

Chapter 12

Static Equilibrium and Elasticity

First Edition

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Preface

A rigid object can have zero resultant force and still have a strong tendency to rotate. A structure can satisfy the equations of force and moment balance while its individual support forces remain undetermined. A rod can be made of a stiff material and nevertheless be quite flexible if it is long and thin. These are not separate curiosities: they are reminders that equilibrium, geometry and material response answer different parts of an engineering-physics problem.

This chapter develops two connected ideas. First, we apply force and torque balance to extended bodies that remain at rest. We learn how to choose a useful moment origin, replace a distributed load by an equivalent resultant when that replacement is legitimate, determine support reactions in statically determinate models, and distinguish tipping from sliding. Second, we relax the perfectly rigid-body idealisation and introduce stress, strain and linear elastic response. That additional physics explains why geometry and material modulus both matter, and why equilibrium equations alone do not determine every redundant support force. A rigid crosshead may rotate: equal end displacement requires an actual rotational constraint, not simply the word rigid.

The chapter is a physics foundation for technical and engineering students. It is not a substitute for a course in structural analysis, mechanics of materials, geotechnical engineering or a design code. Numerical loads, moduli and geometries used below are stated teaching parameters unless explicitly described otherwise. They are not safety ratings, certified material data or design recommendations.

The Guideline

Treat a static-equilibrium problem as a model before treating it as algebra. Isolate the body, identify every external force and couple, state what each ideal support can exert, choose axes and a moment origin, and only then write balance equations. If deformations influence the unknown reactions, equilibrium must be supplemented by geometry and a constitutive model.

Pedagogical boundary

The core develops planar statics and simple uniform elastic response. It does not derive beam-bending theory, stress concentrations, fracture criteria, fatigue life, Euler buckling loads or code-based safety factors. Those topics require additional field equations, geometry and empirical information. The advanced section compares guided and freely rotating crossheads, develops three-support compatibility and contact engagement, and uses a restrained hinged member to distinguish equilibrium from stability.

Conventions and notation

Vectors are bold. In planar problems, positive x is to the right, positive y is upward and positive torque about the z axis is counterclockwise unless another convention is stated. A torque about origin O is

$$\boldsymbol{\tau}_O = \mathbf{r} \times \mathbf{F},$$

where $\mathbf{r}$ points from O to the force application point. The origin and axis are part of the torque definition.

For axial deformation we use a tension-positive signed convention:

$$\sigma = \frac{F_{\perp}}{A}, \quad \varepsilon = \frac{\Delta L}{L_0}.$$

Thus tensile stress and strain are positive and compressive values are negative. The Young modulus E_Y is a positive material parameter in the small-strain linear elastic regime. For simple shear we write $\tau_{\text{shear}} = G_s \gamma$ to avoid confusing shear stress with torque. For hydrostatic compression, a positive pressure increase $\Delta p > 0$ produces $\Delta V < 0$, and the positive bulk modulus is

$$B = -\frac{\Delta p}{\Delta V/V_0}.$$

Symbol	Meaning	SI unit
$\mathbf{F}$	force	N
$\boldsymbol{\tau}_O$	torque about origin O	N m
M	total mass	kg
$\mathbf{R}_{\text{CM}}$	centre-of-mass position	m
A	cross-sectional or loaded area	m ²
$L_0, \Delta L$	reference length and length change	m
$\sigma, \tau_{\text{shear}}$	normal and shear stress	Pa
ε, γ	axial and shear strain	1
E_Y, G_s, B	Young, shear and bulk moduli	Pa
k	specimen or structural stiffness	N m ⁻¹
u_{el}	elastic energy per reference volume	J m ⁻³

Unless stated otherwise, numerical mechanics examples use $g = 9.81 \text{ m s}^{-2}$ as a rounded calculation value. The chapter's elastic parameters are stipulated teaching values rather than claims about a named commercial material.

Chapter 12

Static Equilibrium and Elasticity

Learning goals

After studying this chapter, you should be able to isolate a rigid body, choose support reactions that match its constraints, and solve force and moment balance with a second-origin check. You should be able to locate a distributed-load resultant, assess sliding and tipping independently, and reject inadmissible contact forces.

For deformable bodies, you should be able to distinguish stress from force, strain from extension, and modulus from specimen stiffness; calculate axial, shear and bulk response within their stated models; and interpret elastic energy and loading–unloading graphs. You should also be able to infer stiffness and modulus from synthetic data with explicit uncertainty assumptions, and combine equilibrium, compatibility and constitutive response in the developed advanced support models.

12.1 From motion to static equilibrium

An instrument support, a laboratory stand and a ladder all present the same first question: can the external actions on the chosen body balance? A structure at rest must resist translation and rotation. Chapters 5 and 10 supplied the dynamical laws for those motions; statics uses their zero-acceleration limit. The apparent simplicity of the equations makes the modelling especially important. A mathematically correct solution for a support that can pull is not a valid solution for a contact pad that can only push.

We begin with a rigid-body idealisation. Distances between material points are treated as fixed while external loads are applied slowly enough for acceleration to be neglected. This does not assert that real bodies never deform. It says that the deformation is small enough not to change the geometry used in the first balance calculation. The elasticity sections will then ask how loads and deformations are related. In a redundant support system, that relation can be needed even to determine the reactions.

Definition 12.1: Static equilibrium

A body is in static equilibrium when it is at rest in the chosen inertial frame and the resultant external force and external torque both vanish:

$$\sum_i \mathbf{F}_i = \mathbf{0}, \quad \sum_i \boldsymbol{\tau}_{O,i} = \mathbf{0}. \quad (12.1)$$

The external actions include applied couples as well as forces. The word *static* specifies the state of rest; balance alone does not stop a body that is already moving.

For a planar system in the xy plane, choose $+z$ out of the page. Counterclockwise moments are

then positive. Each force contributes

$$\tau_{z,O} = xF_y - yF_x, \quad \sum F_x = 0, \quad \sum F_y = 0, \quad \sum \tau_{z,O} = 0. \quad (12.2)$$

The coordinates x, y locate the force application point relative to O . The expression explains why a horizontal force can create a moment: its vertical offset multiplies its horizontal component. A lever arm is the perpendicular distance from the origin to the force's *line of action*, not necessarily the distance to its application point. Moving a force along its own line leaves its torque unchanged in a rigid-body balance. Moving it to a parallel line generally requires adding a couple.

12.1.1 Choosing the body before drawing its forces

A free-body diagram is a statement of the system boundary. If the selected body is a beam, a support's force on the beam belongs on its diagram; the equal-and-opposite force of the beam on the support belongs on a different diagram. Drawing both on the beam would cancel a real external interaction. Internal forces between parts cancel only when both parts are included in the chosen system. A body-only diagram is therefore often clearer than a detailed sketch of the assembly: remove the surroundings and replace each interaction by its allowed force or couple.

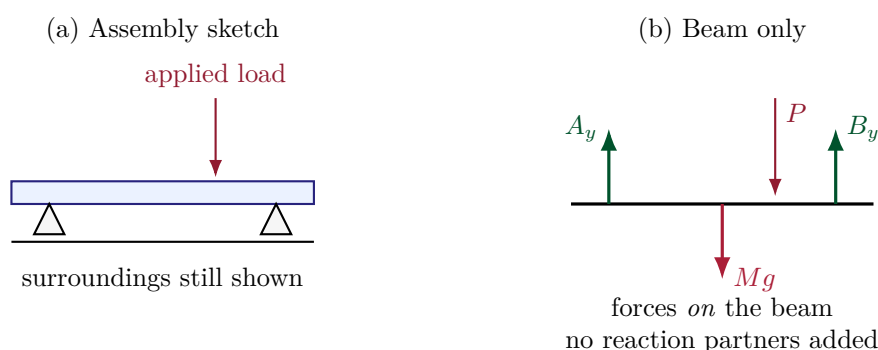


Figure 12.1: An assembly sketch and the free-body diagram of the isolated beam. The weight and support reactions are external to the beam system. Force lengths are schematic; the pin's horizontal component is omitted here only because this introductory loading has no horizontal action.

This habit also prevents double counting gravity. One can retain the body's distributed weight or replace it by its single resultant, but should not include both. The same rule applies when a distributed external load is replaced by a resultant. The reduced diagram represents the same total force and torque; it does not add a new physical load.

12.1.2 A couple can rotate without translating

Let $+\mathbf{F}$ act at $\mathbf{r}_1$ and $-\mathbf{F}$ at $\mathbf{r}_2$. Their vector sum is zero, whereas their torque is

$$\boldsymbol{\tau}_O = \mathbf{r}_1 \times \mathbf{F} - \mathbf{r}_2 \times \mathbf{F} = (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{F}. \quad (12.3)$$

Only the relative separation appears. A *couple* therefore has an origin-independent torque of magnitude Fd , where d is the perpendicular separation of the parallel force lines. If the forces are collinear, $d = 0$. This distinction is why a wrench turns a fitting even when the applied forces have zero resultant.

Example 12.1: A bar under a pure couple

An initially stationary rigid bar is acted on by two forces, each of magnitude 120 N. Compare an equal-and-opposite collinear pair with an equal-and-opposite pair whose parallel lines of action are separated by 0.60 m. Describe both translation and rotation.

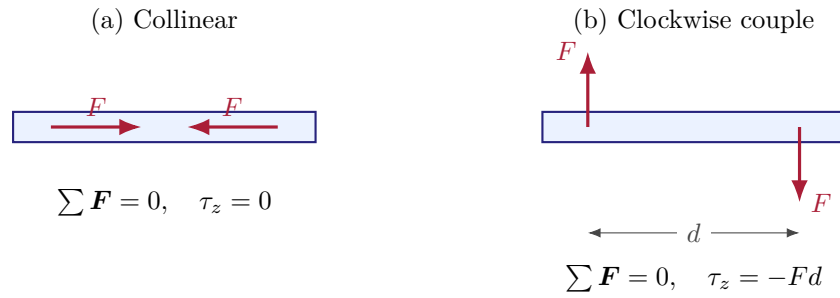


Figure 12.2: Equal and opposite forces need not balance moments. In panel (b), each force gives a clockwise moment about the centre. The distance d measures the separation of their vertical lines of action.

Model and strategy. Isolate the bar and retain these two external forces. Its centre-of-mass acceleration depends on their sum; its rotational tendency depends on their moment. For the collinear pair, the force sum and torque are both zero, so the pair changes neither motion in the rigid-body idealisation. Internal stress can still exist, which is a different question. For the separated pair, the resultant force is again zero, but the couple magnitude is

$$|\tau_z| = Fd = (120)(0.60) = 72.0 \text{ N m}.$$

The arrangement in Fig. 12.2(b) gives $\tau_z = -72.0 \text{ N m}$. The bar's centre of mass does not accelerate because of this pair, while an unconstrained bar has angular acceleration. Its numerical angular acceleration cannot be found without the appropriate moment of inertia.

Independent check. About either application point one force has zero lever arm and the other has lever arm 0.60 m . About the midpoint each has lever arm 0.30 m and the moments add: $-120(0.30) - 120(0.30) = -72.0 \text{ N m}$. Thus both choices give the same couple.

12.1.3 Why the torque origin may be chosen freely

The freedom to choose a convenient origin has a precise qualification. Let O' be displaced from O by $\mathbf{a}$, so every position changes from $\mathbf{r}_i$ to $\mathbf{r}'_i = \mathbf{r}_i - \mathbf{a}$. Summing moments gives

$$\tau_{O'} = \sum_i (\mathbf{r}_i - \mathbf{a}) \times \mathbf{F}_i + \sum_j \mathbf{C}_j \quad (12.4)$$

$$= \tau_O - \mathbf{a} \times \sum_i \mathbf{F}_i. \quad (12.5)$$

Here $\mathbf{C}_j$ denotes any applied couple, already independent of origin. If the force sum vanishes, both torque sums are identical. If it does not vanish, the torque sum generally changes. Consequently, changing origins is legitimate in equilibrium; setting moments to zero about one point does not by itself establish force balance.

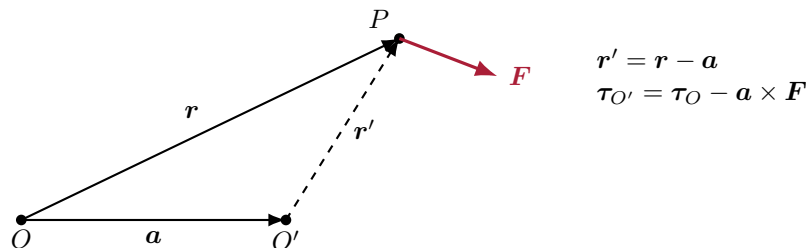


Figure 12.3: Translating the moment origin changes the position vector, not the actual force or its application point. For many forces, replace the single $\mathbf{F}$ in the identity by the resultant external force.

Example 12.2: A misleading zero-moment argument

A horizontal rigid member receives a 100 N downward force at $x = 1.00$ m and a 60.0 N upward force at $x = 3.00$ m. Coordinates refer to its left reference point. Find the resultant force and moments about $x = 0$ and $x = 2.00$ m. Determine where a single additional vertical force would have to act to balance both.

The two existing forces give $\sum F_y = -100 + 60 = -40.0$ N. With counterclockwise positive,

$$\tau_0 = -100(1.00) + 60(3.00) = 80.0 \text{ N m}.$$

About $x = 2.00$ m the first force lies to the left and the second to the right:

$$\tau_2 = (-1.00)(-100) + (1.00)(60) = 160 \text{ N m}.$$

The translation identity checks the sign: $\tau_2 = \tau_0 - (2.00)(-40.0) = 160 \text{ N m}$. A moment origin has not changed the external loading.

Force balance requires an additional 40.0 N upward force. If it acts at $x = a$, moment balance about zero requires $80.0 + 40.0a = 0$, hence $a = -2.00$ m. A contact restricted to $0 \leq x \leq 4.00$ m cannot supply that single-force replacement. The result identifies a geometry problem, not an algebra problem: a support outside that interval, an additional couple, or a changed loading arrangement is needed.

As a final check, the new force has lever arm -4.00 m about $x = 2.00$ m and contributes -160 N m , cancelling the previously calculated moment.

Guided checks

Diagnostic X12-01. Does zero resultant force prove static equilibrium? No: a couple can still produce angular acceleration. Even when force and torque both vanish, a body already moving need not be at rest. The balance conditions and the initial state answer different parts of the question.

12.2 Centre of gravity, contact and tipping

A body's weight acts throughout its volume. In a uniform gravitational field the distributed action has the same external force and torque as one weight acting at the centre of mass. For particles m_i with total mass M ,

$$\mathbf{r}_{\text{cm}} = \frac{\sum_i m_i \mathbf{r}_i}{M}, \quad \sum_i \mathbf{r}_i \times m_i \mathbf{g} = M \mathbf{r}_{\text{cm}} \times \mathbf{g}. \quad (12.6)$$

For a continuous body, replace the sum by $M^{-1} \int \mathbf{r} dm$. The common vector $\mathbf{g}$ can be taken outside the sum only because the field is uniform across the body. This gives the coincidence of centre of mass and centre of gravity in the present near-surface model. A sufficiently nonuniform gravitational field requires the actual distributed forces; that distinction will become useful in Chapter 13.

The location of the mass centre depends on the mass distribution, not on how much total mass is present alone. A heavy component placed near an edge can matter more for tipping than a lighter component near the centre. The same statement underlies balancing a spacecraft's instrument platform before mounting it, although an actual spacecraft load environment requires additional forces and constraints. Here the teaching example uses a laboratory platform in ordinary gravity.

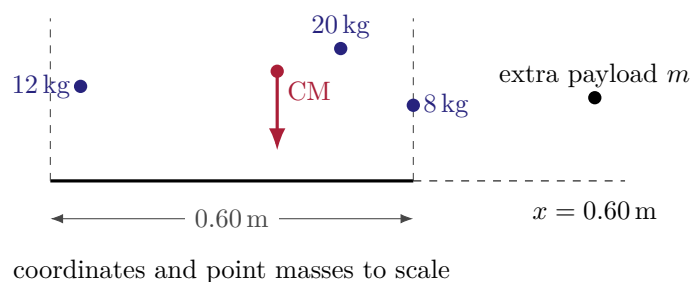


Figure 12.4: The three stipulated component masses determine the mass centre of a rigid platform. The support interval extends from -0.30 to $+0.30$ m. The overhanging payload changes the centre of mass; it is assumed attached rigidly, with its height irrelevant to the horizontal-platform threshold.

Example 12.3: Balancing a composite instrument platform

A rigid platform of negligible mass carries point-like components: 12.0 kg at $(x, y) = (-0.250, 0.250)\text{ m}$, 20.0 kg at $(0.180, 0.350)\text{ m}$ and 8.00 kg at $(0.300, 0.200)\text{ m}$. Its horizontal support interval is $-0.300 \leq x \leq 0.300\text{ m}$. Find the initial centre of mass and the greatest added payload at $x = 0.600\text{ m}$ consistent with the gravity-line criterion. Assume the platform and attachment remain rigid.

The total initial mass is 40.0 kg . Applying the weighted mean separately to the coordinates gives

$$x_{\text{cm}} = \frac{-3.00 + 3.60 + 2.40}{40.0} = 0.0750\text{ m}, \quad y_{\text{cm}} = \frac{3.00 + 7.00 + 1.60}{40.0} = 0.290\text{ m}.$$

The vertical weight line lies inside the support interval. The initial normal resultant can therefore act directly below this mass centre. There is no horizontal force, so friction is not required in this ideal horizontal configuration.

Adding mass m at $x = 0.600\text{ m}$ changes the horizontal coordinate to $(3.00 + 0.600m)/(40.0 + m)$. At the limiting right-hand edge it equals 0.300 m :

$$3.00 + 0.600m = 0.300(40.0 + m), \quad m = 30.0\text{ kg}.$$

This is a threshold, with no margin against perturbation. For smaller positive m the line remains inside; for larger m it lies outside. The payload's height does not enter this particular horizontal, gravity-only criterion, but would matter under a platform tilt or an applied horizontal force.

Independent check. About the right edge, the initial components supply moments proportional to $12(0.550) + 20(0.120) + 8(0) = 9.00\text{ kg m}$. The payload supplies the opposite moment $30(0.300) = 9.00\text{ kg m}$. The common factor g cancels.

12.2.1 A support region is also a physical constraint

A nonadhesive contact can exert compression, not tension. Its normal pressure is nonnegative over the contact area, so the resultant normal force must act within the convex support region. In a two-dimensional section that region is an interval. Under gravity alone on a horizontal surface, force balance makes the total support resultant vertical; moment balance requires its line to coincide with the vertical through the mass centre. When that line reaches an edge, the opposite side has no remaining normal loading. Beyond the edge, the old contact region cannot provide the required moment without pulling.

On a tilted plane, friction gives the support resultant a tangential component. It is then clearer to work in coordinates tied to the plane: s is positive uphill and n is normal outward. The mass centre of a symmetric housing lies at $s = 0, n = h$; the contact interval is $-b/2 \leq s \leq b/2$. Gravity is still

vertically downward in the world frame. Its components in the rotating plane frame are $-Mg \sin \theta$ along s and $-Mg \cos \theta$ along n .

Let the normal resultant act at $s = q$. Force balance requires $N = Mg \cos \theta$ and uphill friction $f = Mg \sin \theta$. Taking moments about the centre of the base gives $qN + hMg \sin \theta = 0$, hence

$$q = -h \tan \theta, \quad |q| \leq \frac{b}{2}, \quad |f| \leq \mu_s N. \quad (12.7)$$

The required normal contact moves downhill. Its limiting position and the friction inequality are separate tests, giving

$$\tan \theta_{\text{tip}} = \frac{b}{2h}, \quad \tan \theta_{\text{slide}} = \mu_s. \quad (12.8)$$

These expressions assume a centred mass, a rigid base, no additional external force and quasi-static loading. They describe the loss of the original equilibrium, not the subsequent sliding or rotational motion.

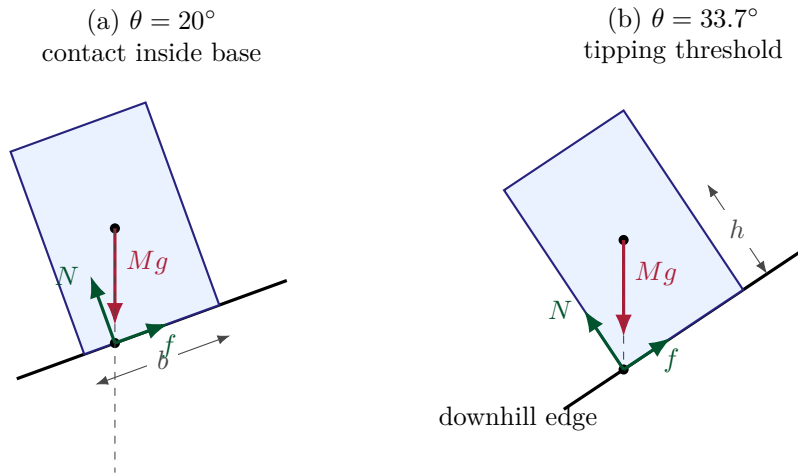


Figure 12.5: A housing tilts *with* its inclined platform. The weight arrows remain vertical in world coordinates. Geometry uses $b : h = 0.60 : 0.45$; the rectangle height is drawn as $2h$. At the threshold the vertical mass-centre line meets the downhill edge. Reaction arrows are schematic, with tangential friction acting uphill.

Example 12.4: A housing on a slowly tilted platform

A rectangular instrument housing has support width $b = 0.60$ m and a centred mass at height $h = 0.45$ m above its base. The whole platform is slowly tilted through angle θ . The stipulated coefficient of static friction is $\mu_s = 0.80$. Which ideal limit occurs first, tipping or sliding? Check the friction requirement at the first limit.

The relevant system is the housing; gravity and the platform's contact forces are its external actions. No fastener, cable or adhesion is included. From the edge-contact condition,

$$\theta_{\text{tip}} = \tan^{-1} \left(\frac{0.60}{2(0.45)} \right) = 33.7^\circ.$$

From the friction condition,

$$\theta_{\text{slide}} = \tan^{-1}(0.80) = 38.7^\circ.$$

The tipping threshold is reached first. At this angle $f/N = \tan \theta = 2/3 = 0.667$, below the stipulated maximum 0.80. Thus friction can still provide the required tangential force when the normal contact reaches the edge.

The geometry can be checked without plane components. The world-horizontal displacement of the mass centre from the downhill edge is $(b/2) \cos \theta - h \sin \theta$. It vanishes at $\tan \theta = b/(2h)$,

reproducing the same result. The criterion does not depend on mass because both the restoring and overturning gravitational moments contain the same factor Mg . These angles are consequences of the stated ideal model. They do not include vibration, compliance, anchoring, an off-centre mass or strength limits. A more massive geometrically identical housing would have the same ideal angles, while relocating its mass centre would change them.

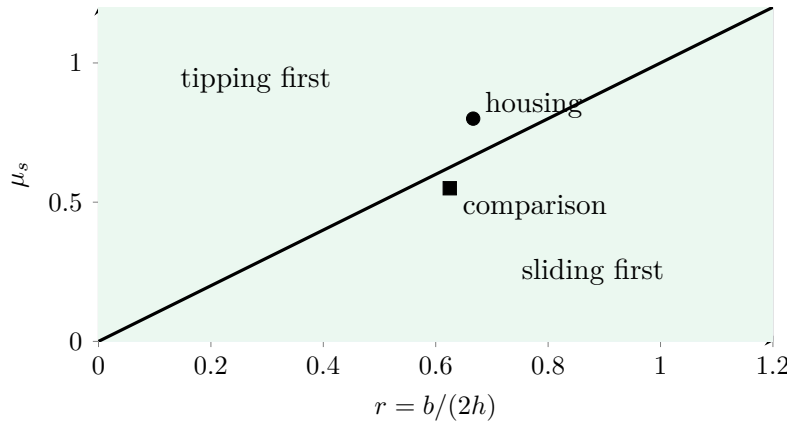


Figure 12.6: The smaller of r and μ_s determines the first quasi-static loss of equilibrium on a gradually tilted plane. Above the diagonal, $r < \mu_s$ and tipping occurs first; below it sliding occurs first. On the diagonal the two ideal thresholds coincide. Both axes are dimensionless.

12.2.2 A horizontal push is a different loading path

A body on a horizontal surface may tip because a horizontal force creates an overturning moment. Its gravity line need not move before tipping; instead, the required support resultant moves. Let a force P act horizontally to the right at height z_P , with weight $W = Mg$ and centred base width b . Force balance gives $N = W$ and leftward friction of magnitude P . Moment balance about the base centre gives $qW = Pz_P$, where q is now the rightward location of the normal resultant. Therefore

$$P_{\text{tip}} = \frac{Wb}{2z_P}, \quad P_{\text{slide}} = \mu_s W. \quad (12.9)$$

The relevant height is the height of the push, not automatically the mass-centre height. The two loading paths, platform tilt and horizontal push, should not be combined by substituting symbols without re-deriving their moments.

Example 12.5: Changing the height of an applied push

A 40.0 kg housing rests on a horizontal surface. Its base width along the push is 0.400 m, its mass centre is centred, and $\mu_s = 0.450$. A horizontal force is increased slowly, first at height 0.600 m and then, in a separate trial, at height 0.200 m. Determine the first ideal failure of the contact in each trial. Use $g = 9.81 \text{ m s}^{-2}$.

The weight is $W = 392.4 \text{ N}$; the sliding threshold in either trial is $0.450(392.4) = 176.6 \text{ N}$. At the greater height,

$$P_{\text{tip}} = \frac{392.4(0.400)}{2(0.600)} = 130.8 \text{ N}.$$

Tipping occurs first. The required friction ratio there is $130.8/392.4 = 0.333$, within the available 0.450.

At the lower height, $P_{\text{tip}} = 392.4 \text{ N}$, exceeding the sliding threshold. Sliding therefore occurs first, at 176.6 N . The normal resultant at that instant is $q = Pz_P/W = 0.0900 \text{ m}$ from the centre, still inside the half-width 0.200 m .

The crossover height follows from $Wb/(2z_P) = \mu_s W$: $z_P = b/(2\mu_s) = 0.444 \text{ m}$. A push above that height favours tipping first; one below it favours sliding first. A limit check supports the result: a push applied arbitrarily close to the base has negligible overturning moment, so its ideal tipping threshold grows without bound, although sliding remains possible.

Guided checks

Diagnostic X12-02. If a housing has not slipped, is it necessarily protected against tipping? No. The friction inequality controls tangential-force feasibility; the location of the normal resultant controls the moment feasibility of the contact. Either constraint can be the first one exhausted.

12.3 Support models and distributed loading

A reaction is the force or couple that the surroundings exert to enforce an ideal constraint. Its direction follows from the constraint, rather than from a memorised arrow. A smooth roller on a horizontal surface has one vertical reaction. On an inclined surface the reaction is normal to that surface. A two-dimensional ideal pin supplies two force components but no moment because it allows rotation. A cable supplies tension along its own direction, with $T \geq 0$. A fixed clamp can supply two force components and one couple moment.

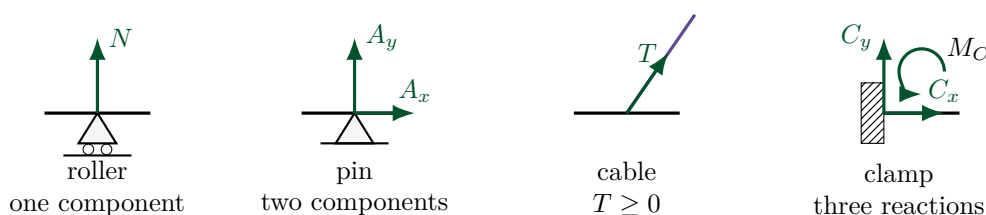


Figure 12.7: Ideal planar support models. The arrows show positive unknown directions, not guarantees of the final signs. A normal contact cannot pull, a cable cannot push, and a pin or clamp may exert either sign of force if its anchorage permits it. A clamp also supplies a reaction couple.

Counting unknowns is a useful preliminary test, but equation independence and support geometry matter too. One planar rigid body provides at most three independent scalar equilibrium equations. A cantilever with one clamp has three reaction unknowns and can be statically determinate. A beam with one pin and one noncollinear roller usually also has three. Two pins commonly introduce four force unknowns: balance alone leaves one redundant reaction. The needed additional relation can come from deformation compatibility. A deficient arrangement, such as reactions all acting along one line while an unbalanced couple is present, may fail to constrain a motion even though the simple unknown count looks promising.

