s by itself does not establish all of those conditions.

Take-home message

Before using a rotational formula, state the system, frame, origin or axis, contact regime, and external forces or impulses. After solving, test the constraint, units, directions, and energy account. A compact formula is useful only after those choices are consistent.

11.10 Exercises and problems

Use $g = 9.81 \text{ m/s}^2$ unless a task states otherwise. In planar problems take $+x$ right, $+y$ up and positive angular components counterclockwise about $+z$ out of the page. State any different local convention. Values are stipulated teaching inputs; the timing dataset in Problem 24 is synthetic. Problems 1–9 preserve the original diagnostic IDs X11-01–X11-09. The complete reasoned solutions follow the entire problem set.

Before calculating, identify the body or system, the origin for angular quantities, the contact model and the conservation law's conditions. A feasibility failure is a physical conclusion, not an instruction to use an invalid rolling formula. Advanced tasks use the preceding advanced section; the other tasks require the core only.

A. Diagnostic reasoning and model selection

1. A student says, “In rolling without slipping, the bottom point has zero velocity, so its acceleration is also zero.” Diagnose the statement. For a wheel rolling at constant CM speed, give the bottom point's acceleration direction and magnitude.
2. A solid sphere ($\kappa = 2/5$) is assumed to roll down a 25° fixed incline where $\mu_s = 0.12$. Test the no-slip assumption before using the rolling acceleration formula. If it fails, explain what must change in the model.
3. A force's line of action passes through origin O but not through origin O' . A student claims the torque is zero “because the lever arm is zero.” Repair the claim and state how the torques are related.
4. A free particle moves uniformly along a straight line that misses origin O . A student argues that its angular momentum must change because its distance from O changes. Use geometry or the cross product to explain why this is wrong.
5. Compare two identical short impulses applied to a free platform: one through the CM and one with offset b . Which changes CM momentum? Which changes spin angular momentum about the CM? State the units of angular impulse.
6. A rotating person pulls two masses inward while external torque is negligible. A student writes, “Angular momentum is constant, so rotational kinetic energy is constant.” Diagnose the statement and identify the energy source if the final kinetic energy is larger.
7. A proposed gyroscope has Ω_p comparable to ω_s . Why is the ideal slow-precession estimate then not self-consistent? Which physical approximation has failed?
8. **Advanced.** A translating rigid body has no spin about its CM. Must its angular momentum about an arbitrary laboratory origin be zero? Use the orbital-plus-spin decomposition.
9. In the pulled-spool model, explain why guessing that friction points left because the spool moves right is unreliable. What two tests must be performed after solving for f_x ?

B. Rolling and vector calculations

10. A wheel of radius 0.25 m rolls right without slip at $V_{\text{CM}} = 2.0 \text{ m/s}$. Find ω_z and the velocity vector of the rightmost rim point.
11. A uniform solid sphere rolls from rest down a 25° incline. Find its no-slip acceleration and the minimum μ_s . Is $\mu_s = 0.12$ sufficient?
12. A uniform solid sphere of radius 0.12 m starts from rest and rolls without slip through a vertical drop of 0.75 m. Find its CM speed and angular speed at the bottom.
13. At one instant, $\mathbf{r} = (0.20\hat{\mathbf{i}} + 0.30\hat{\mathbf{j}} - 0.10\hat{\mathbf{k}}) \text{ m}$ and $\mathbf{F} = (5\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) \text{ N}$. Find τ_O .
14. Two equal opposite forces of magnitude 20 N form a couple. Their parallel lines of action are separated by 0.30 m. Find the couple-torque magnitude and explain why changing the origin does not change it.
15. A particle of mass 0.080 kg moves at 15 m/s along $+x$ on the line $y = 0.25 \text{ m}$. Find $\mathbf{L}_O$. What is the answer about an origin on the path?
16. Two particles have $(m_1, \mathbf{r}_1, \mathbf{v}_1) = (2 \text{ kg}, 1 \text{ m}\hat{\mathbf{i}}, 3 \text{ m/s}\hat{\mathbf{j}})$ and $(m_2, \mathbf{r}_2, \mathbf{v}_2) = (1 \text{ kg}, 2 \text{ m}\hat{\mathbf{j}}, -2 \text{ m/s}\hat{\mathbf{i}})$. Find total $\mathbf{L}_O$.

C. Angular impulse and conservation

17. A free platform has $M = 6.0 \text{ kg}$ and $I_{\text{CM}} = 2.4 \text{ kg m}^2$. An impulse $8.0\hat{\mathbf{i}} \text{ N s}$ is applied at $\mathbf{r} = 0.30\hat{\mathbf{j}} \text{ m}$. Initially at rest, what are its CM velocity and angular velocity immediately after the impulse?
18. A rotating platform has fixed inertia 5.0 kg m^2 plus two 1.0 kg masses. The masses move symmetrically from $r_i = 1.0 \text{ m}$ to $r_f = 0.20 \text{ m}$. If $\omega_i = 2.0 \text{ rad/s}$ and external torque is negligible, find ω_f and the ratio K_f/K_i . Assume the masses have no radial velocity at either endpoint.
19. A 0.10 kg particle moving at 10 m/s strikes tangentially and sticks to the rim of a stationary uniform disk of mass 4.0 kg and radius 0.50 m on a frictionless axle. Find the final angular speed and the fraction of the initial kinetic energy lost.
20. An ideal spacecraft body has inertia $I_b = 120 \text{ kg m}^2$ and an internal reaction wheel has $I_w = 5.0 \text{ kg m}^2$. Initially both are at rest. The wheel is spun to $\Omega_{\text{rel}} = 6.0 \text{ rad/s}$ relative to the body. Neglect external torque. Find the body's inertial angular velocity.
21. A gyroscope has $M = 1.5 \text{ kg}$, $d = 0.060 \text{ m}$, $I_s = 0.0080 \text{ kg m}^2$ and $\omega_s = 300 \text{ rad/s}$. Estimate Ω_p , the precession period and Ω_p/ω_s .
22. A spool has $M = 2.0 \text{ kg}$, $R = 0.18 \text{ m}$, $r = 0.090 \text{ m}$, $\kappa = 0.60$, $F = 10 \text{ N}$ and $\mu_s = 0.10$. The string leaves the top of the hub horizontally to the right. Find a , f_x and α_z , and check no slip.

D. Computation, data and changing constraints

23. A solid disk ($\kappa = 1/2$) is placed on a 30° incline with $\mu_s = 0.15$. Use the regime criterion to decide whether the pure-rolling solution is feasible. What minimum μ_s is required?
24. **Synthetic-data analysis.** A body rolls from rest through path length $s = 1.50 \text{ m}$ on a 15° incline. Three measured times in a constructed practice dataset are 1.32, 1.34 and 1.33 s. Using the mean time and constant-acceleration model, infer a and κ . Which common rigid-body shape is closest to the inferred κ ?

