

Chapter 13

Gravitation

First Edition

Hamzeh Khanpour

hamzeh.khanpour@agh.edu.pl

Faculty of Physics and Applied Computer Science

AGH University of Krakow

Physics I

Physics for Technical and Engineering Students -
Volume I

Mechanics, Fluids, Waves, and Thermodynamics

1 January 2026

Contents

Preface	ii
Conventions and notation	iii
13 Gravitation	1
13.1 Motivation and roadmap	1
13.2 Newtonian gravitation and superposition	2
13.3 Spherical sources and local gravity	4
13.4 Gravitational potential and spherical interiors	9
13.5 Circular orbits and energy	19
13.6 Kepler’s laws and angular momentum	21
13.7 Tides and extended sources	35
13.8 Limits of Newtonian gravity	39
13.9 Engineering, computation and measurement connections	40
13.10 Advanced Topics for Technical and Engineering Students	43
13.11 Chapter summary and working formula sheet	49
13.12 Exercises and problems	50
13.13 Complete reasoned solutions	55
13.14 Bridge to Chapter 14	65
References	66

Preface

A satellite can remain above a planet even though gravity is continuously pulling it inward. A person standing on the surface can often use a nearly constant value of g , while a spacecraft only a few planetary radii away must account for a gravitational acceleration that changes strongly with position. These are not separate theories. They are different regimes of the same Newtonian inverse-square interaction.

This chapter makes that connection explicit. It begins with Newton's law of gravitation and vector superposition, then replaces the local constant- g picture by the spatially varying gravitational field of spherical sources. From that field we build gravitational potential energy, escape conditions, circular orbits, Kepler's laws and tidal effects. The final advanced section removes the test-particle approximation, introduces the reduced-mass two-body problem and shows how a numerical orbit should be validated by conservation laws and timestep refinement rather than by visual appearance alone.

The opening question is deliberately simple to state: *between two unequal attracting masses, where can a small test mass experience zero net gravitational field?* Solving it requires more than setting two positive magnitudes equal. One must first decide where the two force directions oppose, define distances from the correct sources and test the answer against symmetry and limiting cases. That modelling discipline will recur throughout the chapter.

All numerical values labelled as stipulated or synthetic are teaching inputs rather than measurements. The value of the gravitational constant used for calculations is $G = 6.67430 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$, with the 2022 CODATA standard uncertainty $1.5 \times 10^{-15} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$. Unless stated otherwise, calculations idealise sources as point masses or spherically symmetric bodies and neglect atmosphere, rotation, nonsphericity and third-body perturbations.

The Guideline

Before using a gravitation formula, identify the source mass, the target or field point, the displacement vector and the model boundary. In orbital problems distinguish radius from the source centre from altitude above its surface. In energy problems state the zero of potential. In numerical work, test invariants and convergence before trusting a trajectory.

Pedagogical boundary

The core of this chapter is Newtonian. General relativity is mentioned only to mark the model boundary; relativistic orbital equations, black-hole calculations and cosmology lie outside this chapter. The pendulum as a local measurement of g is a forward application for the oscillations chapter and is not assumed here.

Conventions and notation

Vectors are bold, such as $\mathbf{g}$ and $\mathbf{r}$. A vector from a source at $\mathbf{r}_i$ to a field point at $\mathbf{r}$ is

$$\mathbf{s}_i = \mathbf{r} - \mathbf{r}_i, \quad s_i = |\mathbf{s}_i|.$$

The field of a positive source mass points *toward* the source, so the vector law contains a minus sign. The symbol $g = |\mathbf{g}|$ denotes a non-negative magnitude; a Cartesian component such as g_x may have either sign.

For a spherically symmetric source of radius R , the radial coordinate r is measured from the centre. Altitude is $h = r - R$. The source parameter $\kappa = GM$ is convenient in orbital calculations and has SI unit $\text{m}^3 \text{s}^{-2}$; it should not be confused with the reduced mass $\mu = Mm/(M + m)$ introduced in the advanced section.

Symbol	Meaning	SI unit
G	Newtonian gravitational constant	$\text{m}^3 \text{kg}^{-1} \text{s}^{-2}$
M, m	source/body masses	kg
$\mathbf{g}$	gravitational field / acceleration due to gravity	m s^{-2}
R	radius of a spherical source	m
r	distance from source centre	m
h	altitude, $h = r - R$	m
Φ	gravitational potential, U/m	J kg^{-1}
κ	source parameter GM	$\text{m}^3 \text{s}^{-2}$
U	gravitational potential energy	J
E	mechanical energy of the stated Newtonian system	J
L	orbital angular momentum about the chosen origin	$\text{kg m}^2 \text{s}^{-1}$
a	semimajor axis of a Kepler ellipse	m
e	orbital eccentricity	1
T_{orb}	orbital period	s
μ	reduced mass $Mm/(M + m)$	kg

The global potential convention in this chapter is $U(\infty) = 0$ whenever the isolated inverse-square two-body potential is used. Near a surface one may still use $\Delta U \simeq mg \Delta y$ as a local approximation, but that approximation should not be confused with a different physical potential.

Chapter 13

Gravitation

13.1 Motivation and roadmap

Gravity is weak between laboratory-sized objects, yet it dominates the large-scale mechanics of planets, satellites and stars because mass is always attractive in the Newtonian model and macroscopic bodies contain enormous amounts of it. The same force that gives a dropped object a downward acceleration also bends an orbit continuously toward its central source.

The local formula $\Delta U \simeq mg\Delta y$ and the phrase “take $g = 9.8 \text{ m s}^{-2}$ ” were useful in earlier mechanics because the vertical distances were small compared with a planetary radius. This chapter asks when that simplification is justified and what replaces it when the scale becomes comparable with the distance to the source centre.

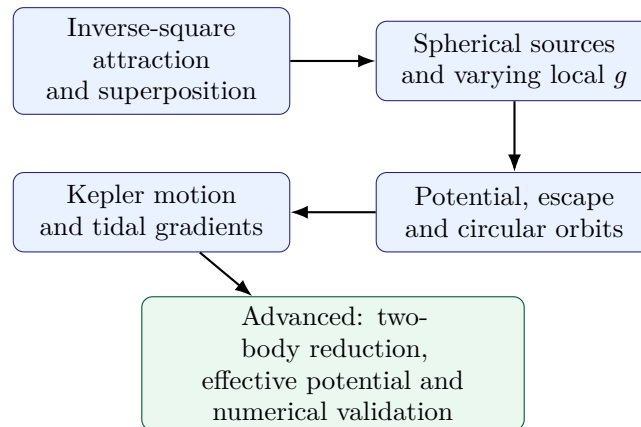


Figure 13.1: Chapter roadmap. The core moves from a vector inverse-square law to orbital mechanics. The advanced branch removes the test-particle approximation and adds numerical conservation diagnostics.

Learning goals

After studying the core and extension pathway, you should be able to:

- add gravitational fields from several point or spherical sources with correct vector directions;
- decide when a constant- g approximation is accurate enough for a stated tolerance;
- derive and use $U = -GMm/r$ with an explicit potential zero and determine escape conditions;
- derive circular-orbit speed, period and energy without introducing a fictitious extra “centripetal force”;
- connect Kepler’s equal-area law with angular momentum conservation;
- distinguish a total gravitational acceleration from the differential acceleration responsible for tides;
- identify assumptions whose failure requires a richer physical model;
- in the advanced pathway, reduce the isolated two-body problem and validate a numerical orbit by invariants and refinement.

Guided checks

Opening question. Two positive source masses M_1 and M_2 are fixed on a line. Is a zero-field point possible between them? Is it necessarily halfway between them? Could a point obtained by equating magnitudes outside the two masses be physically valid? Keep these questions in mind; Example 13.1 returns to them after the vector law is established.

13.2 Newtonian gravitation and superposition

13.2.1 Pairwise Newtonian attraction

Newton’s law of universal gravitation models two point masses m_1 and m_2 separated by a distance $r > 0$ as exerting equal and opposite attractive forces with magnitude

$$F = G \frac{m_1 m_2}{r^2}. \quad (13.1)$$

The inverse square is an experimentally established feature of the Newtonian model, not something determined by dimensional analysis alone. Dimensional analysis does provide an important check:

$$[G] \frac{[m]^2}{[r]^2} = \frac{\text{m}^3 \text{ kg}^2}{\text{kg s}^2 \text{ m}^2} = \text{kg m s}^{-2} = \text{N}.$$

For field calculations it is useful to separate the property of the source from the mass of a small test body. Let a source M_i be at position $\mathbf{r}_i$ and evaluate the field at $\mathbf{r}$. Define $\mathbf{s}_i = \mathbf{r} - \mathbf{r}_i$. Then

$$\mathbf{g}_i(\mathbf{r}) = -GM_i \frac{\mathbf{s}_i}{s_i^3}, \quad \mathbf{r} \neq \mathbf{r}_i. \quad (13.2)$$

The minus sign makes the field point opposite to the source-to-field displacement, hence toward the source. A test mass m experiences

$$\mathbf{F} = m\mathbf{g}. \quad (13.3)$$

Because m cancels from $\mathbf{F}/m$, the gravitational field is a property of the specified sources and observation position within the Newtonian test-particle model.

Definition 13.1: Gravitational field

The Newtonian gravitational field $\mathbf{g}(\mathbf{r})$ is the gravitational force per unit test mass at position $\mathbf{r}$, evaluated for a test body whose presence does not significantly change the prescribed source configuration.

13.2.2 Superposition and extended sources

For several fixed point sources, the total field is the vector sum

$$\mathbf{g}(\mathbf{r}) = - \sum_i GM_i \frac{\mathbf{r} - \mathbf{r}_i}{|\mathbf{r} - \mathbf{r}_i|^3}. \quad (13.4)$$

The magnitudes generally cannot be added. Each source contribution has its own direction. An extended mass distribution can be subdivided into mass elements dm and integrated,

$$\mathbf{g}(\mathbf{r}) = -G \int \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} dm(\mathbf{r}'), \quad (13.5)$$

provided the Newtonian continuum model is appropriate. For a spherically symmetric distribution, the exterior field simplifies exactly to that of the total mass concentrated at the centre; Section 13.3 uses this result.

Caution

A gravitational-field arrow is not a trajectory. It indicates the instantaneous acceleration direction of a freely moving test particle placed at that point. The actual path also depends on the particle's current velocity.

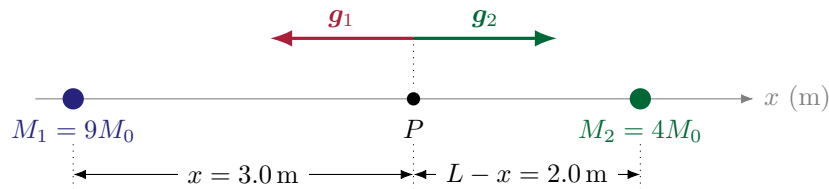


Figure 13.2: Two unequal fixed point sources separated by $L = 5.0$ m. At the cancellation point P , the two field contributions have equal magnitudes and opposite directions. The arrows are displaced vertically for clarity and represent fields at P . Positions are drawn on one linear scale.

Example 13.1: Equilibrium point between two sources

Two fixed source masses $M_1 = 9M_0$ and $M_2 = 4M_0$ are separated by $L = 5.0$ m. Treat them as point sources. Find the point on the line where the gravitational field is zero.

Region first. Between the masses the two fields are oppositely directed. To the left of both sources they both point right; to the right of both they both point left. Therefore a finite cancellation point can occur only in $0 < x < L$ for this configuration.

Magnitude equation. If x is the distance from M_1 ,

$$\frac{GM_1}{x^2} = \frac{GM_2}{(L-x)^2}.$$

Taking positive square roots because distances are positive,

$$\frac{\sqrt{M_1}}{x} = \frac{\sqrt{M_2}}{L-x}, \quad x = \frac{L\sqrt{M_1}}{\sqrt{M_1} + \sqrt{M_2}}.$$

Thus

$$x = 3.0 \text{ m}$$

from the $9M_0$ source, or 2.0 m from the $4M_0$ source.

Checks. The cancellation point is closer to the smaller mass, as it must be. If $M_1 = M_2$, the expression reduces to $x = L/2$. The result is independent of the test mass because the question concerns the field.

Guided checks

X13-01. A student solves $GM_1/x^2 = GM_2/(x-L)^2$, obtains a real root with $x > L$, and declares it a zero-field point. What is missing? *The direction test.* To the right of both positive sources, both fields point left; equal magnitudes would add rather than cancel. Squaring an equation can erase that directional information.

Method

For several gravitational sources: (1) choose the field point, (2) write a displacement from each source to that point, (3) use the signed/vector form of the law, (4) add components, and (5) test the direction against attraction before accepting the algebra.

13.3 Spherical sources and local gravity**13.3.1 Exterior field of a spherically symmetric source**

For a spherically symmetric source of total mass M and radius R , the exterior field for $r \geq R$ is the same as that of a point mass at the centre:

$$\mathbf{g}(r) = -\frac{GM}{r^2} \hat{\mathbf{r}}, \quad r \geq R, \quad (13.6)$$

where $\hat{\mathbf{r}}$ points outward. The magnitude is $g(r) = GM/r^2$. This exterior result does not state what the field is inside an arbitrary body. Internal density structure matters there.

At the surface,

$$g_s = \frac{GM}{R^2}. \quad (13.7)$$

At altitude $h \geq 0$, $r = R + h$ and therefore

$$\frac{g(R+h)}{g_s} = \left(1 + \frac{h}{R}\right)^{-2}. \quad (13.8)$$

This ratio is a useful way to test whether a local constant- g model is acceptable without needing the numerical values of G or M .

13.3.2 The constant- g approximation has a tolerance

For $u = h/R \ll 1$,

$$(1+u)^{-2} = 1 - 2u + 3u^2 + \mathcal{O}(u^3). \quad (13.9)$$

Thus the first fractional change in g is of order $2h/R$. The phrase “ h is small” should therefore mean small relative to R and small enough for the required accuracy.

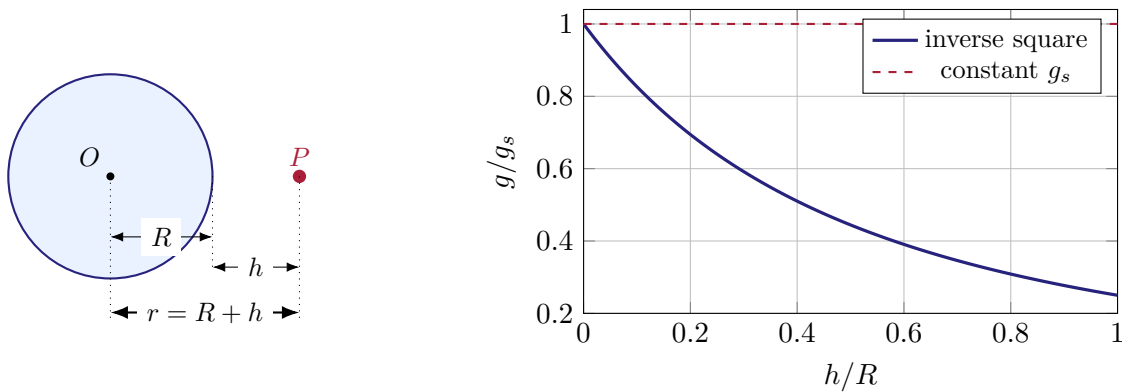


Figure 13.3: Radius r starts at the centre; altitude h starts at the surface. The graph is a calculated dimensionless comparison, $g/g_s = (1 + h/R)^{-2}$; the horizontal approximation overestimates the exterior field.

Example 13.2: Test constant gravity over a height interval

Suppose a spherical source is described by Eq. (13.8). First compare the exact value at $h/R = 0.020$ with the constant- g approximation. Then find the largest h/R for which the constant value g_s overestimates the exact value by less than 1%, where the relative error is defined using the *exact* value in the denominator.

At $u = 0.020$,

$$\frac{g}{g_s} = \frac{1}{1.02^2} = 0.96117.$$

The stated relative error is

$$\epsilon = \frac{g_s - g}{g} = \frac{1}{g/g_s} - 1 = (1+u)^2 - 1,$$

so $\epsilon = 0.0404$, or 4.04%.

For a 1% tolerance,

$$(1+u)^2 - 1 < 0.010 \quad \Rightarrow \quad u < \sqrt{1.010} - 1 = 4.99 \times 10^{-3}.$$

Hence the interval must satisfy

$$h < 0.00499R.$$

For the stipulated illustration $R = 6.40 \times 10^6$ m, this corresponds to about 31.9 km. The number is a model-tolerance result, not a measurement uncertainty.

Example 13.3: Mean density inferred from surface data

For an ideal spherical body, combine $g_s = GM/R^2$ with $M = (4\pi/3)R^3\bar{\rho}$. Then

$$\bar{\rho} = \frac{3g_s}{4\pi GR}. \quad (13.10)$$

Using the stipulated values $g_s = 9.80 \text{ m s}^{-2}$ and $R = 6.37 \times 10^6 \text{ m}$ gives

$$M = 5.96 \times 10^{24} \text{ kg}, \quad \bar{\rho} = 5.50 \times 10^3 \text{ kg m}^{-3}.$$

These two surface observables constrain total mass and mean density under the spherical model. They do *not* uniquely determine how density varies with depth; many internal profiles can have the same total mass and radius.

Guided checks

X13-02. A student says “ $h = 20 \text{ km}$ is small, so constant g is exact enough.” What information is missing? A length is not small by itself: compare h/R , state the observable being approximated and specify an allowed error with a defined denominator. Also check that $g(R+0) = g_s$.

13.3.3 Local-field accuracy depends on the error denominator

For a spherical source, altitude h is measured from its surface and the distance entering gravity is $r = R + h$. A calculation should name the reference surface and the quantity whose approximation is being tested. A spacecraft at altitude $h = R$ is at radius $2R$, so the field is one quarter of its surface value. Substituting h for r would be especially severe near the surface, where that error makes the field diverge as $h \rightarrow 0$ despite a finite exact value.

Set $u = h/R \geq 0$. The exact field is $g(h) = g_s/(1+u)^2$. Two sensible fractional comparisons are

$$\epsilon_{g,\text{exact}} = \frac{g_s - g(h)}{g(h)} = (1+u)^2 - 1 = 2u + u^2, \quad (13.11)$$

$$\epsilon_{g,\text{surface}} = \frac{g_s - g(h)}{g_s} = 1 - (1+u)^{-2}. \quad (13.12)$$

The first measures bias of the approximation relative to the value it is trying to predict. The second measures fractional reduction relative to the surface reference. Neither is inherently invalid; they answer different questions. An instruction such as “gravity changes by one percent” is incomplete until the denominator and reference point are stated. The two expressions agree to first order in u but differ at finite altitude.

For an allowed fraction ε with $0 < \varepsilon < 1$, the exact inequalities are

$$\epsilon_{g,\text{exact}} \leq \varepsilon \iff h \leq R(\sqrt{1+\varepsilon} - 1), \quad (13.13)$$

$$\epsilon_{g,\text{surface}} \leq \varepsilon \iff h \leq R[(1-\varepsilon)^{-1/2} - 1]. \quad (13.14)$$

The familiar estimate $h/R \lesssim \varepsilon/2$ is their small-tolerance limit. Figure 13.4 shows where the two finite-altitude definitions depart from that shared first-order estimate. An exact bound costs little extra algebra and avoids silently applying a first-order formula outside its accuracy range. A model-bias tolerance should also be kept distinct from measurement uncertainty in M , R or position. The formulas above compare two mathematical models with the same parameters.