12.3.1 Replacing a distributed load by its resultant

Let a downward load intensity $w(x)$ act along a horizontal member. Its units are N m^{-1} , so a short interval dx carries force $w(x)dx$. For a nonnegative load with total $W > 0$ over $a \leq x \leq b$, force and moment preservation require

$$W = \int_a^b w(x) dx, \quad \bar{x} = \frac{\int_a^b xw(x) dx}{W}. \quad (12.10)$$

The single downward force W at $\bar{x}$ is equivalent for *external rigid-body balance*. The integral is the continuum counterpart of the mass-weighted mean. A graph of $w(x)$ provides the total load from its area, but locating that area's centroid requires its first moment, not merely finding the visually highest part of the graph.

For uniform w over $0 \leq x \leq L$, $W = wL$ and $\bar{x} = L/2$. The replacement should not be used to claim identical bending or deformation throughout the member: local loading information has been removed. Such calculations require an internal-force and deformation model beyond this chapter's global statics treatment.

Method

Choose the body and frame; replace each support by its permitted reactions; retain all external loads with their actual lines of action. Reduce distributed loads only when total force and moment are the target. Choose a useful moment origin, solve the independent equations, and then check a second origin, dimensions and physical admissibility. A negative unknown is a direction correction for a bilateral constraint, but can invalidate a unilateral contact.

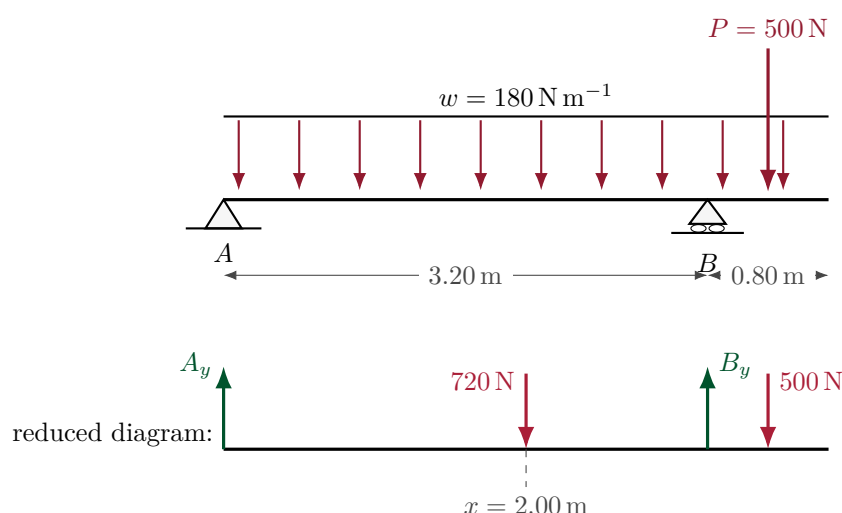


Figure 12.8: The sensor beam and its load-resultant diagram. Distances share the coordinate origin at A . The upper load arrows and the lower 720 N resultant belong to alternative representations, not simultaneous forces on one body. The vertical scale and force lengths are schematic.

Example 12.6: The sensor beam and a second moment origin

A 4.00 m horizontal beam is pinned at A , $x = 0$, and rests on a horizontal roller at B , $x = 3.20$ m. A uniform downward load 180 N m^{-1} acts over the entire beam, including the overhang. A 500 N downward instrument load acts at $x = 3.60$ m. Find all support reactions and verify the result about both supports.

The beam is one rigid body. The unknowns are A_x, A_y, B_y ; the roller supplies no horizontal force or couple. The uniform load is $W = (180)(4.00) = 720 \text{ N}$ at $x = 2.00$ m. Horizontal balance gives $A_x = 0$. Moments about A eliminate both pin components:

$$3.20B_y - 720(2.00) - 500(3.60) = 0, \quad B_y = 1012.5 \text{ N}.$$

Vertical balance gives $A_y = 720 + 500 - 1012.5 = 207.5 \text{ N}$. The roller reaction is positive, so its assumed contact is admissible. The fact that B_y exceeds the instrument load alone is reasonable: it also supports much of the distributed load.

Now take moments about B . The pin is 3.20 m to its left, the uniform-load resultant is 1.20 m

to its left, and the instrument is 0.40 m to its right:

$$-3.20A_y + 720(1.20) - 500(0.40) = -664 + 864 - 200 = 0 \text{ N m.}$$

The vertical-force residual is $207.5 + 1012.5 - 720 - 500 = 0 \text{ N}$. Both residuals vanish before rounding. Replacing the moment origin has changed each lever arm but has not changed the equilibrium. Reporting reactions to 1 N gives $A_y \simeq 208 \text{ N}$ and $B_y \simeq 1013 \text{ N}$; retaining guard digits in the checks avoids artificial rounding residuals.

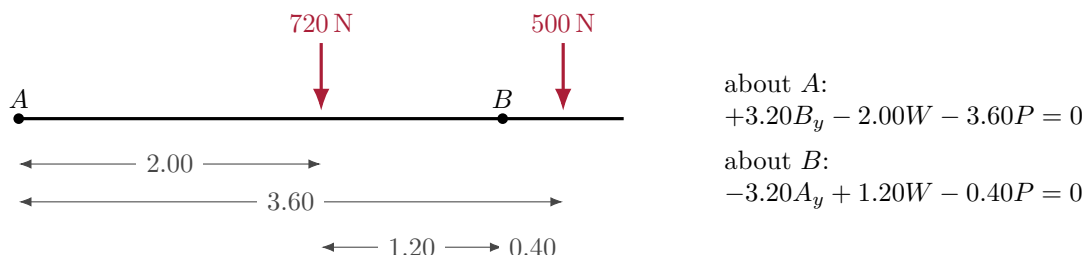


Figure 12.9: Two torque ledgers for one beam. All dimension labels are in metres. A downward force to the left of an origin makes a counterclockwise moment; the same downward force to its right makes a clockwise moment. The algebra preserves these signs.

12.3.2 A triangular load needs its own centroid

For an intensity increasing linearly from zero at the left to w_0 at the right,

$$w(x) = w_0 \frac{x}{L}, \quad 0 \leq x \leq L.$$

Integration gives $W = w_0 L/2$. Its first moment is $\int_0^L x(w_0 x/L) dx = w_0 L^2/3$, so

$$\bar{x} = \frac{w_0 L^2/3}{w_0 L/2} = \frac{2L}{3}. \quad (12.11)$$

The resultant lies closer to the larger-load end. For a triangle decreasing from left to right, the centroid is $L/3$ from the left. The two cases have equal total loads but different torques. A midpoint placement preserves the force and loses the moment, so it can give plausible-looking but incorrect reactions.

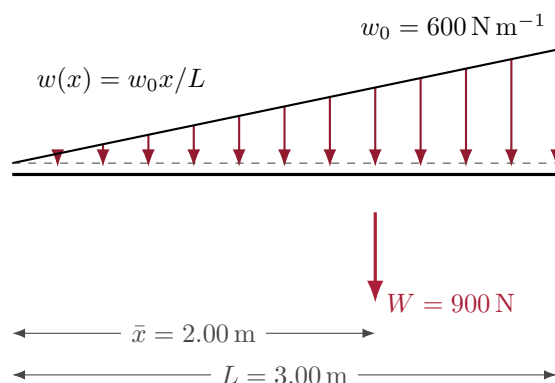


Figure 12.10: A linearly increasing load is represented in global balance by its triangular-area resultant at $2L/3$ from the zero-intensity end. Load-intensity height and force-arrow length use separate schematic vertical scales; the horizontal coordinates are quantitative.

Example 12.7: A nonuniform load on a short support member

A 3.00 m horizontal member is pinned at its left end and rests on a roller at its right end. A downward distributed load increases linearly from zero to 600 N m^{-1} across the full length. An additional 300 N point load acts 0.500 m from the left. Determine the reactions and quantify the error from placing the triangular-load resultant at the midpoint.

The distributed force is $W = \frac{1}{2}(600)(3.00) = 900 \text{ N}$ at $\bar{x} = 2.00 \text{ m}$. With $A_x = 0$, moments about the left end give

$$3.00B_y - 900(2.00) - 300(0.500) = 0, \quad B_y = 650 \text{ N}.$$

Then $A_y = 900 + 300 - 650 = 550 \text{ N}$. Both vertical reactions are admissible. About the right end the independent moment check is $-550(3.00) + 900(1.00) + 300(2.50) = 0 \text{ N m}$.

Using the midpoint would instead give $B_y = [900(1.50) + 300(0.500)]/3.00 = 500 \text{ N}$ and $A_y = 700 \text{ N}$. Those mistaken values preserve total vertical force but give an actual moment residual $3(500) - 900(2) - 300(.5) = -450 \text{ N m}$ about the left support. The centroid error is therefore exposed by the true load's moment, even though both erroneous reactions are positive.

12.3.3 A reaction sign can mark loss of contact

The force direction selected on a diagram is initially a sign convention. Its interpretation depends on the support. An anchored ideal pin may supply a downward force; an ordinary contact pad under a beam cannot. It is therefore misleading to reject all negative reactions or to accept all of them. First identify the constraint, then test the calculated sign against it.

Example 12.8: Finding the lift-off load

Use the 4.00 m beam and 180 N m^{-1} full-length load from Example 12.6, but change the supports: A is now a nonadhesive upward-only pad at $x = 0$, while B is an anchored pin at $x = 3.20 \text{ m}$. A variable downward load $P \geq 0$ acts at $x = 3.60 \text{ m}$. Find the admissible load interval and assess $P = 2500 \text{ N}$.

Moments about A and vertical balance give

$$B_y = \frac{720(2.00) + 3.60P}{3.20} = 450 + 1.125P, \quad A_y = 720 + P - B_y = 270 - 0.125P,$$

where forces are in newtons. The absence of horizontal load gives $B_x = 0$. Since the pad at A must have $A_y \geq 0$, the assumed two-support contact remains possible only for $0 \leq P \leq 2160 \text{ N}$. At $P = 2160 \text{ N}$, $A_y = 0$ and $B_y = 2880 \text{ N}$. The pad is just unloaded. At $P = 2500 \text{ N}$ the formal solution has $A_y = -42.5 \text{ N}$ and $B_y = 3262.5 \text{ N}$. The pad cannot provide that downward pull. The former contact configuration fails; merely reversing its arrow would assert a different support. The subsequent motion or new equilibrium requires another model, such as rotation about B and possible contact elsewhere.

The threshold can be checked directly about B : the uniform load gives $720(1.20) = 864 \text{ N m}$ counterclockwise, while the point load gives $P(0.40)$ clockwise. Equality yields $P = 2160 \text{ N}$. If A were the anchored pin of the original example, a negative A_y could instead be admissible; the threshold found here belongs specifically to the revised pad model.

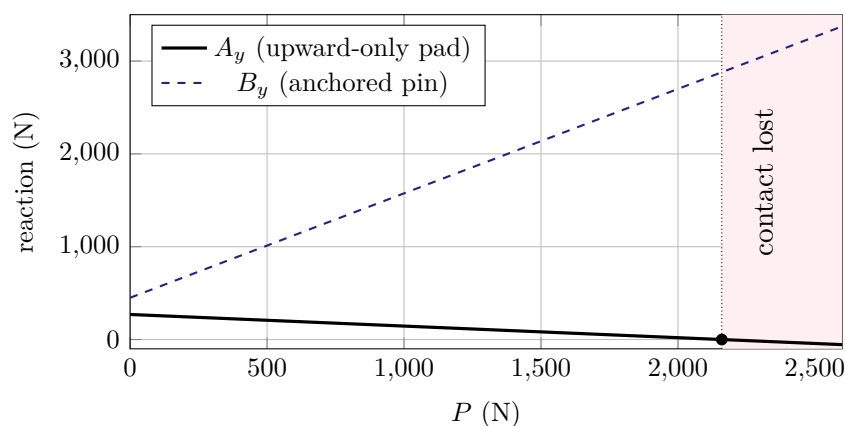


Figure 12.11: The reaction envelope for the support configuration in Example 12.8. The dotted line at 2160 N marks unloading of the pad. The algebraic lines beyond it describe the old constraints, whose pad reaction is physically impossible. Solid and dashed curves distinguish the two supports in grayscale.

12.4 Cables and ladders: geometry controls the reactions

A cable's force direction is known from its geometry even when its magnitude is unknown. Its tension can therefore be resolved into components before solving equilibrium. If a cable is nearly horizontal, only a small fraction of its tension is vertical. A large tension may then be needed to supply a modest supporting vertical force. Treating the full tension as vertical loses precisely the geometric effect that a cable problem is intended to teach.

Example 12.9: The cable-supported beam variation

A horizontal 4.00 m beam is pinned at its left end. Its weight is 400 N at $x = 2.00$ m, and an instrument of weight 600 N is attached at $x = 3.20$ m. A cable attached at the right end runs *upward and to the right* toward an external anchor, making 35.0° with the beam. Find its tension and the pin components. Then determine the smallest cable angle allowed by a stipulated tension ceiling of 2000 N, retaining the same beam geometry.

The beam, instrument load and cable attachment are represented by external forces on the rigid beam. Counterclockwise moments about the pin give

$$4.00T \sin \alpha - 400(2.00) - 600(3.20) = 0, \quad T \sin \alpha = 680 \text{ N}.$$

For $\alpha = 35.0^\circ$, $T = 680 / \sin 35.0^\circ = 1185.5$ N, approximately 1.19 kN. The rightward horizontal component is $T \cos 35.0^\circ = 971.1$ N. Force balance therefore gives

$$A_x = -971.1 \text{ N}, \quad A_y = 400 + 600 - 680 = 320 \text{ N}.$$

The horizontal sign follows from the specified anchor on the right. A cable running up-left would require an equal-magnitude rightward pin component. Both have the same vertical component and moment about the pin.

For the proposed ceiling, $680 / \sin \alpha \leq 2000$, giving $\sin \alpha \geq 0.340$. In the stated range $0 < \alpha \leq 90^\circ$, the smallest angle is 19.9° . As $\alpha \rightarrow 90^\circ$, $T \rightarrow 680$ N and $A_x \rightarrow 0$; as $\alpha \rightarrow 0^+$, both tension and horizontal pin force diverge. These limits independently check the angle dependence. The tension ceiling is a stipulated model input, not a derived material strength rating.

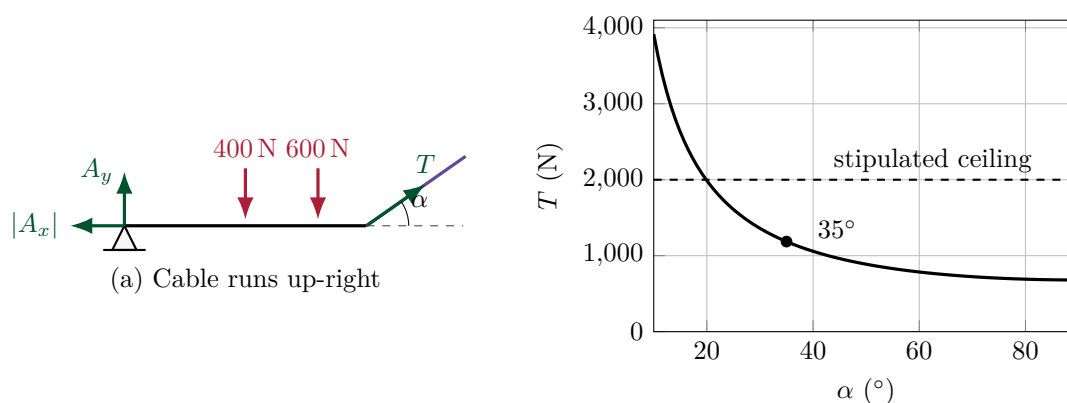


Figure 12.12: Cable direction and the quantitative angle sensitivity for the stipulated beam. The curve is $T = 680/\sin \alpha$, with angle evaluated in degrees. The dashed ceiling intersects at 19.9° ; the filled marker gives the 35° example. Lengths in the force sketch are schematic except for the stated cable angle.

12.4.1 A ladder is a contact problem, not just a moment calculation

Consider a rigid ladder with its foot on a rough horizontal floor and its top against a smooth vertical wall. Let its length be L and its angle above the floor be θ , with the wall to the right. The wall exerts a leftward normal force N_w and no tangential force. The floor exerts upward normal force N_f and horizontal friction f . A uniform ladder's weight W_L acts at distance $L/2$ along it; a climber or attached load of weight W_P acts at distance s from the foot.

Vertical balance gives $N_f = W_L + W_P$. Horizontal balance gives $f = N_w$ directed rightward, toward the wall. Moments about the foot eliminate both floor components:

$$N_w L \sin \theta - W_L \frac{L}{2} \cos \theta - W_P s \cos \theta = 0.$$

Thus

$$N_w = \left(\frac{W_L}{2} + \frac{W_P s}{L} \right) \cot \theta, \quad \mu_{s,\min} = \frac{N_w}{W_L + W_P}. \quad (12.12)$$

Friction is not automatically $\mu_s N_f$: it equals the force required for balance until the limit is reached. Both normal forces must also be nonnegative. For positive weights, $0 < \theta < 90^\circ$ and $0 \leq s \leq L$, those signs are correct. Reducing θ increases the wall force and friction demand. At $\theta \rightarrow 0$ the demand diverges, indicating failure of this two-contact arrangement before a horizontal-ladder limit is reached.

Example 12.10: A loaded ladder and the friction threshold

A uniform rigid ladder of length 5.00 m and weight 200 N rests at 60.0° on a rough horizontal floor against a smooth vertical wall. A 600 N load is located 3.50 m along the ladder from its foot. The floor has stipulated $\mu_s = 0.400$. Find all contact forces, test equilibrium, find the minimum permissible angle for this load position, and find the limiting load position at 60.0° . Vertical balance gives $N_f = 800$ N. The moment equation gives

$$N_w = \left(100 + \frac{600(3.50)}{5.00} \right) \cot 60.0^\circ = 300.2 \text{ N}.$$

The floor friction is 300.2 N rightward. Its maximum available value is $0.400(800) = 320$ N, so static balance is feasible. The required minimum coefficient is $300.2/800 = 0.375$. The force signs also pass: both contacts push on the ladder.

At fixed $s = 3.50$ m, the limiting angle satisfies $520 \cot \theta = 320$, hence $\tan \theta = 520/320$ and $\theta_{\min} = 58.4^\circ$. Angles above this threshold demand less friction in the same ideal geometry.

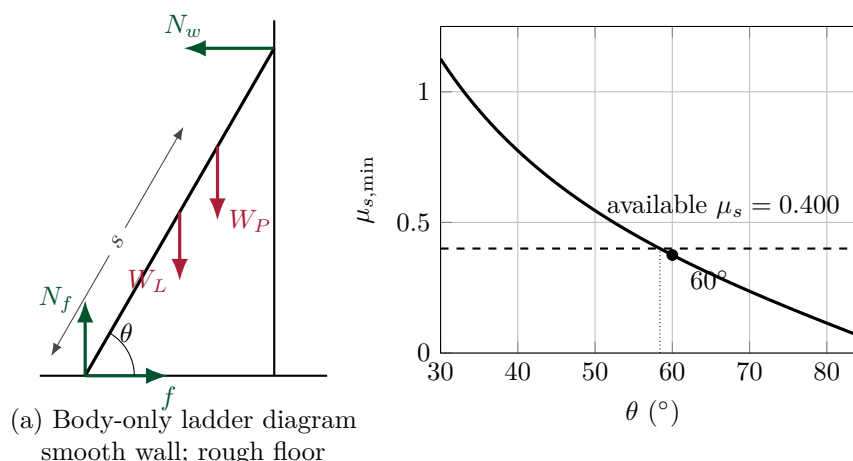


Figure 12.13: The specified loaded ladder and its friction requirement as a function of angle. The forces are all exerted *on* the ladder. The plotted demand is $\mu_{s,\min} = 0.65 \cot \theta$ for the fixed 3.50 m load position. Its intersection with the dashed available coefficient sets the contact threshold. The drawing uses the quantitative 60° angle; force lengths are schematic.

At fixed 60.0° , set the required wall force equal to 320 N and solve for the position:

$$s_{\max} = \frac{L}{W_P} \left[\mu_s (W_L + W_P) \tan \theta - \frac{W_L}{2} \right] = 3.79 \text{ m.}$$

This lies within the ladder length and exceeds the stated 3.50 m position. About the top contact, the floor friction contributes $fL \sin \theta$, the floor normal contributes $-N_f L \cos \theta$, and the weights contribute $W_L(L/2) \cos \theta + W_P(L - s) \cos \theta$. Substitution gives zero moment, independently checking the solution's force directions. This model assesses contact feasibility; it does not assess member strength, footing deformation or a safe climbing procedure.

Guided checks

Diagnostic X12-03. A balance calculation gives -200 N for an upward-only roller. Is this merely an instruction to reverse the arrow? No. A reversed force would pull the body toward the support; a nonadhesive roller cannot do that. The assumed contact is lost. For an anchored pin, however, a negative component may simply identify the opposite force direction.

Key message

Three tests complete a static analysis: the chosen body's external forces balance, their moments balance, and the calculated reactions belong to the support model. Equilibrium is not by itself a strength or stability certificate. Elasticity supplies the next layer: how the same forces deform the body, and when those deformations affect the reaction calculation.

12.5 Stress, strain and Young's modulus

12.5.1 Why a balanced body can still deform

The earlier rigid-body model deliberately suppressed changes of shape. Its equilibrium equations determine allowable force resultants, but do not tell us how much a body stretches under those forces. A tensile test makes the distinction clear. Opposite end forces can have zero total force and zero

total moment while separating the ends of a specimen. Zero resultant force prevents acceleration of the centre of mass; it does not mean that every part is free of internal force. Elasticity supplies the additional relation between internal loading and deformation.

Imagine cutting a straight bar across a section perpendicular to its axis and retaining the right-hand portion as the body of interest. The material removed on the left formerly exerted forces across that section. Those forces must now appear on the isolated portion's free-body diagram. Let their axial resultant have magnitude N when the bar is in tension. The cut force points left on the right-hand portion and balances the applied end force pointing right. On the left-hand portion the cut force points right. These two cut forces are an action–reaction pair on different bodies. They are not two extra forces to add to the intact bar.

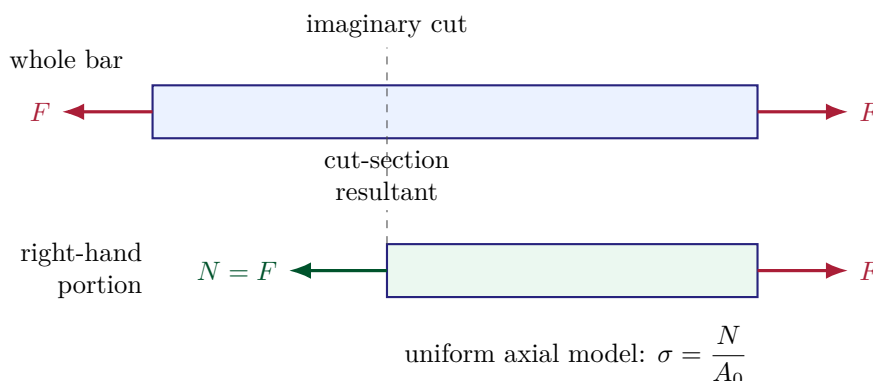


Figure 12.14: An internal force becomes an external force when a portion of the bar is isolated. In the stated uniform axial model, each cut transmits the same tensile resultant. Arrow lengths and specimen dimensions are schematic.

Stress describes how force is distributed over a section. In the uniform axial idealisation the average signed normal stress is

$$\sigma = \frac{N}{A_0}, \quad 1 \text{ Pa} = 1 \text{ N m}^{-2}, \quad (12.13)$$

where A_0 is the reference cross-sectional area. We assign $N > 0$ to tension and $N < 0$ to compression. This sign describes the internal loading mode; it is not simply the sign of the force's global x component. For example, the leftward cut force on the right-hand portion in Fig. 12.14 represents positive tensile stress. A consistent cut-body convention is therefore essential.

More generally, if normal stress varies over a section, the axial resultant is $N = \int_A \sigma \, dA$. Dividing by area gives its average, not its value at every point. Our uniform model is suitable for a straight member carrying an approximately central axial load, away from clamps, abrupt area changes and load-application regions. An eccentric force adds a bending moment; local stress near a notch requires information absent from this average-stress model. The same qualification matters in compression: the axial calculation does not decide whether a slender member remains straight.

12.5.2 Signed deformation and a constitutive relation

Let L_0 denote a stress-free reference length at the stated temperature and let L be the loaded length measured between the same material marks. The signed extension and engineering axial strain are

$$\delta = \Delta L = L - L_0, \quad \varepsilon = \frac{\Delta L}{L_0}. \quad (12.14)$$

Strain is dimensionless because both lengths use the same unit. Tension gives $\delta > 0$ and $\varepsilon > 0$; compression gives negative values. Percent strain is $100\varepsilon\%$. Thus $\varepsilon = 0.001$ means a change of one part in a thousand, or 0.1%, not a strain of one millimetre independent of gauge length.

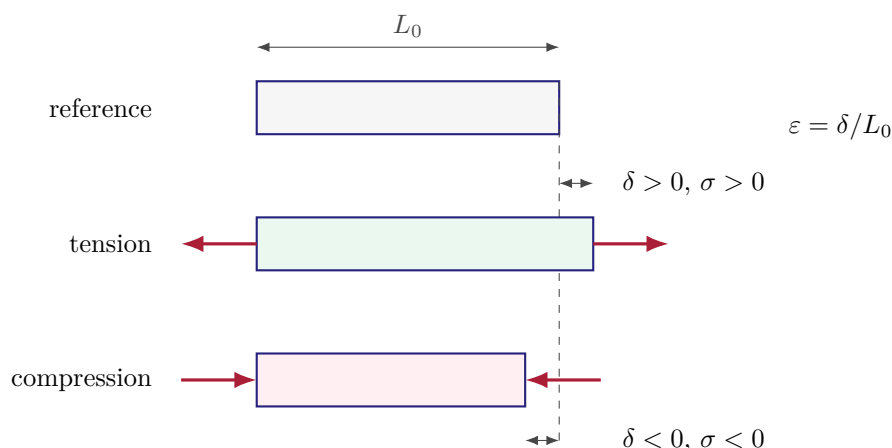


Figure 12.15: Tension-positive conventions for normal stress and engineering strain. Reference marks define L_0 before loading. The visible length changes are exaggerated; no lateral deformation model is implied by the rectangular silhouettes.

The definitions of stress and strain do not by themselves predict material behaviour. For approximately uniform, sufficiently small axial strain in a linear elastic range we adopt

$$\sigma = E_Y \varepsilon, \quad E_Y > 0. \quad (12.15)$$

Young's modulus E_Y is the slope of the stress–strain line in that range. It has the same unit as stress. At fixed stress a larger modulus produces a smaller strain. At fixed strain it requires a larger stress. This is a constitutive model, inferred from material response under specified conditions, rather than a further equilibrium equation. A numerical modulus alone supplies no elastic-limit, strength or failure information.

Example 12.11: Compression with a tension-positive convention

A straight short specimen has reference length $L_0 = 0.250$ m, area $A_0 = 200$ mm² and stipulated $E_Y = 120$ GPa. Opposite central forces compress it with magnitude 18.0 kN. Neglect its weight and assume it remains straight in its small-strain linear elastic range. Find the signed stress, strain, length change and loaded length. Explain why equilibrium does not imply zero strain. **Model and strategy.** Isolate a cut portion to establish the axial loading, then apply the signed constitutive relation. Compression means $N = -18.0 \times 10^3$ N. The reference area is 200×10^{-6} m², so

$$\sigma = \frac{-18.0 \times 10^3}{200 \times 10^{-6}} = -90.0 \text{ MPa}, \quad \varepsilon = \frac{-90.0 \times 10^6}{120 \times 10^9} = -7.50 \times 10^{-4}.$$

The length change is $\delta = L_0 \varepsilon = -0.1875$ mm, hence $L = 249.8125$ mm, reported as approximately 249.81 mm at the displayed precision. The shortening magnitude is 0.188 mm.

Interpretation and check. The strain magnitude is 0.0750%, consistent with the small-deformation geometry assumed here. The two end forces cancel as external resultants, while internal compressive stress persists. Making the area twice as large at the same force would halve both stress and shortening. The calculation checks a stipulated straight axial model; it does not establish a buckling or strength limit.