25. **Advanced.** A rigid body has $M = 3.0 \text{ kg}$, $\mathbf{R}_{\text{CM}} = (2\hat{\mathbf{i}} - \hat{\mathbf{j}}) \text{ m}$, $\mathbf{V}_{\text{CM}} = (\hat{\mathbf{i}} + 2\hat{\mathbf{j}}) \text{ m/s}$ and $\mathbf{L}_{\text{CM}} = 4\hat{\mathbf{k}} \text{ kg m}^2/\text{s}$. Find $\mathbf{L}_O$. What remains if the origin is shifted to the CM?
26. **Advanced.** In principal axes, $\mathbf{I} = \text{diag}(2, 5, 6) \text{ kg m}^2$ and $\boldsymbol{\omega} = (3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}) \text{ rad/s}$. Find $\mathbf{L}$, and calculate the angle between $\mathbf{L}$ and $\boldsymbol{\omega}$.
27. **Advanced.** A system has total linear momentum $\mathbf{P} = 10\hat{\mathbf{i}} \text{ kg m/s}$ and angular momentum $\mathbf{L}_O = 6\hat{\mathbf{k}} \text{ kg m}^2/\text{s}$. The new origin is displaced by $\mathbf{a} = 0.40\hat{\mathbf{j}} \text{ m}$. Find $\mathbf{L}_{O'}$.
28. For steady ideal precession at arbitrary fixed tilt $0 < \theta < \pi$, start from $Mgd \sin \theta = \Omega_p L \sin \theta$. Explain why the rate is independent of θ within this approximation but why the approximation itself can still fail as the physical motion changes.
29. In the pulled-spool result, determine the friction direction for (a) $r/R = 0.30, \kappa = 0.50$ and (b) $r/R = 0.70, \kappa = 0.50$. What happens at $r/R = 0.50$?
30. A wheel rolls without slip on a conveyor belt moving right at speed U . With counterclockwise ω_z positive, derive the one-dimensional no-slip relation between $V_{\text{CM}}, \omega_z, R$ and U . Recover the stationary-ground result as $U \rightarrow 0$.
31. Estimate the spin angular momentum of a bicycle wheel modelled as a thin hoop of mass 1.2 kg and radius 0.35 m when the bicycle moves at 8.0 m/s without slip. State the main modelling limitations.
32. Design a low-height rolling experiment capable of distinguishing a solid disk from a hoop using arrival time. State the direct measurements, inferred quantity, at least three systematic effects, and one observation that would specifically test for slipping.

E. Synthesis and advanced reasoning

33. A uniform disk rolls down an incline and then enters a horizontal rough region. Explain, without assuming the friction direction in advance, under what circumstances static friction can be zero during uniform rolling on the horizontal surface.
34. A short impulse $\mathbf{J}$ is applied at point A of a composite body. Its line of action passes through a fixed inertial origin O but not through the body's CM. Compare the angular impulse about O and the CM. Why can one be zero while the other is not? Treat the lever arms as fixed during the short event.
35. A torque-free symmetric rotor changes its principal-axis inertia from I to $I/3$ while remaining in the scalar principal-axis regime. By what factors do ω and K change? Identify the required energy source.
36. **Advanced.** Starting from the component definition of the inertia tensor, show that the inertia tensor is symmetric and explain why orthogonal principal axes can be chosen. Do not derive Euler's equations.

F. Reading graphs and diagnosing models

37. A rigid rotor about a fixed principal axis has $I = 2.0 \text{ kg m}^2$ and initial $L_z = 2.0 \text{ kg m}^2/\text{s}$. Its net external torque is the piecewise-linear history in Figure 11.32. Find (a) the net angular impulse from 0 to 4 s, (b) L_z at $t = 3$ and 4 s, and (c) the final kinetic energy. (d) Explain whether the negative torque reverses the spin during the plotted interval.
38. Figure 11.33 gives synthetic velocity histories before a body of mass $M = 3.0 \text{ kg}$ and radius $R = 0.20 \text{ m}$ reaches pure rolling on a stationary horizontal surface. The exact curves are

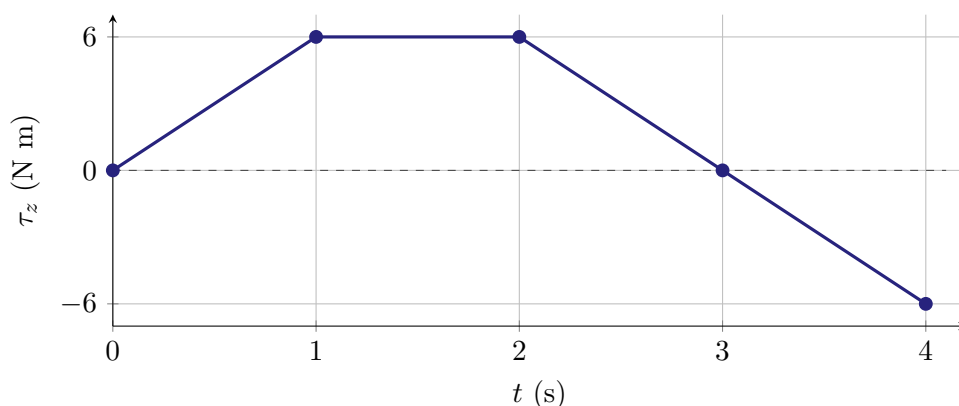


Figure 11.32: C11-P01. Stipulated net-torque history for Problem 37. Straight segments connect the displayed vertices; the angular impulse is its signed area. The graph gives torque, rather than angular momentum or power.

$V_{\text{CM}} = 2.0 - 0.50t$ and $R|\omega_z| = 1.0t$, with velocities in m/s and t in seconds. The spin is clockwise. Assume contact friction is the only horizontal force and the only CM torque; its magnitude is constant while the contact slips. Find (a) the time and speed at the onset of rolling, (b) the friction magnitude and μ_k , and (c) I_{CM} and κ . (d) Calculate the mechanical kinetic-energy loss up to rolling. Explain why extending the two lines beyond their intersection would contradict the assumed contact model.

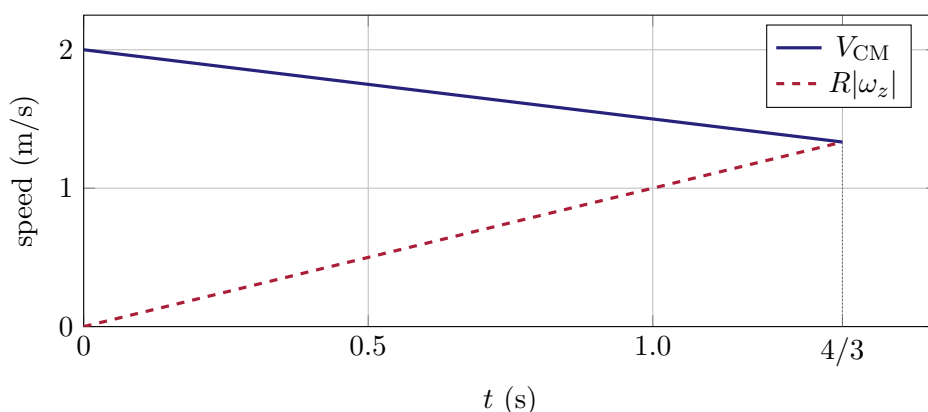


Figure 11.33: C11-P02. Exact synthetic speed histories for Problem 38, shown only through the onset of pure rolling. The solid and dashed curves have different slopes because translation and spin evolve independently during slipping.