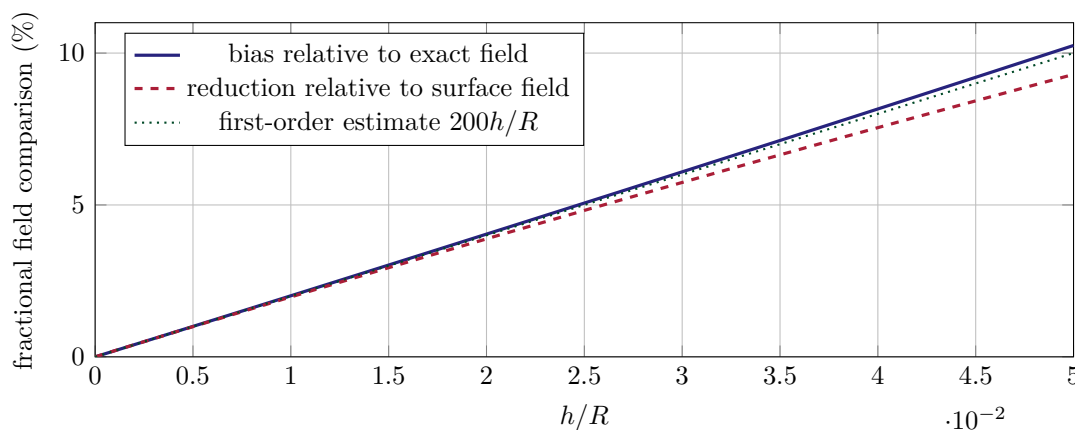


Figure 13.4: Two precisely defined fractional comparisons of constant surface gravity with the exact exterior field. Both increase with altitude and share a first-order slope, but their finite-altitude values differ. Curves are calculated from the ideal spherical model; the plotted percentages are approximation biases, not confidence intervals.

Example 13.4: An altitude limit for a specified force accuracy

Statement. An ideal spherical body has $R = 6.40 \times 10^6$ m and $\kappa = GM = 4.00 \times 10^{14} \text{ m}^3 \text{ s}^{-2}$. (a) Calculate the gravitational field at altitude 100 km and the percentage by which a constant- g_s force model overestimates it, using the exact value in the denominator. (b) Find the largest nonnegative altitude for a 1.00% allowed overestimate. (c) Repeat the altitude bound if “one percent” instead means the decrease divided by the surface field.

Model and strategy. The object’s mass cancels from a relative force error because $F = mg$. Evaluate the exact radius first and solve the corresponding dimensionless inequality; do not identify a length with an accuracy merely because it sounds small.

Solution. The surface field is $9.765625 \text{ m s}^{-2}$. At $h = 1.00 \times 10^5$ m, $u = 0.015625$ and

$$g = \frac{4.00 \times 10^{14}}{(6.50 \times 10^6)^2} = 9.46746 \text{ m s}^{-2}.$$

The requested overestimate is $100(2u + u^2) = 3.14941\%$. For part (b),

$$h_{\max} = R(\sqrt{1.01} - 1) = 3.19204 \times 10^4 \text{ m} \simeq 31.9 \text{ km}.$$

For part (c),

$$h_{\max} = R[(.99)^{-1/2} - 1] = 3.22420 \times 10^4 \text{ m} \simeq 32.2 \text{ km}.$$

Check and interpretation. The exact field at zero altitude is g_s and both errors vanish. The second bound is slightly larger because it divides the same field difference by the larger surface value. These limits concern the model’s force accuracy over a height interval; they neither specify experimental error bars nor guarantee the accuracy of every observable calculated using constant gravity.

13.3.4 Gravitational field and apparent weight on a rotating body

The gravitational force on a test mass is mg . A scale measures a support force, which also depends on the object’s acceleration. These quantities coincide for a supported object at rest in an inertial frame, but a person fixed to a rotating surface follows a circular path around the rotation axis. The required centripetal acceleration changes the support reading even if the gravitational source is treated as perfectly spherical.

Use an ideal spherical body of radius R , gravitational parameter $\kappa = GM$, and angular speed Ω . Let λ be geocentric latitude, with $\lambda = 0$ at the equator and $\lambda = \pi/2$ at the north pole. The distance to the rotation axis is $s = R \cos \lambda$. Denote by $\hat{s}$ the outward cylindrical radial direction, by $\hat{r}$ the outward spherical radial direction, and by $\hat{n}$ the local direction of increasing latitude. Geometry gives

$$\hat{s} = \cos \lambda \hat{r} - \sin \lambda \hat{n}, \quad \mathbf{a} = -\Omega^2 R \cos \lambda \hat{s}.$$

Newton's second law in the inertial frame is $m\mathbf{g} + \mathbf{N} = m\mathbf{a}$, where $\mathbf{N}$ denotes the total support force. With $g_s = \kappa/R^2$,

$$\frac{\mathbf{N}}{m} = (g_s - \Omega^2 R \cos^2 \lambda) \hat{r} + \Omega^2 R \sin \lambda \cos \lambda \hat{n}. \quad (13.15)$$

There is a northward support component in the northern hemisphere. A radial platform needs this component as friction or another lateral constraint. If the platform is perpendicular to the local effective gravity and provides all support along its normal, an ideal scale reads $N = mg_{\text{eff}}$, with

$$g_{\text{eff}} = \sqrt{(g_s - \Omega^2 R \cos^2 \lambda)^2 + (\Omega^2 R \sin \lambda \cos \lambda)^2}. \quad (13.16)$$

The effective downward vector is $-\mathbf{N}/m$. A rotating-frame derivation adds the centrifugal acceleration $+\Omega^2 s \hat{s}$ to gravity and gives the same result. That centrifugal term is a frame-dependent inertial term; it does not change the Newtonian gravitational field generated by the source.

The support geometry and latitude dependence are shown in Fig. 13.5. At the equator, provided $\Omega^2 R < g_s$, $g_{\text{eff}} = g_s - \Omega^2 R$. At the pole there is no centripetal acceleration and $g_{\text{eff}} = g_s$. Away from those special latitudes the effective vertical is slightly tilted from the radial direction. Its tilt magnitude obeys

$$\tan \delta = \frac{\Omega^2 R |\sin \lambda \cos \lambda|}{g_s - \Omega^2 R \cos^2 \lambda}.$$

The spherical model deliberately omits oblateness, latitude-dependent radius, topography and nonuniform mass structure. Numerical values calculated here describe the stipulated ideal body, not a precision prediction of terrestrial gravity. When more accuracy is required, those effects must enter the source and geometry as additional model terms.

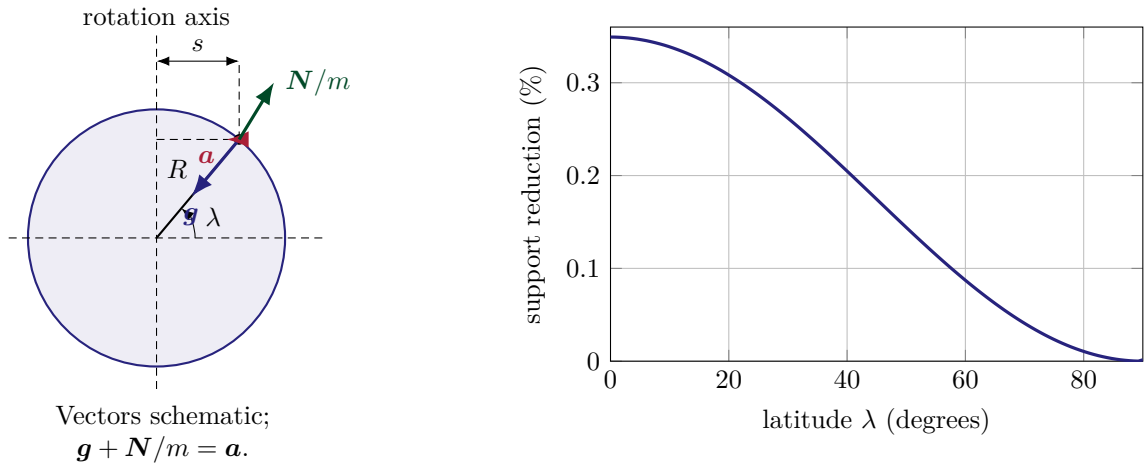


Figure 13.5: Support on a rotating ideal sphere. The meridional diagram shows the inertial centripetal acceleration towards the rotation axis and the support required in addition to gravity; the acceleration arrow is exaggerated relative to the numerical model, while the drawn vectors satisfy their stated sum. The calculated curve is $100(1 - g_{\text{eff}}/g_s)$ for $R = 6.40 \times 10^6$ m, $\kappa = 4.00 \times 10^{14}$ m³ s⁻² and $\Omega = 7.30 \times 10^{-5}$ s⁻¹. The actual gravitational field magnitude remains g_s in this spherical model.

Example 13.5: A scale reading at latitude sixty degrees

Statement. An 80.0 kg person remains fixed on the stipulated rotating spherical body with $R = 6.40 \times 10^6$ m, $\kappa = 4.00 \times 10^{14}$ m³ s⁻² and $\Omega = 7.30 \times 10^{-5}$ s⁻¹. At geocentric latitude 60.0°, (a) calculate the gravitational-force magnitude, (b) calculate radial and northward support components, (c) find the total reading of a scale aligned perpendicular to effective gravity and its vertical tilt, and (d) compare the equatorial reading. Assume the stipulated spherical gravitational model and neglect all other forces.

Model and strategy. Use inertial force balance for the centripetal acceleration. At an oblique latitude the total support is a vector; a radial normal component alone is not its magnitude.

Solution. The gravitational-force magnitude is $mg_s = 80(9.765625) = 781.250$ N. The rotation scale is $b = \Omega^2 R = 0.0341056$ m s⁻². At 60°,

$$N_r = m(g_s - b/4) = 780.567888 \text{ N}, \quad N_n = mb \frac{\sqrt{3}}{4} = 1.18145264 \text{ N}.$$

The total support is $N = \sqrt{N_r^2 + N_n^2} = 780.568782$ N, with

$$\delta = \tan^{-1}(N_n/N_r) = 0.0867217^\circ.$$

A radial platform would require the lateral support in addition to its radial normal force. At the equator the total reading is $m(g_s - b) = 778.521552$ N.

Check and interpretation. Removing rotation makes the support exactly mg_s at every latitude. The ideal scale reading is smallest at the equator and returns to mg_s at the pole. Guard digits were retained to reveal the small difference between radial and total support; these are stipulated calculations, not claims of sub-millinewton experimental precision. An orbiting scale can read zero because scale and occupant share free-fall acceleration even though gravity remains substantial.

13.4 Gravitational potential and spherical interiors

13.4.1 Potential from work

For a test mass m outside a spherical source M , the radial force is

$$F_r = -\frac{GMm}{r^2}.$$

Because the force is conservative, the potential-energy change satisfies $dU = -F_r dr$. Choosing $U(\infty) = 0$,

$$U(r) - U(\infty) = -\int_{\infty}^r F_r(r') dr' \quad (13.17)$$

$$= \int_{\infty}^r \frac{GMm}{r'^2} dr' \quad (13.18)$$

$$= -\frac{GMm}{r}. \quad (13.19)$$

Therefore

$$\boxed{U(r) = -\frac{GMm}{r}}, \quad U(\infty) = 0. \quad (13.20)$$

The negative sign does not mean “negative energy is unphysical.” It records the chosen zero: a bound pair at finite separation has lower potential energy than the infinitely separated reference state. Differentiation checks the sign,

$$F_r = -\frac{dU}{dr} = -\frac{GMm}{r^2}. \quad (13.21)$$

For small vertical changes $\Delta y \ll R$ near a spherical surface,

$$\Delta U = GMm \left(\frac{1}{R} - \frac{1}{R + \Delta y} \right) \simeq \frac{GMm}{R^2} \Delta y = mg_s \Delta y,$$

so the familiar local expression is the leading approximation to the global inverse-square potential difference.

13.4.2 Why a spherical shell has a constant interior potential

The exterior point-mass rule is much stronger than a statement about the direction of a force. Spherical symmetry makes the field radial, but symmetry alone does not determine its radial dependence. The dependence follows from the inverse-square interaction and the way contributions from the extended source add. A thin spherical shell provides a useful exact test: its interior field vanishes even though a point inside is generally closer to one part of the shell than to another. The nearby material is individually more influential, but its solid-angle distribution and the contributions from the opposite side balance. Deriving the potential makes that cancellation explicit without resolving every force into components.

Let a shell of radius a carry total mass M uniformly over its surface. Its surface density is $\sigma = M/(4\pi a^2)$. Place a field point P a distance r from the centre and use the line OP as the polar axis. A narrow latitude belt between θ and $\theta + d\theta$ has area $2\pi a^2 \sin \theta d\theta$, mass

$$dM = \frac{M}{2} \sin \theta d\theta,$$

and a common distance from P . The geometry in Fig. 13.6 gives

$$s(\theta) = \sqrt{a^2 + r^2 - 2ar \cos \theta}.$$

Potential per unit test mass is denoted Φ , with $\Phi(\infty) = 0$. Since each belt contributes $-GdM/s$, the full shell gives

$$\Phi(r) = -\frac{GM}{2} \int_0^\pi \frac{\sin \theta d\theta}{\sqrt{a^2 + r^2 - 2ar \cos \theta}}. \quad (13.22)$$

For $r > 0$, differentiation of $s^2 = a^2 + r^2 - 2ar \cos \theta$ gives $s ds = ar \sin \theta d\theta$. The nearest and farthest belts have $s = |r - a|$ and $s = r + a$, respectively. Therefore

$$\Phi(r) = -\frac{GM}{2ar} \int_{|r-a|}^{r+a} ds = -\frac{GM}{2ar} [r + a - |r - a|]. \quad (13.23)$$

The absolute value carries the distinction between an interior and an exterior field point; replacing it by $r - a$ everywhere would erase the interior solution. Evaluating it correctly produces

$$\Phi(r) = \begin{cases} -GM/a, & 0 \leq r \leq a, \\ -GM/r, & r \geq a. \end{cases} \quad (13.24)$$

At $r = 0$ the intermediate expression divides by zero, but the physical integral does not: all source elements are at distance a , so $\Phi(0) = -GM/a$. This is also the interior limit of the piecewise result. A removable singularity in an intermediate formula must not be mistaken for a physical singularity.

For a radial potential, $\mathbf{g} = -\Phi'(r)\hat{\mathbf{r}}$. Away from the ideal shell surface,

$$\mathbf{g}(r) = \begin{cases} \mathbf{0}, & r < a, \\ -GM\hat{\mathbf{r}}/r^2, & r > a. \end{cases} \quad (13.25)$$

This is the shell theorem in potential form. A nonzero negative potential is compatible with zero field: potential measures energy relative to a chosen reference, whereas field measures its spatial

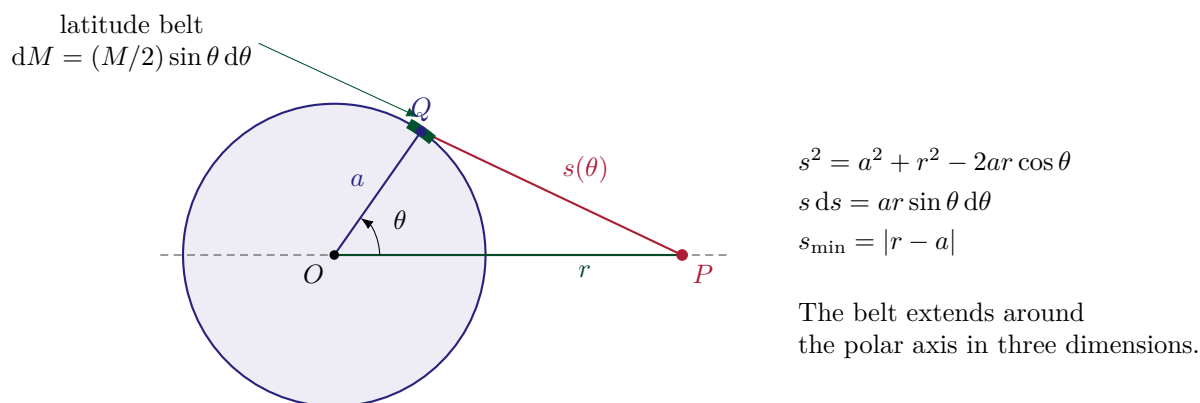


Figure 13.6: Angular construction for the exact shell-potential integral. The marked arc is a meridional cross section of a complete latitude belt. All mass in that belt has the same distance s from the axial field point. The diagram is schematic; the integration, rather than the drawn scale, determines the result.

gradient. A stationary test object inside the shell experiences no gravitational force from the shell, but moving it from the interior to infinity requires positive work. Conversely, the interior is not a region where the interaction has disappeared.

Figure 13.7 separates the potential from its radial slope. At $r = a$, the potential is continuous and has a corner. Its radial derivative jumps because the source is an ideal surface mass. The two one-sided fields differ; a unique classical derivative at the surface is not obtained by simply differentiating through the corner. A physical shell of finite thickness replaces the jump by a field variation across that thickness. This distinction will matter when continuity checks are used below: a volume density with no surface sheet has different matching conditions from an infinitesimally thin shell.

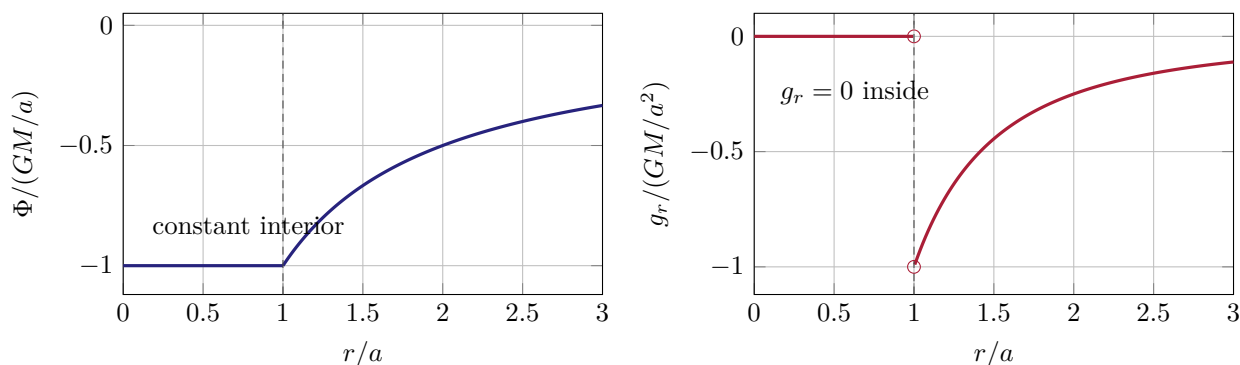


Figure 13.7: Calculated potential and signed radial field of an ideal thin shell. The potential is continuous; its slope is discontinuous at $r = a$. Open field markers show the distinct one-sided limits and avoid assigning a differentiable field value to the ideal mass sheet. Negative g_r means attraction towards the centre.

Example 13.6: A force-free interior still has binding energy

Statement. An ideal uniform thin shell has stipulated mass $M = 4.00 \times 10^{20}$ kg and radius $a = 4.00 \times 10^5$ m. A 500 kg test object is transported quasistatically from the centre to radius $2a$. Find (a) its initial and final potential energies, (b) the net external work, and (c) the shell's gravitational field at the centre and just outside its surface. Ignore shell deformation, test-object back reaction and other forces. Use $G = 6.67430 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$.

Model and strategy. The shell theorem determines the potential at both endpoints. Quasistatic transport leaves no change in kinetic energy, so the net external work equals ΔU . The

ideal shell is a mathematical source model; passage through it means a limiting path through an opening whose effect on the ideal field is neglected.