12.5.3 Engineering measures and the small-strain boundary

Engineering strain always uses the original length in its denominator. An alternative accumulates each infinitesimal length increment relative to the current length:

$$\varepsilon_{\text{true}} = \int_{L_0}^L \frac{d\ell}{\ell} = \ln \frac{L}{L_0} = \ln(1 + \varepsilon), \quad \varepsilon > -1. \quad (12.16)$$

The integral follows directly from the antiderivative of $1/\ell$. It is a kinematic definition for uniform axial stretching, not a claim that a material obeys a particular stress law at large strain. Expanding the logarithm near zero gives $\varepsilon_{\text{true}} = \varepsilon - \varepsilon^2/2 + \dots$. Consequently the absolute difference starts at second order, while the fractional discrepancy is of order $|\varepsilon|/2$.

Engineering stress uses A_0 , whereas the current-area normal stress is $\sigma_{\text{true}} = N/A$. The two stresses differ if lateral deformation changes the area. Holding area constant is an approximation, not a general consequence of elasticity. In a uniform constant-volume stretch, $AL = A_0L_0$ and therefore $\sigma_{\text{true}} = (1 + \varepsilon)\sigma$. A compressible material needs its own lateral-response information instead. These distinctions prevent an unqualified continuation of the elementary $\sigma = E_Y\varepsilon$ line into a large-deformation regime.

Figure 12.16 quantifies a geometrical approximation without assigning a material or failure threshold. The small-strain approximation can be satisfactory while the linear constitutive law is already inadequate. Conversely a reversible material may undergo large deformation for which the reference-geometry formulas require refinement. The admissible interval must therefore be judged by both geometry and measured response, with an error tolerance appropriate to the question.

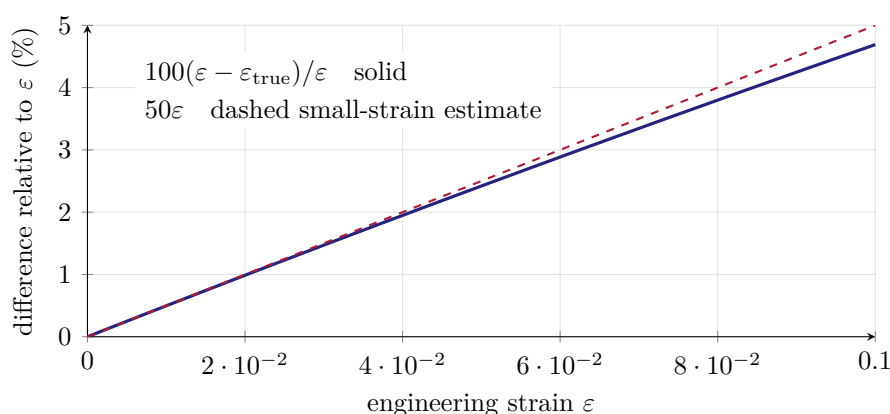


Figure 12.16: Analytical difference between uniform engineering and logarithmic strain in tension. The relative difference tends to zero as strain tends to zero. The plotted interval is a comparison of definitions, not a certified linear-elastic range for any material.

Example 12.12: How small is a strain-measure correction?

A uniform specimen stretches from 100.0 mm to 102.0 mm. Calculate its engineering and logarithmic strains and their relative difference. If its reference area is 20.0 mm^2 and the tensile force is 1.00 kN, find engineering stress. For this comparison only, assume volume remains constant and find current-area stress. Repeat the strain comparison at $\varepsilon = 0.001$ and explain what the comparisons establish.

Solution. The change is 2.0 mm, giving $\varepsilon = 0.0200$ and $\varepsilon_{\text{true}} = \ln(1.0200) = 0.0198026$. Relative to engineering strain, their difference is 0.9869%. This agrees with the approximately 1% ordinate at $\varepsilon = 0.02$ in Fig. 12.16. The stress based on the original area is

$$\sigma = \frac{1000}{20.0 \times 10^{-6}} = 50.0 \text{ MPa}.$$

Constant volume gives $A = A_0/1.02 = 19.6078 \text{ mm}^2$, so $\sigma_{\text{true}} = 51.0 \text{ MPa}$. The smaller current area increases stress by 2%.

At $\varepsilon = 0.001$, $\ln(1.001) = 0.0009995003$; the strain-definition discrepancy is only 0.04997%. Its leading estimate $100\varepsilon/2 = 0.05\%$ checks both scale and sign. These computations establish differences between stated measures. They provide no evidence that either loading level lies within the specimen's elastic or proportional range. That separate question requires response data.

12.5.4 Material modulus versus specimen stiffness

For a uniform prismatic specimen with constant N , insert the definitions into Eq. (12.15):

$$\frac{N}{A_0} = E_Y \frac{\delta}{L_0}, \quad \delta = \frac{NL_0}{E_Y A_0}, \quad k_{\text{axial}} = \frac{E_Y A_0}{L_0}. \quad (12.17)$$

Thus $N = k_{\text{axial}}\delta$. The positive stiffness has units Nm^{-1} because $E_Y A_0$ is a force. Its reciprocal $C = L_0/(E_Y A_0)$ is compliance, or displacement per unit force. Both tension and compression obey the signed relation within the declared linear model.

The stress–strain slope is a material property of this idealisation. The force–extension slope belongs to a particular specimen. Increasing area supplies more material in parallel to transmit the same load, while increasing length supplies more deformable material in series. Consequently a thicker specimen is stiffer and a longer one is more compliant. Geometry can outweigh a comparison based on modulus alone. This is why a measurement of force and extension determines a stiffness first; recovering a modulus also requires independently known gauge length and area.

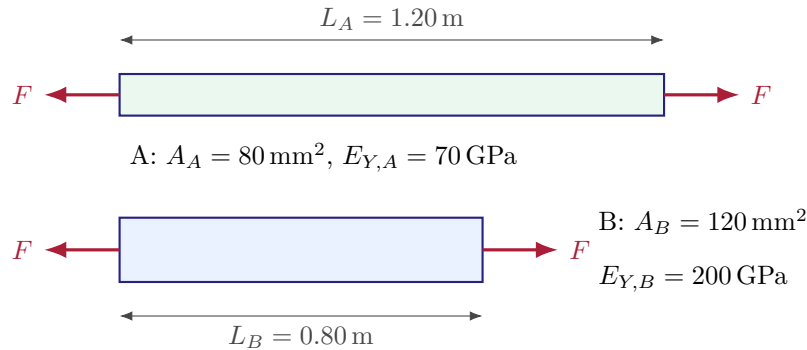


Figure 12.17: Two independent axial specimens with stipulated dimensions and moduli. Reference lengths have a common drawing scale; thickness is only a visual cue and does not represent cross-sectional area to scale. Each rod carries 12.0 kN tension independently.

Example 12.13: Two rods: comparing material and geometry

Rod A has $L_A = 1.20 \text{ m}$, $A_A = 80 \text{ mm}^2$ and stipulated $E_{Y,A} = 70 \text{ GPa}$. Rod B has $L_B = 0.80 \text{ m}$, $A_B = 120 \text{ mm}^2$ and stipulated $E_{Y,B} = 200 \text{ GPa}$. Each carries the same tensile load $F = 12.0 \text{ kN}$ in its small-strain linear elastic range. Compare stress, strain, extension and specimen stiffness. Explain the stiffness ratio and check the corresponding force–extension plot.

Strategy. The rods are independent uniform axial specimens. Their common force does not imply common stress because their areas differ. Convert mm^2 to m^2 before substitution. For A,

$$\sigma_A = 150 \text{ MPa}, \quad \varepsilon_A = \frac{150 \times 10^6}{70 \times 10^9} = 2.14286 \times 10^{-3}, \quad \delta_A = 2.57 \text{ mm}.$$

For B,

$$\sigma_B = 100 \text{ MPa}, \quad \varepsilon_B = 5.00 \times 10^{-4}, \quad \delta_B = 0.400 \text{ mm}.$$

The stiffnesses are $k_A = 4.67 \text{ MN m}^{-1}$ and $k_B = 30.0 \text{ MN m}^{-1}$. Algebraically,

$$\frac{k_B}{k_A} = \frac{E_{Y,B}}{E_{Y,A}} \frac{A_B}{A_A} \frac{L_A}{L_B} = \frac{200}{70} \frac{120}{80} \frac{1.20}{0.80} = 6.42857.$$

The modulus ratio alone is only 2.85714. Rod B is also thicker and shorter, increasing its specimen stiffness further.

Graph and limits. At $F = 12 \text{ kN}$ the horizontal load line in Fig. 12.18 meets B at 0.400 mm and A at 2.571 mm. Their extension ratio is the reciprocal stiffness ratio. Doubling either area at fixed material and length doubles stiffness and halves extension without changing modulus. Doubling gauge length doubles extension without changing strain at fixed stress. The strains are small, but acceptance of the linear range remains a stated material assumption.

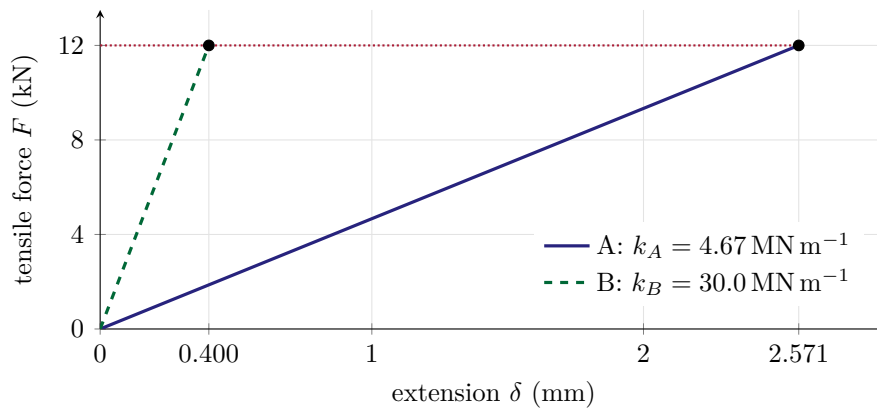


Figure 12.18: Analytical force–extension curves for the two stipulated rods, restricted to the declared 0–12 kN loading interval. The common-force intersections reveal compliance directly. This plot’s slopes are specimen stiffnesses, rather than Young moduli.

Guided checks

Diagnostic X12-04. Replacing a rod by one of the same material and length but twice the area does not double its Young modulus. The stress at the same force halves; the strain and extension halve; and $k_{\text{axial}} = E_Y A_0 / L_0$ doubles. Material response and specimen geometry enter different parts of the calculation.

12.6 Shear and bulk response

Axial extension is one loading mode. Tangential loading changes shape through relative motion of layers; pressure loading produces a volume change. Their stress and strain measures must be matched to the physical mode. Substituting a Young modulus into every deformation formula would discard this distinction. The three moduli all have units of pressure, but dimensional agreement is not sufficient to make them interchangeable.

12.6.1 Simple shear: displacement, angle and moment balance

Consider a block whose lower face is clamped and whose top face moves sideways. Its reference height is h , measured perpendicular to the lower face. Define simple-shear engineering strain by

$$\gamma = \frac{\Delta x}{h} = \tan \phi, \quad \phi = \arctan \gamma, \quad (12.18)$$

where ϕ is the change in angle of an initially vertical material line. For this specified simple-shear geometry, $\gamma = \Delta x/h$ is an exact kinematic definition. The approximation is $\phi \simeq \gamma$ for small angles in radians. Writing $\tan \gamma \simeq \gamma$ would incorrectly use the same symbol for the ratio and the angle. The displacement must connect the undeformed and deformed positions of the *same* top corner, rather than measure the width of the shifted face.

The average shear stress and linear constitutive relation are

$$\tau_{\text{shear}} = \frac{F_{\parallel}}{A_0}, \quad \boxed{\tau_{\text{shear}} = G_s \gamma}, \quad \Delta x = \frac{F_{\parallel} h}{G_s A_0}. \quad (12.19)$$

Here A_0 is the sheared face area and $G_s > 0$ is the stipulated shear modulus. This specimen formula is a reduced, shear-dominated approximation with small deformation and approximately uniform shear; end effects and bending deformation are neglected. The angle relation is geometrical; the stress relation is physical and requires the stated range.

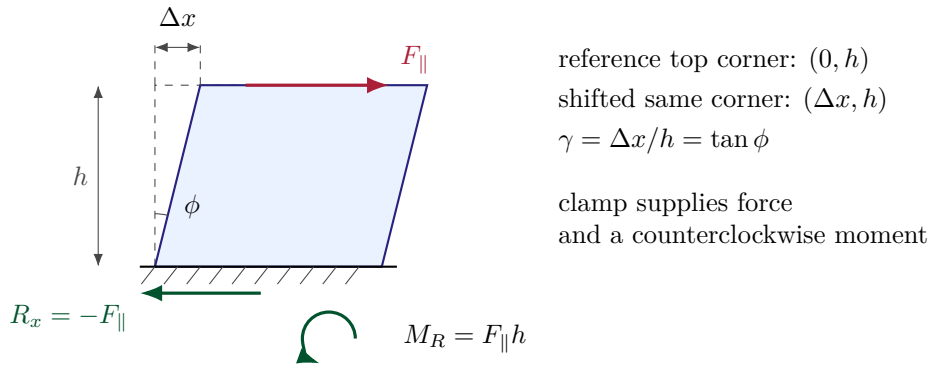


Figure 12.19: Correct simple-shear displacement measured from the same reference corner. The dashed outline is undeformed. The top-force/bottom-force pair alone has a clockwise moment; the clamp also supplies the counterclockwise reaction moment shown. The display distortion is exaggerated. The reference-overlay construction draws on the method in I. Neutelings, *Mechanical stress*, TikZ.net; geometry and equilibrium are constructed independently here.

The moment reaction is scientifically consequential. A force to the right on the top face and an equal force to the left on the bottom face cancel in translation, but form a clockwise couple of magnitude $F_{\parallel} h$. For the clamped block, the clamp must supply a counterclockwise moment as well as a horizontal force. Alternatively, an exactly uniform pure-shear element requires complementary tangential tractions on its side faces which balance that couple. A sketch containing only two separated opposite arrows does not represent a freely floating block in static equilibrium. Fig. 12.19 explicitly chooses a clamp model so that global force, moment and displacement constraints agree. Those global balances are necessary, but do not establish an exactly uniform internal shear field; the reduced deformation approximation remains an additional assumption.

12.6.2 Bulk compression and a useful sign check

Pressure acts normally inward on each surface element of a body. Under a small nearly uniform increase $\Delta p > 0$ the fractional volume change is $\varepsilon_V = \Delta V/V_0$. Define a positive bulk modulus by

$$\boxed{\Delta p = -B \frac{\Delta V}{V_0}, \quad B > 0.} \quad (12.20)$$

The minus sign says that increasing pressure decreases volume. It is essential because tensile-positive strain and compressive-positive pressure use opposite signs. For finite changes this linear form treats B as approximately constant over a small range. The local differential form is $B = -V dp/dV$ under stated thermodynamic conditions. In a fluid, temperature or heat-exchange conditions can matter; no universal large-compression line is inferred here.

Opposite inward pressures can have zero total force while compressing a body, just as opposite axial forces can stretch it. Uniform pressure acts in all directions and is therefore different from a force pair pressing only the ends of a bar. For an isotropic homogeneous specimen, hydrostatic compression gives equal directional strains and no preferred distortion. A static fluid can support pressure but cannot sustain the same fixed-shape shear response as an elastic solid; a tangential disturbance makes it flow rather than defining a nonzero static shear modulus.

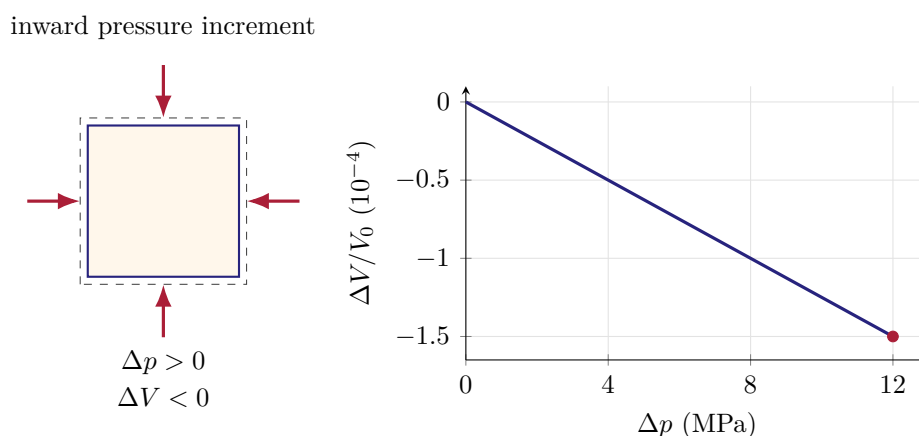


Figure 12.20: Hydrostatic compression, with schematic inward normal arrows and exaggerated shrinkage at left. At right the analytical small-compression model with stipulated $B = 80.0$ GPa predicts negative volume strain. The endpoint belongs to the bulk part of Example 12.14.

Example 12.14: Compare axial, shear and bulk deformation

Consider three independent stipulated specimens, each in its small-deformation linear range. (a) An axial bar has $F = 30.0$ kN, $A_0 = 300$ mm², $L_0 = 0.500$ m and $E_Y = 100$ GPa. (b) A clamped shear block has $F_{\parallel} = 18.0$ kN, $A_0 = 600$ mm², $h = 0.200$ m and $G_s = 25.0$ GPa. (c) A body of volume $V_0 = 0.0150$ m³ has $B = 80.0$ GPa and experiences a uniform pressure increase $\Delta p = 12.0$ MPa. Find the associated stresses, strains and dimensional changes. For the shear block also determine the angle and clamp reactions.

Axial solution. $\sigma = F/A_0 = 100$ MPa. Hence $\varepsilon = \sigma/E_Y = 1.00 \times 10^{-3}$ and $\delta = L_0\varepsilon = 0.500$ mm.

Shear solution. $\tau_{\text{shear}} = 18.0 \times 10^3 / (600 \times 10^{-6}) = 30.0$ MPa. Thus $\gamma = \tau_{\text{shear}}/G_s = 1.20 \times 10^{-3}$ and $\Delta x = h\gamma = 0.240$ mm. The angle is $\phi = \arctan(0.0012) = 0.0011999994$ rad = 0.06875° . The small-angle approximation differs here by less than one part in a million. The clamp supplies $R_x = -18.0$ kN and a counterclockwise moment $M_R = F_{\parallel}h = 3.60$ kN m. Force and moment balance are both satisfied.

Bulk solution. Equation (12.20) gives

$$\frac{\Delta V}{V_0} = -\frac{12.0 \times 10^6}{80.0 \times 10^9} = -1.50 \times 10^{-4}, \quad \Delta V = -2.25 \times 10^{-6} \text{ m}^3 = -2.25 \text{ cm}^3.$$

Interpretation. All three strain measures are dimensionless, but they describe different geometrical changes. The dimensional outputs are respectively a length, a transverse shift and a volume. Positive pressure gives negative volume change. The numerical moduli belong to

three separate models and are not being asserted to form one material's isotropic modulus triple.

Guided checks

Diagnostic X12-05. The minus sign in the bulk law is required because $\Delta p > 0$ and $\Delta V < 0$ for stable compression. Removing it while keeping the same definitions would assign a negative modulus to that response. Changing to a positive *compression magnitude* is possible, but then its definition and the accompanying formula must both change.

12.6.3 Lateral contraction and the isotropic bridge

A tensile bar commonly becomes narrower while becoming longer. Define Poisson's ratio in a uniaxial small-strain test with free lateral surfaces by $\nu = -\varepsilon_{\text{lateral}}/\varepsilon_{\text{axial}}$. A positive ν therefore means lateral contraction in tension. Its value is dimensionless and need not be inferred from Young's modulus alone. The area approximation used above neglects lateral changes of order strain; the volume change to first order is the sum of the three directional strains.

For a homogeneous isotropic linear elastic model there are only two independent elastic constants. The corresponding relations are

$$G_s = \frac{E_Y}{2(1 + \nu)}, \quad B = \frac{E_Y}{3(1 - 2\nu)}. \quad (12.21)$$

The bulk relation can be understood without a full tensor course. Superpose equal compressive normal stresses $-p$ in three directions. In the x direction, its own normal stress gives $-p/E_Y$ while each transverse compressive stress gives $+\nu p/E_Y$. Thus $\varepsilon_x = -(1 - 2\nu)p/E_Y$ and likewise in y, z . Their sum is $\Delta V/V_0 = -3(1 - 2\nu)p/E_Y$, which yields the stated B .

For the shear relation, a small square in pure shear has a diagonal that extends and the perpendicular diagonal that shortens. In axes rotated through 45° , the shear loading is equivalent to normal stresses $+\tau_{\text{shear}}$ and $-\tau_{\text{shear}}$. Their strain difference is $2(1 + \nu)\tau_{\text{shear}}/E_Y$; this equals the engineering shear strain γ . Therefore $\tau_{\text{shear}}/\gamma = E_Y/[2(1 + \nu)]$. This argument explicitly adds isotropy, superposition and infinitesimal strain. It is a bridge between modes, not a licence to impose the relations on an anisotropic composite or on unrelated table entries.

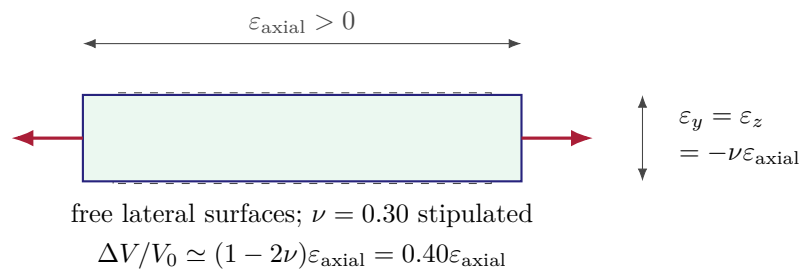


Figure 12.21: A uniaxial tensile specimen with free lateral surfaces contracts sideways for stipulated $\nu = 0.30$. The reference outline is dashed. Axial and lateral strains share one display exaggeration factor of 160, corresponding to $\varepsilon_{\text{axial}} = 0.001$ and $\varepsilon_{\text{lateral}} = -0.0003$. The first-order volume change depends on all three directional strains.

Example 12.15: A consistent isotropic material model

Stipulate an isotropic linear elastic specimen with $E_Y = 100$ GPa and $\nu = 0.25$. Its reference length is 0.500 m and its square section is 10.0 mm on a side. A central tensile force of 5.00 kN is applied while the lateral faces remain free. Find axial and lateral strains, length and width changes, first-order volume strain, shear modulus and bulk modulus. Would independently substituting $G_s = 25$ GPa preserve this same model?

Solution. The area is 100 mm^2 , so $\sigma = 50.0$ MPa and $\varepsilon_x = 5.00 \times 10^{-4}$. Each lateral strain is $\varepsilon_y = \varepsilon_z = -0.25\varepsilon_x = -1.25 \times 10^{-4}$. Consequently the length increases by 0.250 mm and each 10.0 mm width decreases by 0.00125 mm, or $1.25 \mu\text{m}$. To first order,

$$\frac{\Delta V}{V_0} = \varepsilon_x + \varepsilon_y + \varepsilon_z = 2.50 \times 10^{-4}.$$

The two other moduli are $G_s = 100/[2(1.25)] = 40.0$ GPa and $B = 100/[3(0.50)] = 66.7$ GPa.

Consistency check. Taking $G_s = 25$ GPa with $E_Y = 100$ GPa would imply $\nu = E_Y/(2G_s) - 1 = 1$, contradicting the stipulated 0.25 and giving a negative B in the isotropic formula. Thus the replacement does not preserve this stable isotropic model. Numerical properties drawn from different idealised specimens, temperatures or approximate tables need not form a consistent triple. Only when isotropy and the same test state are declared should these cross-mode constraints be imposed.

12.7 Elasticity, proportionality and plasticity

12.7.1 Three claims that require different evidence

Elasticity means recovery of deformation on removal of load within the specified conditions. Linearity means proportionality of stress and strain. Plasticity means that a part of the deformation remains after unloading in the stated macroscopic model. These definitions concern different features of a loading history. A single loading curve cannot by itself establish full recovery: unloading observations are needed. Likewise a straight segment reveals proportionality over that segment, not a universal law up to fracture.

A proportionality limit identifies the boundary of an approximately straight relation at a stated tolerance. The elastic limit identifies the boundary beyond which unloading leaves persistent strain under the stated test conditions. The two need not coincide. Time-dependent recovery, hysteresis and temperature dependence can make a real test more complicated; here we use two deliberately simple models to isolate the distinction. Neither curve is measured data and neither supplies numerical thresholds for a named material.

First define dimensionless stress $s = \sigma/\sigma_0$ and strain $e = \varepsilon/\varepsilon_0$, with arbitrary positive reference scales. Consider a reversible toy law

$$s(e) = \begin{cases} e, & 0 \leq e \leq 1, \\ e + \frac{1}{2}(e - 1)^2, & 1 < e \leq 2. \end{cases} \quad (12.22)$$

Loading and unloading follow this *same* path. The law and its slope are continuous at $e = 1$, and its slope is positive throughout the domain. Proportionality ends there, but recovery remains complete even for loading to $e = 2$. Calling the whole path plastic merely because it curves would confuse definitions.

For a contrasting toy with plastic deformation, adopt the normalized elastic–perfectly plastic loading law $s = e$ for $0 \leq e \leq 1$ and $s = 1$ for $1 \leq e \leq 2.5$. After loading to $e_m = 2.5$, unload along $s = e - 1.5$ until stress reaches zero. The unloading slope is the original elastic slope, but its zero

occurs at residual strain $e_r = 1.5$. This explicitly specified path, rather than an arbitrarily placed label on a smooth curve, establishes irreversibility. It has different threshold behaviour from the reversible toy; no single universal metal curve is implied.

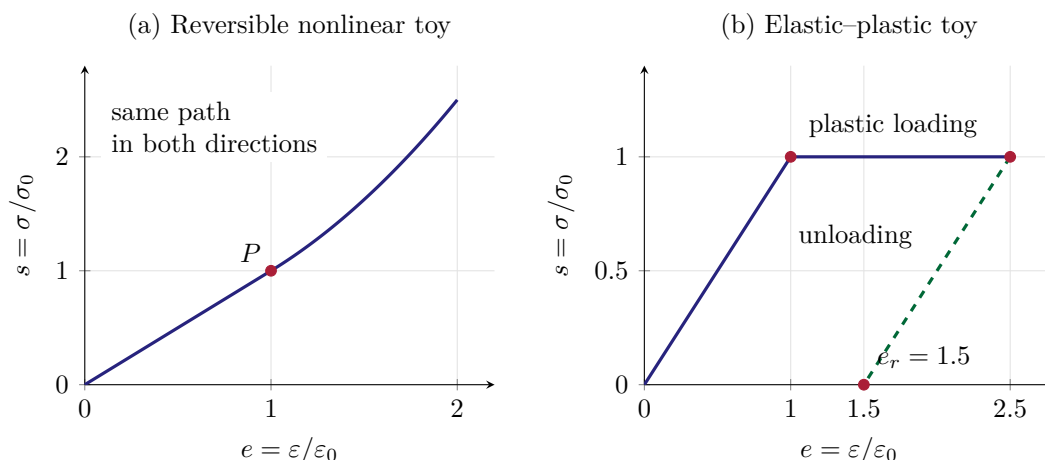


Figure 12.22: Two quantitative normalized teaching models, each with a specified loading and unloading law. In (a) P marks the end of exact proportionality, yet the whole path is reversible. In (b) loading beyond the plateau's start leaves a calculated residual strain. The models are not empirical material curves.

Example 12.16: Reading a loading–unloading diagram

Use Fig. 12.22 to assess three statements: (i) any departure from the initial straight line implies plasticity; (ii) the initial Young modulus describes the whole loading response; (iii) a nonzero strain at zero stress after unloading indicates a plastic component in the elastic–plastic toy. Then calculate the residual strain and recoverable strain after loading model (b) to $e_m = 2.5$. Use stipulated scales $\varepsilon_0 = 0.001$ and $\sigma_0 = 100$ MPa to give dimensional values.