39. An upper-hub tangent pull acts rightward on a spool with $M = 2.0 \text{ kg}$, $\kappa = 0.50$, $\mu_s = 0.080$ and $F = 10 \text{ N}$. Write $\lambda = r/R$. Figure 11.34 plots the required f_x/F in the ideal rolling model and the available static-friction bounds for these inputs. (a) Derive the interval of λ in which pure rolling is feasible. (b) Give the friction direction at $\lambda = 0.20, 0.50, 0.80$. (c) At $\lambda = 0.50$, find a and explain why zero friction can be compatible with rolling. (d) At $\lambda = 0.80$, find the largest pull compatible with the static-friction limit, keeping the other inputs fixed.
40. Two protocols start with $I_i = 8.0 \text{ kg m}^2$ and $\omega_i = 1.50 \text{ rad/s}$ and finish with $I_f = 3.0 \text{ kg m}^2$. Protocol A has negligible external torque; protocol B keeps angular speed fixed using a motor. Figure 11.35 compares the corresponding rotational-energy functions. (a) Derive both curves. (b) Find the final angular speeds and rotational energies. (c) Determine the change in energy for each protocol and the external angular impulse required in B. Assume radial motions have ended at both endpoints; discuss why the two curves represent different constraints.

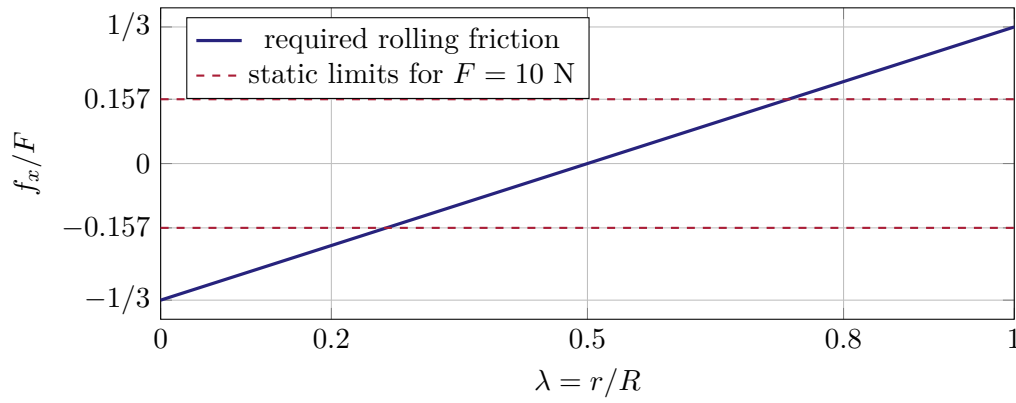


Figure 11.34: C11-P03. Required signed friction and static bounds for Problem 39. A required point lying outside the horizontal bounds invalidates the no-slip prediction; it does not prescribe the slipping acceleration.

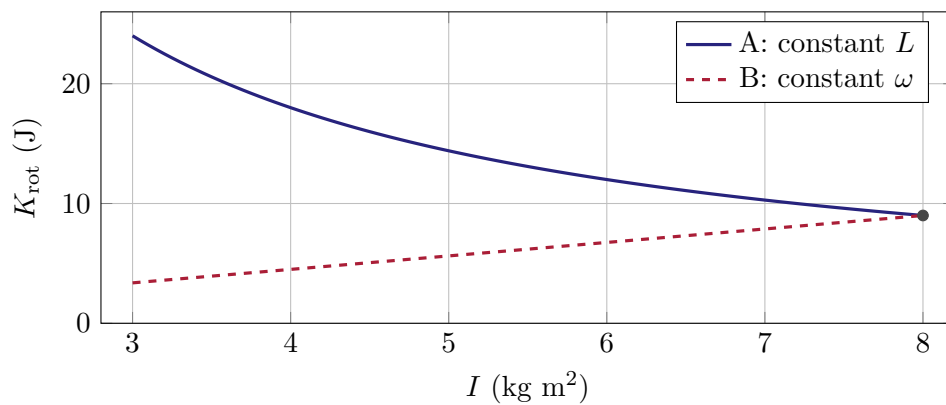


Figure 11.35: C11-P04. Two analytical energy paths for Problem 40. Both begin at the marked common state; the physical rearrangement proceeds from right to left. Energy is a state function of the specified rotational state, while the maintained constraint determines which states are reached.

41. **Advanced.** A body with $M = 2.0 \text{ kg}$ translates at $3.0\hat{i} \text{ m/s}$ along $y = b = 0.40 \text{ m}$, while its constant CM-relative angular momentum is $\mathbf{L}_{\text{CM}} = 1.80\hat{k} \text{ kg m}^2/\text{s}$. Consider fixed origins O' on the vertical line $x = 0$, with height y_0 above laboratory origin O . Figure 11.36 plots $L_{O',z}$ against y_0 . (a) Derive the plotted relation and locate the origin giving zero total angular momentum. (b) Determine the value about O and about an origin on the CM path. (c) Does the zero crossing mean that the body has stopped spinning? Explain the physical cancellation.

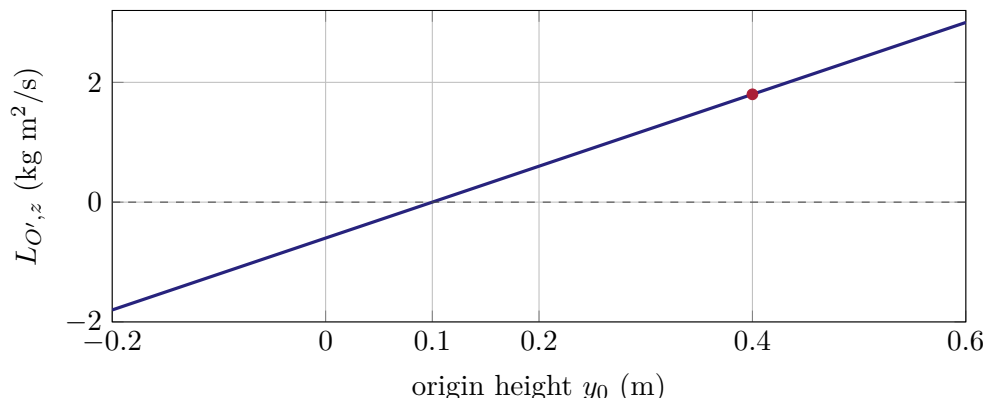


Figure 11.36: C11-P05. Total angular momentum about a family of fixed origins for Problem 41. The marked point corresponds to an origin on the CM path. The varying part is orbital; the CM-relative spin does not change as the origin is relabelled.

42. A rigid wheel of radius $R = 0.20 \text{ m}$ rolls without slipping on a horizontal belt. Figure 11.37 gives the horizontal velocities of its points versus their height h above the belt at one instant. The exact line runs from $(h, v_x) = (0, 1.0)$ to $(0.40, -0.20)$ in metres and m/s. (a) Infer the belt speed, CM velocity and signed ω_z . (b) Find the top point's velocity and the complete velocity of the rightmost point. (c) If the belt speed, CM velocity and angular velocity are constant, find the bottom material point's acceleration. Explain how a top point can move left while the centre moves right.