Solution. First, $GM = 2.66972 \times 10^{10} \text{ m}^3 \text{ s}^{-2}$. Then

$$U_i = -\frac{GMm}{a} = -3.33715 \times 10^7 \text{ J}, \quad U_f = -\frac{GMm}{2a} = -1.668575 \times 10^7 \text{ J}.$$

Thus $W_{\text{ext}} = U_f - U_i = +1.668575 \times 10^7 \text{ J}$, or $1.67 \times 10^7 \text{ J}$ at three significant figures. The force is zero throughout the interior, so none of this work is accumulated before reaching the shell. Beyond the shell, the object moves against attraction and the required work is positive. At the centre $\mathbf{g} = \mathbf{0}$; immediately outside the surface the inward magnitude is

$$g(a^+) = \frac{GM}{a^2} = 0.1668575 \text{ m s}^{-2}.$$

Check and interpretation. The potential energy becomes less negative as radius increases. Both energy terms scale as GMm/a , which has joule units, whereas the field scales as GM/a^2 . The interior field cannot be inferred from the value of the potential alone. The quoted work is a net reversible work between endpoints; it is not a propulsion-energy budget.

13.4.3 An arbitrary radial density profile: what enclosed mass determines

A spherical body can be assembled from concentric thin shells. Let its radius be R and its volume density be $\rho(s)$, a nonnegative function of radius s . Define

$$M_{\text{enc}}(r) = 4\pi \int_0^r \rho(s)s^2 ds, \quad M = 4\pi \int_0^R \rho(s)s^2 ds. \quad (13.26)$$

For an interior field point, shells with $s < r$ act as central point masses; shells with $s > r$ contribute constant potentials and zero fields at that point. Consequently,

$$\mathbf{g}(r) = -\frac{GM_{\text{enc}}(r)}{r^2} \hat{\mathbf{r}}, \quad 0 < r < R. \quad (13.27)$$

The formula uses the mass enclosed by the field-point radius, not the total mass. Replacing $M_{\text{enc}}(r)$ by M throughout the body would create an artificial $1/r^2$ divergence at the centre. For a density continuous at the centre with finite $\rho(0)$, $M_{\text{enc}}(r) = (4\pi/3)\rho(0)r^3 + o(r^3)$ and $g(r) \simeq (4\pi G\rho(0)/3)r$ tends to zero. If the assumed density itself diverges, the central limit must be examined for that particular model.

The corresponding potential includes both the enclosed mass and the outer shells:

$$\Phi(r) = -G \left[\frac{M_{\text{enc}}(r)}{r} + 4\pi \int_r^R \rho(s)s ds \right], \quad 0 < r \leq R. \quad (13.28)$$

This expression is more informative than $-GM_{\text{enc}}/r$ alone. The latter omits the binding-energy contribution of the outer material. To check the field, differentiate the bracket. The $M'_{\text{enc}}(r)/r = 4\pi\rho(r)r$ term cancels the derivative of the outer-shell integral, leaving

$$\frac{d\Phi}{dr} = \frac{GM_{\text{enc}}(r)}{r^2}.$$

This cancellation is a direct consistency test between the potential and the shell theorem. At $r = R$, the integral vanishes, $M_{\text{enc}}(R) = M$, and the interior potential joins the exterior value $-GM/R$.

Figure 13.8 shows how different enclosed-mass profiles generate different interior fields. For finite volume density with no surface mass sheet, M_{enc} is continuous at the boundary. The potential and its first radial derivative therefore match continuously, while higher derivatives can change when the density changes discontinuously. Two bodies with the same total mass and radius have the same exterior Newtonian field even if their interior profiles differ. Exterior observations alone cannot distinguish those profiles under exact spherical symmetry; an interior probe or additional physical information is needed.

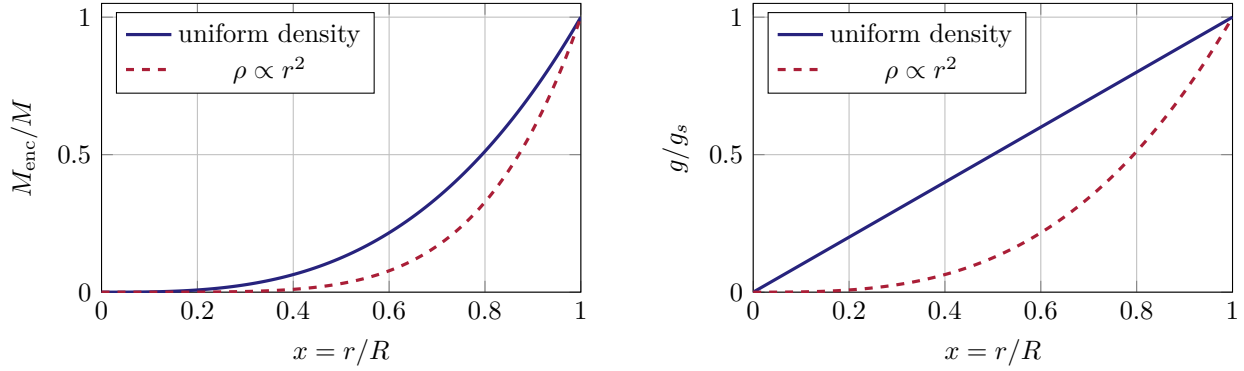


Figure 13.8: Calculated interior mass and field for two ideal spherical bodies with identical M and R . The outward-increasing model has $\rho(r) = 5Mr^2/(4\pi R^5)$, so $M_{\text{enc}}/M = x^5$. Its interior field is weaker than that of the uniform body, despite their identical surface and exterior fields. The ordinate g is the inward field magnitude.

13.4.4 The uniform sphere: field, potential and boundary checks

For a uniform sphere, $\rho = 3M/(4\pi R^3)$ and $M_{\text{enc}}(r) = M(r/R)^3$. The inward field magnitude is linear inside and inverse square outside:

$$g(r) = \begin{cases} GMr/R^3, & 0 \leq r \leq R, \\ GM/r^2, & r \geq R. \end{cases} \quad (13.29)$$

The field is strongest at the surface. Moving inward reduces the contributing enclosed mass faster than it reduces the squared distance. It is therefore incorrect to extrapolate the exterior law towards the centre of a uniform body.

Integrate the interior field inward from the known surface potential, rather than choosing a new unrelated reference:

$$\Phi(r) - \Phi(R) = \int_R^r \frac{GMs}{R^3} ds = \frac{GM}{2R^3}(r^2 - R^2).$$

Using $\Phi(R) = -GM/R$ gives

$$\Phi(r) = \begin{cases} -\frac{GM}{2R} \left(3 - \frac{r^2}{R^2} \right), & 0 \leq r \leq R, \\ -GM/r, & r \geq R. \end{cases} \quad (13.30)$$

At the centre $\Phi(0) = -3GM/(2R)$, while $g(0) = 0$. At the surface both potential formulas give $-GM/R$, and both derivatives give GM/R^2 . These checks test the integration constant, the sign, and the matching condition independently. Dimensional checks confirm that GM/R is an energy per unit mass and GM/R^2 an acceleration.

The continuous matching and central limits are visible in Fig. 13.9. The linear interior field is also a useful ideal oscillator model. In a narrow straight tunnel through a uniform, nonrotating sphere, with drag and the tunnel's missing mass neglected, displacement x from the centre obeys $\ddot{x} = -(GM/R^3)x$.

This statement concerns signed displacement on a line, rather than the nonnegative radius r . A body released from rest at the surface accelerates inward, passes the centre with maximum speed, and would oscillate with angular frequency $\sqrt{GM/R^3}$. Real planets have nonuniform density, rotation, pressure and thermal constraints; the mathematical result tests a model rather than prescribing a feasible transport system.

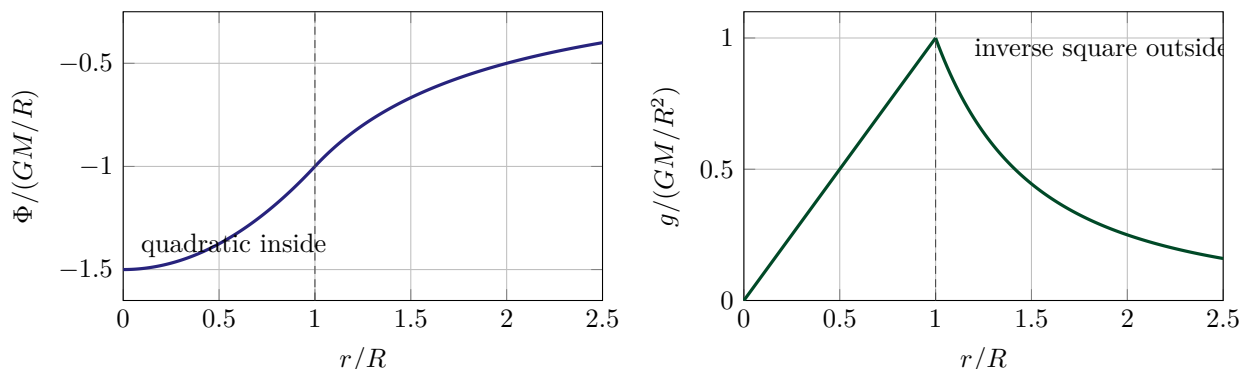


Figure 13.9: Exact normalized potential and field magnitude for a uniform sphere. The potential and its first derivative match at the surface. The field magnitude is continuous and reaches its maximum there, but its radial slope changes when the density drops to zero. A vanishing central field accompanies the most negative potential.

Example 13.7: Identical exterior gravity, different interior motion

Statement. Two spherical bodies have the stipulated parameters $R = 6.40 \times 10^6 \text{ m}$ and $GM = \kappa = 4.00 \times 10^{14} \text{ m}^3 \text{ s}^{-2}$. Body A has uniform density. Body B has $\rho(r) = Cr^2$ for $0 \leq r \leq R$, with C chosen to give the same total mass. (a) Find C , the enclosed mass and interior potential of B. (b) Compare the two field magnitudes at $r = R/2$. (c) In an ideal narrow radial tunnel, find the speed at the centre of a test body released from rest at the surface of each body. Neglect rotation, drag and the tunnel's removed mass.

Model and strategy. Normalize each density using the mass integral, apply the enclosed-mass field, and integrate the potential with the shared boundary value $\Phi(R) = -\kappa/R$. Finally use specific-energy conservation, $v^2/2 + \Phi = \text{constant}$.

Solution. For B,

$$M = 4\pi C \frac{R^5}{5}, \quad C = \frac{5M}{4\pi R^5}, \quad M_{\text{enc}} = M \left(\frac{r}{R} \right)^5.$$

Thus $g_B = \kappa r^3/R^5$ and

$$\Phi_B(r) = -\frac{\kappa}{R} + \int_R^r \frac{\kappa s^3}{R^5} ds = -\frac{\kappa}{4R} \left[5 - \left(\frac{r}{R} \right)^4 \right].$$

The shared surface field is $g_s = \kappa/R^2 = 9.765625 \text{ m s}^{-2}$. At half radius,

$$g_A = \frac{1}{2}g_s = 4.8828125 \text{ m s}^{-2}, \quad g_B = \frac{1}{8}g_s = 1.220703125 \text{ m s}^{-2}.$$

For A, $\Phi_A(0) = -3\kappa/(2R)$; for B, $\Phi_B(0) = -5\kappa/(4R)$. Therefore

$$v_{A,0} = \sqrt{\frac{\kappa}{R}} = 7.906 \text{ km s}^{-1}, \quad v_{B,0} = \sqrt{\frac{\kappa}{2R}} = 5.590 \text{ km s}^{-1}.$$

Check and interpretation. Both interior potentials and fields agree with the same exterior values at R . B places relatively more mass at large radius, leaving less enclosed mass at a given interior point. Its centre-to-surface potential difference is smaller. Exterior gravity and total mass do not uniquely fix an interior trajectory. The expression for C has units kg m^{-5} , as required by $\rho = Cr^2$.

13.4.5 A local energy approximation has its own tolerance

Force accuracy and energy accuracy should be checked separately. For a mass m raised from R to $R + h$, the exact gravitational potential-energy increase is

$$\Delta U = GMm \left(\frac{1}{R} - \frac{1}{R+h} \right) = \frac{mg_s h}{1 + h/R}. \quad (13.31)$$

The local result $\Delta U_{\text{local}} = mg_s h$ treats the force as its maximum value along the upward path, so it overestimates the work. For $h > 0$ its bias relative to the exact energy change is particularly simple:

$$\epsilon_{U,\text{exact}} = \frac{\Delta U_{\text{local}} - \Delta U}{\Delta U} = \frac{h}{R}. \quad (13.32)$$

At $h = 0$ both energy changes vanish, so the ratio is formally $0/0$; its continuous limit is zero. This boundary detail is harmless if stated, but numerical code should avoid evaluating the quotient directly at the origin.

An energy tolerance ε therefore allows $h \leq \varepsilon R$, while the same numerical tolerance for the field at the upper endpoint allows only $h \leq R(\sqrt{1 + \varepsilon} - 1)$. The energy averages the decreasing force over the path. It need not have the same error as the force at the most distant point. In the small-height limit, the endpoint-force error is approximately twice the integrated-work error.

The exact result also supplies a controlled series:

$$\Delta U = mg_s h \left[1 - \frac{h}{R} + \left(\frac{h}{R} \right)^2 - \dots \right], \quad |h/R| < 1.$$

Keeping the first correction gives $\Delta U_1 = mg_s h(1 - h/R)$. For $0 < h < R$, this corrected approximation underestimates the exact change, with relative error magnitude $(h/R)^2$. Indeed, dividing ΔU_1 by the exact result gives $(1 - u)(1 + u) = 1 - u^2$. This exact error identity is stronger than simply calling the correction “second order”. Outside the stated range the truncated expression should not be used as a global energy law: for $h > R$ it even becomes negative.

Figure 13.10 compares these energy approximations with the separate endpoint-field error. The term $mg_s h$ is a local difference of potential energies, not the absolute potential $-GMm/(R + h)$. One may choose a local zero at the surface, but that adds a constant and does not alter forces or energy differences. Mixing a surface-zero local expression with an infinity-zero escape calculation would introduce an incorrect energy offset.

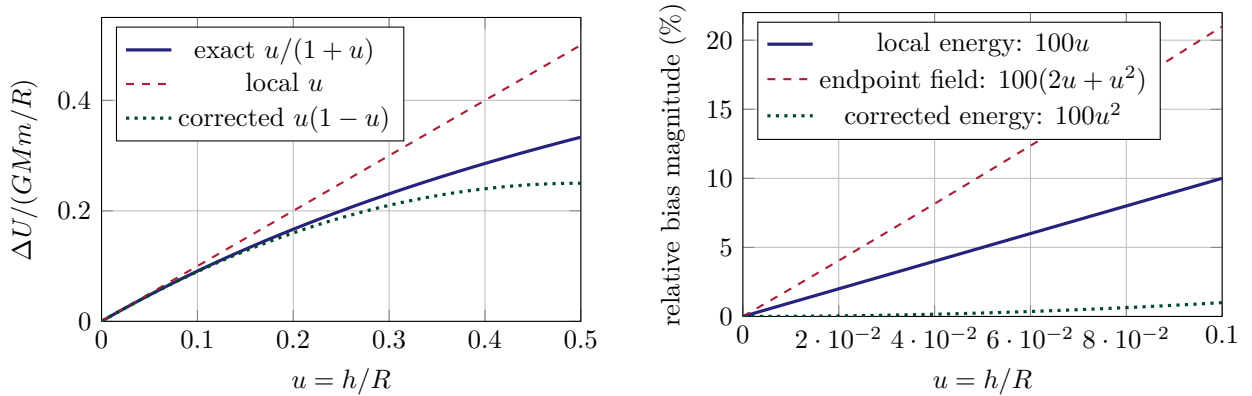


Figure 13.10: Calculated lift energy and the distinct biases of local energy and endpoint-field approximations. The corrected energy curve lies below the exact energy for $0 < u < 1$. Every relative bias in the right panel uses its own exact observable in the denominator; values at $u = 0$ mean continuous limits.

Example 13.8: Lifting a payload: energy bias versus force bias

Statement. Under the same spherical parameters $R = 6.40 \times 10^6$ m and $\kappa = 4.00 \times 10^{14}$ m³ s⁻², a 1200 kg payload is moved from the surface to altitude 400 km. (a) Find the exact increase of gravitational potential energy. (b) Evaluate $mg_s h$ and its relative bias. (c) Evaluate the first-corrected result $mg_s h(1 - h/R)$ and its relative bias. (d) Compare the altitude limits for a 1% error in local lift energy and in endpoint gravitational force. Ignore atmosphere and rotation; no final orbital velocity is specified.

Model and strategy. Use endpoint potential energies for the exact work, then compare each approximate result with that exact energy difference. A payload's lift energy does not include a circular orbit's kinetic-energy requirement or propulsion losses.

Solution. Here $u = h/R = 0.0625$ and $g_s = 9.765625$ m s⁻². Therefore

$$\Delta U = \frac{1200(9.765625)(4.00 \times 10^5)}{1.0625} = 4.4117647 \times 10^9 \text{ J.}$$

The local result is 4.6875000×10^9 J and overestimates the exact change by 6.25%. The corrected result is 4.39453125×10^9 J; its relative underestimate is $u^2 = 0.00390625$, or 0.390625%. A 1% local-energy tolerance allows $h \leq 0.01R = 64.0$ km; the endpoint-force tolerance allows only $h \leq 31.9$ km with the exact force in its denominator.

Check and interpretation. The exact answer lies between the local and corrected approximations. All energy expressions carry the common scale $mg_s h$. No kinetic energy was requested, so this work cannot be called the complete cost of placing a satellite in orbit. The accuracy of a force approximation should not be substituted for a separately calculable energy error.

13.4.6 Potential superposition, gradients and a ring-axis extension

Potential is often the simpler quantity to construct for an extended source because it is a scalar. For discrete stationary masses, using a common zero at infinity,

$$\Phi(\mathbf{r}) = -G \sum_i \frac{M_i}{|\mathbf{r} - \mathbf{r}_i|}, \quad \mathbf{g}(\mathbf{r}) = -\nabla \Phi(\mathbf{r}). \quad (13.33)$$

The gradient contains both the change of magnitude and the direction of steepest increase. Gravity points towards decreasing potential. For a displacement $d\mathbf{r}$, $d\Phi = \nabla \Phi \cdot d\mathbf{r} = -\mathbf{g} \cdot d\mathbf{r}$. Moving perpendicular to the local field is therefore tangent to an equipotential to first order; remaining at constant potential over a finite displacement requires following the equipotential surface.

When potentials from several sources are added, their additive constants must use the same reference convention. Field superposition always survives adding a constant, but comparing energies at separated points requires consistency. Potentials from positive masses with an infinity zero are nonpositive in the isolated Newtonian models considered here. A cancellation of their fields does not generally cancel their potentials. The interior of a shell already demonstrates this, and a ring gives a useful nonspherical extension with a simple axial integral.

Consider a uniform thin ring of radius a and total mass M_r , lying in the plane $z = 0$. Every ring element is the same distance $\sqrt{a^2 + z^2}$ from an axial point. Writing $\kappa_r = GM_r$,

$$\Phi(z) = -\frac{\kappa_r}{\sqrt{a^2 + z^2}}, \quad g_z(z) = -\frac{\kappa_r z}{(a^2 + z^2)^{3/2}}. \quad (13.34)$$

These equations apply on the symmetry axis. They must not be used off-axis simply by replacing z with a spherical distance. At the ring centre $g_z = 0$, but $\Phi = -\kappa_r/a$; at large positive z , the potential approaches $-\kappa_r/z$ and the axial field approaches $-\kappa_r/z^2$. Thus the distant source acts approximately like a point mass, while its near field retains its geometry.