Reasoned classification. Statement (i) is false: model (a) departs from its initial line after $e = 1$ but retraces its complete path. No residual deformation occurs. Statement (ii) is false for both full loading paths. The initial slope applies to the proportional segment; the nonlinear portion of (a) has a changing slope and the plateau of (b) has zero loading slope. Statement (iii) is correct within model (b), whose unloading path is explicitly specified. In a real test one would additionally distinguish permanent plastic strain from delayed recovery or a shifted instrument zero.

Calculation. In model (b), unloading from $(e, s) = (2.5, 1)$ has unit slope. Removing one stress unit removes one strain unit, leaving $e_r = 2.5 - 1 = 1.5$. Thus $\varepsilon_m = 0.0025$, $\varepsilon_r = 0.0015$, and the recovered component is 0.0010. The initial modulus corresponding to the scales is $E_Y = \sigma_0/\varepsilon_0 = 100$ GPa.

Check. Substitution into the unloading law gives $s = 1.5 - 1.5 = 0$ at the residual endpoint. The distinction between the two paths follows from their unloading rules, not from a guessed material name, colour or shape. No strength or fracture stress can be inferred from either toy.

Guided checks

Diagnostic X12-06. Proportional and elastic limits need not denote the same stress. The reversible nonlinear model supplies a counterexample: proportionality ends at $e = 1$, but unloading from any state shown still returns to the reference state. A loading-only graph lacking an unloading rule is insufficient evidence for that recovery claim.

12.8 Elastic energy and axial assemblies

12.8.1 Work as a graph area

For a specimen loaded quasi-statically from its unloaded state, external work is the integral of force with respect to displacement. In the ideal reversible linear model this work becomes stored elastic energy:

$$U_{\text{el}} = \int_0^\delta k\delta' d\delta' = \frac{1}{2}k\delta^2 = \frac{1}{2}N\delta. \quad (12.23)$$

The integral is the area of a triangle because force grows from zero to its final value. Multiplying the final force by the entire displacement would instead use the area of a rectangle and double the energy. In compression N and δ are both negative under our convention, so their product and the stored energy remain positive. This is a useful sign check rather than a reason to discard the convention.

For a uniform reference volume $V_0 = A_0L_0$, dividing by V_0 gives

$$u_{\text{el}} = \frac{U_{\text{el}}}{V_0} = \frac{1}{2}\sigma\varepsilon = \frac{1}{2}E_Y\varepsilon^2 = \frac{\sigma^2}{2E_Y}. \quad (12.24)$$

A stress–strain area has units $\text{Pa} = \text{J m}^{-3}$, so it represents energy per reference volume. A force–extension area instead has units $\text{N m} = \text{J}$. Figure 12.23 uses one specimen and one model in both panels; matching the axes to the physical quantity is essential. A triangular area drawn under an unrelated experimental fit would not verify this calculation.

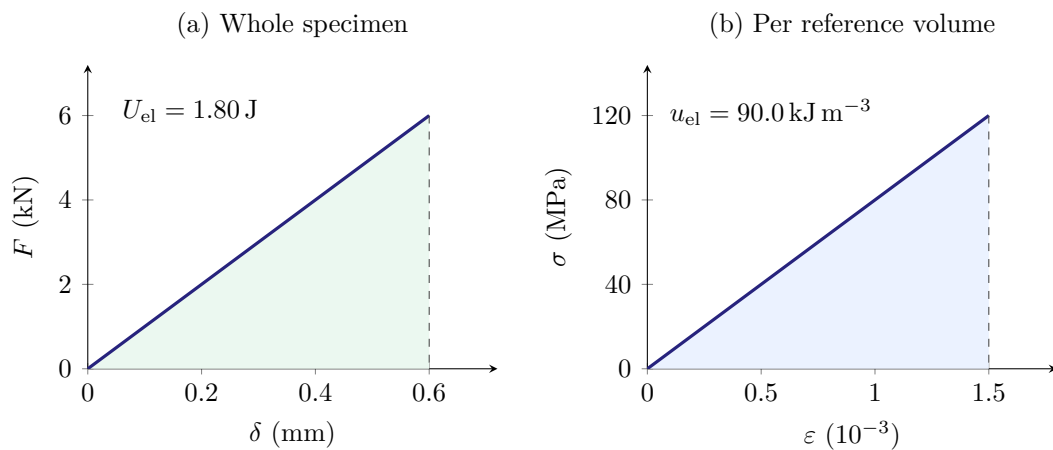


Figure 12.23: The same stipulated linear specimen represented by force–extension and stress–strain areas. Multiplying panel (b)’s energy density by $V_0 = 2.00 \times 10^{-5} \text{ m}^3$ gives panel (a)’s energy. Axis scale factors must be included when converting graphical area to SI energy.

Example 12.17: Three checks of one elastic energy

A stipulated uniform axial specimen has $L_0 = 0.400 \text{ m}$, $A_0 = 50.0 \text{ mm}^2$ and $E_Y = 80.0 \text{ GPa}$. Its tensile stress reaches 120 MPa within its small-strain linear elastic range. Find strain, extension, force and energy. Verify the energy from force–displacement, stress–strain and stiffness descriptions.

Solution. The strain is $\varepsilon = 120 \times 10^6 / (80.0 \times 10^9) = 1.50 \times 10^{-3}$ and the extension is $\delta = 0.400\varepsilon = 0.600 \text{ mm}$. The force is $N = \sigma A_0 = 6000 \text{ N}$, while $k = E_Y A_0 / L_0 = 10.0 \text{ MN m}^{-1}$. The force–displacement integral gives

$$U_{\text{el}} = \frac{1}{2}N\delta = \frac{1}{2}(6000)(0.000600) = 1.80 \text{ J}.$$

The stress–strain area gives $u_{el} = \frac{1}{2}\sigma\varepsilon = 90.0 \text{ kJ m}^{-3}$. Since $V_0 = A_0 L_0 = 2.00 \times 10^{-5} \text{ m}^3$, multiplying this density by volume also gives 1.80 J. Finally $\frac{1}{2}k\delta^2 = \frac{1}{2}(10.0 \times 10^6)(0.000600)^2 = 1.80 \text{ J}$.

Interpretation and limit. The two graph areas in Fig. 12.23 have different units but describe the same loading. Doubling stress within this same linear model doubles extension and quadruples stored energy. A continuation beyond the declared elastic range cannot be assessed from the modulus alone.

For a reversible nonlinear axial law the more general expression is $u = \int_0^\varepsilon \sigma(\varepsilon') d\varepsilon'$ using the appropriate conjugate measures for the adopted small-strain description. It does not reduce to $\frac{1}{2}\sigma\varepsilon$ unless the graph is a straight line from the origin. For a path with plastic deformation, loading work need not all be recoverable; the unloading area measures the returned energy in the stipulated quasi-static model. This is another reason that the loading curve and the unloading rule must be distinguished.

In normalized model (b) of Fig. 12.22, loading to $e = 2.5$ has area $\frac{1}{2}(1)(1) + 1(2.5 - 1) = 2.0$. Unloading recovers a triangle of area 0.5. The difference 1.5 is irreversible work per unit $\sigma_0\varepsilon_0$ and reference volume in this ideal elastic–perfectly plastic model, which assigns no further hardening energy. In the reversible model (a), loading to $e = 2$ stores normalized area $\int_0^2 e de + \frac{1}{2} \int_1^2 (e - 1)^2 de = 13/6$, and the identical unloading path returns that full amount. These exact graph-area checks support the classification independently of visual labels.

12.8.2 Series bars and a spatially varying area

Members connected end to end form a series axial assembly. With no applied load at their joint and negligible member weight, cutting each section establishes the same axial resultant N . Their extensions add because the assembly’s endpoints enclose both members:

$$\delta_{\text{tot}} = N \sum_i \frac{L_i}{E_{Y,i} A_i}, \quad \frac{1}{k_{\text{series}}} = \sum_i \frac{1}{k_i}. \quad (12.25)$$

This is a deformation compatibility statement together with a constitutive law. It is not obtained from force balance alone. The assembly is more compliant than either individual member, because the same force causes two extensions rather than one. This arrangement differs from parallel supports, where a suitable crosshead can impose equal endpoint displacement while allowing unequal forces.

For a gradually varying area $A_0(x)$ and local modulus $E_Y(x)$, divide the member into short intervals. Within the one-dimensional axial approximation, a small reference interval dx extends by $d\delta = N(x)dx/[E_Y(x)A_0(x)]$. Adding those intervals gives

$$\delta = \int_0^{L_0} \frac{N(x)}{E_Y(x)A_0(x)} dx. \quad (12.26)$$

Here $N(x)$ must first be obtained from equilibrium, including distributed axial loads when present. In the simple unloaded-sidewall example below, $N(x) = F$ is constant. The model ignores local three-dimensional stress near a taper, clamps or joints; a gradual taper makes the axial approximation more plausible but does not convert it into an exact full elasticity solution.

Take constant modulus and a linearly increasing reference area,

$$A_0(x) = A_a \left(1 + \alpha \frac{x}{L_0} \right), \quad \alpha > -1.$$

The integral becomes

$$\delta = \frac{FL_0}{E_Y A_a} \frac{\ln(1 + \alpha)}{\alpha} \quad (\alpha \neq 0), \quad \delta \longrightarrow \frac{FL_0}{E_Y A_a} \quad (\alpha \rightarrow 0). \quad (12.27)$$

The limit follows because $\ln(1 + \alpha) = \alpha + O(\alpha^2)$. For $\alpha > 0$, every section is at least as large as A_a , so extension must be smaller than a uniform member of area A_a . It must also exceed a uniform member of the same length and largest area $A_a(1 + \alpha)$. These bounds offer an independent physical check on the integral. Using arithmetic average area generally fails because compliance contains $1/A_0(x)$, not $A_0(x)$.

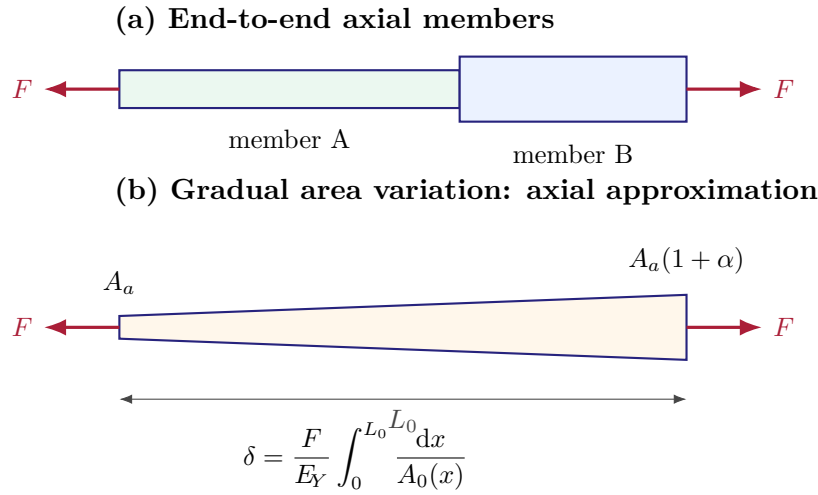


Figure 12.24: Two axial models where compliance adds along the length. The series drawing is schematic. In the taper drawing, silhouette width is a visual cue; the stipulated cross-sectional *area*, rather than that width, varies linearly. All forces act on the common axis.

Example 12.18: Adding compliance and checking a taper integral

(a) Connect rods A and B from Example 12.13 end to end, neglect joint deformation and member weight, and apply 12.0 kN tension to the assembly. Find total extension, equivalent stiffness and stored energy. (b) Separately consider a constant-modulus member with $L_0 = 1.00$ m, $E_Y = 100$ GPa and area increasing linearly from $A_a = 100$ mm² to 200 mm². Under $F = 10.0$ kN, find the axial-model extension, endpoint stresses and energy. Test the integral against uniform-area limits and an average-area estimate.

Part (a). The joint has no external axial load, so the same force passes through each member. Their earlier extensions add: $\delta_{\text{tot}} = 2.57143 + 0.400 = 2.97143$ mm. Thus

$$k_{\text{series}} = \frac{12.0 \times 10^3}{0.00297143} = 4.03846 \text{ MN m}^{-1}.$$

This is smaller than either 4.66667 or 30.0 MN m⁻¹, as expected for series compliance. The energy is $U = \frac{1}{2} F \delta_{\text{tot}} = 17.8286$ J, or 17.8 J. Independently the separate energies are 15.4286 J and 2.400 J, whose sum agrees.

Part (b). The area ratio gives $\alpha = 1$. Equation (12.27) yields

$$\delta = \frac{(10.0 \times 10^3)(1.00)}{(100 \times 10^9)(100 \times 10^{-6})} \ln 2 = 0.693147 \text{ mm}.$$

Stress decreases from 100 MPa at the small end to 50.0 MPa at the large end. Maximum local axial strain is 0.00100, consistent with the geometrical small-strain assumption. Linear loading gives $U = \frac{1}{2} F \delta = 3.46574$ J.

Checks and interpretation. Uniform members with the minimum and maximum areas would extend by 1.00 mm and 0.500 mm. The taper result lies between them. Substituting the arithmetic mean area 150 mm² gives 0.666667 mm, about 3.82% below the integral result; it underweights the compliance of smaller sections. Numerical integration of $F/[E_Y A_0(x)]$

supplies an independent check on the logarithmic expression. As the area ratio tends to one, the factor $\ln(1 + \alpha)/\alpha$ tends to one and recovers the uniform-bar law. Local joint, taper and grip effects remain outside this one-dimensional model.

The same logic explains why equilibrium and deformation must cooperate. A cut establishes force transmission; a strain law turns that force into local deformation; compatibility then determines how deformations combine. The next applied and advanced discussions use these distinctions when extracting a modulus from data and sharing a load between redundant supports. Keeping the model statements separate makes it possible to diagnose whether a discrepancy comes from geometry, force balance, measurement or material response.

12.9 Measurement and model validation of elastic response

The preceding energy and axial-assembly treatment establishes how elastic forces produce displacement and recoverable work. We now ask an inverse question: what can observed forces and displacements establish about a specimen and its material? Equations (12.23) and (12.24) remain useful checks, but a measurement must also distinguish gauge deformation from apparatus motion and account for geometry uncertainty.

12.9.1 A measured stiffness is not yet a material modulus

An extension test first returns a relationship between force and displacement. Converting its slope into a material property requires additional geometric and measurement assumptions. For synthetic observations (δ_i, F_i) , consider the model

$$F_i = k\delta_i + b + r_i. \quad (12.28)$$

Here k is a specimen stiffness, b a force intercept, and r_i the residual at observation i . A displacement zero error can produce an apparent force intercept, just as a force zero error can. Preload, seating, or unmodeled grip motion can also affect the interpretation. The intercept must be considered physically rather than automatically removed because an ideal unloaded specimen has zero force.

For ordinary least squares, assume the extension values can be treated as known and the force deviations have independent, equal-variance scatter. The fitted line minimizes $S = \sum_i (F_i - k\delta_i - b)^2$. Setting the derivatives with respect to b and k to zero gives the normal equations. The first implies $b = \bar{F} - k\bar{\delta}$, where a bar denotes the arithmetic mean. Substitution into the second yields

$$\hat{k} = \frac{\sum_i (\delta_i - \bar{\delta})(F_i - \bar{F})}{\sum_i (\delta_i - \bar{\delta})^2}, \quad \hat{b} = \bar{F} - \hat{k}\bar{\delta}. \quad (12.29)$$

These are estimates within the assumed model. If displacement uncertainty is consequential, ordinary least squares can bias the slope, and a method that treats both measurement uncertainties is needed. A nearly perfect straight line does not by itself validate those assumptions.

Table 12.1: Preserved synthetic axial-test data. These seven constructed points demonstrate analysis; they are not measured properties of a named material.

δ (mm)	.040	.100	.160	.220	.280	.340	.400
F (N)	410	1015	1592	2210	2798	3418	3990

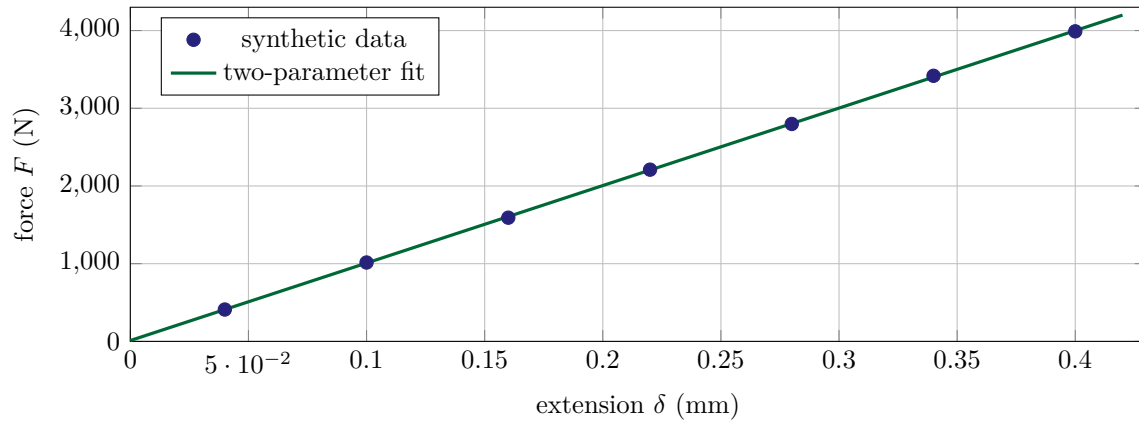


Figure 12.25: The preserved force–extension points and their fitted line. A line that looks satisfactory on this force scale still requires a residual analysis on a much finer vertical scale. Its slope is $9971.43 \text{ N mm}^{-1}$, equivalent to $9.97143 \text{ MN m}^{-1}$.

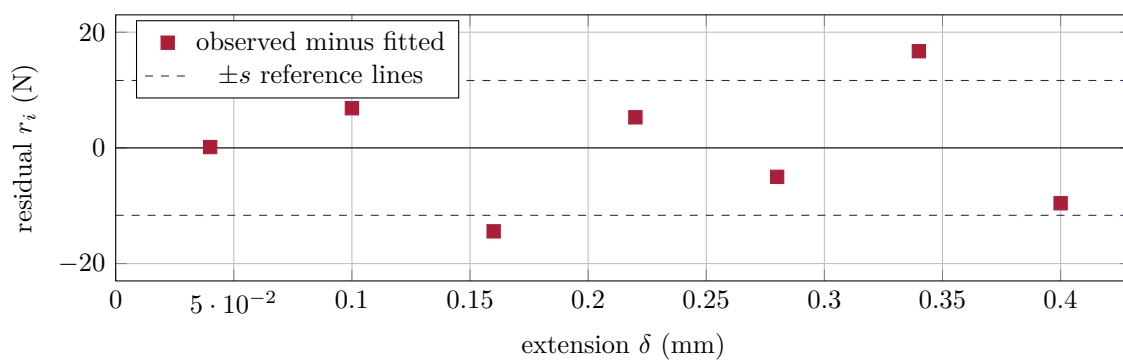


Figure 12.26: Residuals expose deviations hidden by the full force scale. The dashed lines mark one estimated residual standard deviation, not acceptance limits or confidence bounds on an individual point. Seven observations provide only modest evidence about curvature or drift.

Example 12.19: Infer a modulus and its uncertainty

Statement. Fit all seven points of Table 12.1 to $F = k\delta + b$. Take $L_0 = 0.400$ m and $A = 50.0$ mm². Find stiffness, intercept, residual scatter, and inferred modulus. Compare with a line forced through zero. Estimate combined standard uncertainty when independent geometry uncertainties are $u(L_0)/L_0 = 0.25\%$ and $u(A)/A = 1.0\%$, and slope uncertainty is estimated from these residuals.

Calculation. Convert extension to metres before identifying the stiffness unit. Equation (12.29) gives

$$\hat{k} = 9.971428571 \times 10^6 \text{ N m}^{-1}, \quad \hat{b} = 11.0 \text{ N}.$$

The predicted force at .220 mm is 2204.714 N, so that point has residual +5.286 N. The full residual sequence, in N, is

$$(0.143, 6.857, -14.429, 5.286, -5.000, 16.714, -9.571).$$

Its sum is zero to rounding, as required for an intercept fit. The residual sum of squares is 679.142857 N². The descriptive root-mean-square residual is $\sqrt{\sum r_i^2/7} = 9.84989$ N. Estimating the scatter after fitting two parameters instead uses five residual degrees of freedom:

$$s = \sqrt{\frac{\sum r_i^2}{7-2}} = 11.65455 \text{ N}, \quad u(\hat{k}) = \frac{s}{\sqrt{\sum (\delta_i - \bar{\delta})^2}} = 36708.4 \text{ N m}^{-1}.$$

Thus the relative slope standard uncertainty is 0.368136%. These two scatter quantities have different denominators because they answer different questions.

The material inference uses

$$\hat{E}_Y = \frac{\hat{k}L_0}{A} = 79.7714 \text{ GPa}.$$

A through-zero fit minimizes $\sum (F_i - k_0\delta_i)^2$ with $k_0 = \sum \delta_i F_i / \sum \delta_i^2$. It gives $k_0 = 10.0099636$ MN m⁻¹ and $E_{Y0} = 80.0797$ GPa, about 0.386% above the two-parameter value. The difference is small here because the synthetic intercept is small relative to the largest forces; this is not a general reason to suppress an intercept.

Uncertainty. First-order propagation of $E_Y = kL_0/A$ for independent inputs gives

$$\frac{u(E_Y)}{E_Y} = \sqrt{\left(\frac{u(k)}{k}\right)^2 + \left(\frac{u(L_0)}{L_0}\right)^2 + \left(\frac{u(A)}{A}\right)^2} = 0.0109454.$$

Thus $u(E_Y) = 0.873$ GPa; an appropriately rounded report is $E_Y = (79.8 \pm 0.9)$ GPa, with the $\pm$ denoting combined standard uncertainty under the stated model. It is not a confidence interval and does not include unquantified grip or alignment bias. The estimated intercept standard uncertainty is 9.20 N; the fitted 11 N intercept is only about 1.2 times that uncertainty, so these data do not establish a resolved nonzero preload.

Interpretation and checks. Figure 12.26 contains positive and negative deviations without a large monotonic trend. That observation supports using this simple teaching fit, but does not prove perfect linearity. The geometry conversion and residual normal equations supply independent consistency checks; the specimen's constitutive assumptions remain separate physical tests.

12.9.2 Where uncertainty enters the inference

The area term often deserves more attention than the visually striking fit. In the preceding example, the fractional variance contributions are $u_r(k)^2$, $u_r(L_0)^2$, and $u_r(A)^2$, where u_r denotes relative

standard uncertainty. Normalizing their sum gives approximately 11.31%, 5.22%, and 83.47% of the variance, respectively. Figure 12.27 makes the measurement priority visible: greatly improving the force fit alone would leave the dominant area contribution.

If a circular specimen's area is inferred from diameter d , then $A = \pi d^2/4$. A small variation obeys $dA/A = 2dd/d$, so $u_r(A) \simeq 2u_r(d)$ when the diameter measurement dominates the area uncertainty. Consequently 0.5% diameter uncertainty corresponds to 1.0% area uncertainty. Repeating a diameter measurement can estimate repeatability, but cannot remove a shared calibration error or a real variation of diameter along the gauge section.

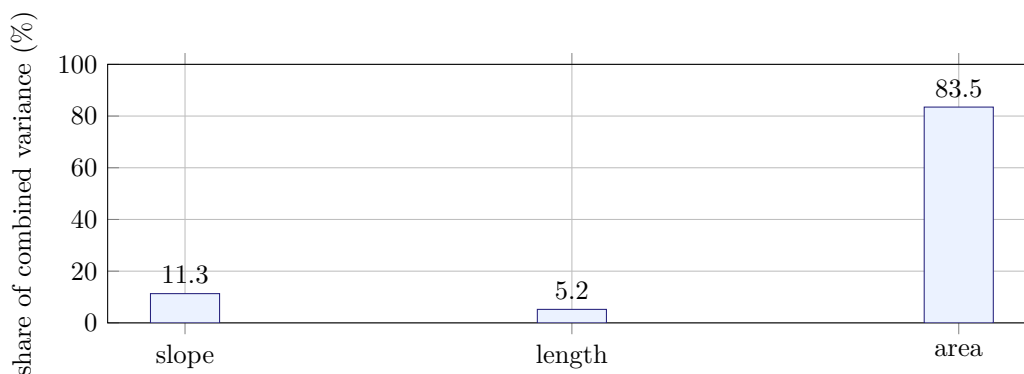


Figure 12.27: Illustrative variance budget for Example 12.19. These are contributions to squared uncertainty, not percentages of the modulus. Their interpretation assumes independent inputs and excludes physical model bias.

Guided checks

Diagnostic X12-07. Does a slope uncertainty of 0.37% establish a material-modulus uncertainty of 0.37%? No. Gauge length and cross-sectional area enter the conversion, and their uncertainties must be propagated. Misalignment or displacement measured outside the gauge section can create bias that is absent from the residual-based slope uncertainty. Precise repeated observations of a biased quantity remain biased.

12.9.3 Machine and grip compliance as a model error

An extensometer mounted on the gauge section measures something different from the displacement of the loading machine's crosshead. Suppose a specimen has stiffness k_s and the combined machine/grip system has stiffness k_g . Treat them as two linear elements in series with the same axial force. Their extensions add:

$$\delta_m = \delta_s + \delta_g = F \left(\frac{1}{k_s} + \frac{1}{k_g} \right), \quad \frac{1}{k_m} = \frac{1}{k_s} + \frac{1}{k_g}. \quad (12.30)$$

The measured structural stiffness k_m is smaller than k_s . Fitting that crosshead displacement and applying $E_Y = k_m L_0 / A$ therefore underestimates the material modulus even when every data point lies on an exact straight line. A residual plot cannot diagnose this error by curvature because the two compliant components are both linear in the model.

Example 12.20: Correct an apparent modulus

Statement. A specimen has $L_0 = 0.400$ m, $A = 50.0$ mm², and true stipulated $E_Y = 80.0$ GPa. The series grip/machine stiffness is $k_g = 40.0$ MN m⁻¹. Find the stiffness measured using crosshead displacement, the apparent modulus from that measurement, and the gauge and grip extensions at $F = 4.00$ kN. Derive a correction when k_g is known independently.

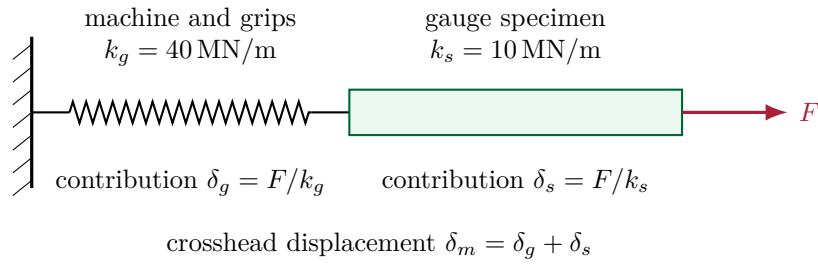


Figure 12.28: A series-compliance model for extension measured by a loading machine. The spring represents the combined grip/machine compliance; its drawn length is schematic. The same force acts through both components, whereas their displacements add. Measuring gauge extension separately changes the inference.

Solution. The specimen stiffness is $k_s = E_Y A / L_0 = 10.0 \text{ MN m}^{-1}$. The reciprocal stiffness rule gives

$$k_m = \frac{k_s k_g}{k_s + k_g} = 8.00 \text{ MN m}^{-1}, \quad E_{Y \text{ apparent}} = \frac{k_m L_0}{A} = 64.0 \text{ GPa}.$$

At the stated force, $\delta_s = 0.400 \text{ mm}$ and $\delta_g = 0.100 \text{ mm}$, so the machine records 0.500 mm . The apparent modulus is 20% below the stipulated material value, despite a perfectly linear response.

Rearranging $1/k_m = 1/k_s + 1/k_g$ gives

$$k_s = \frac{k_m k_g}{k_g - k_m}, \quad E_Y = \frac{k_m k_g L_0}{(k_g - k_m) A}.$$

This correction recovers 80.0 GPa for the supplied values. A positive specimen stiffness requires $k_g > k_m$. If independent measurements violate that inequality, the series model or input estimates need examination.

Sensitivity and interpretation. Logarithmic differentiation gives coefficients $k_g / (k_g - k_m) = 1.25$ for a fractional change in k_m and $-k_m / (k_g - k_m) = -0.25$ for a fractional change in k_g . Their signs show that increasing measured stiffness increases the corrected modulus, while increasing independently assigned grip stiffness reduces the compliance correction. As k_g approaches k_m , the inference becomes sensitive to both measurements. In the limit $k_g \rightarrow \infty$, the grips become rigid and $k_m \rightarrow k_s$, recovering the simple gauge-specimen model.