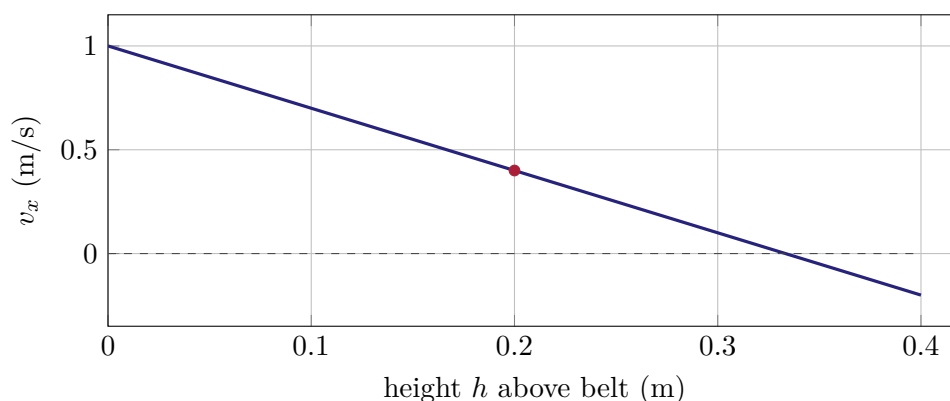


Figure 11.37: C11-P06. Exact horizontal-velocity profile for Problem 42. The CM lies at $h = R$ (marked). This is a spatial profile at one instant, not a time history; its slope diagnoses angular velocity rather than translational acceleration.

11.11 Complete solutions

The solutions follow the problem numbering. Each uses the stated ideal model and keeps contact feasibility, angular-momentum balance and energy accounting distinct. A signed component is not a magnitude; the arrow direction follows the sign after the coordinate system has been stated.

Solutions 1–9: diagnostic reasoning

Problem 1 (X11-01). Zero instantaneous velocity does not imply zero acceleration: velocity can pass through zero while changing. For steady clockwise rolling on stationary ground, $\mathbf{V}_{\text{CM}}$ and $\boldsymbol{\omega}$ are constant. The bottom material point has $\boldsymbol{\rho} = -R\hat{\mathbf{j}}$, so

$$\mathbf{a}_C = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) = \omega^2 R \hat{\mathbf{j}}.$$

Its acceleration points upward and has magnitude $\omega^2 R = V_{\text{CM}}^2/R$. The geometrical contact location at the next instant is occupied by another material point; differentiating a succession of contact labels as though it followed one particle is the mistaken step.

Problem 2 (X11-02). For $\kappa = 2/5$, no slip on a fixed incline requires

$$\mu_s \geq \frac{\kappa}{1 + \kappa} \tan 25^\circ = \frac{2}{7} \tan 25^\circ = 0.13323.$$

The available 0.12 is below the requirement. The pure-rolling model is therefore infeasible. Translation and spin must be evolved independently while slipping, using a kinetic-friction law when relative contact motion occurs. No unique slipping acceleration follows from μ_s alone because μ_k was not specified. Normal contact must also remain valid; on this simple incline $N = Mg \cos 25^\circ > 0$.

Problem 3 (X11-03). The perpendicular lever arm is measured from a specified origin to the force's line of action. It is zero about O , so $\boldsymbol{\tau}_O = 0$, but it is nonzero about the stipulated O' . If $\mathbf{a} = \mathbf{r}_{O'/O}$, then $\mathbf{r}_{P/O'} = \mathbf{r}_{P/O} - \mathbf{a}$, and hence

$$\boldsymbol{\tau}_{O'} = \boldsymbol{\tau}_O - \mathbf{a} \times \mathbf{F} = -\mathbf{a} \times \mathbf{F}.$$

The force itself has not changed; the origin-dependent moment arm has. Shifting the origin parallel to the force would leave the torque unchanged.

Problem 4 (X11-04). Choose the free path as $y = b$, with $\mathbf{v} = v\hat{\mathbf{i}}$ and $\mathbf{r} = x\hat{\mathbf{i}} + b\hat{\mathbf{j}}$. Then

$$\mathbf{L}_O = (x\hat{\mathbf{i}} + b\hat{\mathbf{j}}) \times mv\hat{\mathbf{i}} = -bmv\hat{\mathbf{k}}.$$

The increasing or decreasing part of $\mathbf{r}$ is parallel to momentum and contributes no cross product. Equivalently, $r \sin \phi = b$ remains constant even while r changes. Zero force gives zero torque about every fixed inertial origin, consistently predicting constant angular momentum.

Problem 5 (X11-05). Both impulses change the free platform's CM momentum by the same $\Delta \mathbf{P} = \mathbf{J}$, regardless of application point. The central impulse has zero CM moment arm and produces $\Delta \mathbf{L}_{\text{CM}} = 0$. For an off-centre application, $\Delta \mathbf{L}_{\text{CM}} \simeq \mathbf{r}_{A/\text{CM}} \times \mathbf{J}$, which is nonzero when the lever arm is perpendicular to the impulse. Angular impulse has units $\text{N m s} = \text{kg m}^2/\text{s}$. The distinction concerns spin; about another origin both applications can carry orbital angular impulse.

Problem 6 (X11-06). Under negligible external torque the relevant angular momentum is constant. In the fixed principal-axis model $L = I\omega$, but

$$K_{\text{rot}} = \frac{L^2}{2I}.$$

Moving the masses inward lowers I and increases ω and the endpoint rotational kinetic energy. The person supplies internal work, converting stored internal energy into mechanical kinetic energy. Angular-momentum conservation alone says nothing about conservation of kinetic energy. If radial motion is present during the pull, its kinetic energy must also be included at intermediate times.

Problem 7 (X11-07). The simple estimate treats angular momentum as approximately $I_s\omega_s\hat{s}$ and assumes steady precession much slower than spin. If Ω_p is comparable to ω_s , neglected precessional contributions can be important, so the derivation cannot establish its own assumed spin-dominated motion. Nutation and the transverse inertia may need to be included. A small rate ratio is a useful necessary plausibility indicator in a rotor with comparable inertias; the stronger test compares transverse/precessional angular momentum with $I_s\omega_s$.

Problem 8 (X11-08). No. Zero CM-relative spin sets only the second term of

$$\mathbf{L}_O = \mathbf{R}_{\text{CM}} \times M\mathbf{V}_{\text{CM}} + \mathbf{L}_{\text{CM}}$$

to zero. A translating body's orbital term is generally nonzero about an off-path origin. It vanishes when the CM trajectory passes through the chosen origin or its position is parallel to momentum. The result is origin-dependent without being ambiguous once the origin is stated.

Problem 9 (X11-09). CM motion is not the relative slipping motion of the surfaces. A spool's translation and spin can create opposite slipping tendencies, and the pull's moment changes their balance. Solve the coupled force/torque equations with signed f_x , then interpret its sign. Next check $|f_x| \leq \mu_s N$ and that contact is maintained ($N \geq 0$). For the horizontal upper-tangent model, $N = Mg$ and $f_x = F(r/R - \kappa)/(1 + \kappa)$. Exceeding the bound invalidates the assumed rolling solution; replacing the required value by $\mu_s N$ inside the same no-slip equations is not justified.

Solutions 10–16: rolling and vector calculations

Problem 10. Stationary-ground no slip gives $V_{\text{CM}} + R\omega_z = 0$, so $\omega_z = -2.0/0.25 = -8.0 \text{ rad/s}$, clockwise. At the rightmost point $\boldsymbol{\rho} = R\hat{\mathbf{i}}$,

$$\mathbf{v}_S = 2.0\hat{\mathbf{i}} + (-8.0\hat{\mathbf{k}}) \times (0.25\hat{\mathbf{i}}) = (2.0\hat{\mathbf{i}} - 2.0\hat{\mathbf{j}}) \text{ m/s}.$$

The speed is $\sqrt{2^2 + 2^2} = 2.83 \text{ m/s}$ and its direction is 45° below $+x$. The rotational component is downward, as expected for clockwise spin at the right-hand rim.