For positive z , the field points back towards the plane. Its magnitude first increases from zero and then falls towards zero at infinity. Differentiating $z(a^2 + z^2)^{-3/2}$ gives

$$\frac{d}{dz} \left[\frac{z}{(a^2 + z^2)^{3/2}} \right] = \frac{a^2 - 2z^2}{(a^2 + z^2)^{5/2}}.$$

The shape and sign reversal are shown in Fig. 13.11. The maximum therefore occurs at $z = a/\sqrt{2}$, where $|g_z| = 2\kappa_r/(3\sqrt{3}a^2)$. At small axial displacement,

$$\Phi(z) = -\frac{\kappa_r}{a} + \frac{\kappa_r z^2}{2a^3} + \mathcal{O}(z^4), \quad g_z(z) = -\frac{\kappa_r}{a^3}z + \mathcal{O}(z^3).$$

Constrained motion along the axis is thus approximately harmonic near the centre. That does not establish stability in three dimensions. The ring centre lies in vacuum, where the Newtonian potential satisfies $\nabla^2\Phi = 0$. Here $\nabla^2\Phi = \Phi_{xx} + \Phi_{yy} + \Phi_{zz}$ is the sum of its three Cartesian second derivatives; each $-GdM/s$ contribution has zero sum away from its source. Axial symmetry makes the two transverse second derivatives equal. Since $\Phi_{zz}(0) = \kappa_r/a^3 > 0$, their common value is $-\kappa_r/(2a^3)$. A small transverse displacement is consequently accelerated away from the centre. The same equilibrium is stable for the one-dimensional axial constraint and unstable for free transverse motion. A potential plot along a single line cannot settle full spatial stability.

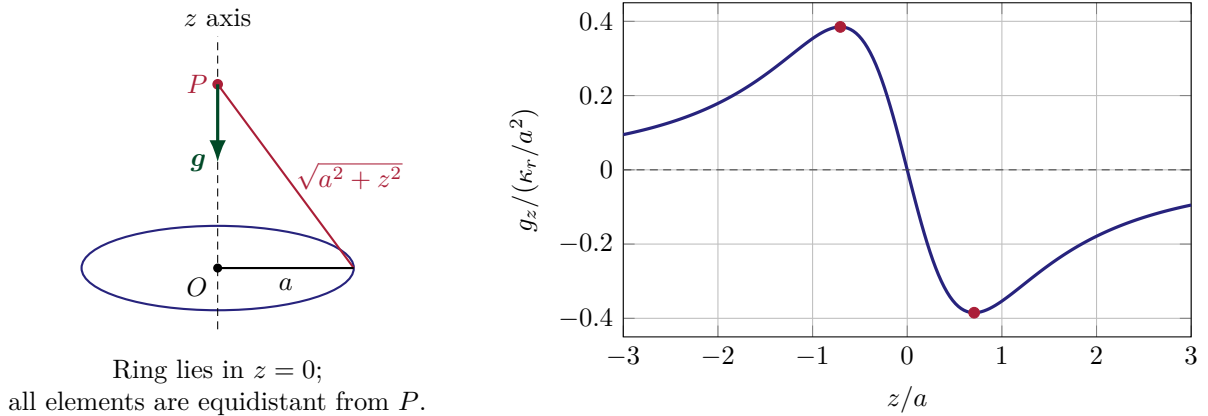


Figure 13.11: An ideal ring's axial field, calculated by differentiating its scalar potential. Attraction points towards the ring plane on both sides. The marked extrema occur at $z/a = \pm 1/\sqrt{2}$ and have magnitude $2/(3\sqrt{3})$ in the plotted units. The perspective ring diagram is schematic and the analytic result applies only on its axis.

Example 13.9: An axial drop towards an ideal ring

Statement. A fixed uniform ring has stipulated radius $a = 1000$ m and gravitational parameter $\kappa_r = GM_r = 1.00 \times 10^4 \text{ m}^3 \text{ s}^{-2}$. A negligible test mass is released from rest at $z_0 = 500$ m and constrained to move on the ring axis. (a) Calculate its initial axial acceleration and potential per unit mass. (b) Find its speed on first reaching $z = 0$. (c) Determine the location and magnitude of the maximum axial field. (d) Find the small-amplitude axial oscillation period and explain whether it guarantees three-dimensional stability. Neglect all other sources and losses; the ideal constraint does no work along the axis.

Model and strategy. Evaluate the exact ring-axis expressions, use conservation of specific energy for the finite drop, differentiate the field for its maximum, and use the linearized field only for the small-amplitude period.

Solution. At release,

$$g_z(z_0) = -\frac{10^4(500)}{(10^6 + 250000)^{3/2}} = -3.57771 \times 10^{-3} \text{ m s}^{-2}, \quad \Phi(z_0) = -8.94427 \text{ J kg}^{-1}.$$

At the centre $\Phi(0) = -10.0000 \text{ J kg}^{-1}$, so

$$v(0) = \sqrt{2[\Phi(z_0) - \Phi(0)]} = 1.45309 \text{ m s}^{-1}.$$

The field magnitude is maximal at $|z| = a/\sqrt{2} = 707.107 \text{ m}$ and equals $3.84900 \times 10^{-3} \text{ m s}^{-2}$. The linearized equation is $\ddot{z} = -(\kappa_r/a^3)z$, giving

$$T_{\text{small}} = 2\pi\sqrt{\frac{a^3}{\kappa_r}} = 1986.92 \text{ s} \simeq 33.1 \text{ min}.$$

Check and interpretation. The test mass gains kinetic energy as its potential decreases and the initial acceleration has the sign required to move towards the plane. The 500 m release is not a small displacement compared with a , so T_{small} is not presented as its exact finite-amplitude period. Axial restoring motion coexists with transverse instability; removing the axial constraint changes the stability question.

13.4.7 Practical checks when using a potential model

A useful potential calculation should survive several independent checks. The energy reference should be stated before values are compared. Its gradient should reproduce the intended signed force components. Symmetries should cancel the components they forbid, while boundary and far-field limits should match the appropriate source model. A length scaling should also be visible: a potential generated by mass M across a characteristic scale L is of order GM/L , whereas a field is of order GM/L^2 . Numerical differentiation can test analytic formulas, but a small finite-difference residual is meaningful only when the step size and floating-point cancellation are controlled.

For a spherical source, inspect $r \rightarrow 0$, $r = R$ and $r \rightarrow \infty$ separately. At a volume-density boundary, check both potential and first derivative; at an ideal surface sheet, expect a derivative jump. For a nonspherical source, identify the coordinate restriction of each formula. The ring-axis expression, for example, supplies an exact one-dimensional slice rather than a complete spatial potential. Finally, distinguish local model accuracy from source-parameter uncertainty: a perfectly evaluated approximation can still predict the wrong physics if the geometry, density or required accuracy was chosen incorrectly.

13.4.8 Escape as an energy threshold

For an isolated test particle with speed v at radius r ,

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}. \quad (13.35)$$

Reaching arbitrarily large r is energetically possible only if $E \geq 0$. For a finite source, the initial direction and angular momentum must also produce a trajectory that avoids impact. An outward radial launch meets that geometrical condition in the isolated model. The minimum launch speed at r_0 corresponds to $E = 0$ and vanishing speed in the $r \rightarrow \infty$ limit:

$$v_{\text{esc}}(r_0) = \sqrt{\frac{2GM}{r_0}}. \quad (13.36)$$

This is not the speed at which gravity “turns off.” Gravity approaches zero continuously as distance grows. Escape is an energy statement made under assumptions such as an isolated spherical source, no drag and a negligible test mass.

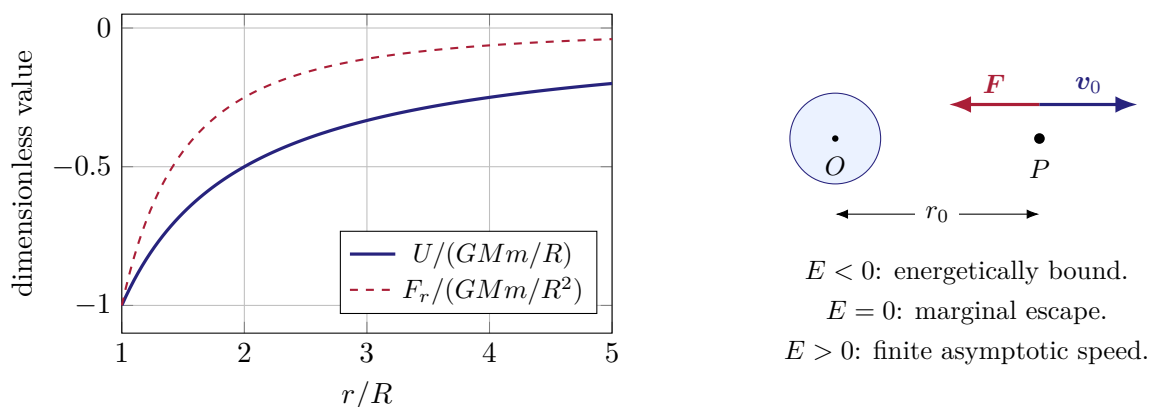


Figure 13.12: Exterior potential and radial force use different normalization scales, explicitly identified in the legend. A radial outward launch illustrates energy classification with $U(\infty) = 0$. A finite source can be struck by a trajectory; nonnegative energy alone does not guarantee collision-free escape.

Example 13.10: Release and escape from a specified radius

Use a stipulated spherical-source parameter $GM = 4.00 \times 10^{14} \text{ m}^3 \text{ s}^{-2}$ and launch radius $r_0 = 8.00 \times 10^6 \text{ m}$. Neglect atmosphere, rotation and third bodies. Compare launch speeds $0.80v_{\text{esc}}$ and $1.05v_{\text{esc}}$.

The escape speed is

$$v_{\text{esc}} = \sqrt{\frac{2(4.00 \times 10^{14})}{8.00 \times 10^6}} = 1.00 \times 10^4 \text{ m s}^{-1}.$$

For energy classification it is enough to use specific energy $\varepsilon = E/m = v^2/2 - GM/r_0$. For $v_1 = 8.00 \text{ km s}^{-1}$,

$$\varepsilon_1 = -1.80 \times 10^7 \text{ J kg}^{-1} < 0,$$

so the state is bound under the stated model. For $v_2 = 10.5 \text{ km s}^{-1}$,

$$\varepsilon_2 = 5.13 \times 10^6 \text{ J kg}^{-1} > 0,$$

so it is unbound.

If the same source is used but the launch radius increases by 10%, the escape speed is multiplied by $1/\sqrt{1.10} = 0.9535$. Escape speed decreases with radius as $r^{-1/2}$, while the field magnitude decreases faster, as r^{-2} .

Guided checks

X13-03. Three claims are proposed: (i) U is negative because gravity is “negative”; (ii) escape occurs where the force becomes exactly zero; (iii) doubling r_0 halves v_{esc} . Repair them. The sign of U follows from the chosen zero and attraction; escape is an energy threshold at infinity; and $v_{\text{esc}} \propto r_0^{-1/2}$, so doubling radius multiplies it by $1/\sqrt{2}$.

13.5 Circular orbits and energy

13.5.1 Circular force balance

Consider a test mass m in a circular orbit of radius r about a spherical source M . The gravitational force is the inward radial force required for circular acceleration. Newton's second law gives

$$\frac{GMm}{r^2} = m \frac{v_c^2}{r}. \quad (13.37)$$

The mass m cancels, giving

$$v_c = \sqrt{\frac{GM}{r}}. \quad (13.38)$$

There is no additional force called “centripetal force” balancing gravity. *Centripetal* describes the radial direction of the net force required by circular motion. In this ideal problem, gravity supplies that net radial force.

The orbital period follows from $v_c = 2\pi r/T_{\text{orb}}$:

$$T_{\text{orb}} = 2\pi \sqrt{\frac{r^3}{GM}}. \quad (13.39)$$

A larger circular orbit therefore has a *smaller* speed but a *longer* period.

13.5.2 Binding energy of a circular orbit

Using Eq. (13.38), the kinetic energy is

$$K = \frac{1}{2}mv_c^2 = \frac{GMm}{2r}.$$

The potential energy is $U = -GMm/r$, so

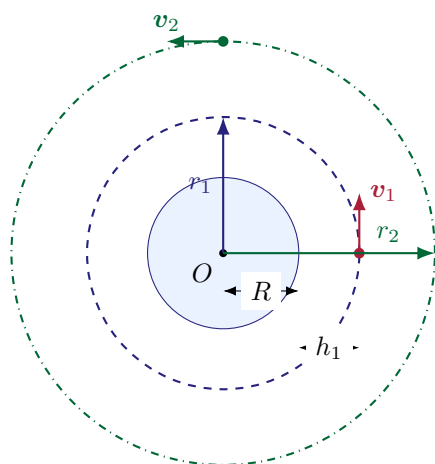
$$E = K + U = -\frac{GMm}{2r}. \quad (13.40)$$

Thus $K = -E$ and $U = 2E$ for this circular test-particle model. The negative total energy identifies the orbit as bound. At the same radius,

$$v_{\text{esc}} = \sqrt{2} v_c. \quad (13.41)$$

Key message

An orbiting body can have a large gravitational acceleration and still appear weightless relative to a nearby freely falling vehicle. Weightlessness in orbit is primarily a condition of common free fall, not an absence of gravity.



$r_i = R + h_i$ in every orbital formula.

For $r_2 > r_1$: $v_2 < v_1$ and $T_2 > T_1$.

The single force in the ideal orbit is gravity. Velocity is tangent; acceleration points inward.

Figure 13.13: Two circular orbits about one spherical source. Radii and velocities are schematic, with radial measurements separated from altitude. The outer satellite shown at the top moves left for the counterclockwise orientation.

Example 13.11: Compare two ideal satellite radii

Two satellites of the same negligible mass move on circular orbits about the same spherical source. Let $r_2 = 4r_1$.

From Eqs. (13.38)–(13.40),

$$\frac{v_2}{v_1} = \sqrt{\frac{r_1}{r_2}} = \frac{1}{2}, \quad \frac{T_2}{T_1} = \left(\frac{r_2}{r_1}\right)^{3/2} = 8, \quad \frac{E_2}{E_1} = \frac{r_1}{r_2} = \frac{1}{4}.$$

Because both energies are negative, E_2 is *less negative*; increasing orbital radius requires an increase in mechanical energy.

For the stipulated source $GM = 4.00 \times 10^{14} \text{ m}^3 \text{ s}^{-2}$, take $r_1 = 8.00 \times 10^6 \text{ m}$ and $r_2 = 3.20 \times 10^7 \text{ m}$. Then

$$v_1 = 7.07 \text{ km s}^{-1}, \quad v_2 = 3.54 \text{ km s}^{-1},$$

$$T_1 = 7.11 \times 10^3 \text{ s}, \quad T_2 = 5.69 \times 10^4 \text{ s}.$$

The energy difference between two circular states does not specify a practical transfer manoeuvre. A transfer trajectory has its own intermediate velocities and energy changes.

Guided checks

X13-04. A student argues: “At a higher circular orbit gravity is weaker, so the satellite must move faster to avoid falling.” Repair the reasoning. The circular speed satisfies $v_c^2 = GM/r$ and therefore decreases with radius. The period still increases because the path circumference grows while the speed falls. At any fixed r , $v_{\text{esc}} = \sqrt{2}v_c$.

13.6 Kepler’s laws and angular momentum

Kepler’s empirical description of planetary motion can be understood within Newtonian central-force mechanics. In the ideal two-body Kepler problem: (1) a bound orbit is an ellipse with the attracting focus occupied by the central body in the test-particle limit; (2) the radius vector sweeps equal areas in equal times; and (3) the square of the period is proportional to the cube of the semimajor axis for orbits about the same central mass.

13.6.1 Equal areas from angular momentum conservation

For a central force, $\mathbf{F}$ is parallel or antiparallel to $\mathbf{r}$, so the torque about the force centre is zero:

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \mathbf{0}.$$

Therefore the angular momentum $\mathbf{L} = m\mathbf{r} \times \mathbf{v}$ is constant. During a short time dt , the swept area is approximately the area of a triangle with base r and transverse displacement $v_{\perp}dt$:

$$dA = \frac{1}{2}rv_{\perp}dt.$$

Since $L = mrv_{\perp}$,

$$\boxed{\frac{dA}{dt} = \frac{L}{2m} = \text{constant}}. \quad (13.42)$$

Equal areas therefore correspond to equal times. They do *not* imply equal changes in polar angle or equal arc lengths.

For an ellipse of semimajor axis a and eccentricity e ,

$$r_p = a(1 - e), \quad r_a = a(1 + e), \quad (13.43)$$

where r_p and r_a are periapsis and apoapsis distances from the focus. At each apse the velocity is transverse, so angular-momentum conservation gives

$$mr_pv_p = mr_av_a, \quad \boxed{\frac{v_p}{v_a} = \frac{r_a}{r_p} = \frac{1+e}{1-e}}. \quad (13.44)$$

13.6.2 Period–size relation and its dynamical development

For a test mass moving in a Newtonian Kepler ellipse about a fixed source mass M ,

$$\boxed{T_{\text{orb}}^2 = \frac{4\pi^2}{GM}a^3}. \quad (13.45)$$

The circular result is recovered when $e = 0$ and $a = r$. The advanced section shows how the denominator changes to $G(M + m)$ for the relative motion when both masses are dynamically important.

13.6.3 The ellipse has a centre and two foci

An ellipse diagram contains several distances that must not be interchanged. Place its geometric centre at $C = (0, 0)$ and align the major axis with the x axis. The curve satisfies

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a \geq b > 0.$$

Its two foci are $F_{\pm} = (\pm c, 0)$. The defining geometric property is that the sum of the distances from a point on the ellipse to the two foci is $2a$. At the top point $(0, b)$ the distances are equal, so $2\sqrt{b^2 + c^2} = 2a$. Consequently,

$$c = \sqrt{a^2 - b^2}, \quad e = \frac{c}{a}, \quad b = a\sqrt{1 - e^2}. \quad (13.46)$$

The eccentricity e is dimensionless. A circle has $c = 0$ and $e = 0$; its two foci coincide with its centre. An ordinary ellipse has $0 < e < 1$. The limit $e \rightarrow 1$ at fixed a is a flattened bound ellipse; it is not a finite-sized parabola. A parabolic limit must be taken while keeping an appropriate distance, such as the periapsis or semilatus rectum, finite and letting a diverge.

For a test-particle gravitational orbit, put the attracting body at $F_+ = (c, 0)$. Nothing is required to occupy F_- . The periapsis is the right-hand end of the ellipse, and its distance from the attracting focus is $a - c = a(1 - e)$. The apoapsis is the left-hand end, a distance $a + c = a(1 + e)$ from that same focus. These distances are not a and b . The identities

$$a = \frac{r_p + r_a}{2}, \quad e = \frac{r_a - r_p}{r_a + r_p}$$

are therefore useful for reconstructing an orbit from observed apsides. All of these are centre-to-body distances in the point-source orbital model. If the source has a physical radius R , a separate comparison $r_p > R$ is needed before interpreting the mathematical ellipse as a complete exterior orbit.

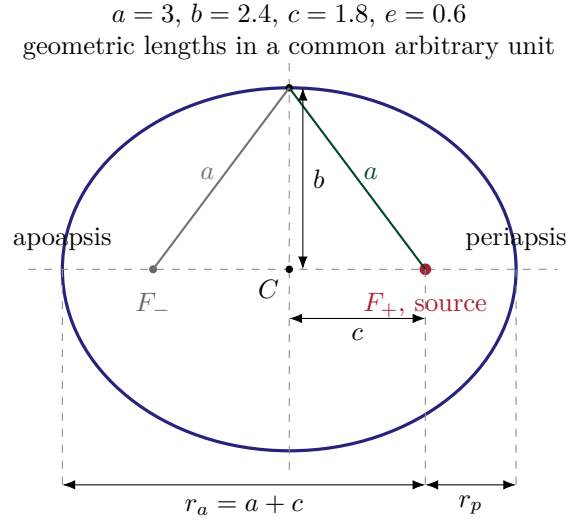


Figure 13.14: An independently reconstructed ellipse with its actual foci. The right-hand focus is the gravitational source; the centre is a different point. The top-point triangle verifies $a^2 = b^2 + c^2$. The ellipse-and-projection construction was informed by Alexandros Tsagkaropoulos, *Elliptical Orbit*, TikZ.net (CC BY-SA 4.0); the focus positions and all gravitational interpretations here were calculated independently.