12.9.4 A proposed investigation with a worked analysis exemplar

A low-load investigation can test the whole inference chain without attempting a destructive strength measurement. This is a proposed teaching activity, not a report of available AGH apparatus or of an experiment already performed. Use an appropriately constrained elastic element and a preselected reversible range. Record gauge length and cross-sectional measurements, force zero, displacement zero, loading order, and repeated values. If crosshead displacement is used, state that choice explicitly; if possible, compare it with direct gauge extension.

Investigation L12-01. (a) Identify quantities directly observed and quantities inferred. (b) Explain a loading/unloading sequence that tests reversibility and zero return. (c) Analyze Table 12.1, showing a fitted line, residuals, units, and uncertainty. (d) Propose the most useful additional measurement before interpreting the fitted modulus as a material property. (e) Give a defensible criterion for restricting the fit interval and for comparing repeated runs.

Worked exemplar and assessment criteria. For (a), force, gauge extension, gauge length, and dimensions are observations; area may be inferred from dimensions, stiffness from the fit, and

modulus from stiffness and geometry. The geometric conversion is justified only for the stated axial gauge model. For (b), take increasing low loads, return through several of the same loads, and revisit the initial zero after unloading. Repeat one intermediate load during the sequence. Agreement of up/down values within their combined measurement uncertainty supports reversibility; an unresolved difference calls for more precise observation. A persistent residual extension or systematic loop is evidence against treating the whole path as loss-free linear elasticity.

For (c), the fully worked fit is Example 12.19; the inferred result is (79.8 ± 0.9) GPa under its stipulated assumptions. That synthetic dataset contains no unloading observations, so it cannot certify reversibility. For (d), compare direct gauge extension with crosshead displacement, as in Example 12.20. Alternatively, measure the cross-section at several gauge positions and improve its calibration because the illustrative area variance dominates. The better choice depends on whether the principal concern is unquantified compliance bias or the quantified geometry uncertainty; an honest answer states that distinction.

For (e), select a contiguous low-load interval whose residuals lack systematic structure larger than the independently estimated repeatability scatter, and whose loading/unloading differences are consistent with reversible response. Merely shortening the interval until the fitted line looks attractive is not a defensible selection rule. For independent repeated estimates E_1, E_2 with standard uncertainties u_1, u_2 , compare their difference with $\sqrt{u_1^2 + u_2^2}$; a difference appreciably larger than a few such units deserves investigation. Shared calibration contributions are correlated and must not be treated as independent in that comparison. A satisfactory student report includes raw observations, unit conversions, model assumptions, residual interpretation, and limitations, not only a final slope.

12.9.5 Computational connection: preserving units and testing a model

Activity C12-01. Use the seven preserved points to reproduce the two-parameter fit. Report the line, residuals, residual RMS, residual standard deviation, slope uncertainty, and inferred modulus. Repeat the fit with extension in millimetres and explain the changed numerical slope. Compare the full fit with the lowest four and highest four points, and explain whether those results establish load-dependent stiffness.

The following compact algorithm uses only arithmetic, sums, and a square root; it is optional reproduction code, not a chapter-build dependency.

```
x = [v*1e-3 for v in [.04,.10,.16,.22,.28,.34,.40]]
y = [410,1015,1592,2210,2798,3418,3990]
n = len(x); xb = sum(x)/n; yb = sum(y)/n
Sxx = sum((v-xb)**2 for v in x)
k = sum((v-xb)*(f-yb) for v,f in zip(x,y))/Sxx
b = yb-k*xb
r = [f-(k*v+b) for v,f in zip(x,y)]
s = (sum(v*v for v in r)/(n-2))**0.5
uk = s/Sxx**0.5
E = k*.400/(50e-6)
```

Worked response. The full-fit outputs are those of Example 12.19. In millimetres, the slope is $9971.428571 \text{ N mm}^{-1}$; multiplying by 1000 mm m^{-1} returns $9.971428571 \times 10^6 \text{ N m}^{-1}$. The physical modulus is unchanged. Fitting the first four points gives $k = 9.961667 \text{ MN m}^{-1}$, while the last four give $9.933333 \text{ MN m}^{-1}$. Their difference is about 0.284% of the full-fit slope. These overlapping subsets have fewer observations and are not independent; small slope differences alone do not establish material nonlinearity. Additional repeats and a specified scatter model are required. The normal-equation checks $\sum r_i \simeq 0$ and $\sum \delta_i r_i \simeq 0$ validate the fit arithmetic, while residual shape and unloading observations test different physical assumptions.

12.10 Advanced Topics for Technical and Engineering Students

12.10.1 Prerequisite bridge: equilibrium, compatibility, and constitutive response

The following enrichment uses force and moment balance, the linear axial relation $F = k\delta$, and elementary simultaneous equations. Its new ingredient is geometric compatibility: connected points cannot move independently. A rigid planar bar can translate and rotate without bending. Therefore rigidity requires its displaced points to remain collinear, but does not require them to have equal vertical displacements. In a small-rotation model, write the downward displacement at coordinate x as

$$d(x) = d_0 + \theta x, \quad (12.31)$$

where d_0 is downward translation of the chosen origin and θ is a small clockwise rotation in radians. The right-hand end moves farther downward when $\theta > 0$. This is the first-order approximation to rigid-body kinematics; second-order horizontal shifts are neglected. A horizontal bar has $\theta = 0$ only if the loads, supports, and constraints make that condition true.

Three statement types must remain distinct. Equilibrium balances external force and moment. Compatibility describes admissible connected displacements. Constitutive response connects each support force to its deformation. A redundant system needs all three, whereas an unguided bar on two vertical supports often has two reaction unknowns and two independent equilibrium equations. Drawing two unequal supports with a horizontal bar cannot replace the missing mechanical constraint.

12.10.2 Guided parallel supports: where the balancing couple comes from

Place rods at $x = -a$ and $x = +a$, with stiffnesses $k_1, k_2 > 0$. Their lower ends attach to a rigid crosshead. An ideal frictionless translational antirotation guide allows downward translation but prevents rotation. It exerts a reaction couple M_g and no net vertical force. This is a mechanical idealization: horizontal contact reactions within the guide can form the couple, and their net work is zero because the permitted motion contains no rotation or horizontal translation. The upper anchors may lie at different heights so that unequal unloaded rod lengths are geometrically consistent.

Compatibility gives $\delta_1 = \delta_2 = d_0$. Vertical equilibrium gives $F_1 + F_2 = P$, and the rod laws give $F_i = k_i d_0$. Therefore

$$d_0 = \frac{P}{k_1 + k_2}, \quad F_1 = P \frac{k_1}{k_1 + k_2}, \quad F_2 = P \frac{k_2}{k_1 + k_2}. \quad (12.32)$$

For a centered downward load and counterclockwise-positive physical moments about the crosshead center, $-aF_1 + aF_2 + M_g = 0$. Thus

$$M_g = a(F_1 - F_2). \quad (12.33)$$

The guide couple is essential when the support forces differ. Without it, the same unequal forces would leave a nonzero moment even though their sum equals the load. Centered loading does not guarantee a horizontal crosshead supported by unequal rods.

Example 12.21: Unequal rods with an explicit guide

Statement. Rod 1 has $E_1 = 70.0 \text{ GPa}$, $A_1 = 300 \text{ mm}^2$, and $L_1 = 1.75 \text{ m}$. Rod 2 has $E_2 = 200 \text{ GPa}$, $A_2 = 160 \text{ mm}^2$, and $L_2 = 4.00 \text{ m}$. They attach at $x = \mp 0.400 \text{ m}$ to the guided crosshead of Fig. 12.29. A centered 30.0 kN downward load acts on it. Find the rod forces, common displacement, and guide couple; check both balances.

Solution. The stipulated axial model gives

$$k_1 = \frac{70.0 \times 10^9 \cdot 300 \times 10^{-6}}{1.75} = 12.0 \text{ MN m}^{-1}, \quad k_2 = \frac{200 \times 10^9 \cdot 160 \times 10^{-6}}{4.00} = 8.00 \text{ MN m}^{-1}.$$

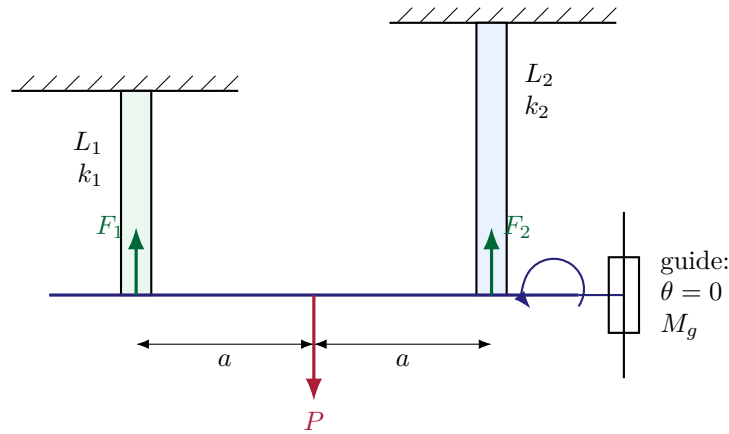


Figure 12.29: A guided crosshead can translate while remaining horizontal. Different upper-anchor levels allow different rod lengths; heights are schematic. The ideal guide supplies a balancing reaction couple and no vertical reaction, so both force and moment equilibrium are satisfied with unequal rod forces.

The guide imposes equal extension, so $d_0 = 30000/(20.0 \times 10^6) = 1.50 \text{ mm}$, $F_1 = 18.0 \text{ kN}$, and $F_2 = 12.0 \text{ kN}$. The force check is $18.0 + 12.0 - 30.0 = 0 \text{ kN}$. The required counterclockwise guide couple is

$$M_g = (0.400)(18.0 - 12.0) = 2.40 \text{ kN m}.$$

The moment check is $-(0.400)(18.0) + (0.400)(12.0) + 2.40 = 0 \text{ kN m}$. Constitutive substitution returns $F_1/k_1 = F_2/k_2 = 1.50 \text{ mm}$. The strains are $0.00150/1.75 = 8.57 \times 10^{-4}$ and $0.00150/4.00 = 3.75 \times 10^{-4}$, small within this stipulated model.

Limits. As $k_2 \rightarrow 0$, rod 1 carries all of P and the guide supplies aP . As $k_2 \rightarrow \infty$, rod 2 carries nearly all of P and the couple approaches $-aP$. Equal stiffness gives equal forces and zero guide couple. These limits show why equal displacement and equal force are distinct conditions.

12.10.3 Removing the guide: rigid does not mean horizontal

Now remove the antirotation guide, keep the two rod positions, and retain a centered load. The only crosshead reactions in the small-displacement planar model are F_1, F_2 . Force and moment equilibrium directly imply $F_1 + F_2 = P$ and $-aF_1 + aF_2 = 0$, hence $F_1 = F_2 = P/2$. Compatibility then determines the motion rather than a stiffness-proportional force split. From $d(-a) = F_1/k_1$ and $d(+a) = F_2/k_2$,

$$d_0 = \frac{P}{4} \left(\frac{1}{k_1} + \frac{1}{k_2} \right), \quad \theta = \frac{P}{4a} \left(\frac{1}{k_2} - \frac{1}{k_1} \right). \quad (12.34)$$

Unequal extension is compatible with a rigid rotating bar. To first order, the bar remains straight as it tilts; it does not bend to accommodate the difference. The rods are taken to remain approximately axial/vertical, with suitable end connections and sufficiently small rotation. The small horizontal movements of their lower ends enter beyond the order retained here.

Example 12.22: The same rods without the guide

Statement. Remove the guide from Example 12.21. Keep the centered 30.0 kN load, rod stiffnesses 12.0 and 8.00 MN m^{-1} , and locations $x = \mp 0.400 \text{ m}$. Find reaction forces, both extensions, center translation, and clockwise rotation. Diagnose the proposed answer $F_1 = 18.0 \text{ kN}$, $F_2 = 12.0 \text{ kN}$ with a horizontal bar.

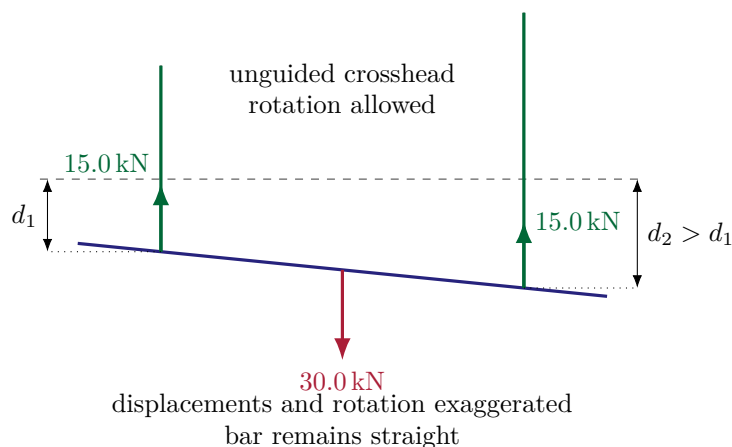


Figure 12.30: Removing the guide changes the mechanical problem. Symmetric reaction positions and a centered load force equal upward reactions, while unequal rod stiffnesses cause unequal extensions and a small rigid-body rotation. The dashed line marks the unloaded crosshead level.

Solution. There is no guide couple. Moment balance about the center gives equal forces, and force balance fixes each at 15.0 kN. Thus

$$d_1 = \frac{15000}{12.0 \times 10^6} = 1.250 \text{ mm}, \quad d_2 = \frac{15000}{8.00 \times 10^6} = 1.875 \text{ mm}.$$

The center translation is their mean, $d_0 = 1.5625 \text{ mm}$. The rotation is their difference divided by $2a$:

$$\theta = \frac{0.001875 - 0.001250}{0.800} = 7.8125 \times 10^{-4} \text{ rad} = 0.0448^\circ.$$

Substitution in $d_0 \mp a\theta$ returns the two extensions, providing a compatibility check. The rotation is small enough for the first-order kinematic model.

Diagnosis. The proposed 18/12 kN force split satisfies vertical balance but leaves a clockwise moment of 2.40 kN m. There is no external couple to balance it. Declaring the bar rigid does not remove that moment or prohibit rotation. Those forces belong to the explicitly guided problem, not to this unguided one. A correct solution must use the same constraints in its diagram, equations, and interpretation.

Guided checks

Diagnostic X12-08. Does a rigid bar force all its support points to have the same vertical displacement? Only if rotation is prevented or vanishes for the complete mechanical model. A rigid bar can translate and rotate with $d(x) = d_0 + \theta x$. Thus a symmetric drawing alone establishes neither equal displacement nor equal force when the stiffnesses or constraints differ.

12.10.4 Three supports: a genuinely redundant planar system

With three vertical supports, an unguided planar bar has three reaction forces but only two relevant equilibrium equations: vertical force and out-of-plane moment. Horizontal force balance is either identically satisfied or determines a separate horizontal reaction; it does not provide a third independent equation for these vertical reactions. Elastic deformation plus rigid compatibility supplies the needed information.

Let support i lie at x_i , have stiffness k_i , and have no initial gap. With all three engaged, its upward

reaction is $R_i = k_i(d_0 + \theta x_i)$. A downward point load P at $x = e$ gives

$$\begin{pmatrix} \sum k_i & \sum k_i x_i \\ \sum k_i x_i & \sum k_i x_i^2 \end{pmatrix} \begin{pmatrix} d_0 \\ \theta \end{pmatrix} = \begin{pmatrix} P \\ Pe \end{pmatrix}. \quad (12.35)$$

This matrix is simply a compact notation for two simultaneous equations. The first row is $\sum R_i = P$ and the second is $\sum x_i R_i = Pe$. Their coefficients have different units, matching displacement and dimensionless rotation. After solving for the motion, recover each reaction and check its allowed sign. A compression-only post cannot pull downward; a calculated negative reaction would require removing that contact and solving the active-support problem again.

The determinant is positive for positive stiffnesses at at least two distinct positions. One useful identity is

$$(\sum k_i)(\sum k_i x_i^2) - (\sum k_i x_i)^2 = \sum_{i < j} k_i k_j (x_i - x_j)^2 > 0.$$

The identity explains what the algebra requires physically: spatially separated supports resist both translation and rotation. If all locations coincide, they cannot supply an independent restoring moment. It also provides a symbolic check on the matrix rather than merely trusting a numerical solver.

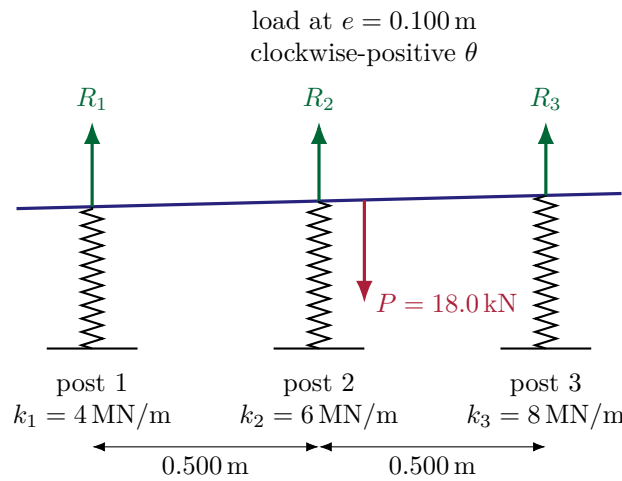


Figure 12.31: A three-post support is statically redundant: force and moment balance must be combined with the linear displacement profile of the rigid bar and each post's stiffness. Spring symbols represent axial post compliance, and the tiny actual rotation is exaggerated.

Example 12.23: Solve a three-post support

Statement. A rigid bar rests on three linear compression-only posts at $x = -0.500, 0, +0.500$ m with stiffnesses $4.00, 6.00, 8.00 \text{ MN m}^{-1}$. Their unloaded tops lie on the same level, and there are no initial gaps. A downward load 18.0 kN acts at $x = +0.100$ m. Neglect bar weight. Find center displacement, rotation, and all post reactions. Check contact admissibility and compare with assigning one-third of the load to every post.

Solution. The coefficients in Eq. (12.35) are $\sum k_i = 18.0 \times 10^6 \text{ N/m}$, $\sum k_i x_i = 2.00 \times 10^6 \text{ N}$, and $\sum k_i x_i^2 = 3.00 \times 10^6 \text{ N m}$. Thus

$$18.0 \times 10^6 d_0 + 2.00 \times 10^6 \theta = 18000,$$

$$2.00 \times 10^6 d_0 + 3.00 \times 10^6 \theta = 1800.$$

Eliminating either unknown gives $d_0 = 0.001008 \text{ m}$ and $\theta = -7.20 \times 10^{-5} \text{ rad}$. The negative clockwise angle means the right end moves slightly less downward. The three displacements are

$$(d_1, d_2, d_3) = (1.044, 1.008, 0.972) \text{ mm}.$$

Multiplying by the corresponding stiffnesses gives

$$(R_1, R_2, R_3) = (4.176, 6.048, 7.776) \text{ kN}.$$

All are positive, so the assumed engaged compression-only contacts are admissible. The force sum is 18.000 kN and the moment is $(-.500)(4.176) + (.500)(7.776) = 1.800 \text{ kN m}$, equal to Pe . Compatibility is checked by $(d_1 + d_3)/2 = d_2$.

Interpretation. Equal 6.00 kN reactions would have zero resultant moment and therefore could not balance this eccentric load. They would also imply extensions 1.500, 1.000, 0.750 mm, whose midpoint fails the rigid-bar condition. Both equilibrium and compatibility reject that guess. The total stored energy is $\sum R_i^2/(2k_i) = 9.0072 \text{ J}$; it also equals $\frac{1}{2}P(d_0 + \theta e)$ for proportional loading from zero, providing an independent work check.

12.10.5 Initial gaps change which supports are active

A support may not engage until a crosshead has moved far enough to close a gap. For a guided horizontal crosshead, let post 1 engage immediately with stiffness k_1 , and let post 2 have initial downward gap $h > 0$ and stiffness k_2 . Take compression-only contact. Then

$$F_1 = k_1 d, \quad F_2 = k_2 \max(d - h, 0), \quad P = k_1 d + k_2 \max(d - h, 0). \quad (12.36)$$

For $P \leq k_1 h$, only post 1 carries load and $d = P/k_1$. Above that threshold both engage, so $d = (P + k_2 h)/(k_1 + k_2)$. Force is continuous at gap closure, while the tangent stiffness rises from k_1 to $k_1 + k_2$. The guide still supplies whatever couple is needed to maintain zero rotation; contact activation does not remove moment balance.

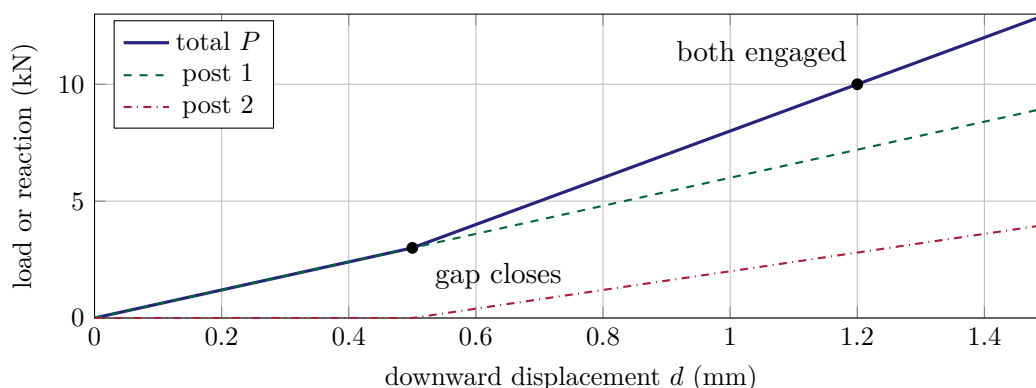


Figure 12.32: An initial 0.500 mm gap produces a piecewise linear response for guided supports with $k_1 = 6.00$ and $k_2 = 4.00 \text{ MN/m}$. The total tangent stiffness changes when post 2 begins carrying compressive force; no force jump is imposed at contact.

Guided calculation G12-01. For Fig. 12.32, find the displacement and reactions at $P = 10.0 \text{ kN}$. The gap closes at $k_1 h = 3.00 \text{ kN}$, so the both-engaged branch applies. It gives $d = (10000 + 4.00 \times 10^6 \times .000500)/(10.0 \times 10^6) = 1.20 \text{ mm}$. Thus $F_1 = 7.20 \text{ kN}$ and $F_2 = 2.80 \text{ kN}$; their sum is 10.0 kN and $d - h = 0.700 \text{ mm} > 0$, validating the selected branch. If posts are at $\mp 0.400 \text{ m}$, the centered load requires a counterclockwise guide couple $0.400(7.20 - 2.80) = 1.76 \text{ kN m}$.

12.10.6 A controlled stability model and its limits

Equilibrium specifies zero net generalized force at one configuration; stability asks what happens after an admissible perturbation. A simple model makes this distinction quantitative without invoking unintroduced beam-bending theory. Consider a rigid rod of length ℓ hinged at its bottom. A torsional spring at the hinge exerts restoring torque $-K_\theta \theta$, with K_θ in N m per radian. A

fixed-direction downward dead load P acts at the upper tip. The rod is initially vertical and the angle θ is its tilt. This model is a hinged rigid link, not an elastic column.

The tip height is $\ell \cos \theta$. Relative to the upright state, its total potential energy is

$$\Pi(\theta) = \frac{1}{2}K_\theta\theta^2 + P\ell(\cos \theta - 1). \quad (12.37)$$

The spring term rises with tilt, whereas the dead load lowers potential energy when the tip descends. Differentiating gives $d\Pi/d\theta = K_\theta\theta - P\ell \sin \theta$. It vanishes at $\theta = 0$ for every P : the perfectly upright rod is always an equilibrium configuration in this ideal model. Expanding $\cos \theta = 1 - \theta^2/2 + \theta^4/24 + \dots$ reveals

$$\Pi(\theta) = \frac{1}{2}(K_\theta - P\ell)\theta^2 + \frac{P\ell}{24}\theta^4 + \mathcal{O}(\theta^6). \quad (12.38)$$

Positive curvature $K_\theta - P\ell > 0$ gives a restoring response to sufficiently small tilt. Negative curvature makes the upright state unstable. At the critical value $P = K_\theta/\ell$, the linearized restoring stiffness vanishes; in this particular controlled model the positive quartic term still gives a local minimum, but the leading restoring torque is cubic. One cannot make a universal statement about zero curvature without examining the higher terms and constraints of the actual system.

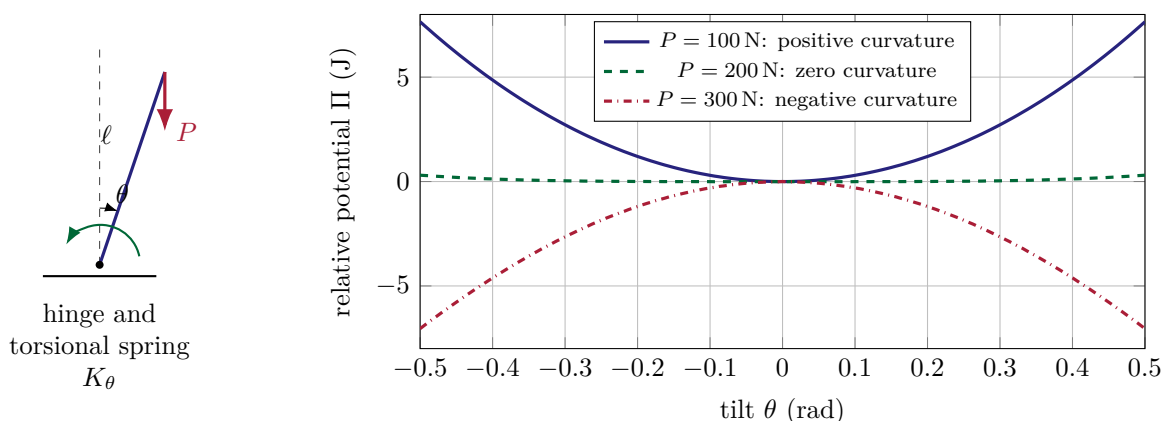


Figure 12.33: The hinged-link geometry (schematic) and potential for $K_\theta = 120 \text{ N m}$ and $\ell = 0.600 \text{ m}$. The tip load keeps its downward direction as the rigid rod tilts. All curves have zero slope at the upright state, but their curvature distinguishes a local minimum from a local maximum. The zero-curvature curve retains a positive quartic rise in this specified model.

Example 12.24: Balance at an unstable configuration

Statement. In the hinged-link model take $K_\theta = 120 \text{ N m}$ and $\ell = 0.600 \text{ m}$. Determine the upright equilibrium's linearized stability for $P = 100, 200, 300 \text{ N}$, and the critical load. Explain whether that critical load is an Euler buckling result for a real slender rod.

Solution. At $\theta = 0$, $K_\theta\theta - P\ell \sin \theta = 0$ for all three loads. The critical curvature change occurs at $P_c = K_\theta/\ell = 200 \text{ N}$. The quadratic curvatures $K_{\text{eff}} = K_\theta - P\ell$ are $60, 0, -60 \text{ N m}$, respectively. Therefore the 100 N configuration has positive linear restoring stiffness, while the 300 N configuration is unstable despite satisfying exact upright equilibrium.

At 200 N the quadratic test is inconclusive by itself. The specified exact potential has leading term $(P\ell/24)\theta^4 = 5\theta^4 \text{ J}$, which is positive for sufficiently small nonzero tilt. The upright state remains a local minimum with vanishing linear stiffness. This higher-order conclusion belongs to this potential and cannot be transferred to a different structure merely because its quadratic coefficient also vanishes.

Boundary and check. The load 200 N is the stability threshold of a rigid hinged link with a stipulated torsional spring. It is not an Euler-column formula. Quantitative buckling of a slender elastic member requires bending stiffness, member length and cross-section, end

conditions, load application, and a deformation model; imperfections may further change the response. The core axial relation $\sigma = E_Y \varepsilon$ alone supplies none of that lateral theory. Differentiating the exact potential twice at zero independently returns all three curvature values shown in Fig. 12.33.