Problem 11. Use the downhill-positive translation equation $Mg \sin \theta - f = Ma$ and the CM torque magnitude $fR = I_{\text{CM}}a/R$. For a solid sphere $I_{\text{CM}} = (2/5)MR^2$, so

$$a_{\text{roll}} = \frac{9.81 \sin 25^\circ}{1.4} = 2.961 \text{ m/s}^2, \quad \mu_{s,\text{min}} = \frac{2}{7} \tan 25^\circ = 0.1332.$$

The supplied 0.12 is insufficient. Thus 2.961 m/s^2 is a hypothetical no-slip acceleration, not the actual acceleration on this surface. A slipping calculation requires further friction information. This qualification is part of the requested answer.

Problem 12. No slip on a fixed ideal surface means static friction transfers energy between translation and spin without dissipating total mechanical energy. With zero initial speed,

$$Mgh = \frac{1}{2}M(1 + 2/5)V^2, \quad V = \sqrt{\frac{2(9.81)(0.75)}{1.4}} = 3.242 \text{ m/s}.$$

The final angular-speed magnitude is $V/R = 3.242/0.12 = 27.02 \text{ rad/s}$. Mass cancels because both gravitational energy and total rolling inertia scale with it. The direction of angular velocity follows the actual downhill direction, which was not otherwise fixed by the statement.

Problem 13. Compute each component from the cross product, retaining the order $\mathbf{r} \times \mathbf{F}$:

$$\tau_x = 0.30(6) - (-0.10)(-4) = 1.4, \quad \tau_y = (-0.10)(5) - 0.20(6) = -1.7,$$

$$\tau_z = 0.20(-4) - 0.30(5) = -2.3.$$

Thus $\boldsymbol{\tau}_O = (1.4\hat{\mathbf{i}} - 1.7\hat{\mathbf{j}} - 2.3\hat{\mathbf{k}}) \text{ N m}$. Every component has dimensions length times force. An additional vector check gives $\boldsymbol{\tau} \cdot \mathbf{r} = \boldsymbol{\tau} \cdot \mathbf{F} = 0$, as a cross product must be perpendicular to both factors.

Problem 14. A couple comprises two equal opposite parallel forces, so the net force vanishes. The separation $d = 0.30 \text{ m}$ is perpendicular to their parallel lines of action. The moment magnitude is therefore

$$\tau = Fd = (20)(0.30) = 6.0 \text{ N m}.$$

For an origin shift $\mathbf{a}$, the net torque changes by $-\mathbf{a} \times \mathbf{F}_{\text{net}} = 0$. Its direction is determined by the two force arrows and their spatial order; only the magnitude was specified. Zero net force does not mean zero torque.

Problem 15. The particle momentum is $\mathbf{p} = (0.080)(15)\hat{\mathbf{i}} = 1.20\hat{\mathbf{i}} \text{ kg m/s}$. At any position on its path,

$$\mathbf{L}_O = (x\hat{\mathbf{i}} + 0.25\hat{\mathbf{j}}) \times 1.20\hat{\mathbf{i}} = -0.300\hat{\mathbf{k}} \text{ kg m}^2/\text{s}.$$

An origin on the path has zero perpendicular offset and hence zero angular momentum. The parallel coordinate cancels in both descriptions. Into-page angular momentum is consistent with $\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}}$.

Problem 16. For the first particle, $\mathbf{r}_1 \times m_1 \mathbf{v}_1 = (1\hat{\mathbf{i}}) \times (6\hat{\mathbf{j}}) = 6\hat{\mathbf{k}} \text{ kg m}^2/\text{s}$. For the second,

$$(2\hat{\mathbf{j}}) \times (-2\hat{\mathbf{i}}) = 4\hat{\mathbf{k}} \text{ kg m}^2/\text{s}.$$

Adding contributions about the same origin gives $\mathbf{L}_O = 10\hat{\mathbf{k}} \text{ kg m}^2/\text{s}$. The second result is positive because both the velocity sign and cross-product order contribute a minus sign. Adding speeds or moment magnitudes without directions would obscure that reasoning.

Solutions 17–22: impulse and conservation

Problem 17. For the free body initially at rest, linear impulse gives

$$\mathbf{V}_{\text{CM}} = \frac{8.0}{6.0}\hat{\mathbf{i}} = 1.333\hat{\mathbf{i}} \text{ m/s}.$$

With the short-event lever arm fixed, the CM spin impulse is

$$\Delta \mathbf{L}_{\text{CM}} = (0.30\hat{\mathbf{j}}) \times (8.0\hat{\mathbf{i}}) = -2.40\hat{\mathbf{k}} \text{ N m s}.$$

Thus $\omega_z = -2.40/2.4 = -1.00 \text{ rad/s}$, clockwise. The CM spin balance is valid for this fixed-mass free system even though its CM translates; it follows from the CM-relative theorem rather than blindly treating the CM as a fixed laboratory point.

Problem 18. The platform and two masses form the torque-free system. Their endpoint inertias are

$$I_i = 5.0 + 2(1.0)(1.0)^2 = 7.0 \text{ kg m}^2, \quad I_f = 5.0 + 2(1.0)(0.20)^2 = 5.08 \text{ kg m}^2.$$

Conservation of the fixed-axis component gives $L = I_i \omega_i = 14.0 \text{ kg m}^2/\text{s}$ and

$$\omega_f = \frac{14.0}{5.08} = 2.756 \text{ rad/s}, \quad \frac{K_f}{K_i} = \frac{I_i}{I_f} = 1.37795.$$

For checking, $K_i = 14.0 \text{ J}$ and $K_f = L^2/(2I_f) = 19.29 \text{ J}$. The extra 5.29 J is supplied by internal work in the ideal endpoint account. Absence of radial velocity at both endpoints is needed to equate these rotational energies with the endpoints' full mechanical kinetic energies.

Problem 19. Choose the axle as the origin and positive spin in the incoming particle's moment direction. The axle impulse has zero moment about that axis. The short collision therefore conserves this angular-momentum component, even though linear momentum is not conserved for the disk-plus-particle system because of the axle impulse. Initially,

$$L_i = mvR = (0.10)(10)(0.50) = 0.500 \text{ kg m}^2/\text{s}.$$

After capture $I_f = \frac{1}{2}(4.0)(0.50)^2 + 0.10(0.50)^2 = 0.525 \text{ kg m}^2$, giving

$$\omega_f = \frac{0.500}{0.525} = 0.9524 \text{ rad/s}.$$

The initial particle energy is $\frac{1}{2}(0.10)(10)^2 = 5.00 \text{ J}$; the final energy is $L_i^2/(2I_f) = 0.2381 \text{ J}$. Consequently

$$\frac{K_i - K_f}{K_i} = 0.95238 = 95.24\%.$$

Energy lost from mechanical kinetic energy appears in deformation, heat and other internal modes during the inelastic capture. The equality of the numerical speed and fractional-loss decimals here is incidental to the selected parameters.