13.6.4 A local mathematical bridge: from force to conic orbit

The equal-area law establishes how angular momentum constrains the motion, but it does not by itself establish the shape of the path. We now derive that shape from the inverse-square force. Work in the fixed orbital plane with polar coordinates (r, θ) centred on the source. Choose increasing θ in the direction of motion and put $\theta = 0$ at periapsis. Define

$$\kappa = GM, \quad \ell = \frac{L}{m} = r^2 \dot{\theta}, \quad \varepsilon = \frac{E}{m} = \frac{v^2}{2} - \frac{\kappa}{r}.$$

The symbol ℓ has units $\text{m}^2 \text{s}^{-1}$; it is not the altitude h . The derivation assumes $\ell \neq 0$, so that angular position can be used as an independent variable. A purely radial trajectory requires its own one-dimensional treatment.

The polar unit vectors rotate as the body moves. Their derivatives are $\dot{\hat{r}} = \dot{\theta} \hat{\theta}$ and $\dot{\hat{\theta}} = -\dot{\theta} \hat{r}$. Differentiating $\mathbf{r} = r \hat{r}$ twice gives

$$\mathbf{v} = \dot{r} \hat{r} + r \dot{\theta} \hat{\theta}, \quad \mathbf{a} = (\ddot{r} - r \dot{\theta}^2) \hat{r} + (r \ddot{\theta} + 2 \dot{r} \dot{\theta}) \hat{\theta}.$$

The transverse acceleration vanishes for a central force. Multiplying that statement by r shows that $d(r^2 \dot{\theta})/dt = 0$, recovering ℓ conservation. The radial equation is

$$\ddot{r} - r \dot{\theta}^2 = -\frac{\kappa}{r^2}. \quad (13.47)$$

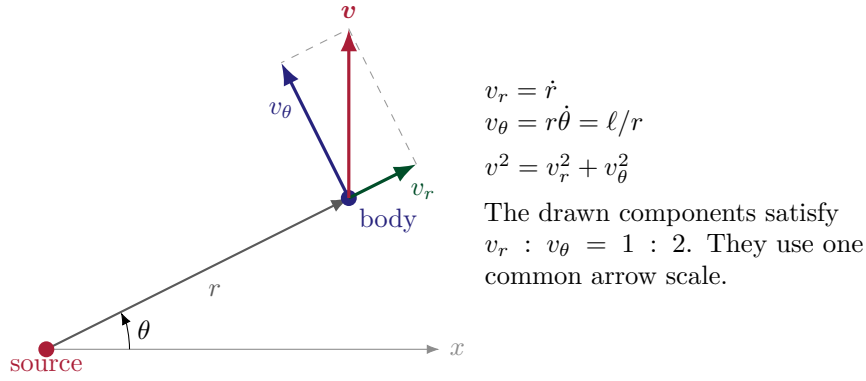


Figure 13.15: Local polar velocity components at a generic point. The resultant velocity is tangent to the trajectory, while the transverse direction is perpendicular to the source-to-body radius. These directions coincide only where $v_r = 0$, including the apsides.

The term $r\dot{\theta}^2$ appears because the radial basis rotates; it is not an additional gravitational interaction. To eliminate time, introduce $u(\theta) = 1/r$ and use a prime for $d/d\theta$. The chain rule and $\dot{\theta} = \ell u^2$ give

$$\dot{r} = -\ell u', \quad \ddot{r} = -\ell^2 u^2 u'', \quad r\dot{\theta}^2 = \ell^2 u^3.$$

Substitution into Eq. (13.47), followed by division by $-\ell^2 u^2$, yields Binet's equation for this force:

$$u'' + u = \frac{\kappa}{\ell^2}. \quad (13.48)$$

A constant particular solution is κ/ℓ^2 , and the general homogeneous solution is a combination of sine and cosine. Orienting the angular origin at periapsis absorbs the phase into θ , so the solution can be written

$$\boxed{r(\theta) = \frac{p}{1 + e \cos \theta}}, \quad p = \frac{\ell^2}{\kappa}. \quad (13.49)$$

The constant p is the *semilatus rectum*: it is the focus-to-orbit distance at $\theta = \pm\pi/2$. It has dimensions of length. The constant $e \geq 0$ is set by the initial position and velocity and becomes the geometric eccentricity. For an ellipse, $r_p = p/(1 + e)$ and $r_a = p/(1 - e)$; comparing with the earlier apse relations gives $p = a(1 - e^2)$.

We can verify the energy classification without assuming an ellipse. Since $v_r = -\ell u'$ and $v_\theta = \ell u$,

$$\varepsilon = \frac{\ell^2}{2} [(u')^2 + u^2] - \kappa u.$$

Inserting $u = (1 + e \cos \theta)/p$ and $u' = -e \sin \theta/p$ cancels all angle-dependent terms, leaving

$$\varepsilon = \frac{\kappa^2}{2\ell^2} (e^2 - 1), \quad \boxed{e^2 = 1 + \frac{2\varepsilon\ell^2}{\kappa^2}}. \quad (13.50)$$

The cancellation is a direct algebraic check of the orbit and force models. The combination in the square root is dimensionless. For nonzero ℓ , $\varepsilon < 0$ gives $0 \leq e < 1$ and a bound ellipse, $\varepsilon = 0$ gives $e = 1$ and a parabola, and $\varepsilon > 0$ gives $e > 1$ and a hyperbola. The circle is the $e = 0$ member of the bound family. These results concern the ideal uninterrupted exterior trajectory; a physical surface can intercept it.

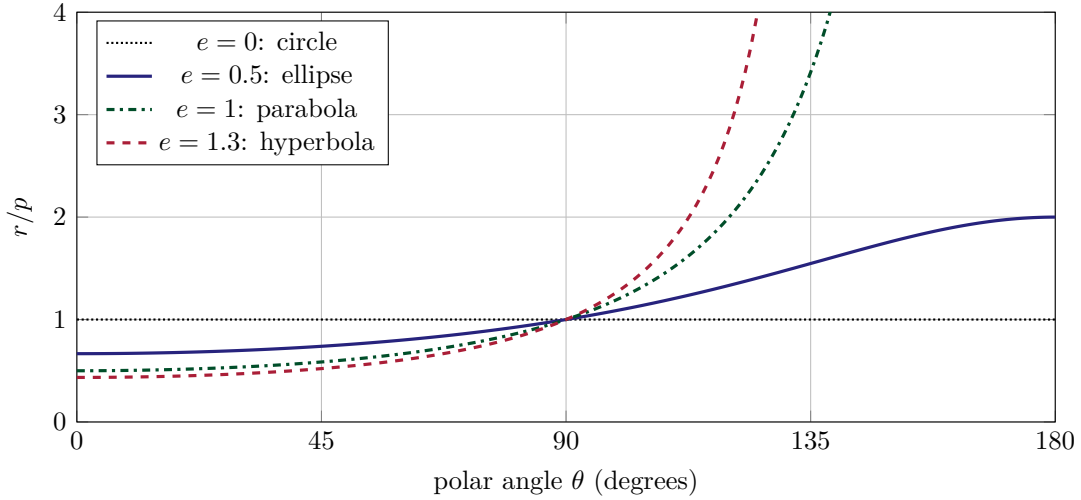


Figure 13.16: A quantitative comparison at fixed semilatus rectum p . Every curve passes through $r/p = 1$ at 90° . The ellipse has a finite apoapsis at 180° ; the parabola has $r \rightarrow \infty$ there. The hyperbola has an asymptote at $\theta_\infty = \arccos(-1/1.3) = 140.3^\circ$, beyond the displayed segment. Open curves are clipped at $r/p = 4$; clipping is not a physical endpoint. The other half follows by reflection in θ .

Example 13.12: Find velocity components away from an apse

An outward-moving body is on an ellipse with $a = 2.00 \times 10^7$ m and $e = 0.400$, about the stipulated source $\kappa = 4.00 \times 10^{14} \text{ m}^3 \text{ s}^{-2}$. At $r = 1.50 \times 10^7$ m, determine its polar angle, radial and transverse speeds, total speed and flight-path angle above the local transverse direction.

Solution. First calculate $p = a(1 - e^2) = 1.68 \times 10^7$ m and $\ell = \sqrt{\kappa p} = 8.198 \times 10^{10} \text{ m}^2 \text{ s}^{-1}$. The conic equation gives

$$\cos \theta = \frac{p/r - 1}{e} = 0.300, \quad \theta = 72.54^\circ$$

on the outward branch, $0 < \theta < \pi$. Differentiation of the conic gives $v_r = (\kappa e / \ell) \sin \theta$; angular momentum gives $v_\theta = \ell / r$. Hence

$$v_r = 1.862 \text{ km s}^{-1}, \quad v_\theta = 5.465 \text{ km s}^{-1}, \quad v = \sqrt{v_r^2 + v_\theta^2} = 5.774 \text{ km s}^{-1}.$$

The flight-path angle is $\gamma = \arctan(v_r / v_\theta) = 18.81^\circ$. The independent energy relation below also gives $v^2 = \kappa(2/r - 1/a) = 3.333 \times 10^7 \text{ m}^2 \text{ s}^{-2}$. On the inward branch the same radius occurs at $360^\circ - 72.54^\circ$: the radial velocity reverses sign while v_θ and the speed remain unchanged. Thus specifying radius and speed alone does not identify a unique orbital position or direction of travel.

13.6.5 Energy, the vis-viva relation and Kepler's third law

For an ellipse, insert $\ell^2 = \kappa a(1 - e^2)$ into Eq. (13.50). The factor $(1 - e^2)$ cancels:

$$\varepsilon = -\frac{\kappa}{2a}, \quad E = -\frac{\kappa m}{2a}. \quad (13.51)$$

The mechanical energy depends on the semimajor axis, not separately on the eccentricity. Two ellipses with the same a can have very different closest approaches and maximum speeds while sharing the same total energy. Their angular momenta differ because $\ell^2 = \kappa a(1 - e^2)$. Energy and angular momentum together, rather than energy alone, determine the size and shape.

There is a second derivation that is useful when apsides are known. At each apse $v_r = 0$, so energy conservation gives

$$\frac{\ell^2}{2r_p^2} - \frac{\kappa}{r_p} = \frac{\ell^2}{2r_a^2} - \frac{\kappa}{r_a}.$$

For unequal apsides, canceling the common difference yields $\ell^2 = 2\kappa r_p r_a / (r_p + r_a)$. Substituting back gives $\varepsilon = -\kappa / (r_p + r_a) = -\kappa / (2a)$. The circular limit follows continuously, although division by $r_a - r_p$ in this intermediate step is not valid at the exact circular point. This is an example of why the domain of an algebraic cancellation matters even when the final formula has a regular limit.

Using $\varepsilon = v^2/2 - \kappa/r$ now gives the vis-viva relation

$$v^2 = \kappa \left(\frac{2}{r} - \frac{1}{a} \right). \quad (13.52)$$

It predicts the *speed*, not the radial or transverse component separately. At fixed a , smaller r means a larger speed. For $a = r$ in a circular orbit it reduces to $v^2 = \kappa/r$. For an unbound orbit one can extend the formula by defining a *signed energy parameter* $a = -\kappa/(2\varepsilon) < 0$; that signed parameter should not be mistaken for a negative geometric length. A parabola has $\varepsilon = 0$ and no finite a in this convention; its local speed is the escape threshold $v = \sqrt{2\kappa/r}$.

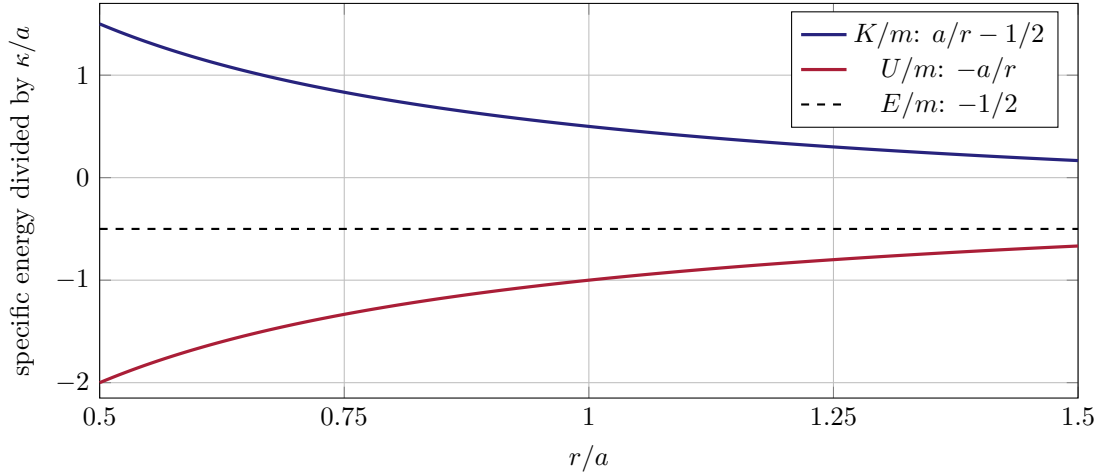


Figure 13.17: Exchange of kinetic and potential energy along an ellipse with $e = 0.5$, whose accessible radii are $0.5 \leq r/a \leq 1.5$. All quantities use the same signed energy scale κ/a . The left boundary is periapsis and the right boundary apoapsis. The sum of the two varying curves is the horizontal total-energy line. The kinetic curve is total kinetic energy per mass, not just radial kinetic energy.

The period can be derived without assuming uniform angular motion. The full ellipse area is $A = \pi ab$. Its constant areal rate is $\dot{A} = \ell/2$, so the time to sweep the complete ellipse is

$$T = \frac{\pi ab}{\ell/2} = \frac{2\pi ab}{\ell}.$$

With $b = a\sqrt{1 - e^2}$ and $\ell = \sqrt{\kappa a(1 - e^2)}$,

$$T = 2\pi \sqrt{\frac{a^3}{\kappa}}. \quad (13.53)$$

Eccentricity cancels from the period, just as it canceled from the energy, but for a different geometric reason: the shrinking ellipse area and areal rate have the same $\sqrt{1 - e^2}$ factor. Thus two ellipses of equal a around the same source have the same period although their motion around the path is distributed differently in time. Dimensional analysis checks a^3/κ as a squared time. In the

exact two-body problem, the corresponding relative orbit uses $\kappa = G(M + m)$; that replacement is discussed in the advanced section.

The circular speed at the instantaneous radius is a useful reference, but an eccentric orbit need not have that speed. Subtracting $v_c(r)^2 = \kappa/r$ from vis-viva gives

$$v^2 - v_c(r)^2 = \kappa \left(\frac{1}{r} - \frac{1}{a} \right).$$

The eccentric body is faster than the corresponding local circular body when $r < a$ and slower when $r > a$. At the two crossings of $r = a$ their speeds agree, but their velocity directions generally do not: the eccentric body has a nonzero radial component there. Equal speed at the same radius therefore does not imply equal orbit. This observation also resolves the two prograde impulses in an outward transfer: the transfer body is fast relative to the inner circle at periapsis and slow relative to the outer circle at apoapsis.

There is a useful time-average check, derived below once the eccentric anomaly is introduced. Kepler's equation implies $dt = (1 - e \cos \psi) d\psi / n$, while $r = a(1 - e \cos \psi)$. For one complete revolution,

$$\left\langle \frac{1}{r} \right\rangle_t = \frac{1}{T} \int_0^T \frac{dt}{r} = \frac{1}{Tna} \int_0^{2\pi} d\psi = \frac{1}{a}.$$

Consequently the time-averaged specific potential and kinetic energies are

$$\left\langle \frac{U}{m} \right\rangle_t = -\frac{\kappa}{a}, \quad \left\langle \frac{K}{m} \right\rangle_t = \frac{\kappa}{2a}.$$

The relation $2\langle K \rangle_t + \langle U \rangle_t = 0$ is the gravitational virial relation for this bound Kepler orbit. It is a relation between averages over time. For an eccentric orbit, $2K + U$ is generally nonzero at an individual point, even though it vanishes at every point on the circular orbit. Sampling uniformly in true angle or eccentric anomaly will not automatically reproduce a time average; the correct time weight must accompany the sampling. In a numerical trajectory stored at uniform time steps, an ordinary mean of sufficiently well-resolved samples approximates the time average directly.

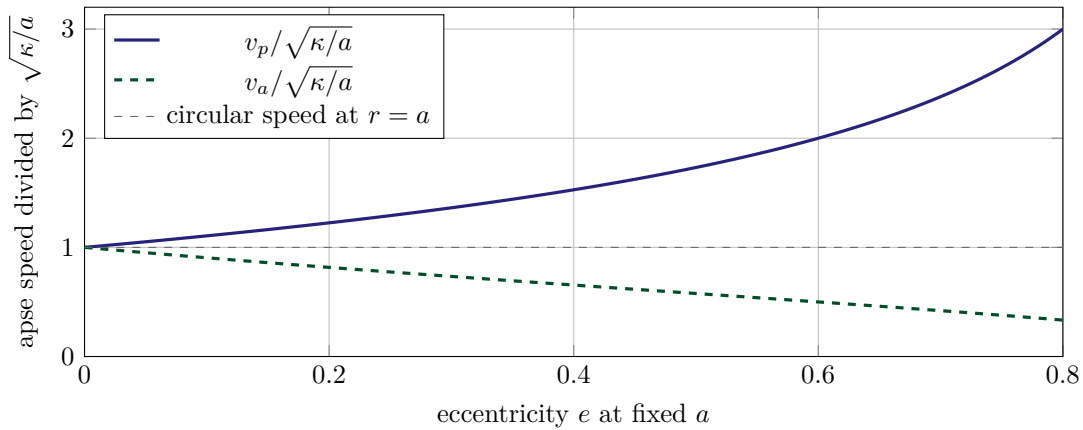


Figure 13.18: At fixed semimajor axis, the periapsis speed rises and the apoapsis speed falls with eccentricity, while energy and period stay fixed. Their product is $v_p v_a = \kappa/a$ and their ratio is $(1 + e)/(1 - e)$. A physical source may intercept the higher-eccentricity ellipses before the mathematical limit $e \rightarrow 1$ is approached.

Example 13.13: Reconstruct an ellipse from two measured apsides

In the stipulated model $\kappa = GM = 4.00 \times 10^{14} \text{ m}^3 \text{ s}^{-2}$, an orbit has $r_p = 8.00 \times 10^6 \text{ m}$ and $r_a = 2.40 \times 10^7 \text{ m}$. Find a, e, b, c , the specific angular momentum, the two apse speeds, the specific energy and the period. State whether the orbit clears a source of radius $R = 6.40 \times 10^6 \text{ m}$.

Solution. The geometry gives

$$a = 1.60 \times 10^7 \text{ m}, \quad e = 0.500, \quad c = 8.00 \times 10^6 \text{ m}, \quad b = 1.386 \times 10^7 \text{ m}.$$

The orbit relations above give $\ell^2 = \kappa a(1-e^2)$, where $\ell = L/m$ is the specific angular momentum. Thus

$$\ell = 6.928 \times 10^{10} \text{ m}^2 \text{ s}^{-1}, \quad v_p = \frac{\ell}{r_p} = 8.660 \text{ km s}^{-1}, \quad v_a = \frac{\ell}{r_a} = 2.887 \text{ km s}^{-1}.$$

Their ratio is 3.00, equal to r_a/r_p , as conservation requires. The specific mechanical energy and period are

$$\varepsilon = -\frac{\kappa}{2a} = -1.250 \times 10^7 \text{ J kg}^{-1}, \quad T = 2\pi\sqrt{\frac{a^3}{\kappa}} = 2.011 \times 10^4 \text{ s} = 5.585 \text{ h}.$$

The energy is negative, but that alone would not establish surface clearance. Here $r_p - R = 1.60 \times 10^6 \text{ m} > 0$, so the entire ideal ellipse lies outside the declared source surface. The source radius does not replace r_p in the orbital formulas.