12.10.7 Connection to oscillations

A stable elastic configuration can also have a small dynamic response. If the hinged model has effective rotational inertia I and no damping, the linearized motion about the upright state is $I\ddot{\theta} + (K_\theta - P\ell)\theta = 0$. Its squared angular frequency is $(K_\theta - P\ell)/I$ when positive. This forward connection explains why the same curvature governs restoring static behavior and small oscillations. It does not turn a static force–extension fit into a complete dynamic measurement: inertia, damping, admissible coordinates, and constraints must also be specified. The oscillations chapter develops those ingredients systematically.

12.11 Synthesis and formula guide

An equilibrium calculation has three distinct levels. First, specify the body and its supports. The force and moment equations then determine whether that model can balance the applied actions. Finally, test whether the calculated reactions can be supplied: an ordinary contact cannot pull, a cable cannot push, and static friction has a finite range. A zero algebraic residual does not repair an inadmissible contact model.

Deformation supplies a different kind of information. A specimen stiffness combines material modulus with geometry; a force–extension slope is therefore not automatically a material property. When several elastic members share a load, the shape and motion of their connections determine compatibility. A rigid bar can translate and rotate. Equal displacement at separated attachment points requires suppression of rotation or an appropriate special loading condition.

Formula summary

Planar rigid body, initially at rest. With rightward x , upward y , and counterclockwise-positive moments,

$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum (x_i F_{y,i} - y_i F_{x,i}) + \sum M_i = 0.$$

Changing the moment origin by $\mathbf{a}$ gives $\boldsymbol{\tau}_{O'} = \boldsymbol{\tau}_O - \mathbf{a} \times \sum \mathbf{F}$. Under force balance the residual moment is origin independent.

Distributed downward load. For a single-sign load density $w(x)$,

$$W = \int w(x) dx, \quad \bar{x} = \frac{\int xw(x) dx}{W}.$$

The replacement preserves the total external force and moment, not the local internal loading.

Admissibility. A unilateral normal contact has $N \geq 0$; a cable has $T \geq 0$; a static friction force must satisfy $|f_s| \leq \mu_s N$. For a symmetric housing of support width b and centre-of-mass height h ,

$$\tan \theta_{\text{tip}} = \frac{b}{2h}, \quad \tan \theta_{\text{slide}} = \mu_s.$$

These are thresholds of the stated unrestrained, quasistatically tilted contact model.

Formula summary

Small, approximately uniform axial deformation. With tension-positive signs and reference geometry A, L_0 ,

$$\sigma = \frac{F_{\perp}}{A}, \quad \varepsilon = \frac{\Delta L}{L_0}, \quad \sigma = E_Y \varepsilon, \quad k = \frac{E_Y A}{L_0}.$$

For a slowly varying axial member within the local linear model, $\delta = \int N(x)/[E_Y(x)A(x)] dx$.

Shear and bulk. Engineering shear strain is $\gamma = \Delta x/h = \tan \phi$; only at small distortion is $\gamma \simeq \phi$ in radians. The linear relations are

$$\tau_{\text{shear}} = G_s \gamma, \quad \Delta p = -B \frac{\Delta V}{V_0}.$$

The small-strain, homogeneous isotropic elastic model additionally gives $G_s = E_Y/[2(1+\nu)]$ and $B = E_Y/[3(1-2\nu)]$; these are not universal constraints on unrelated tabulated measurements.

Reversible linear energy.

$$U_{\text{el}} = \frac{1}{2} F \delta = \frac{1}{2} k \delta^2, \quad u_{\text{el}} = \frac{1}{2} \sigma \varepsilon = \frac{\sigma^2}{2E_Y}.$$

The density is per reference volume. For a nonlinear reversible relation, integrate the actual stress–strain curve; for irreversible loading distinguish recoverable energy from work dissipated.

Guided axial load sharing. For two engaged supports with suppressed crosshead rotation and no vertical guide force,

$$\delta = \frac{P}{k_1 + k_2}, \quad F_i = k_i \delta.$$

Moment balance still applies: the guide must supply the needed reaction couple. An unguided bar instead requires force balance, moment balance, and its translation–rotation compatibility.

The chapter’s measurement sequence follows the same separation. A regression estimates a slope under a statistical model. Independent geometry converts that slope to a modulus. Grip compliance, misalignment, changing area, hysteresis, and drift can invalidate the physical inference even when the regression is numerically precise. Residual plots and loading–unloading comparisons test different parts of that inference.

Equilibrium is a condition at a configuration; stability concerns nearby admissible configurations. A positive second derivative of the appropriate total potential energy gives local restoring behaviour in a smooth unconstrained coordinate. Contact boundaries, vanishing curvature, or a missing deformation mode require a more careful model. This distinction prepares both the gravitational reasoning of the next chapter and the oscillation physics developed later.

Bridge to Chapter 13: gravitation

Here a nearly uniform gravitational field allowed the weight of an extended body to act at its centre of mass for total force and moment calculations. Chapter 13 develops the gravitational interaction itself: universal attraction, gravitational fields, and potential energy. The same model-building habits extend from laboratory objects to planets and satellites, but a field that varies appreciably across an object requires integrating its gravitational loading rather than assuming one constant g . Force balance describes a supported configuration; orbital motion instead requires a nonzero resultant force. Potential-energy reasoning connects both settings and provides another way to assess equilibria and their stability.

12.12 Exercises and problems

Use $g = 9.81 \text{ ms}^{-2}$. Moduli, friction coefficients and spring stiffnesses are stipulated teaching inputs. Use SI units internally and the small-deformation linear-elastic model wherever stated. Draw the body being analysed, state the positive moment direction, and test the relevant contact or constitutive conditions.

Problems 1–8 diagnose concepts across the chapter, including elastic response and advanced guided compatibility; Problems 9–16 develop equilibrium. Problems 17–24 develop elastic response; Problems 25–30 use the advanced material. Problems 31–32 are computational and experimental investigations. Problems 33–42 combine representations and models. Every item and subpart has a reasoned solution in Section 12.13.

- Pure-couple diagnostic.** Two horizontal forces $+85 \text{ N}$ and -85 N act on a plate along parallel lines separated by 0.42 m . Use Figure 12.2 to draw this particular body-only diagram. (a) Find resultant force and couple magnitude. (b) Explain what changes if the lines become collinear. (c) Explain why translating the moment origin cannot remove a nonzero couple.
- Tipping versus sliding.** A rectangular device has base width 0.50 m in the downslope direction and centre-of-mass height 0.40 m above the centre of its base. It rests on a slowly tilted plane with $\mu_s = 0.55$. (a) Adapt the support-resultant construction in Figure 12.5 and find ideal tipping and sliding angles. (b) Identify which occurs first. (c) Mark the gravity-line intersection with the base at the first threshold and explain whether it has reached the downhill edge.
- Unilateral reaction.** A weightless beam is supported at A , $x = 0$, and B , $x = 2.0 \text{ m}$, by upward-only contacts. A 600 N downward load acts at $x = 3.0 \text{ m}$. (a) Solve the formal reaction equations. (b) Interpret the sign at A . (c) Explain why merely setting the inadmissible reaction to zero does not produce another static solution of the same loading.
- Material versus geometry.** At fixed tensile load, initial length L_0 and Young modulus E_Y , a rod's area is doubled. State, with reasons, what happens to stress, strain, extension and Young modulus. Repeat for doubling E_Y at fixed load and geometry. Assume small linear-elastic strain and unchanged axial loading.
- Bulk sign.** A specimen experiences a positive pressure increase. Explain why a stable positive bulk modulus requires $\Delta V < 0$. What would positive ΔV mean physically in this convention? Distinguish pressure, a positive compression magnitude, from tension-positive axial stress.
- Proportionality and elasticity.** Sketch a loading–unloading response that is nonlinear but fully reversible. Label axes, a linear range, a proportionality threshold and both loading paths. Explain why plastic strain need not occur immediately beyond a proportionality limit. State which observation, rather than curvature alone, would demonstrate residual strain.
- Fit diagnosis.** An axial force–extension fit has a statistically small slope uncertainty, but residuals increase systematically with extension. Give two physical or experimental reasons not to report the resulting E_Y as a reliable linear-elastic modulus. Explain what a residual graph contributes beyond the fitted slope and its statistical uncertainty.
- Compatibility concept.** Two rods with unequal axial stiffnesses support a rigid crosshead constrained by an ideal antirotation guide to translate vertically without rotating. The guide transmits a couple but no resultant vertical force. Explain why $F_1 + F_2 = P$ is insufficient. State geometric and constitutive conditions closing the model. Explain why rigidity of an *unguided* crosshead would not, on its own, impose equal rod extensions.
- Two equal-and-opposite 150 N vertical forces form a couple with perpendicular separation 0.32 m . Find its magnitude. Does moving the origin change it? State what geometric information is

required to give its signed moment.

10. **Two point loads.** A simply supported 5.0 m weightless beam has supports at its ends. Downward point loads of 600 N and 400 N act at $x = 2.0$ m and $x = 4.0$ m from the left support. (a) Draw a dimensioned body-only diagram, indicating which reaction is eliminated by moments about the left support. (b) Find both vertical reactions. (c) Check moments about the right support and admissibility of upward-only contacts.
11. Two point masses 4.0 kg and 6.0 kg lie on a light bar at $x = 0.20$ m and $x = 1.00$ m. Find the centre-of-mass coordinate. In a uniform gravitational field, where may their combined weight be placed for total torque calculations? Explain why the qualifier *uniform* matters.
12. A rigid rectangular housing has base width 0.40 m and centre-of-mass height 0.30 m above the centre of its base. Find its ideal tipping angle on a frictional incline, assuming sliding is prevented. Identify the geometric event at this threshold and one condition under which this result would cease to describe the housing.
13. A uniform distributed load $w = 250 \text{ N m}^{-1}$ acts only from $x = 1.0$ m to $x = 4.0$ m. (a) Draw its intensity graph and shade the area representing force. (b) Replace it by a statically equivalent resultant for overall force and moment balance. State magnitude and line of action. (c) Explain why this replacement does not preserve internal loading at every section.
14. **Overhanging beam.** A 6.0 m weightless beam has vertical supports at $x = 0$ and $x = 5.0$ m. A uniform downward load 100 N m^{-1} acts over all 6.0 m; a 300 N downward point load acts at $x = 5.5$ m. (a) Draw the body-only diagram and replace the distributed load appropriately. (b) Find support reactions. (c) Check moments about both supports and admissibility of upward-only contacts.
15. **Cable-supported sign.** A horizontal 3.0 m sign is pinned at its left end. Its own weight is 300 N at the midpoint; an additional 500 N downward load acts at $x = 2.4$ m. A cable at its right end exerts tension 40° above the sign, directed upward and rightward toward its anchor. (a) Draw the sign free-body diagram and resolve tension. (b) Find tension and pin-reaction components. (c) Explain why the cable's horizontal component makes no moment about the pin in this zero-thickness horizontal geometry.
16. For Problem 10, redo the calculation taking moments first about the right support rather than the left. Show equations, lever arms and signs. Explain why a different origin cannot change a correctly computed reaction.
17. A uniform rod carries 20.0 kN tension. Its area is 250 mm^2 , initial length 1.00 m and $E_Y = 160 \text{ GPa}$. (a) Find stress, strain and extension. (b) Locate this point on a labelled linear stress-strain sketch and identify the material-property slope. (c) Check small strain.
18. Rod A has $L = 0.60$ m, $A = 60 \text{ mm}^2$, $E_Y = 60 \text{ GPa}$; Rod B has $L = 0.90$ m, $A = 100 \text{ mm}^2$, $E_Y = 150 \text{ GPa}$. Each carries 8.0 kN tension. Find both extensions and axial stiffnesses. Identify the stiffer specimen and explain why this comparison is not a measurement of modulus alone.
19. A block under simple shear has $F_{\parallel} = 12.0 \text{ kN}$ over $A = 400 \text{ mm}^2$, height 0.150 m and $G_s = 30 \text{ GPa}$. Find shear stress, engineering shear strain and top-face displacement. Explain why the opposite face requires a balancing traction or support force.
20. A body has $V_0 = 0.020 \text{ m}^3$ and $B = 50 \text{ GPa}$. A pressure increase 8.0 MPa is applied. Find ΔV , express it in cm^3 , and check the small fractional-volume-change assumption.
21. A uniform linear specimen has $E_Y = 100 \text{ GPa}$, $A = 200 \text{ mm}^2$ and $L_0 = 0.50$ m. Find axial stiffness. If the force-extension slope is measured without knowing A or L_0 , which quantity has

been determined directly? State units distinguishing it from modulus.

22. A linear specimen is loaded quasi-statically from zero to $F = 5.0 \text{ kN}$ at extension 0.40 mm . (a) Use a force–extension graph of the kind in Figure 12.23 to find energy. (b) If $A = 100 \text{ mm}^2$ and $L_0 = 0.50 \text{ m}$, find stress, strain and energy density. (c) Verify that reference volume times energy density gives the same energy and explain why $F\delta$ is not the work in this loading history.
23. An axial test gives $F = 2.00 \text{ kN}$ at $\Delta L = 0.200 \text{ mm}$ for $L_0 = 0.500 \text{ m}$ and $A = 40.0 \text{ mm}^2$ in a confirmed linear range, with corrected zero and no additional measured grip motion. Infer stiffness and modulus. Explain why these qualifications are necessary when inferring a slope from one nonzero point.
24. Two rods of the same material have small linear-elastic extensions. Rod B has twice Rod A's area and half its length. Find k_B/k_A and, under the same force, $\Delta L_B/\Delta L_A$. Explain both geometric ratios.
25. **Advanced: guided parallel supports.** Two axial supports have $k_1 = 6.0 \text{ MN m}^{-1}$ and $k_2 = 4.0 \text{ MN m}^{-1}$. A rigid crosshead carries $P = 20.0 \text{ kN}$ and is held horizontal by an ideal antirotation guide carrying a couple but no vertical resultant. (a) Find common displacement and forces. (b) Check compatibility and equilibrium. (c) Use Figure 12.29 to explain the guide's role in moment balance.
26. **Advanced: equation counting.** A rigid plate in a planar model is supported by four independent vertical elastic posts at distinct horizontal coordinates. Horizontal effects are absent and no antirotation guide is imposed. How many independent force/moment equations are available for these reactions? Explain why equilibrium is insufficient, and describe deformation, stiffness and contact information closing the model. Distinguish this planar problem from noncollinear supports in three dimensions.
27. **Advanced: stiffness limits.** For the *guided, nonrotating* parallel-support model, $F_i = Pk_i/(k_1 + k_2)$. (a) Show that $F_1/F_2 = k_1/k_2$. (b) Evaluate $k_2 \rightarrow 0$ and $k_2 \rightarrow \infty$ at fixed P and $k_1 > 0$. (c) Explain both limits, including the displacement, without confusing an infinitely stiff support with one carrying no force.
28. **Advanced: load extrapolation.** In Example 12.21, retain its ideal antirotation guide, rod data and linear stiffness model, but increase load to 45.0 kN . Rod 1 has $E_{Y,1} = 70 \text{ GPa}$, $A_1 = 300 \text{ mm}^2$, $L_1 = 1.75 \text{ m}$; rod 2 has $E_{Y,2} = 200 \text{ GPa}$, $A_2 = 160 \text{ mm}^2$, $L_2 = 4.00 \text{ m}$. Find displacement, forces, stresses and strains. State what must still be checked before trusting extrapolation.
29. **Advanced: uncertainty.** For $E_Y = kL_0/A$, independent relative standard uncertainties are $u(k)/k = 0.5\%$, $u(L_0)/L_0 = 0.2\%$, $u(A)/A = 1.0\%$. Derive and calculate the first-order relative uncertainty. Identify the dominant contribution. Explain what changes for correlated inputs.
30. **Advanced: equilibrium and stability.** Explain why a perfectly straight compressed slender member can satisfy force balance without proving lateral stability. Name at least three additional modelling ingredients for quantitative buckling. Explain why axial stress alone does not settle this question.
31. **Computational equilibrium check.** Write a calculation for Problem 14 solving reactions and evaluating $\sum F_y$ and $\sum \tau$ about both supports. Supply inputs, algorithm, outputs with units, and verification strategy. Report residuals, explain exact and floating-point expectations, and propose an incorrect reaction pair detected by moments even when force balance passes.
32. **Measurement investigation: proposed, not conducted.** Design a non-destructive low-load extension study. Identify direct and inferred quantities, repeatability and zero-offset checks and

two plausible model biases. Give a criterion for shortening the linear fit interval. Explain how to separate specimen extension from apparatus motion and assess reversibility. No equipment or actual measured dataset is assumed.

33. **Ladder with unilateral contacts.** Figure 12.34(a) shows a 5.00 m uniform ladder of mass 20.0 kg at 60.0° to a rough horizontal floor. A person of mass 60.0 kg stands 3.00 m along it from the foot. The wall is smooth; floor $\mu_s = 0.400$. (a) Draw the body-only diagram using permitted contact forces. (b) Find wall normal, floor normal and floor friction. (c) Test contacts and find the minimum angle at which the same person position can remain static.
34. **Triangular load graph.** The 4.00 m weightless simply supported beam in Figure 12.34(b) has downward intensity $w(x) = (600 \text{ N m}^{-1}) x / (4.00 \text{ m})$, $0 \leq x \leq 4.00 \text{ m}$. (a) Calculate total force as graph area. (b) Integrate its first moment and find equivalent position. (c) Find reactions and check moments about both ends. (d) Diagnose putting the resultant at the midpoint solely because the beam is symmetric.

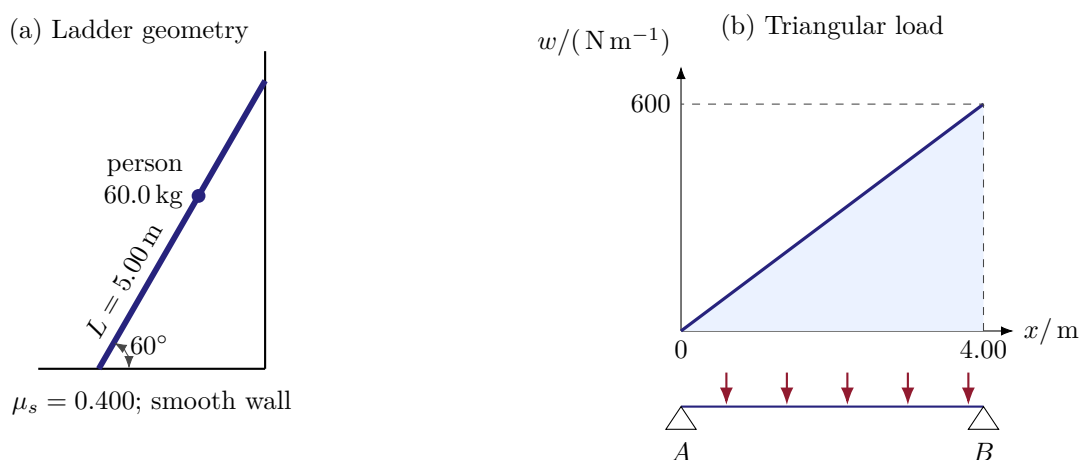


Figure 12.34: Data for Problems 33 and 34. The ladder gives positions before reactions are introduced. The triangular area's horizontal first moment determines the resultant position.

35. **Cable-angle graph.** Retain Problem 15's sign and loads, varying cable angle θ from 10° to 80° , with tension upward and rightward. Figure 12.35 gives tension predicted by moments. (a) Derive the relationship, estimate tensions at 25° , 40° , 60° from the graph and refine numerically. (b) A stipulated limit is $T \leq 1.00 \text{ kN}$; find the admissible angular interval. (c) Find how both pin-reaction components vary and explain the small-angle limit.

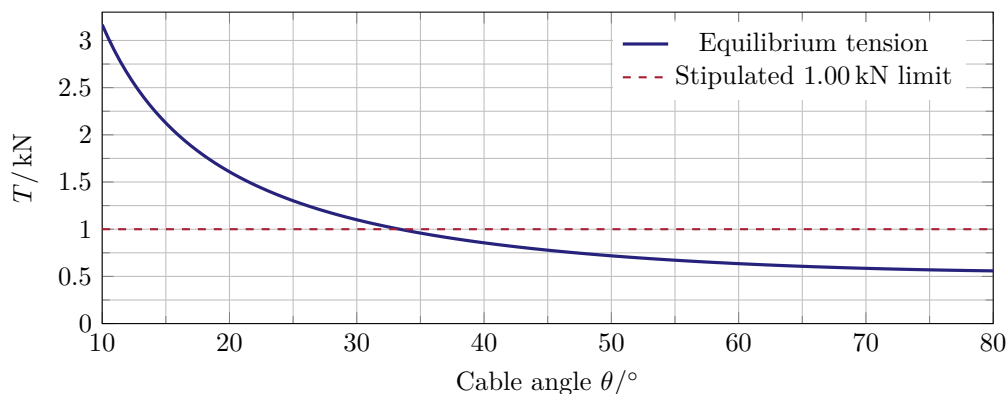


Figure 12.35: Tension demand as the cable approaches horizontal. The horizontal line is a stipulated exercise limit, not a certified rating. Use its intersection to infer allowed angles in Problem 35.

36. **Reading reaction admissibility.** A 600 N downward load can be placed at s on a weightless beam supported at $x = 0, 2.00$ m. Both contacts push upward only. Figure 12.36 shows formal reactions for $-1.00 \leq s \leq 3.00$ m. (a) Identify which line belongs to each support and derive both. (b) Read the admissible interval. (c) Evaluate $s = 3.00$ m and relate it to Problem 3. (d) Interpret zero-reaction endpoints, distinguishing exact balance from robust two-contact support.

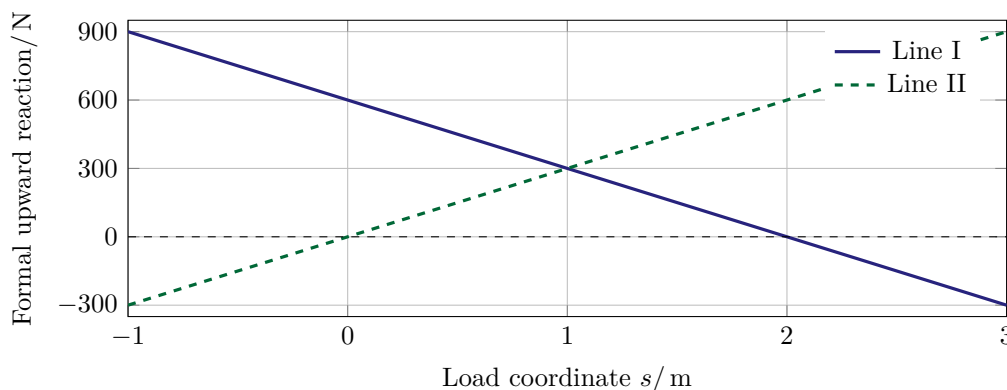


Figure 12.36: Formal two-contact equations. A negative plotted value cannot be supplied by an upward-only support; infer the admissible interval in Problem 36.

37. **Axial segments in series.** Two collinear rods connected end to end by a light connector carry 10.0 kN tension. Segment 1 has $L_1 = 0.800$ m, $A_1 = 100$ mm², $E_{Y,1} = 70.0$ GPa; segment 2 has $L_2 = 1.20$ m, $A_2 = 80.0$ mm², $E_{Y,2} = 200$ GPa. (a) Identify the common quantity. (b) Find each extension, total extension and effective stiffness. (c) Find energy by summing segment energies and integrating the total force–extension relation. (d) Compare with guided parallel rods.
38. **Apparatus compliance.** The synthetic exact curve in Figure 12.37(a) gives actuator force against *total* travel for a specimen and elastic grips in series. It passes through the origin and reaches 5.00 kN at 0.750 mm total travel. Specimen $L_0 = 0.500$ m, $A = 50.0$ mm²; modulus is unknown. Combined grip stiffness is $k_g = 20.0$ MN m⁻¹. (a) Infer assembly stiffness and apparent modulus assigning all travel to the specimen. (b) Correct for grip compliance and infer specimen modulus. (c) Calculate the apparent modulus's fractional bias. Explain why a straight repeatable curve can still give biased modulus.
39. **A nonuniform rod.** Figure 12.37(b) specifies $A(x) = A_0(1 + x/L)$, with $A_0 = 100$ mm², $L = 1.00$ m. A smoothly varying rod has $E_Y = 70.0$ GPa and carries $F = 7.00$ kN. Neglect local end/taper effects in a one-dimensional axial model. (a) Integrate strain to find extension. (b) Find extreme stresses and strains. (c) Compare with replacing area by its length average; give the error's size and sign. (d) Find energy and identify a model assumption this result does not test.
40. **Three posts and a rotating rigid bar.** Figure 12.38(a) shows a weightless rigid bar on vertical compression springs at $x = 0, 1.00, 2.00$ m. All have $k = 2.00$ MN m⁻¹, initially just touch, and push but cannot pull. A 12.0 kN downward load acts at $x = 1.50$ m. The bar may translate and rotate slightly. (a) Use compatibility $\delta(x) = a + bx$, downward positive, and equilibrium to find a, b and reactions. (b) Check contact and small rotation. (c) Find load-point displacement and energy; verify using work of a load increased proportionally from zero.

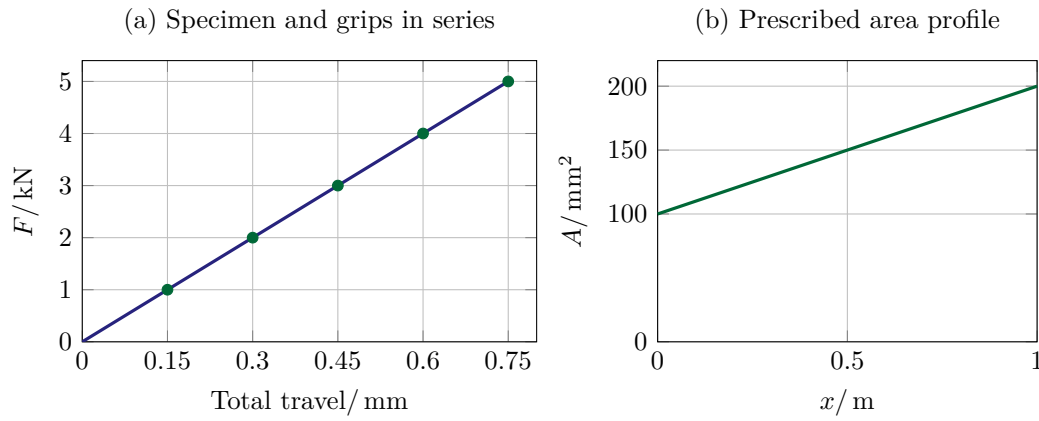


Figure 12.37: Data for Problems 38 and 39. Panel (a) is synthetic, not measured; panel (b) gives area, not stress or extension.

41. **Gap closure.** Figure 12.38(b) shows a plate held horizontal by an ideal antirotation guide carrying any required couple and no vertical resultant. Compression spring 1, $k_1 = 4.00 \text{ MN m}^{-1}$, initially touches; spring 2, $k_2 = 6.00 \text{ MN m}^{-1}$, initially has gap $d = 1.00 \text{ mm}$. (a) Derive piecewise $P(\delta)$ including the contact threshold. (b) For $P = 6.00 \text{ kN}$, find displacement and forces. (c) Find energy using actual compressions and the area under the piecewise graph. (d) Explain why $(k_1 + k_2)\delta = P$ is wrong before closure.

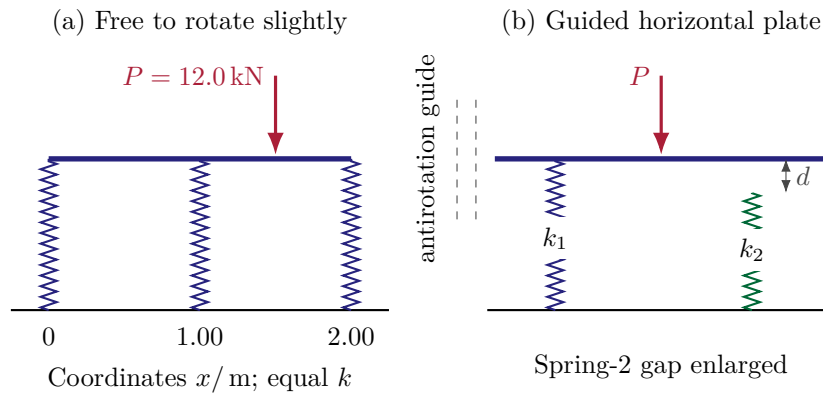


Figure 12.38: Reference configurations for Problems 40 and 41, before load application. A free rigid bar may develop a displacement gradient; an antirotation guide imposes equal plate displacement. An open gap transmits no spring force.