Problem 20. The body and wheel are one system with zero initial total angular momentum. The stipulated wheel speed is relative to the body, so its inertial angular velocity is $\omega_w = \omega_b + \Omega_{\text{rel}}$. Hence

$$120\omega_b + 5(\omega_b + 6) = 0, \quad \omega_b = -\frac{30}{125} = -0.240 \text{ rad/s}.$$

The wheel's inertial rate is 5.760 rad/s : $120(-0.240) + 5(5.760) = 0$ provides a direct conservation check. The body counter-rotates, and the motor supplies energy internally. Confusing relative and inertial wheel rates would give a different, incorrect result.

Problem 21. For the qualified spin-dominated steady-precession model,

$$\Omega_p = \frac{(1.5)(9.81)(0.060)}{(0.0080)(300)} = 0.3679 \text{ rad/s}.$$

A full precession cycle requires angle 2π , so $T_p = 2\pi/\Omega_p = 17.08 \text{ s}$ and

$$\frac{\Omega_p}{\omega_s} = 1.2263 \times 10^{-3}.$$

The rate is slow relative to spin. This satisfies a necessary plausibility check but does not by itself establish spin-angular-momentum dominance, because no transverse inertia or nutation data were supplied. The numerical estimate should retain that qualification rather than certify an arbitrary real gyroscope's motion.

Problem 22. The geometry has $\lambda = r/R = 0.090/0.18 = 0.50$. With positive α_z counterclockwise, write

$$F + f_x = Ma, \quad -rF + Rf_x = \kappa MR^2\alpha_z, \quad a = -R\alpha_z.$$

Solving gives

$$a = \frac{10(1.50)}{2.0(1.60)} = 4.6875 \text{ m/s}^2, \quad f_x = \frac{10(0.50 - 0.60)}{1.60} = -0.625 \text{ N},$$

$$\alpha_z = -\frac{4.6875}{0.18} = -26.04 \text{ rad/s}^2.$$

Static friction points left and spin accelerates clockwise. The available magnitude is $0.10(2.0)(9.81) = 1.962 \text{ N}$, exceeding 0.625 ; $N = Mg > 0$, so no slip is feasible. As a torque check, $-rF + Rf_x = -1.0125 \text{ N m}$ equals $I\alpha_z$ with $I = 0.03888 \text{ kg m}^2$.

Solutions 23–32: data and changing constraints**Problem 23.** For a disk $\kappa = 1/2$, so

$$\mu_{s,\min} = \frac{1/2}{3/2} \tan 30^\circ = \frac{1}{3} \tan 30^\circ = 0.19245.$$

Since $0.15 < 0.19245$, pure rolling cannot be maintained by the stated static friction. The minimum required coefficient is approximately 0.192 for this ideal model. The nominal rolling acceleration $g \sin 30^\circ / 1.5$ is not a valid prediction for the stated contact. A kinetic coefficient would be required to calculate slipping motion.

Problem 24. The constructed times have mean $\bar{t} = (1.32 + 1.34 + 1.33)/3 = 1.330$ s. For constant acceleration from rest,

$$s = \frac{1}{2}at^2 \quad \Rightarrow \quad a = \frac{2(1.50)}{1.330^2} = 1.69597 \text{ m/s}^2.$$

The rolling-force model gives

$$\kappa = \frac{g \sin 15^\circ}{a} - 1 = \frac{(9.81) \sin 15^\circ}{1.69597} - 1 = 0.49709.$$

This is closest to the uniform solid disk/cylinder value 0.5. The sample standard deviation is 0.010 s; the scatter estimates repeatability, not every systematic uncertainty. Arrival time alone does not show no slip: simultaneous angular tracking, known ramp geometry and a reproducible release are needed before interpreting κ as a property of the body's mass distribution.

Problem 25. The body momentum is $\mathbf{P} = 3(\hat{i} + 2\hat{j}) = (3\hat{i} + 6\hat{j})$ kg m/s. Its orbital moment is

$$(2\hat{i} - \hat{j}) \times (3\hat{i} + 6\hat{j}) = (12 + 3)\hat{k} = 15\hat{k} \text{ kg m}^2/\text{s}.$$

Add the supplied spin term to obtain $\mathbf{L}_O = 19\hat{k}$ kg m²/s. About the CM the orbital term vanishes, leaving $\mathbf{L}_{\text{CM}} = 4\hat{k}$ kg m²/s. As an independent shift check, choose $\mathbf{a} = \mathbf{R}_{\text{CM}}$; subtracting $\mathbf{a} \times \mathbf{P}$ removes exactly the $15\hat{k}$ orbital contribution.

Problem 26. Multiplying the diagonal tensor by the angular-velocity column gives

$$\mathbf{L} = (2 \cdot 3\hat{i} + 5 \cdot 4\hat{j}) = (6\hat{i} + 20\hat{j}) \text{ kg m}^2/\text{s}.$$

The angle ψ follows from the dot product rather than a scalar-inertia assumption:

$$\cos \psi = \frac{3(6) + 4(20)}{\sqrt{3^2 + 4^2} \sqrt{6^2 + 20^2}} = \frac{98}{5\sqrt{436}}, \quad \psi = 20.1707^\circ.$$

The component directions are 53.1301° for $\boldsymbol{\omega}$ and 73.3008° for $\mathbf{L}$, giving the same difference. Distinct inertias rescale different components, so the two vectors need not be parallel.

Problem 27. The shift formula uses total linear momentum:

$$\mathbf{L}_{O'} = \mathbf{L}_O - \mathbf{a} \times \mathbf{P}.$$

Here $(0.40\hat{j}) \times (10\hat{i}) = -4.0\hat{k}$ kg m²/s, so

$$\mathbf{L}_{O'} = (6 - (-4))\hat{k} = 10\hat{k} \text{ kg m}^2/\text{s}.$$

The sign is best checked by retaining the cross-product order. This change of reference changes the orbital contribution, not the physical state of the system.

Problem 28. For the stated nonzero tilt, $\sin \theta \neq 0$, so division of $Mgd \sin \theta = \Omega_p L \sin \theta$ gives $\Omega_p = Mgd/L$. With $L \simeq I_s \omega_s$, the ideal estimate is independent of tilt. The same geometry factor occurs on both sides because torque and angular-momentum turning both involve the horizontal projection of the tilted axis. At $\theta = 0$ or π , this equation becomes $0 = 0$ and cannot determine a precession rate. Even away from these endpoints, startup, appreciable nutation or a large precessional contribution can violate the assumptions; cancellation is not a stability proof.

Problem 29. For a positive rightward pull, $F/(1 + \kappa) > 0$, so the sign of required friction is the sign of $\lambda - \kappa$. With $\kappa = 0.50$, case (a) has $0.30 - 0.50 < 0$ and friction points left. Case (b) has $0.70 - 0.50 > 0$ and friction points right. At $\lambda = 0.50$, $f_x = 0$: the pull produces precisely the translational-to-angular acceleration ratio needed for no slip without a tangential contact force. In each nonzero case the static bound still must be checked before accepting the rolling result.

Problem 30. At the bottom, $\boldsymbol{\rho} = -R\hat{\mathbf{j}}$ and the rotational velocity is $\omega_z \hat{\mathbf{k}} \times (-R\hat{\mathbf{j}}) = R\omega_z \hat{\mathbf{i}}$. No relative slipping requires the body's contact velocity to match the belt's laboratory velocity:

$$\boxed{V_{\text{CM}} + R\omega_z = U.}$$

For $U = 0$, this becomes $V_{\text{CM}} = -R\omega_z$, so rightward rolling has negative (clockwise) spin. For a moving belt the contact need not be at rest in the laboratory. The relation alone is kinematic; force and torque equations are still needed to determine acceleration.