13.6.6 Where is the body at a specified time?

The conic equation determines the path, but it does not say where the body is after a prescribed interval. Equal increments of the polar angle θ do not generally represent equal times. An efficient bridge between position and time uses an auxiliary circle and an angle called the *eccentric anomaly*. Denote that angle by ψ here, reserving E for mechanical energy. The name “anomaly” is traditional orbital terminology for an angle; it does not signify an error in the model.

With the ellipse centre as the Cartesian origin, write

$$x = a \cos \psi, \quad y = b \sin \psi, \quad b = a\sqrt{1-e^2}. \quad (13.54)$$

The point $(a \cos \psi, a \sin \psi)$ lies on the auxiliary circle; shrinking its vertical coordinate by b/a gives the orbital point. The attracting focus remains at $(ae, 0)$, so the radius components measured from it are $x_F = a(\cos \psi - e)$ and $y_F = a\sqrt{1-e^2} \sin \psi$. Taking their magnitude gives

$$r = a(1 - e \cos \psi), \quad \cos \theta = \frac{\cos \psi - e}{1 - e \cos \psi}, \quad \sin \theta = \frac{\sqrt{1-e^2} \sin \psi}{1 - e \cos \psi}. \quad (13.55)$$

Both sine and cosine should be used to select the correct quadrant; an inverse cosine alone cannot distinguish the upper and lower halves of an orbit.

The area swept about the focus obeys $dA = (x_F dy_F - y_F dx_F)/2$. Inserting the parameterization and using $\sin^2 \psi + \cos^2 \psi = 1$ yields

$$\frac{dA}{d\psi} = \frac{ab}{2}(1 - e \cos \psi).$$

Starting the clock at periapsis, integration gives $A(\psi) = ab(\psi - e \sin \psi)/2$. Angles in this equation are in radians. The constant areal rate then gives

$$\boxed{\psi - e \sin \psi = n(t - t_p)}, \quad n = \sqrt{\frac{\kappa}{a^3}} = \frac{2\pi}{T}. \quad (13.56)$$

This is Kepler’s equation. Its right-hand side, $\mathcal{M} = n(t - t_p)$, is the *mean anomaly*, an angle that increases uniformly in time. Neither ψ nor θ generally increases uniformly. Their distinction is essential when drawing an equal-time orbit or building a simulation reference solution.

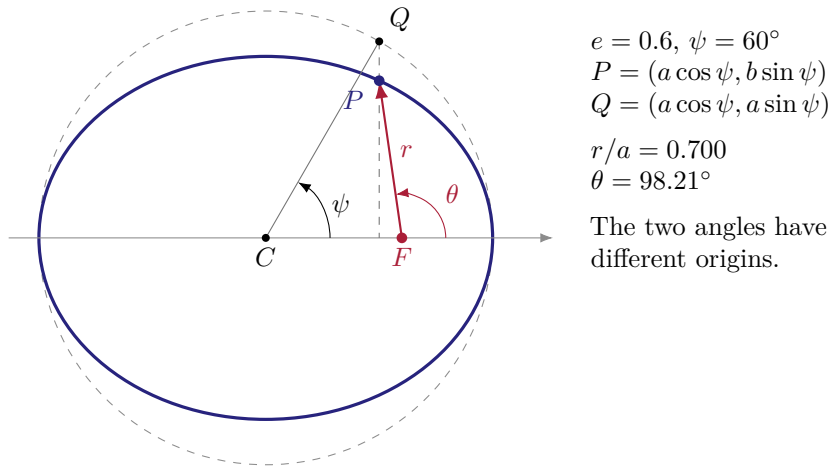


Figure 13.19: The eccentric anomaly is measured at the ellipse centre, while the true polar angle is measured at the source focus. The auxiliary circle is a geometric construction and is not the body's orbit. The projection preserves x and rescales y by b/a ; all plotted geometric lengths use one common scale.

For $e < 1$, $d(\psi - e \sin \psi)/d\psi = 1 - e \cos \psi > 0$, so Kepler's equation has exactly one solution in each unwrapped revolution for a given $\mathcal{M}$. Newton's method uses

$$\psi_{j+1} = \psi_j - \frac{\psi_j - e \sin \psi_j - \mathcal{M}}{1 - e \cos \psi_j}.$$

For moderate eccentricities $\psi_0 = \mathcal{M}$ is often a useful starting value. A bracketed solver on $[0, 2\pi]$ is a reliable alternative for $\mathcal{M} \in [0, 2\pi]$; uniqueness does not mean that every unsafeguarded Newton iteration must converge. Verify the final equation residual and then reconstruct the position. This procedure supplies a check independent of numerical time integration of Newton's vector equation.

Two limiting checks make the timing construction easier to interpret. For $e = 0$, the ellipse becomes a circle, the focus coincides with the centre, and $\psi = \mathcal{M} = \theta$. Uniform motion is then consistent with equal angles in equal times. For any ellipse, periapsis has $\psi = 0$ and apoapsis has $\psi = \pi$. Since the sine term vanishes at both apsides, periapsis-to-apoapsis takes exactly half a period. That fact does not make the first and second quarter-orbits equal in angular extent. The body spends more of that half-period in the outer, slower part of the path. The derivative $d\psi/dt = n/(1 - e \cos \psi)$ is largest at periapsis and smallest at apoapsis, consistent with the geometrical construction and the energy calculation.

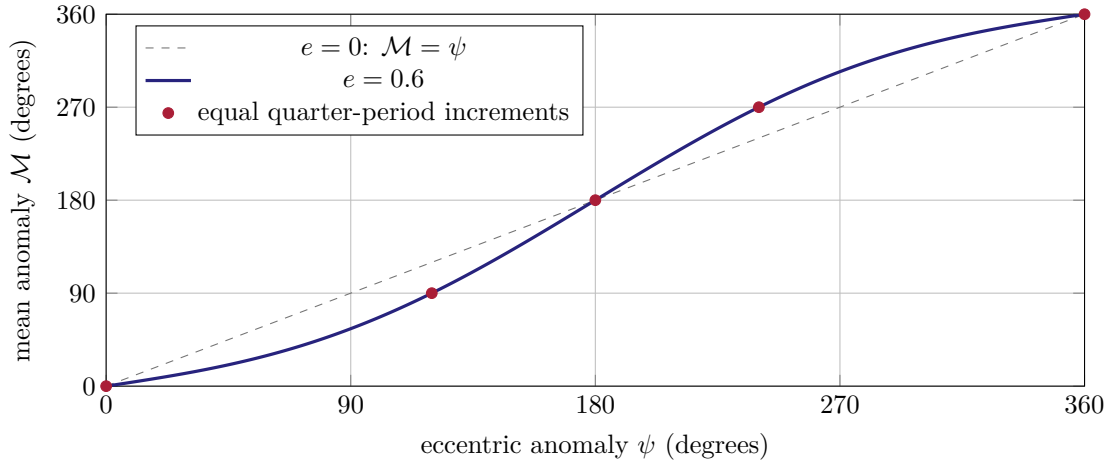


Figure 13.20: Calculated nonuniformity of the eccentric anomaly at $e = 0.6$. Uniform mean-anomaly increments of 90° give eccentric anomalies 0° , 119.824° , 180° , 240.176° and 360° . The formula converts the radian quantity $e \sin \psi$ to degrees on these plotted axes. The quarter-period points solve Kepler's equation; they are not equally spaced on the horizontal axis.

Example 13.14: Construct a position one quarter of a period after periapsis

An ellipse has $e = 0.600$. Find the position at $t - t_p = T/4$ in coordinates measured from the attracting focus, expressed in units of a . Determine the radius and true polar angle, and explain why the polar angle is not 90° .

Solution. The mean anomaly is $\mathcal{M} = 2\pi(T/4)/T = \pi/2$. Solve $\psi - 0.6 \sin \psi = \pi/2$. Starting from $\psi_0 = \pi/2$, Newton iterations give $\psi_1 = 2.170796$, followed by convergence to $\psi = 2.091328966 \text{ rad} = 119.8243^\circ$. Substituting that value in the original equation gives $\psi - 0.6 \sin \psi = 1.570796327$, verifying the time condition. The focus-centred coordinates are

$$\frac{x_F}{a} = \cos \psi - 0.6 = -1.0973423, \quad \frac{y_F}{a} = 0.8 \sin \psi = 0.6940435.$$

Therefore $r/a = 1.2984054$ and $\theta = \text{atan2}(y_F, x_F) = 147.6876^\circ$. The second-quadrant angle is required because $x_F < 0$ and $y_F > 0$. The swept area is $ab(\pi/2)/2 = \pi ab/4$, exactly one quarter of the ellipse area. The unusually large angular advance occurs because the body moves rapidly through periapsis. Equal time means equal *area*, so a uniform 90° polar-angle construction would give the wrong timing even though its points could all lie on the correct ellipse.

13.6.7 An impulse selects a new conic, subject to surface clearance

During an ideal instantaneous impulse, the position is continuous but the velocity changes. The subsequent coast is a new Kepler orbit with the new energy and angular momentum. Start at radius r_0 with zero radial velocity and a prograde transverse speed $v_0 = qv_c$, where $v_c = \sqrt{\kappa/r_0}$ and $q > 0$. Then

$$\varepsilon = \frac{\kappa}{r_0} \left(\frac{q^2}{2} - 1 \right), \quad \ell = q\sqrt{\kappa r_0}, \quad p = q^2 r_0, \quad e = |q^2 - 1|. \quad (13.57)$$

The last identity follows by inserting the first two into the energy–eccentricity relation. The absolute value matters: eccentricity cannot be negative, and below the circular speed the starting point is the apoapsis, not the periapsis.

For $0 < q < 1$ the body initially accelerates inward relative to a circular path and follows a bound ellipse with $r_a = r_0$ and $r_p = r_0 q^2 / (2 - q^2)$. At $q = 1$ it remains circular. For $1 < q < \sqrt{2}$, the

starting point is periapsis and the bound ellipse has $r_p = r_0$ and $r_a = r_0 q^2 / (2 - q^2)$. At $q = \sqrt{2}$ the orbit is parabolic; for $q > \sqrt{2}$ it is hyperbolic. In the last two cases r_0 remains periapsis, but there is no finite apoapsis. These statements assume an outward continuation from the initially transverse state, positive angular momentum, and the declared spherical exterior force law.

If the source has radius $R < r_0$, a subcircular launch avoids the surface only if

$$\frac{r_0 q^2}{2 - q^2} > R \iff q^2 > \frac{2R}{r_0 + R}. \quad (13.58)$$

Equality is a grazing trajectory in the ideal spherical model. Below that value, the mathematical exterior ellipse intersects the surface before it completes a revolution. Continuing the point-source exterior trajectory through the body would be a different, invalid physical model. Atmospheric drag can matter even when $r_p > R$; this simple clearance test declares only a geometric surface condition.

The speed-based launch classification has a specific scope. If the initial radial velocity is nonzero, the launch point is not an apse, and one must calculate ε and ℓ from the actual velocity components before reconstructing e and the turning radii. If the transverse velocity is retrograde, the same magnitude-based energy and shape can occur, but the orbit is traversed in the opposite direction. If $q = 0$ exactly, the angular momentum vanishes and the body moves on a radial line; the conic formula used here, which divided by ℓ^2 , does not apply. A negative energy still indicates a finite radial turning radius in the point-source energy model, but it does not produce a regular closed ellipse that can be followed through a finite physical source.

Near the parabolic threshold, rounded arithmetic can change the sign of a very small computed ε . For example, using a rounded decimal approximation to $\sqrt{2}$ in $q^2/2 - 1$ need not give exactly zero. Retain enough precision for the requested classification, report a tolerance when appropriate, and use the algebraic threshold for a value stipulated as exactly $\sqrt{2}$. This is a calculation issue, not evidence of a physical discontinuity in the local force.

Example 13.15: Classify five transverse launches and test collision

At $r_0 = 8.00 \times 10^6$ m about the stipulated source $\kappa = 4.00 \times 10^{14} \text{ m}^3 \text{ s}^{-2}$, consider launches with $q = 0.80, 1.00, 1.20, \sqrt{2}, 1.60$. Classify their conics. For bound noncircular cases find both apsides. Use $R = 6.40 \times 10^6$ m to decide whether either ellipse intersects the surface. Find the asymptotic speed of the hyperbolic case.

Solution. Here $v_c = 7.071 \text{ km s}^{-1}$ and $\kappa/r_0 = 5.00 \times 10^7 \text{ J kg}^{-1}$. The results from Eq. (13.57) are

q	$\varepsilon \text{ (} 10^6 \text{ J kg}^{-1}\text{)}$	e	conic
0.80	−34.0	0.36	ellipse, launch at apoapsis
1.00	−25.0	0	circle
1.20	−14.0	0.44	ellipse, launch at periapsis
$\sqrt{2}$	0	1	parabola
1.60	+14.0	1.56	hyperbola

For $q = 0.80$, $r_p = 3.765 \times 10^6$ m and $r_a = 8.00 \times 10^6$ m: the predicted periapsis is inside the source, so the exterior coast terminates at surface contact. For $q = 1.20$, $r_p = 8.00 \times 10^6$ m and $r_a = 2.057 \times 10^7$ m, both outside the source. The subcircular clearance threshold is $q > \sqrt{2R/(r_0 + R)} = 0.9428$, confirming the collision result. For the hyperbola, potential energy tends to zero at infinity, so $v_\infty = \sqrt{2\varepsilon} = 5.292 \text{ km s}^{-1}$. The parabolic case instead has $v_\infty = 0$: it reaches arbitrarily large radii with a speed tending to zero, without a finite outer turning point.

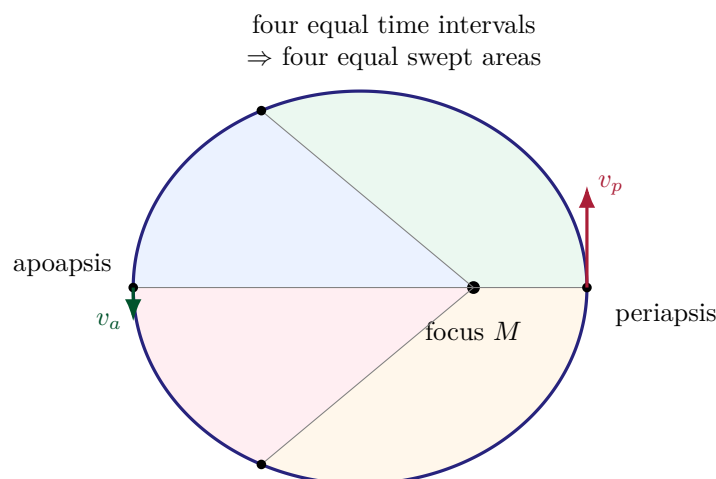


Figure 13.21: F13-05: calculated equal-time area construction for a Kepler ellipse with $a = 1$ and $e = 0.5$ in normalized units. The sector boundaries come from equal increments of mean anomaly; each curved sector is sampled at 101 points. Velocity-arrow lengths are in the exact ratio $v_p/v_a = 3$.

Example 13.16: Compare periapsis and apoapsis motion

Take a test-particle ellipse with $a = 1.20 \times 10^7$ m, $e = 0.40$ around the stipulated source $GM = 4.00 \times 10^{14} \text{ m}^3 \text{ s}^{-2}$.

The apse radii are

$$r_p = 7.20 \times 10^6 \text{ m}, \quad r_a = 1.68 \times 10^7 \text{ m}.$$

Angular momentum gives immediately

$$\frac{v_p}{v_a} = \frac{r_a}{r_p} = \frac{1.68}{0.72} = 2.33.$$

As an independent check, the vis-viva relation obtained from energy,

$$v^2 = GM \left(\frac{2}{r} - \frac{1}{a} \right), \quad (13.59)$$

gives

$$v_p = 8.82 \text{ km s}^{-1}, \quad v_a = 3.78 \text{ km s}^{-1},$$

with the same ratio. In the circular limit $e \rightarrow 0$, $r_p = r_a = a$ and the distinction between periapsis and apoapsis disappears.

Guided checks

X13-05. A diagram marks positions every 30° and labels each sector “equal time.” Why is that generally wrong for an ellipse? Equal time means equal swept area, not equal polar angle. Near periapsis the body moves faster and covers a larger angular interval in the same time than near apoapsis.

13.6.8 An ideal coplanar two-impulse orbit transfer

The energy difference between circular endpoints does not define how to move between them. A useful example is the Hohmann transfer between two prograde, coplanar circular orbits of radii $r_1 < r_2$ about the same spherical source. Assume instantaneous transverse impulses, negligible drag and perturbations, and an unpowered Kepler coast between the impulses. This is an ideal dynamics

calculation, rather than a claim about a particular spacecraft, propulsion system or mission.

Choose the transfer ellipse to be tangent to the inner circle at its periapsis and to the outer circle at its apoapsis. Its semimajor axis is

$$a_t = \frac{r_1 + r_2}{2}, \quad e_t = \frac{r_2 - r_1}{r_2 + r_1}.$$

The two circular speeds and the transfer-ellipse endpoint speeds are

$$\begin{aligned} v_1 &= \sqrt{\frac{\kappa}{r_1}}, & v_2 &= \sqrt{\frac{\kappa}{r_2}}, \\ v_{t,p} &= \sqrt{\kappa \left(\frac{2}{r_1} - \frac{1}{a_t} \right)}, & v_{t,a} &= \sqrt{\kappa \left(\frac{2}{r_2} - \frac{1}{a_t} \right)}. \end{aligned} \quad (13.60)$$

The first prograde impulse raises the speed at r_1 to enter the transfer ellipse. The coast then reduces the speed as the radius increases. At r_2 the transfer speed is lower than the outer circular speed, so a second prograde impulse circularizes the orbit. The two positive speed increments are

$$\begin{aligned} \Delta v_1 &= v_1 \left(\sqrt{\frac{2r_2}{r_1 + r_2}} - 1 \right), \\ \Delta v_2 &= v_2 \left(1 - \sqrt{\frac{2r_1}{r_1 + r_2}} \right), \quad \Delta v_{\text{total}} = \Delta v_1 + \Delta v_2. \end{aligned} \quad (13.61)$$

The coast covers half the ellipse, so

$$t_t = \pi \sqrt{\frac{a_t^3}{\kappa}}. \quad (13.62)$$

The sum here is the sum of impulse *magnitudes*; the impulses occur at different positions and their vectors point in different directions in one fixed Cartesian frame. Summing those vectors would not calculate the propulsion requirement. No optimization claim is required to derive these formulas.

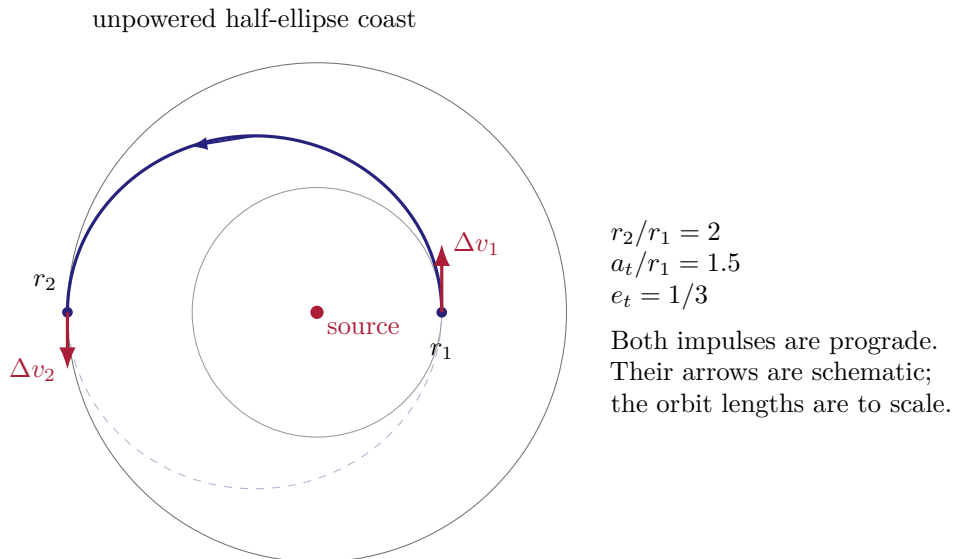


Figure 13.22: An outward two-impulse transfer with $r_2 = 2r_1$. The transfer ellipse has the source at a focus, and its apsides are tangent to the two circular orbits. Only the upper half is traversed during this transfer; the dashed lower half shows the continuation that would occur without the second impulse.