42. **Equilibrium and instability.** A massless rod of length $L = 1.00 \text{ m}$ is hinged at its base and carries mass $m = 2.00 \text{ kg}$ at the top. An ideal horizontal restoring device follows the top's height and exerts $F_x = -kx$, with $x = L \sin \theta$; its potential is $\frac{1}{2}kx^2$. Stipulated effective stiffness is $k = 30.0 \text{ N m}^{-1}$. Angle θ is measured from upright vertical. (a) Derive $U(\theta) - U(0)$ including gravity and show that $\theta = 0$ is an equilibrium. (b) Use Figure 12.39 and a small-angle expansion to decide stability and calculate angular stiffness. (c) Repeat for $k = 10.0 \text{ N m}^{-1}$. (d) At $k = mg/L$, inspect the first nonzero higher-order term. (e) For $k = 30.0 \text{ N m}^{-1}$, compare exact energy at $\theta = 0.100 \text{ rad}$ with the quadratic approximation.

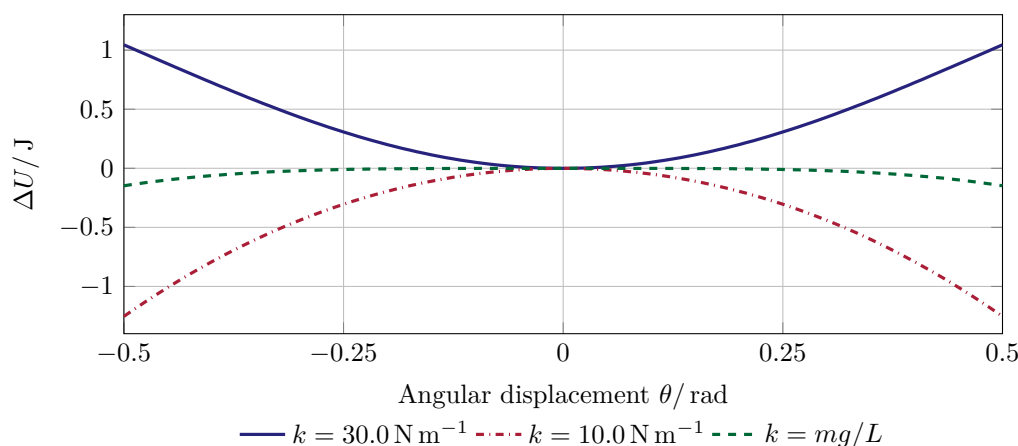


Figure 12.39: Exact potential changes for Problem 42, referenced to upright. The critical curve has zero quadratic curvature but decreases at fourth order.

12.13 Complete reasoned solutions

The solutions below separate equilibrium, geometry and constitutive response. Unrounded values are retained until the final display. A contact that would have to pull is rejected when its model only permits pushing. Numerical results apply to the stipulated models; they are not strength or design certifications.

Solution 1: a couple cannot be removed by an origin change

Figure 12.40(a) supplies the completed force-pair sketch. **(a)** Draw the two parallel horizontal arrows on distinct lines, with their equal magnitudes and opposite directions visible. Their vector sum is zero. Taking moments about any point on one line eliminates that force and leaves the other with perpendicular arm 0.42 m. The magnitude is

$$C = Fd = 85(0.42) = 35.7 \text{ N m}.$$

Its sign depends on which line carries the rightward force. The pair has no translational resultant but still has a rotational effect. **(b)** For collinear lines $d = 0$, so both force and moment vanish. **(c)** Under an origin shift $\mathbf{a}$, the total torque changes by $-\mathbf{a} \times \sum \mathbf{F}$. This is zero for the pair. Moving an origin changes individual lever arms, but their summed couple remains 35.7 N m.

Solution 2: competing contact thresholds

Figure 12.44(a) shows the actual gravity-line construction at the first threshold, with its intersection and downhill edge marked. **(a)** Resolve weight along and normal to the incline. While static, required friction is $f = mg \sin \theta$ and normal resultant $N = mg \cos \theta$. Thus $f/N = \tan \theta$, and sliding is first possible at

$$\theta_{\text{slide}} = \arctan \mu_s = \arctan 0.55 = 28.81^\circ.$$

The vertical gravity line crosses the base at a downslope displacement $h \tan \theta$ from its centre. Tipping requires that intersection to reach the edge, $h \tan \theta = b/2$. Therefore

$$\theta_{\text{tip}} = \arctan \frac{0.50}{2(0.40)} = 32.01^\circ.$$

(b) Increasing tilt encounters 28.81° first, so ideal sliding precedes ideal tipping. **(c)** At sliding onset the intersection is $0.40(0.55) = 0.220$ m from the base centre. The edge lies 0.250 m away, leaving 0.030 m of support width. Mark both distances on the sketch. After sliding begins, a static tipping comparison no longer describes the subsequent motion without a dynamic friction model.

Solution 3: algebraic balance and contact admissibility

(a) Let upward reactions be positive and counterclockwise moments positive. Balance gives

$$R_A + R_B = 600, \quad 2.0R_B - 600(3.0) = 0.$$

Hence $R_B = 900$ N and $R_A = -300$ N. (b) The latter is a downward force on the beam. A support underneath the beam that only pushes upward cannot supply it; the assumed two-contact static state is inadmissible. (c) If R_A is set to zero, force balance requires $R_B = 600$ N, but the load still produces $-600(1.0) = -600$ N m about B . There is no static solution with only that contact and unchanged loading. The beam tends to rotate or requires a different restraint such as an explicitly added hold-down; one cannot silently replace a contact by a pin.

Solution 4: separate material response from specimen geometry

In the axial linear regime,

$$\sigma = \frac{F}{A}, \quad \epsilon = \frac{\sigma}{E_Y}, \quad \delta = L_0 \epsilon = \frac{FL_0}{E_Y A}.$$

Doubling area at fixed F, L_0, E_Y halves stress, strain and extension. Young modulus is unchanged because the material relation has not changed; the assembled axial stiffness doubles. The stress reduction is geometric, not evidence of a modified material.

Doubling E_Y at fixed force and geometry leaves stress unchanged, because F/A is unchanged. Strain and extension halve. Here the stipulated modulus itself doubles, and stiffness again doubles. The identical extension ratio arises from different changes. A force–extension graph cannot distinguish these changes without geometric information; its slope is $E_Y A/L_0$, whereas a stress–strain slope is E_Y .

Solution 5: the minus sign in the bulk modulus

For a small volume change the local bulk relation is

$$B = -\frac{\Delta p}{\Delta V/V_0}.$$

Pressure is positive for compression. With $\Delta p > 0$ and a stable positive B , the denominator must be negative, hence $\Delta V < 0$: increasing pressure decreases volume. Positive ΔV would instead mean expansion as applied pressure rises and would give a negative incremental modulus under this convention. It would invalidate the ordinary positive-modulus stable-compression model rather than merely require dropping the minus sign.

Tension-positive axial stress uses a different scalar sign convention: uniaxial compression gives negative axial stress and strain. Hydrostatic pressure is a positive compression magnitude. Mixing these conventions makes a physically correct compression look algebraically inconsistent.

Solution 6: reversible nonlinearity

Figure 12.43 gives the completed sketch, including opposite loading and unloading arrows on the same path. One explicit sketch is a force–extension curve with

$$F(\delta) = \begin{cases} k\delta, & 0 \leq \delta \leq \delta_p, \\ k\delta + c(\delta - \delta_p)^2, & \delta > \delta_p, \end{cases} \quad k > 0, \quad c > 0.$$

The first branch is a straight line; beyond δ_p the curve bends upward and remains increasing. Here k has units N m^{-1} and c has units N m^{-2} . Draw loading arrows away from the origin and unloading arrows back *on the same curve*. Label δ_p as the end of the strictly proportional range.

The model assigns one force to each extension independent of loading direction. Unloading returns to $\delta = 0$ at $F = 0$ without residual strain, although it passed through a nonlinear range. Elastic reversibility and linear proportionality are different conditions. A nonzero unloaded extension after removing force, with offset and temperature effects excluded, would demonstrate residual deformation. Hysteresis would indicate path dependence but would require further interpretation before assigning all its area to permanent plastic strain.

Solution 7: precision within a wrong model

Define residuals as observed force minus the fitted straight-line force. Plotting residuals against extension should expose structure hidden by a small uncertainty on the fitted slope. A repeated upward curvature can indicate that the specimen has entered a nonlinear constitutive range: a single linear modulus no longer describes all points. Alternatively, changing grip or fixture compliance can alter measured travel as force increases, even when the specimen itself remains linear.

A small formal slope uncertainty quantifies scatter around the assumed regression model; it does not establish that the model, sensor interpretation or fit interval is correct. Repeating loading/unloading and fitting shorter intervals can distinguish repeatable curvature from random noise. Comparing a gauge-length sensor with actuator travel can isolate apparatus motion. An area error can additionally bias the modulus even with straight residuals, but by itself a constant area error does not explain curvature in a force–extension residual plot.

Solution 8: what closes the guided crosshead model

Vertical equilibrium supplies one equation for two rod forces: $F_1 + F_2 = P$. Many force pairs satisfy it. The antirotation guide imposes common axial end displacement,

$$\delta_1 = \delta_2 = \delta, \quad F_i = k_i \delta_i.$$

The first statement is geometric compatibility; the second is a constitutive stiffness relation. Combining them gives

$$P = (k_1 + k_2)\delta, \quad F_i = \frac{Pk_i}{k_1 + k_2}.$$

The guide's reaction couple completes moment balance when unequal reactions would otherwise rotate the crosshead; its zero vertical resultant leaves the displayed force equation intact.

A free rigid crosshead can translate *and rotate*. For small rotation its vertical displacement is $a + bx$, so ends at different coordinates generally have different displacements. Rigidity preserves distances within the member; it does not force a horizontal orientation. Equal extensions in the guided model follow from the stated antirotation constraint, not from drawing a rigid member horizontally.

Solution 9: perpendicular separation of vertical forces

The couple arm is the perpendicular distance between the vertical lines of action, so $C = 150(0.32) = 48.0 \text{ N m}$. The resultant force is zero. Under an origin translation the moment change is the shift crossed with that resultant, hence zero. The magnitude remains 48.0 N m .

To assign a signed planar moment one must state a positive rotation sense and which line carries the upward force. For example, upward force on the right and downward force on the left produce a counterclockwise couple in the usual x -right, y -up coordinates. Reversing the force placements reverses the sign without changing magnitude.

Solution 10: reactions from a point-load diagram

Figure 12.40(b) gives the requested dimensioned free-body diagram and solved reactions. (a) Draw the beam from $x = 0$ to 5.0 m , upward reactions at the ends, and downward 600 and 400 N arrows

at 2.0 and 4.0 m. Include distances rather than inferring them from a schematic scale. The left reaction has zero arm about the left support. **(b)** Vertical and left-origin moment balance give

$$R_A + R_B = 1000, \quad 5.0R_B - 600(2.0) - 400(4.0) = 0.$$

Therefore $R_B = 560$ N and $R_A = 440$ N. **(c)** About the right end,

$$-5.0R_A + 600(3.0) + 400(1.0) = -2200 + 1800 + 400 = 0 \text{ N m}.$$

Force balance also closes exactly. Both reactions are positive and therefore permitted by upward-only contacts. These checks establish global equilibrium and contact signs; they do not predict deflection or the beam's strength.

Solution 11: centre of mass and centre of gravity

The total mass is 10.0 kg and

$$x_{\text{CM}} = \frac{4.0(0.20) + 6.0(1.00)}{10.0} = 0.680 \text{ m}.$$

It lies between the masses and nearer the heavier one, providing a geometric check. In uniform gravity the combined downward force is $W = (10.0)(9.81) = 98.1$ N and may be applied through $x = 0.680$ m for total moments:

$$Wx_{\text{CM}} = g[m_1x_1 + m_2x_2] = 66.708 \text{ N m}.$$

The common factor g is what identifies this weight-resultant location with the mass centroid. If gravity varies appreciably across the system, the individual weights need not be proportional to masses with the same factor, and the centre of gravity must instead be obtained from their actual force distribution.

Solution 12: the gravity line reaches an edge

With sliding prevented, the gravity line intersects the base a distance $h \tan \theta$ downhill from the centre. The static support resultant can remain within the base until that distance reaches $b/2$. Thus

$$\tan \theta_{\text{tip}} = \frac{0.40}{2(0.30)} = \frac{2}{3}, \quad \theta_{\text{tip}} = 33.69^\circ.$$

At the threshold, the weight line and support resultant pass through the downhill edge. Beyond it an ordinary compressive pressure distribution over the base cannot balance the weight moment.

The result assumes a rigid housing, the stated centre of mass, an effectively planar contact, and no extra applied moments. A shifted payload changes the gravity line; deformation changes support geometry; an added tie-down can supply a restraining moment. If sliding were not prevented, its threshold would also have to be tested.

Solution 13: area and first moment of a rectangular load graph

The shaded load graph and its centroid are shown in Figure 12.42(a). **(a)** Plot $w = 250 \text{ N m}^{-1}$ between $x = 1$ and 4 m and zero elsewhere. The rectangle's area has units of force. **(b)** Its force and first moment about $x = 0$ are

$$W = \int_1^4 250 \, dx = 750 \text{ N}, \quad M = \int_1^4 250x \, dx = 1875 \text{ N m}.$$

Consequently $\bar{x} = M/W = 2.50$ m, also the rectangle's horizontal centroid. Replace it by 750 N downward at that coordinate. **(c)** This replacement preserves the total force and moment on the complete chosen body. A cut through the loaded interval includes only part of the distributed load, whereas the single resultant may lie wholly on one side of the cut. It therefore cannot be used as a universal replacement for the original load when calculating section-dependent internal forces.

Solution 14: distributed loading on an overhang

Figure 12.41(a) supplies the solved free-body diagram, with the uniform-load resultant and separate overhang load. **(a)** The uniformly loaded full length has resultant $W = 100(6.0) = 600 \text{ N}$ at $x = 3.0 \text{ m}$. The separate point load remains at $x = 5.5 \text{ m}$; it lies 0.5 m beyond support B . **(b)** Take upward and counterclockwise positive:

$$R_A + R_B = 900, \quad 5.0R_B - 600(3.0) - 300(5.5) = 0.$$

Then $R_B = 690 \text{ N}$ and $R_A = 210 \text{ N}$. **(c)** The left-origin residual is $3450 - 1800 - 1650 = 0 \text{ N m}$. About B , the distributed resultant is to the left and produces a positive moment, but the overhang load is to the right:

$$-5.0(210) + 600(2.0) - 300(0.5) = -1050 + 1200 - 150 = 0 \text{ N m}.$$

The force residual is $210 + 690 - 900 = 0 \text{ N}$. Both supports push upward, satisfying the unilateral-contact signs. Omitting the overhang or reversing its right-origin moment sign would fail the second check.

Solution 15: a cable and a two-component pin

The completed sign diagram is Figure 12.41(b); its rightward cable component is balanced by a leftward pin reaction. **(a)** The chosen body is the sign, not the cable-anchor assembly. Draw weights at 1.5 and 2.4 m , pin components A_x, A_y , and the rightward/upward cable force at 3.0 m . Components are $T_x = T \cos 40^\circ$, $T_y = T \sin 40^\circ$. **(b)** About the pin,

$$3.0T_y - 300(1.5) - 500(2.4) = 0, \quad T_y = 550 \text{ N}.$$

Hence

$$T = \frac{550}{\sin 40^\circ} = 855.648 \text{ N} \simeq 856 \text{ N},$$

$$A_x = -550 \cot 40^\circ = -655.464 \text{ N} \simeq -655 \text{ N}, \quad A_y = 300 + 500 - 550 = 250 \text{ N}.$$

Use unrounded T when calculating A_x ; early rounding to 856 N gives an avoidable discrepancy. The positive tension is admissible. **(c)** The horizontal cable component acts along the sign axis, through the pin's height. Its perpendicular moment arm is zero. A cable attachment displaced vertically would change that conclusion and require its actual coordinates.

Solution 16: a second origin gives the same reactions

About the right support in Problem 10, upward R_A at a leftward arm gives a clockwise moment. Both downward loads to the left give counterclockwise moments:

$$-5.0R_A + 600(5.0 - 2.0) + 400(5.0 - 4.0) = 0.$$

Thus $5.0R_A = 2200 \text{ N m}$, giving $R_A = 440 \text{ N}$. Vertical balance then gives $R_B = 1000 - 440 = 560 \text{ N}$. Substitution about the left end produces $560(5.0) - 600(2.0) - 400(4.0) = 0$.

More generally, changing origin adds a term proportional to the net force. Once force balance is imposed, that term vanishes. The two moment equations therefore express the same global mechanical requirement, although one may eliminate a more convenient unknown. Disagreement signals an omitted load, wrong arm, sign error or failure to impose force balance.

Solution 17: stress, strain and an extension check

Figure 12.42(b) locates the numerical point and shows the material-modulus slope. (a) Convert $250 \text{ mm}^2 = 250 \times 10^{-6} \text{ m}^2$. Then

$$\sigma = \frac{20.0 \times 10^3}{250 \times 10^{-6}} = 80.0 \times 10^6 \text{ Pa} = 80.0 \text{ MPa},$$

$$\epsilon = \frac{80.0 \times 10^6}{160 \times 10^9} = 5.00 \times 10^{-4}, \quad \delta = L_0 \epsilon = 0.500 \text{ mm}.$$

(b) Plot strain horizontally and stress vertically. The point is $(5.00 \times 10^{-4}, 80.0 \text{ MPa})$, lying on $\sigma = E_Y \epsilon$. Its slope in pascals is E_Y , independent of specimen length and area within this constitutive model. (c) The fractional extension is $0.000500 = 0.0500\%$, so geometric small strain is consistent. That check does not independently show that the material remains below its proportionality or yield threshold; the problem stipulates linear response.

Solution 18: material and geometry both change

For each uniform rod, stiffness is $k = E_Y A/L$ and extension is F/k . In SI units,

$$k_A = \frac{(60 \times 10^9)(60 \times 10^{-6})}{0.60} = 6.00 \times 10^6 \text{ N m}^{-1},$$

$$k_B = \frac{(150 \times 10^9)(100 \times 10^{-6})}{0.90} = 16.667 \times 10^6 \text{ N m}^{-1}.$$

Thus $\delta_A = 8000/k_A = 1.333 \text{ mm}$ and $\delta_B = 8000/k_B = 0.480 \text{ mm}$. Rod B is the stiffer specimen because it requires greater force for a given small extension. Its modulus is also larger, but area and length differ as well, so attributing the full stiffness ratio to modulus would be incorrect.

The strains are $\delta_A/L_A = 2.22 \times 10^{-3}$ and $\delta_B/L_B = 5.33 \times 10^{-4}$, both geometrically small. Their respective stresses are 133.3 and 80.0 MPa. Without material limits these numbers cannot establish permissible operating loads.

Solution 19: a small shear distortion

For the stated uniform simple-shear model,

$$\tau_{\text{shear}} = \frac{12.0 \times 10^3}{400 \times 10^{-6}} = 30.0 \text{ MPa}, \quad \gamma = \frac{\tau_{\text{shear}}}{G_s} = 1.00 \times 10^{-3}.$$

Engineering shear strain is exactly $\gamma = \Delta x/h = \tan \phi$ in the stated simple-shear geometry, hence $\Delta x = (0.150)(0.00100) = 0.150 \text{ mm}$. Only the angle relation $\phi \simeq \gamma$ uses the small-angle approximation: $\phi = \arctan 0.00100 \simeq 0.00100 \text{ rad}$.

The top-face tangential force must be balanced by a force on the opposite supported face or by equivalent boundary tractions; otherwise the entire block would accelerate. Maintaining simple shear also requires boundary conditions that prevent a net rigid-body rotation and account for the torque of the applied tractions. The shear modulus characterises local distortion under this stipulated stress state; it is not derived from force balance alone.

Solution 20: pressure compresses the volume

The small-change bulk relation gives

$$\frac{\Delta V}{V_0} = -\frac{\Delta p}{B} = -\frac{8.0 \times 10^6}{50 \times 10^9} = -1.60 \times 10^{-4}.$$

Therefore $\Delta V = (0.020)(-1.60 \times 10^{-4}) = -3.20 \times 10^{-6} \text{ m}^3$. Since $1 \text{ m}^3 = 10^6 \text{ cm}^3$, this is -3.20 cm^3 .

The negative sign denotes contraction. Its fractional magnitude is 0.0160%, which is small enough for the stated first-order geometric relation. A nearly unchanged volume can coexist with an appreciable pressure increase when B is large. The calculation assumes the quoted incremental bulk modulus applies over the pressure range; it does not infer that range from the small fractional change alone.

Solution 21: the measured slope is specimen stiffness

With $A = 200 \times 10^{-6} \text{ m}^2$,

$$k = \frac{E_Y A}{L_0} = \frac{(100 \times 10^9)(200 \times 10^{-6})}{0.50} = 40.0 \times 10^6 \text{ N m}^{-1}.$$

An axial force–extension slope directly measures k in N m^{-1} , assuming the displacement belongs to the specimen and offsets have been handled. To recover Young modulus requires A and L_0 through $E_Y = kL_0/A$. Its units are N m^{-2} , or Pa.

A thicker or shorter specimen of the same material can give a steeper force–extension slope. Consequently, knowing the slope alone does not establish the material modulus. A stress–strain graph has already incorporated the geometric normalisation and is a different representation.

Solution 22: area under a linear loading curve

(a) The force rises from zero, so the work is the triangular graph area:

$$U = \int_0^\delta kq \, dq = \frac{1}{2} F \delta = \frac{1}{2} (5000)(0.000400) = 1.00 \text{ J}.$$

(b) Stress is $\sigma = 5000/(100 \times 10^{-6}) = 50.0 \text{ MPa}$; strain is $\epsilon = 0.000400/0.50 = 8.00 \times 10^{-4}$. The elastic energy density is

$$u = \frac{1}{2} \sigma \epsilon = 2.00 \times 10^4 \text{ J m}^{-3} = 20.0 \text{ kJ m}^{-3}.$$

(c) The reference volume is $V_0 = AL_0 = 5.00 \times 10^{-5} \text{ m}^3$, so $uV_0 = 1.00 \text{ J}$, agreeing with direct work.

Using $F\delta$ would assume the *final* force acted through the entire extension. In the stated quasi-static linear history its mean value is $F/2$. Energy is recoverable under reversible elasticity; loading losses would require a different path or dissipation model.

Solution 23: an inverse elastic calculation

Convert $\Delta L = 2.00 \times 10^{-4} \text{ m}$ and $A = 40.0 \times 10^{-6} \text{ m}^2$. Then

$$k = \frac{2000}{2.00 \times 10^{-4}} = 10.0 \text{ MN m}^{-1}, \quad E_Y = \frac{kL_0}{A} = 125 \text{ GPa}.$$

A second route computes $\sigma = 50.0 \text{ MPa}$ and $\epsilon = 0.000200/0.500 = 4.00 \times 10^{-4}$; their ratio is again 125 GPa.

One nonzero point gives a slope only if the linear relation passes through the corrected origin. A preload, zero offset, nonlinear range or additional grip travel would change that inference. The problem explicitly excludes these ambiguities. In an experiment, multiple points and residual/unloading checks should establish those conditions rather than assuming them from a single ratio.

Solution 24: ratios from the stated dimensions

The material modulus cancels in the stiffness ratio:

$$\frac{k_B}{k_A} = \frac{E_Y(2A_A)/(L_A/2)}{E_Y A_A/L_A} = 4.$$

Under the same force, extension is inversely proportional to stiffness, so $\delta_B/\delta_A = 1/4$. Doubling area halves the stress under the same force and therefore halves strain; halving length then halves extension again.

These rods have the *stated* dimensional ratios. They are not geometrically similar in the strict uniform-scaling sense: area and length do not change by the same linear scale factor. The fourfold stiffness difference is entirely geometric within the shared material model, with no change to Young modulus.

Solution 25: guided load sharing

(a) The guide imposes $\delta_1 = \delta_2 = \delta$ and each support obeys $F_i = k_i\delta$. Thus

$$\delta = \frac{20.0 \times 10^3}{(6.0 + 4.0) \times 10^6} = 2.00 \text{ mm}, \quad F_1 = 12.0 \text{ kN}, \quad F_2 = 8.0 \text{ kN}.$$

(b) The two computed extensions are $F_1/k_1 = F_2/k_2 = 0.00200 \text{ m}$, and $F_1 + F_2 - P = 0$. (c) With support coordinates x_1, x_2 and load coordinate x_P , the guide couple M_g satisfies

$$F_1x_1 + F_2x_2 - Px_P + M_g = 0.$$

For symmetric supports $x_1 = -a$, $x_2 = a$ and centred load, $M_g = a(F_1 - F_2)$. Equal displacement does not make this couple vanish when stiffnesses differ. A free crosshead would instead generally rotate; imposing both zero guide couple and horizontal orientation would overconstrain the unequal-support example.

Solution 26: planar vertical reactions have two equilibrium equations

There are four unknown vertical forces. Only the vertical force balance and one independent planar moment balance constrain them:

$$\sum_i R_i = P, \quad \sum_i x_i R_i = Px_P$$

for a single point load. Horizontal balance is identically $0 = 0$ in this vertical-only model. Taking moments about another planar origin does not add an independent equation after force balance has been imposed. Equilibrium therefore leaves two reaction freedoms.

Closure requires post stiffnesses, unloaded heights/gaps and rigid-plate compatibility. For small rotation, post-end displacements follow $\delta_i = a + bx_i$; constitutive laws give reactions from those displacements. Contact conditions reject a compression-only post with negative predicted compression and require the active-support set to be reconsidered. In three dimensions with vertical forces at noncollinear (x_i, y_i) coordinates, vertical balance and two horizontal-axis moment equations can be independent, giving three equations; that is a different model, not an extra equation available in the planar case.

Solution 27: interpret stiffness limits before cancelling quantities

(a) For finite positive k_1, k_2 and nonzero P ,

$$\frac{F_1}{F_2} = \frac{Pk_1/(k_1 + k_2)}{Pk_2/(k_1 + k_2)} = \frac{k_1}{k_2}.$$

(b,c) As $k_2 \rightarrow 0$, $F_2 \rightarrow 0$, $F_1 \rightarrow P$, and $\delta \rightarrow P/k_1$: the second support becomes unable to carry finite load at the finite compatible displacement. The ratio diverges; it should not be evaluated by dividing by zero at the endpoint.

As $k_2 \rightarrow \infty$, $\delta = P/(k_1 + k_2) \rightarrow 0$, $F_1 \rightarrow 0$ and $F_2 \rightarrow P$. The product $k_2\delta$ tends to a finite force even though displacement tends to zero. A very stiff support is therefore not a force-free support. These conclusions depend on the nonrotating guide and zero relative offsets. Removing the guide changes compatibility and generally changes the load-sharing limit.

Solution 28: extrapolation checks more than equilibrium

The stipulated rod stiffnesses are $k_1 = (70 \times 10^9)(300 \times 10^{-6})/1.75 = 12.0 \text{ MN m}^{-1}$ and $k_2 = (200 \times 10^9)(160 \times 10^{-6})/4.00 = 8.00 \text{ MN m}^{-1}$. Therefore

$$\delta = \frac{45000}{20.0 \times 10^6} = 2.25 \text{ mm}, \quad F_1 = 27.0 \text{ kN}, \quad F_2 = 18.0 \text{ kN}.$$

The stresses are $\sigma_1 = 90.0 \text{ MPa}$ and $\sigma_2 = 112.5 \text{ MPa}$; strains are $\epsilon_1 = 1.286 \times 10^{-3}$ and $\epsilon_2 = 5.625 \times 10^{-4}$. Multiplication by the respective lengths returns the same 2.25 mm.

The force residual is zero and strains are geometrically small. The material proportionality ranges, guide behaviour, connections and load application must nevertheless remain consistent with the ideal model. Small strain alone does not establish linearity or permissible stress. If these were compression members, lateral stability would also need consideration; the tensile-rod computation does not supply a buckling result.