Problem 31. The hoop model gives $I = MR^2 = (1.2)(0.35)^2 = 0.147 \text{ kg m}^2$ and rolling gives $|\omega| = 8.0/0.35 = 22.857 \text{ rad/s}$. Thus

$$|L_{\text{spin}}| = I|\omega| = MRV = (1.2)(0.35)(8.0) = 3.36 \text{ kg m}^2/\text{s}.$$

This is spin angular momentum about the wheel's symmetry axis through its CM, not the complete bicycle's moment about an arbitrary origin. The model neglects hub/spoke mass distribution, deformation, steering and nonideal contact. Actual wheel inertia must be measured or calculated from its distributed mass before using the estimate quantitatively.

Problem 32. A defensible exemplar uses the same rigid-body radius, the same ramp and a release from rest, with each body verified as the intended mass distribution. Choose a low vertical drop $h = 0.10 \text{ m}$ on a 10° ramp; the path is $s = h/\sin 10^\circ = 0.576 \text{ m}$. Provided static friction is sufficient, the predicted times are

$$t_d = \sqrt{\frac{2s(1.5)}{g \sin 10^\circ}} = 1.01 \text{ s}, \quad t_h = \sqrt{\frac{2s(2.0)}{g \sin 10^\circ}} = 1.16 \text{ s}.$$

The roughly 0.16 s separation suggests calibrated video timing or photogates rather than hand timing alone. Direct measurements are ramp distance, angle, elapsed time and image positions/marker angles. Inferred quantities include $a = 2s/t^2$ and $\kappa = g \sin \theta/a - 1$. Useful controls include a common release gate, repeated trials and a checked frame rate.

At least three relevant systematic effects are release push, ramp-angle or length calibration, trigger delay, rolling resistance, and departures from ideal disk/hoop geometry. A specific slip test compares video-derived V_{CM} with $R|\omega_z|$ throughout the run; systematic mismatch reveals relative contact motion. Before release the thresholds are $\mu_{\text{min},d} = \tan 10^\circ/3 = 0.0588$ and $\mu_{\text{min},h} = \tan 10^\circ/2 = 0.0882$. A report merits full credit when it separates direct/inferred quantities, supplies a quantitative discrimination estimate, names three mechanisms with their effect, and includes an explicit slip test. Different low-height dimensions are acceptable if feasibility and timing resolution are justified.

Solutions 33–36: synthesis

Problem 33. On an ideal rigid horizontal surface, after any entry transient has ended, uniform rolling requires $a_{\text{CM}} = 0$ and $\alpha_z = 0$. If there is no applied horizontal force, no applied driving/braking moment and no rolling-resistance torque, the force and CM-torque equations admit $f_x = 0$. Normal force balances weight and the no-slip kinematic relation persists. Roughness supplies an available static-friction range; it does not force friction to equal its limiting magnitude. A discontinuous junction, deformation, drag or a motor changes the model and can require a nonzero contact force.

Problem 34. During a sufficiently short event, $\Delta \mathbf{L}_O \simeq \mathbf{r}_{A/O} \times \mathbf{J}$. Since the impulse's line of action passes through O , $\mathbf{r}_{A/O}$ is parallel to $\mathbf{J}$ and the cross product is zero. About the CM, however,

$$\Delta \mathbf{L}_{\text{CM}} \simeq \mathbf{r}_{A/\text{CM}} \times \mathbf{J} = -\mathbf{R}_{\text{CM}/O} \times \mathbf{J}$$

can be nonzero. The last equality follows from $\mathbf{r}_{A/\text{CM}} = \mathbf{r}_{A/O} - \mathbf{R}_{\text{CM}/O}$. The resulting change in CM orbital angular momentum about O compensates the spin change, leaving the total angular impulse about O zero. The two origin-labelled statements are consistent descriptions of one event.

Problem 35. With fixed nonzero angular momentum, $I\omega$ is unchanged. Reducing inertia from I to $I/3$ gives $\omega_f = 3\omega_i$. Then

$$K_f = \frac{1}{2} \frac{I}{3} (3\omega_i)^2 = 3K_i.$$

Internal work supplies the increase $\Delta K = 2K_i$ in the ideal rotational endpoint account. The result assumes the principal-axis model remains valid and does not imply that every conceivable rearrangement conserves angular momentum about every axis. It also does not exclude radial kinetic energy during the rearrangement.

Problem 36. For a fixed discrete mass distribution,

$$I_{ab} = \sum_i m_i (\rho_i^2 \delta_{ab} - \rho_{i,a} \rho_{i,b}).$$

Both δ_{ab} and $\rho_{i,a} \rho_{i,b}$ are unchanged under $a \leftrightarrow b$, so $I_{ab} = I_{ba}$ and the tensor is real symmetric. If $\mathbf{u}$ and $\mathbf{v}$ have distinct eigenvalues λ, μ , symmetry gives

$$\lambda \mathbf{u} \cdot \mathbf{v} = (\mathbf{I}\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (\mathbf{I}\mathbf{v}) = \mu \mathbf{u} \cdot \mathbf{v}.$$

Thus $(\lambda - \mu)\mathbf{u} \cdot \mathbf{v} = 0$, establishing orthogonality for distinct eigenvalues. The real-symmetric spectral theorem supplies a complete orthonormal eigenbasis; within a repeated-eigenvalue subspace an orthonormal basis can be chosen without changing the eigenvalue. These axes are the principal axes. No time-dependent Euler equations are needed for this instantaneous tensor statement.

Solutions 37–42: graph interpretation

Problem 37. (a) The signed areas are a positive triangle of area $\frac{1}{2}(1)(6) = 3 \text{ N m s}$ over 0–1 s, a rectangle of area 6 N m s over 1–2 s, a positive triangle of area 3 N m s over 2–3 s, and a negative triangle of area -3 N m s over 3–4 s. The total angular impulse is 9 N m s .

(b) Add cumulative impulse to the initial moment: $L_z(3) = 2 + 3 + 6 + 3 = 14 \text{ kg m}^2/\text{s}$ and $L_z(4) = 2 + 9 = 11 \text{ kg m}^2/\text{s}$. (c) With constant principal-axis inertia,

$$K_f = \frac{L_z(4)^2}{2I} = \frac{11^2}{4} = 30.25 \text{ J}, \quad \omega_f = L_z/I = 5.50 \text{ rad/s}.$$

(d) The negative torque reduces positive angular momentum by 3 units; it cannot reverse the spin because L_z remains positive throughout. Torque is the slope of the angular-momentum history, not its sign. An independent energy check is $\Delta K = \int \tau_z \omega_z dt = \int (L_z/I) dL_z = (11^2 - 2^2)/(2I) = 29.25 \text{ J}$; adding the initial 1.00 J agrees with the result.

Problem 38. (a) The signed slip velocity is $u = V_{\text{CM}} + R\omega_z = (2 - 0.50t) - t = 2 - 1.50t$. Rolling begins when $u = 0$, at $t_r = 4/3 = 1.333\text{ s}$ and $V_r = 1.333\text{ m/s}$. (b) The negative translation slope gives $f_x = Ma = 3(-0.50) = -1.50\text{ N}$, so $\mu_k = 1.50/(3 \cdot 9.81) = 0.05097$. The initial positive slip therefore produces the expected leftward friction.