Example 13.17: Calculate both impulses and the coast time

Use the stipulated source $\kappa = 4.00 \times 10^{14} \text{ m}^3 \text{ s}^{-2}$. Calculate an ideal outward transfer from $r_1 = 8.00 \times 10^6 \text{ m}$ to $r_2 = 1.60 \times 10^7 \text{ m}$. Find the transfer geometry, four endpoint speeds, both impulse magnitudes, their sum and the coast duration.

Solution. The transfer ellipse has $a_t = 1.20 \times 10^7 \text{ m}$ and $e_t = 1/3$. The circular speeds are $v_1 = 7.0711 \text{ km s}^{-1}$ and $v_2 = 5.0000 \text{ km s}^{-1}$. Vis-viva gives

$$v_{t,p} = 8.1650 \text{ km s}^{-1}, \quad v_{t,a} = 4.0825 \text{ km s}^{-1}.$$

Thus $r_1 v_{t,p} = r_2 v_{t,a}$, an independent angular-momentum check on the coast. The impulse magnitudes are

$$\Delta v_1 = 1.0939 \text{ km s}^{-1}, \quad \Delta v_2 = 0.9175 \text{ km s}^{-1}, \quad \Delta v_{\text{total}} = 2.0114 \text{ km s}^{-1}.$$

Both are prograde even though the final circular speed is smaller than the initial circular speed. The comparison that determines the second impulse is $v_2 - v_{t,a}$ at the *same* outer radius. The coast duration is

$$t_t = \pi \sqrt{\frac{(1.20 \times 10^7)^3}{4.00 \times 10^{14}}} = 6.530 \times 10^3 \text{ s} = 108.83 \text{ min}.$$

This is half the period of the transfer ellipse, not half the period of either endpoint circle. The assumptions describe the coast and velocity changes completely, but they supply no propellant consumption or engine-performance model.

13.6.9 Mechanical energy increments and propulsion are different quantities

An instantaneous impulse occurs at fixed position. Consequently, the gravitational potential energy does not change during it. For an ideal particle of constant mass, let its initial velocity be $\mathbf{v}$ and its velocity increment be $\Delta \mathbf{v}$. The specific mechanical-energy increment is

$$\Delta \varepsilon = \frac{|\mathbf{v} + \Delta \mathbf{v}|^2 - |\mathbf{v}|^2}{2} = \mathbf{v} \cdot \Delta \mathbf{v} + \frac{|\Delta \mathbf{v}|^2}{2}. \quad (13.63)$$

For a prograde transverse impulse this becomes $v\Delta v + (\Delta v)^2/2$. The increment depends on the pre-impulse speed and the direction as well as the impulse magnitude. A retrograde impulse of magnitude d gives $-vd + d^2/2$ when it does not reverse the velocity. It is therefore incorrect to set the required mechanical-energy change equal to $m(\Delta v_{\text{total}})^2/2$.

During the unpowered transfer coast, ε remains $-\kappa/(2a_t)$, although kinetic and potential energies exchange. The two impulses change that constant in sequence from $-\kappa/(2r_1)$ to $-\kappa/(2a_t)$ and finally to $-\kappa/(2r_2)$. Their sum is fixed by the endpoints:

$$\Delta \varepsilon_{\text{endpoints}} = \frac{\kappa}{2} \left(\frac{1}{r_1} - \frac{1}{r_2} \right) > 0.$$

This telescoping energy identity is a useful consistency check. It cannot by itself predict the required Δv of a chosen trajectory. A real rocket changes mass and transfers momentum and energy to exhaust; chemical or electrical input energy and propellant usage require an explicit propulsion model. The constant-mass particle calculation represents changes in the vehicle's orbital mechanical state, not an engine energy budget.

Example 13.18: Reconcile the transfer energy ledger

For the preceding transfer, use an ideal constant particle mass $m = 200 \text{ kg}$. Calculate the mechanical-energy increments at both impulses and their sum. Independently verify the sum from the endpoint circular energies. Test the proposed expression $\frac{1}{2}m(\Delta v_{\text{total}})^2$ for that energy change.

Solution. The three specific energies are

$$\varepsilon_1 = -25.000 \times 10^6 \text{ J kg}^{-1}, \quad \varepsilon_t = -16.667 \times 10^6 \text{ J kg}^{-1}, \quad \varepsilon_2 = -12.500 \times 10^6 \text{ J kg}^{-1}.$$

Therefore $\Delta\varepsilon_1 = 8.3333 \times 10^6 \text{ J kg}^{-1}$ and $\Delta\varepsilon_2 = 4.1667 \times 10^6 \text{ J kg}^{-1}$. Multiplying by m gives

$$\Delta E_1 = 1.6667 \times 10^9 \text{ J}, \quad \Delta E_2 = 0.8333 \times 10^9 \text{ J}, \quad \Delta E = 2.5000 \times 10^9 \text{ J}.$$

Using the unrounded speeds in $m[v_1\Delta v_1 + (\Delta v_1)^2/2]$ and $m[v_{t,a}\Delta v_2 + (\Delta v_2)^2/2]$ reproduces the two increments. The endpoint check is

$$\Delta E = \frac{\kappa m}{2} \left(\frac{1}{r_1} - \frac{1}{r_2} \right) = 2.5000 \times 10^9 \text{ J}.$$

In contrast, $\frac{1}{2}m(2011.415 \text{ m s}^{-1})^2 = 4.046 \times 10^8 \text{ J}$, which is substantially different. That expression omits the velocity–impulse cross terms and treats two separated impulses as one fictitious speed change from rest. Neither calculated orbital-energy quantity supplies the energy consumed by an actual engine.

Guided checks

A fully resolved orbit-model check. Suppose two bodies orbit the same fixed source on ellipses of equal a but different e . Which quantities are equal: energy per mass, period, angular momentum per mass, periapsis speed? The first two are equal because $\varepsilon = -\kappa/(2a)$ and $T = 2\pi\sqrt{a^3/\kappa}$. The angular momentum differs through $\ell = \sqrt{\kappa a(1-e^2)}$, and the periapsis speed differs through $v_p = \sqrt{\kappa/a} \sqrt{(1+e)/(1-e)}$. These statements assume uninterrupted exterior ellipses. Surface clearance must still be checked for each eccentricity.

13.7 Tides and extended sources

Tidal physics begins with a subtraction. An extended body and its centre are both accelerated by the external source. The physically important deformation tendency is governed by the *difference* between the field at a point in the body and the field at the centre, not by the total external acceleration alone.

13.7.1 From a varying field to a tidal tensor

A tidal calculation compares neighboring points in one common reference frame. Let an external point source lie at the origin, and let the centre of a small body be at $\mathbf{R}$. Write $R = |\mathbf{R}|$, $\hat{\mathbf{n}} = \mathbf{R}/R$ and $\kappa = GM$. A point of the small body is at $\mathbf{R} + \boldsymbol{\delta}$. In a freely falling frame whose axes do not rotate relative to inertial axes, the centre’s uniform acceleration has been subtracted. The remaining gravitational acceleration is

$$\Delta \mathbf{g} = \mathbf{g}(\mathbf{R} + \boldsymbol{\delta}) - \mathbf{g}(\mathbf{R}), \quad \mathbf{g}(\mathbf{r}) = -\kappa \frac{\mathbf{r}}{r^3}. \quad (13.64)$$

This is a relative acceleration, not a new force exerted by a second source. A rotating body-fixed frame would require centrifugal and Coriolis terms as well; they are absent from the nonrotating local frame used here.

To calculate the leading variation, resolve the field into Cartesian components $g_i = -\kappa x_i r^{-3}$. A partial derivative with respect to x_j varies that coordinate while keeping the other coordinates fixed. From $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$,

$$\frac{\partial r}{\partial x_j} = \frac{x_j}{r}, \quad \frac{\partial r^{-3}}{\partial x_j} = -3x_j r^{-5}.$$

The product rule then gives

$$\mathsf{T}_{ij} \equiv \left. \frac{\partial g_i}{\partial x_j} \right|_{\mathbf{R}} = \frac{\kappa}{R^3} (3n_i n_j - \delta_{ij}). \quad (13.65)$$

Here δ_{ij} is the Kronecker symbol: it equals one for $i = j$ and zero otherwise. The matrix T collects all nine first derivatives. Multiplying it by an offset combines the effects of the offset's components:

$$\Delta \mathbf{g} = \mathsf{T} \boldsymbol{\delta} + \mathcal{O}\left(\frac{\kappa |\boldsymbol{\delta}|^2}{R^4}\right), \quad |\boldsymbol{\delta}| \ll R. \quad (13.66)$$

The tensor has units s^{-2} , so its product with a displacement has units m s^{-2} . The remainder indicates an acceleration scale; it is not a numerical error bar or a universal tolerance bound.

Choose the local x axis along $\hat{\mathbf{n}}$. Then

$$\mathsf{T} = \frac{\kappa}{R^3} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \Delta \mathbf{g} \simeq \frac{\kappa}{R^3} (2\delta x \hat{\mathbf{i}} - \delta y \hat{\mathbf{j}} - \delta z \hat{\mathbf{k}}). \quad (13.67)$$

A purely radial offset is multiplied by $+2\kappa/R^3$: the near and far sides separate relative to the centre. A purely transverse offset is multiplied by $-\kappa/R^3$: its differential acceleration points toward the source–centre line. These three directions are eigenvectors of the matrix, meaning that the mapping preserves their directions or reverses them without mixing components. For a general offset, radial stretching and transverse compression act together.

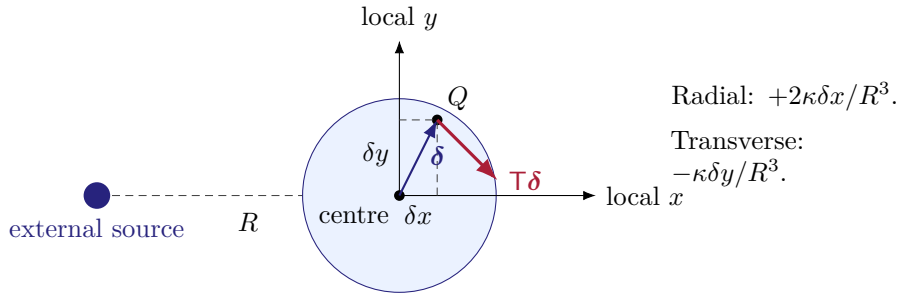


Figure 13.23: An off-axis offset produces both radial stretching and transverse compression in the linear tidal model. The local coordinate components define the geometry; the red arrow is a differential acceleration at Q , not the total attraction toward the distant source. Lengths are schematic.

The trace, the sum of the diagonal elements, is zero:

$$\text{tr } \mathsf{T} = \frac{\kappa}{R^3} (2 - 1 - 1) = 0.$$

This is the divergence of the external field at a point away from the source. It expresses the vacuum character of the point-source field there. It does not imply that the tidal field vanishes: the stretching eigenvalue is balanced by two compression eigenvalues. Nor does it imply that every extended body keeps its volume under a finite deformation. Material stresses, self-gravity and the subsequent motion must still be modeled. Inside a mass distribution, the gravitational-field divergence generally differs from zero.

13.7.2 How accurate is a first-order tidal estimate?

For a radial offset of magnitude d , define $u = d/R$, where $0 < u < 1$. Normalize the signed differential acceleration by the centre-field magnitude $g_0 = \kappa/R^2$. Direct subtraction gives

$$\frac{\Delta g_{x,\text{far}}}{g_0} = 1 - (1 + u)^{-2}, \quad \frac{\Delta g_{x,\text{near}}}{g_0} = 1 - (1 - u)^{-2}. \quad (13.68)$$

The linear predictions are $+2u$ and $-2u$. The near side stretches slightly more strongly in magnitude than the far side, because the inverse-square field is curved rather than exactly linear. Inspect both the signed acceleration and its relative residual in Fig. 13.24; a small absolute remainder need not be negligible relative to a very small tidal signal.

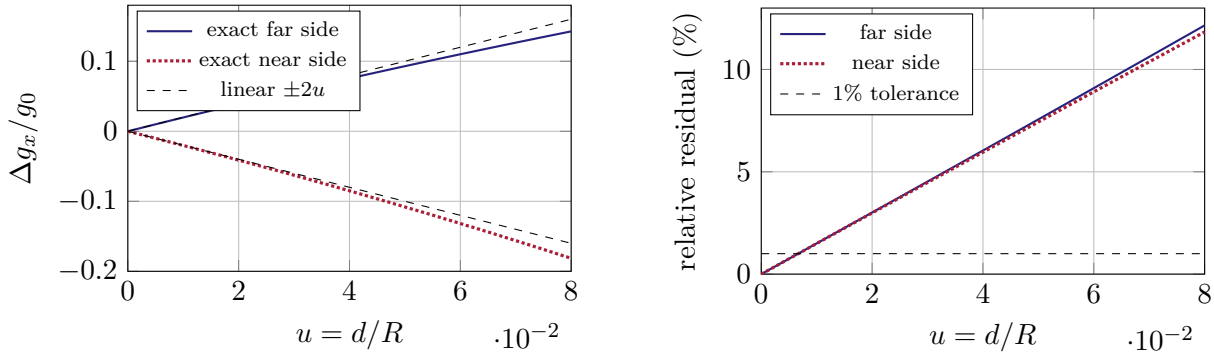


Figure 13.24: Exact and linear radial tidal accelerations, and the absolute error of each linear estimate divided by its own exact tidal magnitude. The residual curves have their continuous zero limit at $u = 0$, where the tidal magnitude itself vanishes. A tolerance refers to the differential acceleration, not the much larger centre acceleration. All curves are calculated from the same point-source model.

The relative residuals follow algebraically from Eq. (13.68):

$$\epsilon_{\text{near}} = \frac{3u - 2u^2}{2 - u}, \quad \epsilon_{\text{far}} = \frac{3u + 2u^2}{2 + u}.$$

Both begin as $3u/2$, although the absolute Taylor remainder is of order $u^2 g_0$. Requiring both radial residuals to be at most 1% means using the slightly larger far-side residual. Its boundary obeys $2u^2 + 2.99u - 0.02 = 0$, giving $u = 0.0066593$. This is a specific radial comparison with a specified denominator; it is not an orientation-independent rule for every tidal approximation.

Example 13.19: An off-axis tidal estimate with a check

An ideal external source has $\kappa = 4.00 \times 10^{14} \text{ m}^3 \text{ s}^{-2}$. A small body's centre is at $(1.00 \times 10^8, 0, 0) \text{ m}$. A point in it is offset by $\boldsymbol{\delta} = (1.00 \times 10^6, 2.00 \times 10^6, 0) \text{ m}$. Find the linear differential acceleration and compare it with direct evaluation of the source field.

Model and directions. The local $+x$ direction points away from the source and $+y$ is transverse. The dimensionless size is $|\boldsymbol{\delta}|/R = 0.0224$, so first order is plausible but requires a quantitative accuracy check. Since $\kappa/R^3 = 4.00 \times 10^{-10} \text{ s}^{-2}$,

$$\Delta \mathbf{g}_{\text{lin}} = (8.00\hat{\mathbf{i}} - 8.00\hat{\mathbf{j}}) \times 10^{-4} \text{ m s}^{-2}.$$

The positive radial component means this point lags outward relative to the centre; the negative transverse component compresses it toward the source–centre axis.

Independent exact check. Evaluate Eq. (13.64) at $\mathbf{R} + \boldsymbol{\delta} = (1.01 \times 10^8, 2.00 \times 10^6, 0) \text{ m}$ and subtract $\mathbf{g}(\mathbf{R}) = (-0.0400, 0, 0) \text{ m s}^{-2}$. This gives

$$\Delta \mathbf{g}_{\text{exact}} = (8.1121\hat{\mathbf{i}} - 7.7602\hat{\mathbf{j}}) \times 10^{-4} \text{ m s}^{-2}.$$

Using vector magnitudes to define one residual,

$$\frac{|\Delta \mathbf{g}_{\text{exact}} - \Delta \mathbf{g}_{\text{lin}}|}{|\Delta \mathbf{g}_{\text{exact}}|} = 0.0236.$$

The linear model is accurate to about 2.36% by that criterion. It is suitable for a 3% teaching estimate, but fails a 1% requirement. A small geometric ratio alone did not decide the answer; the requested observable and its tolerance completed the model test.

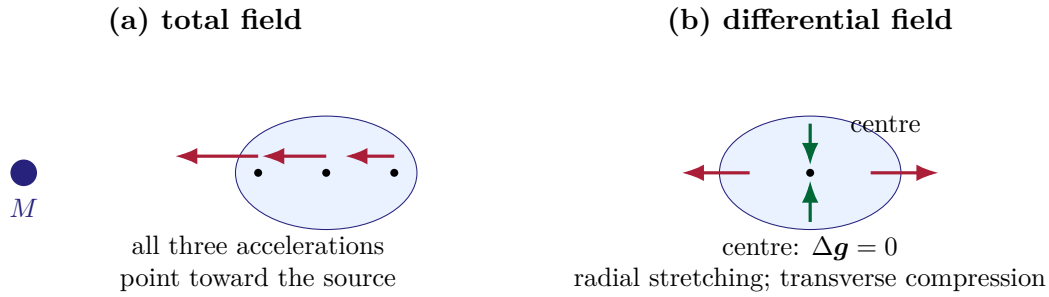


Figure 13.25: In a nonrotating frame translated with the centre, subtract the centre’s external acceleration before interpreting deformation. Arrow lengths are schematic and use separate scales in the two panels. The source lies left of both panels; the differential acceleration vanishes at the centre.

Example 13.20: Exact and first-order differential acceleration

Use the stipulated parameter $GM = 4.00 \times 10^{14} \text{ m}^3 \text{ s}^{-2}$. Let the body centre be at $R = 1.00 \times 10^8 \text{ m}$ and compare points displaced radially by $\delta x = \pm 1.00 \times 10^5 \text{ m}$. At the centre, $g_x = -GM/R^2 = -0.0400 \text{ m s}^{-2}$. At the near-side point $R - \delta x$,

$$\Delta g_{x,\text{near}} = -8.012 \times 10^{-5} \text{ m s}^{-2},$$

while at the far-side point $R + \delta x$,

$$\Delta g_{x,\text{far}} = +7.988 \times 10^{-5} \text{ m s}^{-2}.$$

The first-order magnitude from Eq. (13.67) is

$$\frac{2GM|\delta x|}{R^3} = 8.00 \times 10^{-5} \text{ m s}^{-2}.$$

The small mismatch between the two exact values is a second-order effect because $|\delta x|/R = 10^{-3}$. For a transverse displacement $\delta y = 1.00 \times 10^5 \text{ m}$, the first-order prediction is $\Delta g_y = -4.00 \times 10^{-5} \text{ m s}^{-2}$; direct evaluation gives $-3.999994 \times 10^{-5} \text{ m s}^{-2}$. The agreement tests both the scale and the sign of the local expansion.

Guided checks

X13-06. Why is a drawing with three equal-length inward arrows not a tidal-force diagram? It shows a uniform external acceleration, which would move every part together and produce no first-order relative stretching. Tides require spatial variation and are described by $\Delta \mathbf{g}$ after the centre acceleration is subtracted.