Solution 29: first-order uncertainty and covariance

Linearising the inferred modulus gives

$$\frac{dE_Y}{E_Y} = \frac{dk}{k} + \frac{dL_0}{L_0} - \frac{dA}{A}.$$

For independent standard uncertainties, signs disappear when squaring the three contributions:

$$\frac{u(E_Y)}{E_Y} = \sqrt{0.005^2 + 0.002^2 + 0.010^2} = 0.01136 = 1.14\%.$$

The area variance contributes $0.010^2/(0.005^2 + 0.002^2 + 0.010^2) = 77.5\%$ of the total, so it dominates. Improving the already small length term has limited benefit.

With correlations, add $2 \text{ cov}(k, L_0)/(kL_0) - 2 \text{ cov}(k, A)/(kA) - 2 \text{ cov}(L_0, A)/(L_0A)$ to the relative variance. The signs follow the sensitivities above. First-order propagation also assumes uncertainties small enough that nonlinear transformation effects can be neglected; it does not include systematic errors absent from the input uncertainty model.

Solution 30: straightness is not a stability proof

In the perfectly straight ideal configuration, compressive end forces may be collinear and equal, so total force and moment vanish. Stability asks a different question: after an admissible small lateral displacement, do forces restore the member or amplify the displacement? The unperturbed balance equations do not answer it.

A quantitative model needs bending stiffness from material response and cross-section geometry, length/slenderness, end restraints, and a deformation/perturbation model with specified load application. Initial curvature, eccentricity and imperfections can additionally affect observed behaviour. Checking P/A examines average axial stress in the assumed straight state, but lateral instability can occur while that stress remains within a linear material range. Conversely, passing a global force residual does not supply missing bending information. Detailed beam-buckling theory is outside this chapter; recognising the missing ingredients is the requested conclusion.

Solution 31: computation as a balance audit

Use SI inputs $L = 6$, $x_B = 5$, $w = 100$, $Q = 300$, $x_Q = 5.5$. Replace the distributed load by $W = wL$ at $x_W = L/2$, solve the left-origin moment equation for R_B , then use vertical balance for R_A . The following optional standard-Python calculation reproduces the algebra; it is not required to build the chapter.

```

L, xB, w, Q, xQ = 6.0, 5.0, 100.0, 300.0, 5.5
W, xW = w*L, L/2
RB = (W*xW + Q*xQ)/xB
RA = W + Q - RB
rF = RA + RB - W - Q
rA = RB*xB - W*xW - Q*xQ
rB = -RA*xB + W*(xB-xW) - Q*(xQ-xB)
print(RA, RB, rF, rA, rB)

```

Outputs are $R_A = 210\text{ N}$, $R_B = 690\text{ N}$ and residuals $(0\text{ N}, 0\text{ N m}, 0\text{ N m})$. These particular decimal inputs are exactly represented or combined into exactly represented results by the displayed operations; the reproduced script returns zeros. General floating-point inputs can leave rounding-scale residuals. Compare $|r_F|$ with the sum of absolute force terms and $|r_\tau|$ with the sum of absolute moment terms, rather than comparing quantities with unlike units.

For a deliberate error take $R_A = R_B = 450\text{ N}$. Force balance still passes, but $r_A = 2250 - 1800 - 1650 = -1200\text{ N m}$, while $r_B = -2250 + 1200 - 150 = -1200\text{ N m}$. The second-origin audit catches this wrong split. No spatial discretisation or convergence study is required for this algebraic point/resultant calculation; changing the representation back to an integrated uniform load supplies an independent exact check.

Solution 32: a worked experimental plan and its assessment criteria

This is a proposed investigation, not a report of a conducted laboratory test. Choose a specimen and a low-load interval expected to remain reversible; record the assumptions before interpreting a modulus. Apply force axially with a controllable apparatus. Measure force, local gauge-length extension, initial gauge length and cross-sectional dimensions directly. Record sensor resolution, calibration information, environmental conditions and the loading/unloading sequence. Infer stiffness from the slope of force against local extension, then $E_Y = kL_0/A$ if the uniform axial model is supported.

For repeatability, revisit at least three load levels on repeated loading cycles and compare extensions with their measurement scatter. For an offset check, record nominal zero before loading and after unloading, including force-zero and displacement-zero readings. A fitted intercept can diagnose a zero/preload issue, but should not be hidden by forcing a regression through the origin. The unloaded local extension should return within the established uncertainty/repeatability interval if the test is reversible.

Use a displacement sensor spanning only the specimen gauge length, or independently characterise apparatus compliance and subtract the corresponding force-dependent travel. Actuator travel alone can include grip, frame and seating deformation. Compare actuator and local measurements at the same forces. A difference proportional to force is consistent with an additional elastic series compliance; a changing difference can indicate seating or other nonlinear apparatus behaviour.

Two defensible biases are misalignment, which superposes bending and makes one surface's extension unrepresentative of a uniform axial strain, and area variation, which invalidates a single uniform-area formula. Grip compliance, sensor drift and temperature change are additional plausible biases. Plot residuals against force/extension and loading order. Shorten the fitted interval when repeated curvature exceeds the repeatability band, when unloading fails to retrace the loading path within that band, or when the inferred slope changes systematically as the upper load bound is altered.

An assessable response should specify: the chosen system and low-load assumption; direct measurements with units; how local extension and area are obtained; repeatability and offset procedures; the inference equations; at least two distinct bias mechanisms; residual and unloading comparisons; and a stated criterion for rejecting or restricting the model. Valid outcomes include an approximately constant modulus over a limited interval, a demonstrable apparatus correction, or the conclusion that

the data cannot support a modulus. Invented equipment specifications or a single predetermined successful outcome would not improve this investigation.

Solution 33: solve friction as an unknown before testing its bound

Figure 12.44(b) provides the completed body-only ladder diagram with all permitted contact forces. **(a)** The ladder free-body diagram has two downward weights: $20g$ at $L/2$ and $60g$ at $s = 3.00$ m. The smooth wall supplies only a horizontal normal N_w directed away from the wall. At the floor draw upward normal N_f and friction f toward the wall. There is no wall friction in this model. **(b)** Vertical and horizontal balance give $N_f = 80g = 784.8$ N and $f = N_w$. Taking counterclockwise moments about the foot,

$$N_w L \sin \theta = g[20(L/2) + 60s] \cos \theta.$$

Thus

$$N_w = \frac{9.81[20(2.50) + 60(3.00)]}{5.00} \cot 60^\circ = 260.535 \text{ N},$$

and $f = 260.535$ N toward the wall. **(c)** Both normals are positive. Friction demand is below the limiting magnitude $\mu_s N_f = 313.92$ N; the required ratio is $0.33198 < 0.400$. Static friction is therefore not at its limiting value at 60° .

For the same position along the ladder,

$$\frac{f}{N_f} = \frac{20(2.50) + 60(3.00)}{5.00(80)} \cot \theta = 0.575 \cot \theta.$$

The lowest admissible angle obeys $0.575 \cot \theta_{\min} = 0.400$, hence $\theta_{\min} = \arctan(0.575/0.400) \simeq 55.18^\circ$. Below it the required friction exceeds the stipulated bound. This ideal threshold assumes quasi-static loading and the same contacts and person position; it is not a ladder-use safety instruction.

Solution 34: the load shape, not beam shape, fixes the centroid

(a) The triangular area is $W = \frac{1}{2}(4.00)(600) = 1200$ N. **(b)** With $w(x) = 150x$ N m⁻¹ when x is expressed in metres,

$$M_A = \int_0^4 x(150x) dx = 150 \frac{4^3}{3} = 3200 \text{ N m}, \quad \bar{x} = \frac{3200}{1200} = \frac{8}{3} \text{ m}.$$

The resultant lies two-thirds of the length from the zero-intensity end, nearer the high-intensity end. **(c)** From $4R_B = M_A$, $R_B = 800$ N; force balance gives $R_A = 400$ N. About B , $-4(400) + 1200(4 - 8/3) = 0$ N m. Both contacts remain compressive. **(d)** A midpoint replacement gives an incorrect moment $1200(2) = 2400$ N m, missing 800 N m. The resulting 600/600 N split passes force balance but gives a left-origin moment residual -800 N m for the actual load. The beam's geometric symmetry does not make the load distribution symmetric.

Solution 35: geometry makes nearly horizontal cables demanding

(a) Moment balance still requires $3T \sin \theta = 1650$ N m, hence

$$T(\theta) = 550 \csc \theta \text{ N}.$$

Graph readings of approximately 1.30, 0.86 and 0.64 kN at 25° , 40° , 60° agree with 1.3014, 0.85565, 0.63509 kN from numerical trigonometry. **(b)** Since $0 < \theta < 90^\circ$, sine increases with angle. The stipulated demand limit requires $\sin \theta \geq 0.550$, so $\theta \geq \arcsin 0.550 = 33.37^\circ$. Intersecting with the given domain gives $33.37^\circ \leq \theta \leq 80^\circ$. **(c)** Pin reactions are

$$A_y = 800 - T \sin \theta = 250 \text{ N}, \quad A_x = -T \cos \theta = -550 \cot \theta \text{ N}.$$

Vertical reaction is constant, while the leftward horizontal magnitude decreases as the angle increases. As $\theta \rightarrow 0^+$, T and $|A_x|$ diverge because finite vertical tension component is still required from an increasingly horizontal direction. This is a failure of favourable geometry, not a larger applied weight.

Solution 36: formal reaction lines include nonphysical branches

(a) About A , $2R_B - 600s = 0$, giving $R_B = 300s$ N for s in metres. Then $R_A = 600 - 300s$ N. Line I decreases and belongs to A ; Line II increases and belongs to B . (b) Both contacts require nonnegative reaction: $300s \geq 0$ and $600 - 300s \geq 0$. Their intersection is $0 \leq s \leq 2.00$ m. (c) At $s = 3.00$ m the formal pair is $(-300, 900)$ N, reproducing Problem 3 and violating support A 's contact model. (d) At $s = 0$ the pair is $(600, 0)$ N; at $s = 2$ it is $(0, 600)$ N. A load exactly over the sole active support can balance in the ideal configuration. The zero-reaction contact supplies no restoring margin against a load shift to the outside. The endpoint is a boundary of the static admissible set, not evidence of robust two-contact support under perturbations.

Solution 37: series rods share force and add extension

(a) Equilibrium of the light connector requires equal tensile force magnitude in both segments. Their extensions need not be equal. (b) The segment stiffnesses are $k_1 = 8.75 \text{ MN m}^{-1}$ and $k_2 = 13.333 \text{ MN m}^{-1}$. Thus

$$\delta_1 = \frac{10000}{8.75 \times 10^6} = 1.14286 \text{ mm}, \quad \delta_2 = \frac{10000}{13.333 \times 10^6} = 0.750 \text{ mm}.$$

Total extension is 1.89286 mm and $k_{\text{eff}} = F/(\delta_1 + \delta_2) = 5.28302 \text{ MN m}^{-1}$. Equivalently $1/k_{\text{eff}} = 1/k_1 + 1/k_2$, so the assembly is less stiff than either segment. (c) Segment energies are $\frac{1}{2}F\delta_1 = 5.71429 \text{ J}$ and $\frac{1}{2}F\delta_2 = 3.750 \text{ J}$, summing to 9.46429 J . The assembly graph gives $\frac{1}{2}F(\delta_1 + \delta_2)$, the same value. (d) Series compatibility adds successive end displacements, while force is common. Guided parallel compatibility makes displacement common while forces add. These different geometries explain the different effective-stiffness rules; neither rule follows from material modulus alone.

Solution 38: a precise straight line can measure the wrong compliance

(a) Read the graph slope:

$$k_m = \frac{5000}{0.000750} = 6.66667 \text{ MN m}^{-1}, \quad E_{Y,app} = \frac{k_m L_0}{A} = 66.6667 \text{ GPa}.$$

(b) Series travels add: $\delta_{\text{total}} = F/k_s + F/k_g$, hence $1/k_s = 1/k_m - 1/k_g$. Substitution gives

$$\frac{1}{k_s} = \frac{1}{6.66667 \times 10^6} - \frac{1}{20.0 \times 10^6} = 1.00 \times 10^{-7} \text{ m N}^{-1},$$

so $k_s = 10.0 \text{ MN m}^{-1}$ and $E_{Y,s} = k_s L_0 / A = 100 \text{ GPa}$. At 5.00 kN , grip travel is 0.250 mm and specimen extension 0.500 mm ; their sum reproduces the 0.750 mm graph point. (c) Bias relative to the corrected modulus is $(66.6667 - 100)/100 = -0.33333$, or 33.3% low. Both series elements are linear, so their combined curve is also linear and repeatable. Residual scatter alone cannot identify which component moved. The correction requires independent grip compliance or local specimen extension.

Solution 39: extension weights the inverse area

(a) With no distributed axial loading, tension is constant. Local strain is $F/[E_Y A(x)]$ and the extension of an element is that strain times dx . Therefore

$$\delta = \int_0^L \frac{F dx}{E_Y A_0(1+x/L)} = \frac{FL}{E_Y A_0} \ln 2 = 0.693147 \text{ mm}.$$

(b) The smallest area is at $x = 0$, so stress there is 70.0 MPa and strain 1.00×10^{-3} . At $x = L$, area doubles, giving 35.0 MPa and 5.00×10^{-4} . All local strains are small in the assumed model. (c) Mean area is $1.5A_0$. Its uniform-area estimate is $FL/(1.5E_Y A_0) = 0.666667 \text{ mm}$, with relative error

$$\frac{0.666667 - 0.693147}{0.693147} = -0.03820.$$

It underestimates extension by 3.82%. Integrating inverse area is not equivalent to taking the inverse of mean area; narrow regions contribute more strongly. **(d)** $U = \frac{1}{2}F\delta = 2.42602 \text{ J}$, also obtained by integrating $F^2/[2E_Y A(x)]$ along the length. This verifies the one-dimensional energy relation. It does not test local multiaxial stress near grips or the validity of neglecting taper/end effects; those were assumptions, not results of the integral.

Solution 40: compatibility allows a displacement gradient

(a) Set $x_i = (0, 1, 2) \text{ m}$. In the all-contact trial state, $R_i = k(a + bx_i)$. Force and moment balance about $x = 0$ give

$$k(3a + 3b \text{ m}) = 12000 \text{ N}, \quad k(3a \text{ m} + 5b \text{ m}^2) = 18000 \text{ N m}.$$

Working with SI numerical coordinates, the two equations reduce to $3a + 3b = 0.006$ and $3a + 5b = 0.009$. Thus $a = 0.000500 \text{ m}$ and $b = 0.00150$. The spring compressions are (0.500, 2.00, 3.50) mm and reactions are (1.00, 4.00, 7.00) kN. **(b)** All compressions are positive, so the trial active set is admissible. Reactions sum to 12.0 kN; their moment is $4.00(1) + 7.00(2) = 18.0 \text{ kN m}$. Slope $b = 0.00150$ corresponds to about 0.086° , consistent with small rotation. **(c)** Load-point displacement is $a + 1.50b = 0.00275 \text{ m}$. Summing spring energies gives

$$U = \frac{k}{2}[(0.0005)^2 + (0.0020)^2 + (0.0035)^2] = 16.5 \text{ J}.$$

With proportional loading, all displacements scale with P , so external work is $\frac{1}{2}P\delta(x_P) = \frac{1}{2}(12000)(0.00275) = 16.5 \text{ J}$. Equal spring stiffness does not imply equal reactions under an eccentric load when the bar is free to rotate.

Solution 41: an open contact is a different branch

(a) For $0 \leq \delta \leq d$, spring 2 has no compression and no force. For $\delta \geq d$, its compression is $\delta - d$. Therefore

$$P(\delta) = \begin{cases} k_1\delta, & 0 \leq \delta \leq d, \\ k_1\delta + k_2(\delta - d), & \delta \geq d. \end{cases}$$

The threshold is $P_c = k_1d = 4.00 \text{ kN}$, and both branches agree there. Tangent stiffness changes from 4.00 to 10.0 MN m^{-1} . **(b)** Since $6.00 \text{ kN} > P_c$, use the closed branch:

$$\delta = \frac{P + k_2d}{k_1 + k_2} = 1.20 \text{ mm}, \quad F_1 = 4.80 \text{ kN}, \quad F_2 = 1.20 \text{ kN}.$$

The second spring compresses only 0.200 mm; both forces are positive and sum to P . **(c)** Energy is $U = \frac{1}{2}k_1\delta^2 + \frac{1}{2}k_2(\delta - d)^2 = 2.88 + 0.12 = 3.00 \text{ J}$. The P - δ graph area is 2.00 J under the first branch plus $\frac{1}{2}(4000 + 6000)(0.000200) = 1.00 \text{ J}$ under the second. **(d)** Treating both springs as initially engaged would assign a positive force to an open gap. It would predict $\delta = 0.600 \text{ mm} < d$, contradicting its own assumed contact state. Nor is final energy $\frac{1}{2}P\delta$: the complete loading curve is piecewise with an offset after closure.

Solution 42: curvature and the critical fourth-order term

(a) The mass height is $L \cos \theta$, and the elastic-device extension is the horizontal displacement $L \sin \theta$. With energy referenced to the upright state,

$$\Delta U = mgL(\cos \theta - 1) + \frac{1}{2}kL^2 \sin^2 \theta.$$

Its derivative is $U'(\theta) = \sin \theta[kL^2 \cos \theta - mgL]$, which vanishes at $\theta = 0$. The balance is an equilibrium for every listed stiffness. **(b)** Expanding about zero gives

$$\Delta U = \frac{1}{2}(kL^2 - mgL)\theta^2 + \left(\frac{mgL}{24} - \frac{kL^2}{6}\right)\theta^4 + \mathcal{O}(\theta^6).$$

For $k = 30.0 \text{ N m}^{-1}$, angular stiffness is $K_\theta = 30.0 - 19.62 = 10.38 \text{ N m}$ per radian of angular displacement, with radians dimensionless in SI. Positive curvature makes the upright equilibrium locally stable. The blue curve rises on both sides. **(c)** At $k = 10.0 \text{ N m}^{-1}$, $K_\theta = -9.62 \text{ N m}$; energy decreases for small nonzero angles and the upright state is unstable. **(d)** At $k = mg/L$, the quadratic term vanishes and the quartic coefficient is $-mgL/8 < 0$. Small nonzero displacement lowers energy, so this particular critical state is also unstable; zero quadratic curvature alone is inconclusive without the next term. In fact the exact critical expression is $\Delta U = -\frac{1}{2}mgL(1 - \cos \theta)^2$, independently confirming the sign. **(e)** At $\theta = 0.100 \text{ rad}$ and $k = 30.0$, exact energy is $19.62(\cos 0.1 - 1) + 15 \sin^2 0.1 = 0.0514824 \text{ J}$. The quadratic approximation is $\frac{1}{2}(10.38)(0.1)^2 = 0.0519000 \text{ J}$, about 0.81% high relative to the exact value. The fourth-order correction explains the direction. The specified height-following device is essential to this potential; a fixed-anchor geometric spring would generally have a different force law and energy.

Solved diagrams and graphical constructions

Figures 12.40–12.44 complete the requested sketches in Problems 1, 2, 6, 10, 13–15, 17 and 33. They accompany the corresponding reasoned solutions above. Explicit distances and graph axes carry quantitative information; force-arrow lengths are schematic unless an axis defines their scale.

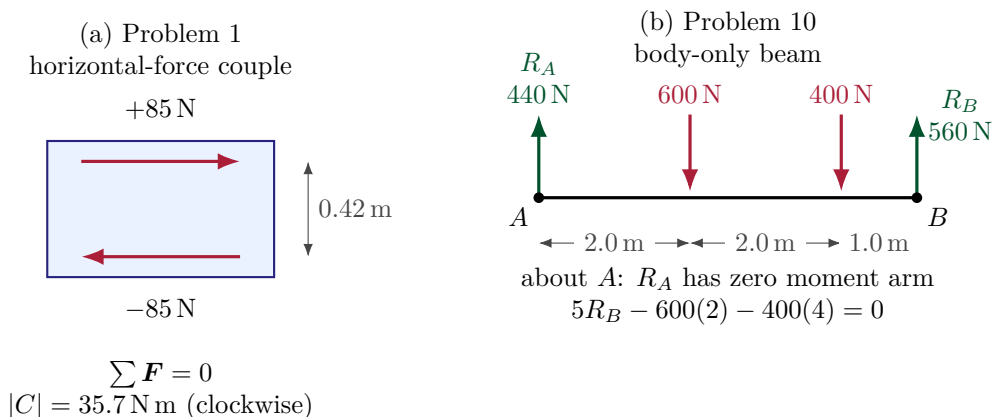


Figure 12.40: Actual solution diagrams for Problems 1 and 10. Panel (a) chooses the upper force rightward and lower force leftward, producing a clockwise couple; collinearity would remove its moment. Panel (b) shows the end reactions and all point-load positions. Both positive reactions satisfy upward-only contact.

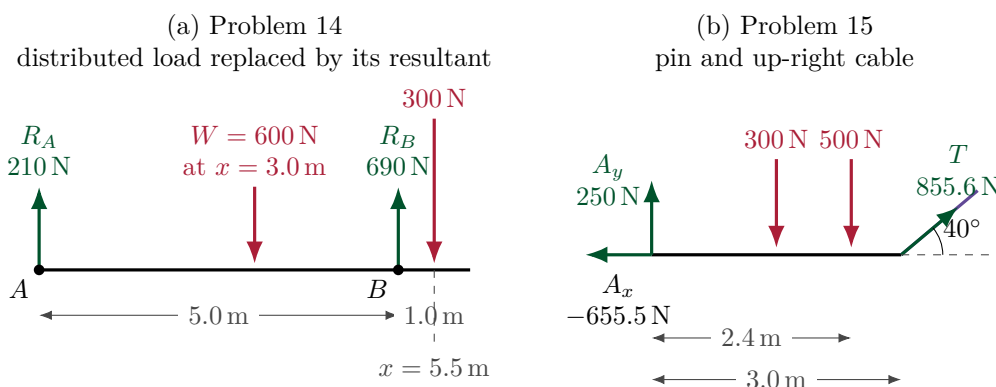


Figure 12.41: Solved body-only diagrams for Problems 14 and 15. The overhang load's lever arm about B is 0.50 m; the uniform-load resultant is 2.0 m left of B. For the sign, $T_y = 550 \text{ N}$ and $T_x = 655.5 \text{ N}$; the resultant tension is shown once, while its components are stated here to avoid double-counting them as extra forces. The cable's horizontal component lies along the pin's zero-thickness beam line.

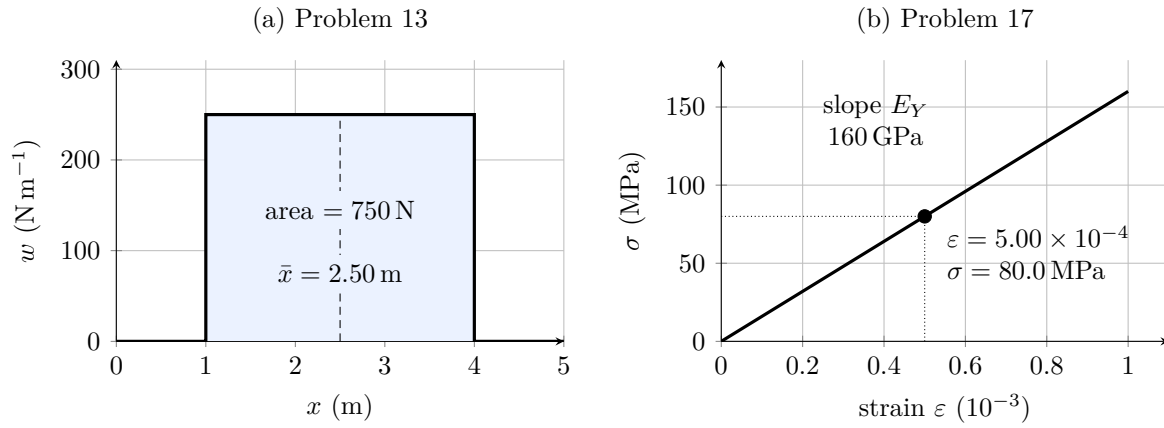


Figure 12.42: Graphical answers to Problems 13 and 17. The rectangular area is force, and its centroid fixes the resultant's line of action. The stress–strain point lies on $\sigma = E_Y \varepsilon$; the horizontal axis displays strain in thousandths, so the displayed slope 160 MPa per 10^{-3} strain corresponds to 160 GPa.

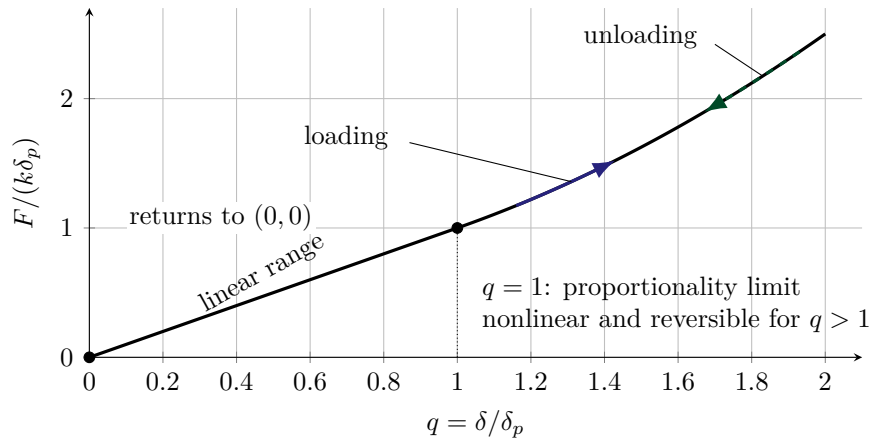


Figure 12.43: A complete reversible nonlinear sketch for Problem 6. This normalized instance of Solution 6 chooses $c\delta_p/k = 0.50$. The blue forward arrow and dashed green backward arrow lie on the *same* force–extension curve. The proportional range ends at $q = 1$, yet unloading still retraces the curve and returns to zero extension at zero force. No separate unloading branch or residual strain is drawn.

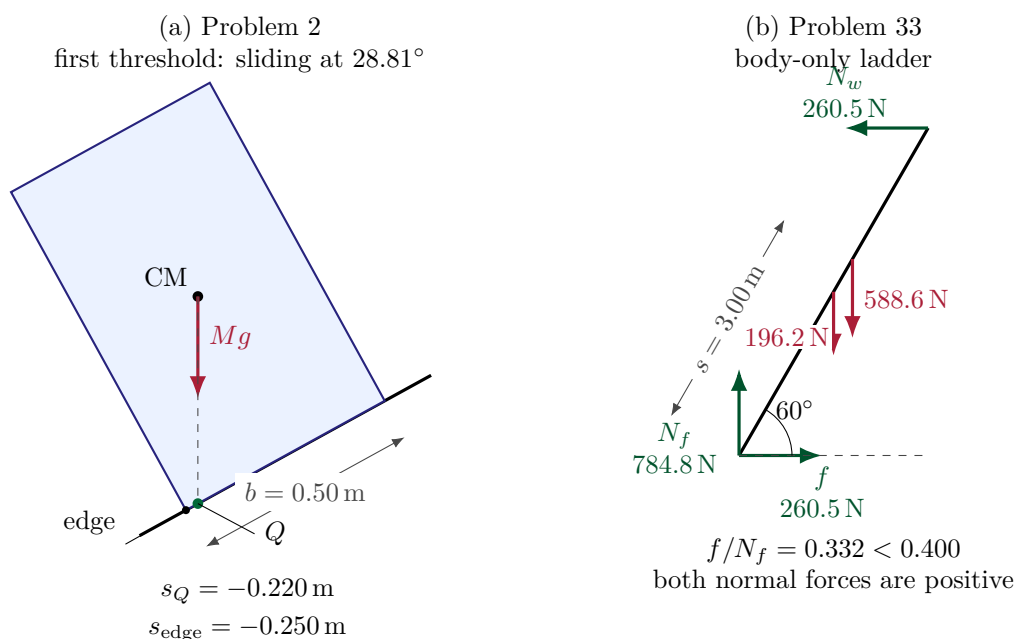


Figure 12.44: The actual constructions requested in Problems 2 and 33. At the housing’s sliding threshold, the world-vertical gravity line intersects its inclined base at Q , 0.030 m inside the downhill edge, so tipping has not yet occurred. The ladder diagram shows only forces on the ladder: leftward wall normal, upward floor normal, rightward floor friction and both downward weights. The person is 3.00 m along the ladder; the uniform ladder’s mass centre is at 2.50 m .

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