(c) Since the spin is clockwise, $\omega_z = -t/R = -5t$ and $\alpha_z = -5.0\text{ rad/s}^2$. The contact torque is $Rf_x = -0.300\text{ N m}$, so

$$I_{\text{CM}} = \frac{Rf_x}{\alpha_z} = 0.060\text{ kg m}^2, \quad \kappa = \frac{0.060}{3(0.20)^2} = 0.50.$$

(d) Initially $K_i = \frac{1}{2}(3)(2)^2 = 6.0\text{ J}$. At rolling $|\omega_r| = V_r/R = 20/3\text{ rad/s}$, giving $K_f = \frac{1}{2}M(1 + \kappa)V_r^2 = 4.0\text{ J}$. The loss is 2.0 J . Independently, frictional dissipation is $\int_0^{t_r} f|u| dt = 1.50 \int_0^{4/3} (2 - 1.50t) dt = 2.0\text{ J}$. Extending the same straight lines past the intersection would make $u < 0$ while leaving friction leftward, contrary to the sliding law. For this otherwise force-free ideal body, zero static friction can maintain the ensuing uniform rolling.

Problem 39. (a) The rolling equations require $f_x = F(\lambda - \kappa)/(1 + \kappa)$. The bound is $\mu_s Mg = (0.080)(2.0)(9.81) = 1.5696\text{ N}$. Hence

$$\left| \frac{\lambda - 0.50}{1.50} \right| \leq \frac{1.5696}{10} = 0.15696, \quad \boxed{0.26456 \leq \lambda \leq 0.73544}.$$

This interval already lies within the physical hub range $0 < \lambda < 1$. (b) The model requires leftward, zero and rightward friction at $\lambda = 0.20, 0.50, 0.80$ respectively. At the two outer values it requires $\mp 2.0\text{ N}$, beyond the available magnitude, so those are signs of the *required hypothetical* rolling force, not a completed prediction of sliding motion.

(c) At $\lambda = 0.50$, $a = 10(1.50)/(2.0 \cdot 1.50) = 5.0\text{ m/s}^2$ and $f_x = 0$. The pull's clockwise hub moment alone gives the angular acceleration matching $a = -R\alpha_z$. (d) At $\lambda = 0.80$, $|f_x| = 0.20F$, so $F_{\text{max}} = 1.5696/0.20 = 7.848\text{ N}$. Equality is marginal static contact; experimental uncertainty would call for a margin below it.

Problem 40. (a) Both start with $L_i = 8(1.50) = 12\text{ kg m}^2/\text{s}$ and $K_i = 9.0\text{ J}$. In A, angular momentum is constant, so $K_A = L_i^2/(2I) = 72/I$ when SI units are used. In B, $\omega = 1.50\text{ rad/s}$ remains fixed, so $K_B = \frac{1}{2}I(1.50)^2 = 1.125I$.

(b) At $I_f = 3.0\text{ kg m}^2$, A gives $\omega_{A,f} = 12/3 = 4.0\text{ rad/s}$ and $K_{A,f} = 24.0\text{ J}$. B gives $\omega_{B,f} = 1.50\text{ rad/s}$ and $K_{B,f} = 3.375\text{ J}$. (c) Therefore $\Delta K_A = +15.0\text{ J}$ and $\Delta K_B = -5.625\text{ J}$. The required external angular impulse in B is

$$J_{\tau,z} = L_{B,f} - L_i = 3(1.50) - 12 = -7.50\text{ N m s}.$$

The motor removes angular momentum while the inertia changes. The negative endpoint energy change is the net result of all work transfers; one should not assign it to the motor alone without modelling the radial actuator. In A the ideal internal actuator supplies the increased endpoint mechanical energy. The two curves impose different constraints; they are not two calculations of the same torque-free rearrangement.

Problem 41. (a) Momentum is $\mathbf{P} = 6\hat{\mathbf{i}}\text{ kg m/s}$. The height relative to O' is $b - y_0$, and the changing x coordinate is parallel to momentum. Thus

$$L_{O',z} = -6(0.40 - y_0) + 1.80 = -0.60 + 6y_0,$$

with SI units understood. Zero total angular momentum occurs at $y_0 = 0.60/6 = 0.10\text{ m}$. (b) At O ($y_0 = 0$), $L_{O,z} = -0.60\text{ kg m}^2/\text{s}$. On the CM path ($y_0 = b$), the orbital moment vanishes and $L_z = +1.80\text{ kg m}^2/\text{s}$.

(c) No physical motion changes when the origin changes. At $y_0 = 0.10$ m, orbital angular momentum is $-1.80 \text{ kg m}^2/\text{s}$ and cancels the unchanged $+1.80 \text{ kg m}^2/\text{s}$ spin. A zero total moment about this origin cannot be interpreted as zero spin. Differentiating the plotted relation gives $dL_z/dy_0 = P_x = 6.0 \text{ kg m/s}$, independently matching the origin-shift law.

Problem 42. (a) With h measured from the belt, the point's relative vertical coordinate is $\rho_y = h - R$. Rigid-body velocity gives

$$v_x(h) = V_{\text{CM}} - \omega_z(h - R).$$

The plotted slope is $(-0.20 - 1.0)/0.40 = -3.0 \text{ s}^{-1}$, hence $\omega_z = +3.0 \text{ rad/s}$, counterclockwise. At $h = R$ the horizontal velocity is the CM speed $V_{\text{CM}} = 0.40 \text{ m/s}$. No slip makes the bottom velocity equal the belt speed, $U = 1.0 \text{ m/s}$; $V_{\text{CM}} + R\omega_z = 0.40 + 0.60 = 1.0$ checks the constraint.

(b) At the top, $v_{T,x} = V_{\text{CM}} - R\omega_z = -0.20 \text{ m/s}$ and $v_{T,y} = 0$. At the rightmost rim point,

$$\mathbf{v}_S = 0.40\hat{\mathbf{i}} + (3\hat{\mathbf{k}}) \times (0.20\hat{\mathbf{i}}) = (0.40\hat{\mathbf{i}} + 0.60\hat{\mathbf{j}}) \text{ m/s}.$$

(c) For constant translation and spin, the bottom material point has centripetal acceleration $\mathbf{a}_C = \omega_z^2 R \hat{\mathbf{j}} = 1.80\hat{\mathbf{j}} \text{ m/s}^2$. A succession of points occupies the belt contact, so equality of instantaneous contact velocities does not mean the body's material point shares the belt material point's acceleration. At the top the leftward rotational contribution -0.60 m/s exceeds the rightward translation 0.40 m/s , making the net velocity leftward without contradiction.

11.12 Bridge to Chapter 12

The next chapter, *Static Equilibrium and Elasticity*, applies the force and torque balances to extended bodies at rest and then introduces the relation between forces and small deformations. A convenient moment origin can eliminate unknown support forces from one equation, but all forces must still satisfy the full balance. This is the same origin-aware reasoning used here, with angular acceleration and the rate of angular momentum set to zero under the appropriate conditions.

Rolling also demonstrates how a constraint can connect degrees of freedom without replacing Newton's laws. Later mechanics uses comparable constraints, but each must be checked for physical admissibility. In gravitation, developed in Chapter 13, central forces and rotational symmetry will again make angular momentum useful.

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