13.8 Limits of Newtonian gravity

Newtonian gravitation is exceptionally successful for many mechanical and engineering calculations, but its equations are a model with assumptions rather than a universal description at every scale. Two questions should be kept separate: *Is the Newtonian model accurate enough for the requested observable?* and *If not, which omitted physics matters?*

For motion, a useful dimensionless indicator is

$$\beta = \frac{v}{c}, \quad (13.69)$$

where c is the speed of light. A small β supports a low-speed approximation, but there is no universal value of β at which all Newtonian predictions suddenly become invalid. The required accuracy and the observable matter. A second gravitational scale comparison is the dimensionless potential depth $GM/(rc^2)$; again, it is an indicator of weak-field character, not a stand-alone replacement theory.

Other Newtonian idealisations can fail much earlier than relativity becomes important. A real source can be nonspherical or rotating; a nominal two-body system can have important third-body perturbations; atmospheric drag or radiation pressure can act; a satellite can have finite size and attitude dynamics; and the test mass can be large enough that source recoil cannot be ignored. The correct response is to identify the neglected effect and formulate a model that contains it.

Which assumption limits the Newtonian calculation?

Motion and field regime

Check v/c , $GM/(rc^2)$ and the requested precision. Relativity needs a separate verified model.

Source geometry

Nonsphericity, rotation and third bodies may matter while all speeds remain small.

Body and system

Comparable masses require both bodies to move. An extended body may respond to gradients.

Other forces

Drag, thrust, radiation and contact forces require an explicit total-force model.

Figure 13.26: A qualitative assumption map. These four checks are complementary and do not define universal accuracy thresholds. A force-model limitation is different from a numerical timestep error.

Example 13.21: Which model assumptions need re-examination?

Compare two descriptions.

Case A. A small sensor package moves at 6.0 km s^{-1} in a stipulated weak-field two-body calculation. Then $v/c \simeq 2.0 \times 10^{-5}$. If metre-level precision over a short teaching simulation is not requested, relativity is unlikely to be the first limitation; source nonsphericity, drag or numerical error could be more relevant depending on the setup.

Case B. A hypothetical calculation is described only by $v = 0.30c$ and a very compact source. Newtonian equations can still be evaluated algebraically, but the indicators warn that one should not interpret the result as a precision physical prediction without a verified relativistic model and additional information about the source geometry.

The lesson is methodological: an applicability indicator tells us what assumption deserves scrutiny. It does not supply the missing correction and does not define a sharp universal boundary.

Guided checks

X13-08. “Newtonian gravity is valid whenever $v < 0.1c$.” Diagnose the statement. There is no universal speed threshold that guarantees a requested accuracy for every gravitational observable. State the tolerance, the field/source regime and other omitted effects; then select an appropriate verified theory.

13.9 Engineering, computation and measurement connections

13.9.1 Orbital dynamics joined to horizon geometry

A circular-orbit calculation gives the platform’s motion, while a tangent construction gives its line of sight to a spherical horizon. Keep those two models distinct. Let O be the source centre, P the platform and Q the horizon point. A ray PQ that just clears the surface is tangent to the radius OQ , so $\angle OQP = 90^\circ$. The right triangle has $OP = r$ and $OQ = R$. If $\alpha = \angle POQ$,

$$\cos \alpha = \frac{R}{r}, \quad s_{\text{slant}} = \sqrt{r^2 - R^2}, \quad s_{\text{arc}} = R\alpha. \quad (13.70)$$

The angle in the arc formula must be in radians. Slant range and surface arc are different distances; neither is an instrument’s ground resolution. The visible cap extends to this central half-angle around the point directly beneath the platform in the ideal spherical, unobstructed geometry.

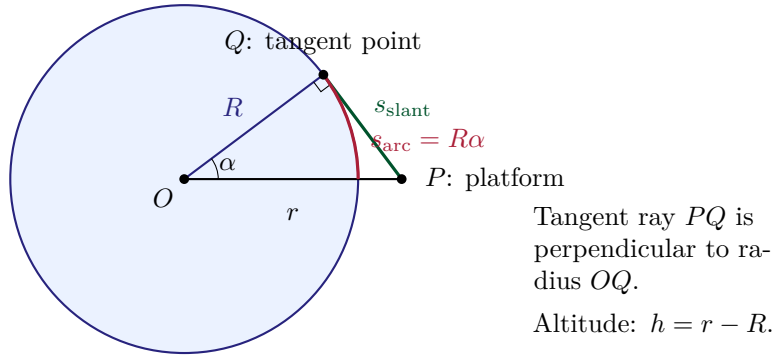


Figure 13.27: A spherical horizon is determined by a tangent, not by a second orbital force. The triangle is drawn with $R/r = 0.8$, the ratio used in the worked example. The red surface arc and the green straight slant range are distinct geometrical quantities. Atmosphere, refraction, terrain and instrument field of view are omitted.

Example 13.22: Time advance and horizon of an ideal sensor orbit

A test-mass platform follows a circular orbit about a spherical source with $\kappa = 4.00 \times 10^{14} \text{ m}^3 \text{ s}^{-2}$, surface radius $R = 6.40 \times 10^6 \text{ m}$, and orbital radius $r = 8.00 \times 10^6 \text{ m}$. Find the period, angular advance in ten minutes, horizon half-angle, surface arc and slant range.

Dynamics. The altitude is $1.60 \times 10^6 \text{ m}$, but it is r , not the altitude, that enters the circular period:

$$T = 2\pi\sqrt{r^3/\kappa} = 7108.6 \text{ s} \simeq 118.5 \text{ min.}$$

For $\Delta t = 600 \text{ s}$, $\Delta\Phi = 2\pi\Delta t/T = 0.53033 \text{ rad} = 30.4^\circ$.

Geometry. Equation (13.70) gives $\alpha = \arccos(0.8) = 0.643501 \text{ rad} = 36.9^\circ$. Thus

$$s_{\text{arc}} = 4.1184 \times 10^6 \text{ m}, \quad s_{\text{slant}} = 4.8000 \times 10^6 \text{ m.}$$

These are consistent with the right-triangle relation $r^2 = R^2 + s_{\text{slant}}^2$. The ten-minute orbital advance and horizon half-angle answer different questions: one is a motion over time, the

other an instantaneous visibility limit. A real sensing calculation additionally needs attitude, instrument, atmospheric and surface models. The stipulated numbers establish a scale, not an operational mission specification.

13.9.2 A synthetic orbital-data inference

No astronomical observation is claimed here. Consider a synthetic circular-orbit dataset constructed with radius $r = 8.00 \times 10^6$ m and an observed repeat period $T = (7110 \pm 10)$ s. Under the circular two-body test-particle model,

$$GM = \frac{4\pi^2 r^3}{T^2}. \quad (13.71)$$

Using the central value gives $GM = 3.998 \times 10^{14} \text{ m}^3 \text{ s}^{-2}$, close to the stipulated generating value $4.00 \times 10^{14} \text{ m}^3 \text{ s}^{-2}$. The inferred quantity is the gravitational parameter, not G and M separately. An independent mass or an independent measurement of G is required to separate them.

If radius uncertainty is important, it enters cubically in Eq. (13.71); period enters quadratically. For small independent uncertainties,

$$\left(\frac{\sigma_{GM}}{GM}\right)^2 \simeq 9 \left(\frac{\sigma_r}{r}\right)^2 + 4 \left(\frac{\sigma_T}{T}\right)^2. \quad (13.72)$$

Systematic departures from circularity are a model bias, not a random timing uncertainty.

13.9.3 Local gravimetry: a model and an uncertainty budget

An orbit constrains a source parameter on a planetary scale. A drop experiment instead probes a local acceleration. If a small object is released from rest and falls a known distance H in time t_f , a constant-field model gives

$$H = \frac{1}{2} g_{\text{loc}} t_f^2, \quad g_{\text{loc}} = \frac{2H}{t_f^2}. \quad (13.73)$$

This estimate requires an actual release-from-rest condition and a consistent start time. A delay between the timing signal and release, or an unknown initial velocity, biases the result rather than appearing automatically as random uncertainty. Air resistance must be negligible or modeled. A ground-fixed apparatus also measures motion relative to a rotating planet; precision gravimetry must distinguish effective local acceleration from the spherical nonrotating field GM/R^2 .

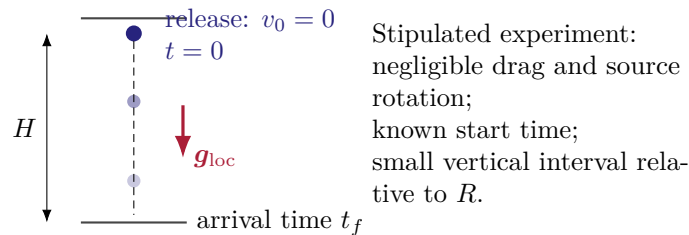


Figure 13.28: A local drop-time gravimetry model. The dots show successive positions schematically; their spacing is not a dataset. The measured interval H and flight time t_f determine an acceleration only after the release and force assumptions are specified.

For small independent standard uncertainties σ_H and σ_t , logarithmic differentiation or the ordinary derivative propagation rule gives

$$\left(\frac{\sigma_g}{g_{\text{loc}}}\right)^2 \simeq \left(\frac{\sigma_H}{H}\right)^2 + 4 \left(\frac{\sigma_t}{t_f}\right)^2. \quad (13.74)$$

The factor two on timing follows from the exponent -2 , not from having two independently measured times. If a time difference is formed from two correlated timestamps, its uncertainty must be evaluated first from their covariance; independence cannot be assumed merely because the labels differ. Report measurement uncertainty and model bias separately.

Example 13.23: Synthetic drop data and what they determine

In a stipulated nonrotating, negligible-drag drop experiment, $H = (20.00 \pm 0.02) \text{ m}$ and $t_f = (2.020 \pm 0.004) \text{ s}$. The uncertainties are independent standard uncertainties. The ideal spherical source has radius $R = 6.40 \times 10^6 \text{ m}$, treated as exact here. Infer the local acceleration and its uncertainty; compare height variation with measurement uncertainty; then infer the gravitational parameter.

Estimate. Direct substitution gives $g_{\text{loc}} = 9.80296 \text{ m s}^{-2}$. The relative standard uncertainty is

$$\sqrt{(0.02/20.00)^2 + 4(0.004/2.020)^2} = 0.0040847.$$

Thus the appropriate rounded result is $g_{\text{loc}} = (9.803 \pm 0.040) \text{ m s}^{-2}$. Timing supplies the larger uncertainty contribution.

Approximation check. The fractional surface-field change over 20 m is approximately $2H/R = 6.25 \times 10^{-6}$, far below the 4.08×10^{-3} measurement uncertainty. This comparison supports a constant-field treatment at the present precision. It does not certify drag, release delay or rotation as negligible in a real apparatus; they were stipulated away.

Source inference. Under the same spherical, nonrotating model, $\kappa = g_{\text{loc}} R^2 = 4.0153 \times 10^{14} \text{ m}^3 \text{ s}^{-2}$. With R exact, it has the same relative uncertainty:

$$\kappa = (4.015 \pm 0.016) \times 10^{14} \text{ m}^3 \text{ s}^{-2}.$$

The experiment determines a local acceleration directly and a gravitational parameter through the assumed source model. It does not distinguish G from M , or reconstruct the internal density profile. A complete assessment checks the estimate, uncertainty propagation, height approximation and limitations as four separate parts of the argument.

13.9.4 Computation should answer a physics question

A numerical orbit calculation needs more than a plotted curve. Record the gravitational parameter or masses, initial position and velocity, integration method, timestep, simulated duration and units. Then compare against an analytic limit such as a circular orbit, monitor energy and angular momentum, and repeat with smaller timesteps. A trajectory that appears to spiral outward may be revealing the integrator rather than the physics.

The advanced section carries out this workflow for a dimensionless two-body example. Its central rule is broadly useful: *validate the algorithm against conserved quantities and refinement before interpreting fine structure in the trajectory.*

13.10 Advanced Topics for Technical and Engineering Students

The core treated a small body moving in a prescribed field. That is often an excellent approximation, but it hides two linked questions important in engineering simulation and celestial mechanics: What changes when both masses move appreciably? How can a numerical trajectory be shown to represent the intended two-body model rather than an integration artifact?

The additional mathematics needed here is relative-coordinate algebra, center-of-mass motion and a local use of the effective radial potential. No relativistic dynamics is introduced.

13.10.1 The isolated two-body reduction

Let masses M and m have inertial positions $\mathbf{r}_M$ and $\mathbf{r}_m$, and define the relative separation

$$\mathbf{r} = \mathbf{r}_m - \mathbf{r}_M, \quad (13.75)$$

with $r = |\mathbf{r}|$. Newton's third law gives

$$M\ddot{\mathbf{r}}_M = +GMm\frac{\mathbf{r}}{r^3}, \quad (13.76)$$

$$m\ddot{\mathbf{r}}_m = -GMm\frac{\mathbf{r}}{r^3}. \quad (13.77)$$

Subtracting the first acceleration from the second yields

$$\ddot{\mathbf{r}} = -G(M+m)\frac{\mathbf{r}}{r^3}. \quad (13.78)$$

The center of mass

$$\mathbf{R}_{\text{CM}} = \frac{M\mathbf{r}_M + m\mathbf{r}_m}{M+m} \quad (13.79)$$

moves uniformly because the total external force is zero. In the CM frame,

$$\mathbf{r}_M = -\frac{m}{M+m}\mathbf{r}, \quad \mathbf{r}_m = +\frac{M}{M+m}\mathbf{r}. \quad (13.80)$$

Thus the relative orbit determines both barycentric orbits.

The reduced mass

$$\mu = \frac{Mm}{M+m} \quad (13.81)$$

lets the two-body energy be written as

$$E_{\text{rel}} = \frac{1}{2}\mu\dot{\mathbf{r}}^2 - \frac{GMm}{r}. \quad (13.82)$$

When $M \gg m$, $\mu \rightarrow m$ and $M+m \rightarrow M$, so the core test-particle equations are recovered.

Theoretical point 13.1: Why the total mass appears in relative motion

Each body accelerates toward the other. The separation closes because both accelerate, so the relative acceleration contains $G(M+m)$ rather than GM alone. This does not mean the gravitational force has changed; the pair force still has magnitude GMm/r^2 .

13.10.2 Effective radial potential and turning points

Angular momentum makes a two-dimensional central-force orbit reducible to a one-dimensional radial problem. Define the specific relative angular momentum $\ell = |\mathbf{r} \times \dot{\mathbf{r}}|$ and specific relative energy

$$\varepsilon = \frac{1}{2}\dot{r}^2 + \frac{\ell^2}{2r^2} - \frac{G(M+m)}{r}. \quad (13.83)$$

The effective potential per unit reduced mass is

$$V_{\text{eff}}(r) = \frac{\ell^2}{2r^2} - \frac{G(M+m)}{r}. \quad (13.84)$$

The $\ell^2/(2r^2)$ term is not a new force law; it is the kinetic energy associated with transverse motion after angular momentum has been used to eliminate the angular coordinate.

A circular relative orbit lies at a stationary point,

$$\frac{dV_{\text{eff}}}{dr} = 0 \quad \Rightarrow \quad r_c = \frac{\ell^2}{G(M+m)}.$$

At that radius $d^2V_{\text{eff}}/dr^2 > 0$, so the Newtonian circular orbit is radially stable to sufficiently small perturbations within the isolated two-body model. For a bound eccentric orbit, periapsis and apoapsis are radial turning points satisfying $\dot{r} = 0$ and therefore $\varepsilon = V_{\text{eff}}(r)$.

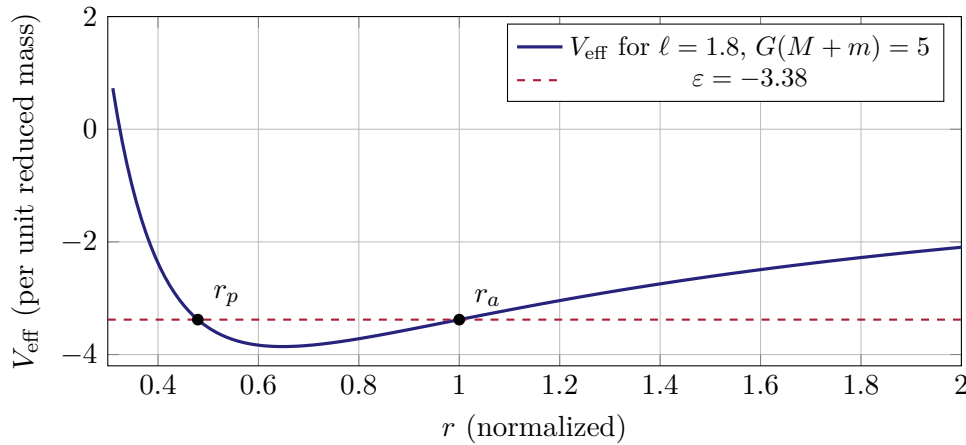


Figure 13.29: Effective radial potential for the numerical example below. The intersections with the conserved specific energy identify the two radial turning points. All quantities use the stated normalized units. The plot is calculated from Eq. (13.84).

13.10.3 A symplectic-style step: velocity Verlet

For the relative equation $\ddot{\mathbf{r}} = \mathbf{a}(\mathbf{r})$, a velocity-Verlet step from $(\mathbf{r}_n, \mathbf{v}_n)$ with timestep Δt is

$$\mathbf{v}_{n+1/2} = \mathbf{v}_n + \frac{\Delta t}{2} \mathbf{a}(\mathbf{r}_n), \quad (13.85)$$

$$\mathbf{r}_{n+1} = \mathbf{r}_n + \Delta t \mathbf{v}_{n+1/2}, \quad (13.86)$$

$$\mathbf{v}_{n+1} = \mathbf{v}_{n+1/2} + \frac{\Delta t}{2} \mathbf{a}(\mathbf{r}_{n+1}). \quad (13.87)$$

It is second-order accurate in the timestep for smooth trajectories away from the point singularity. Its useful long-time behaviour does not remove the need for convergence tests. In a central force, the exact solution conserves energy and angular momentum; the numerical solution should approximate both with errors that behave consistently under refinement.

Example 13.24: Numerical bound two-body orbit

Use normalized units with $G = 1$, masses $M = 4$ and $m = 1$, and initial relative state

$$\mathbf{r}_0 = (1, 0), \quad \mathbf{v}_0 = (0, 1.8).$$

Then $G(M + m) = 5$, specific energy and angular momentum are

$$\varepsilon_0 = \frac{1}{2}(1.8)^2 - 5 = -3.38, \quad \ell_0 = 1.8.$$

The analytic Kepler parameters are

$$a = -\frac{G(M + m)}{2\varepsilon_0} = 0.739645, \quad e = 0.352000,$$

$$r_p = a(1 - e) = 0.479290, \quad r_a = a(1 + e) = 1.000000, \quad T = 2\pi\sqrt{\frac{a^3}{5}} = 1.78743.$$

The initial point is therefore apoapsis.

Velocity Verlet was run for one analytic period using $N = 100, 200, 400, 800$ equal steps. The maximum relative specific-energy error and one-period position-closure error were:

N	Δt	$\max \Delta\varepsilon / \varepsilon_0 $	$ \mathbf{r}(T) - \mathbf{r}_0 $
100	0.0178743	3.112×10^{-3}	8.944×10^{-3}
200	0.00893717	7.852×10^{-4}	2.247×10^{-3}
400	0.00446858	1.967×10^{-4}	5.625×10^{-4}
800	0.00223429	4.921×10^{-5}	1.407×10^{-4}

Halving Δt reduces these errors by approximately a factor of four, consistent with second-order convergence. The relative angular momentum in these runs is conserved to about 4×10^{-15} or better, near floating-point round-off for this implementation.

The barycentric paths follow Eq. (13.80): the heavier mass moves on the smaller orbit, with scale factor $m/(M + m) = 0.2$, while the lighter mass has scale factor $M/(M + m) = 0.8$ relative to the separation orbit.

Guided checks

X13-07. A program produces a visually closed ellipse for one timestep. Is that sufficient validation? No. Check an analytic circular or Kepler benchmark, monitor energy and angular momentum, inspect periapsis with adequate temporal resolution, and refine Δt . A visually plausible orbit can have a significant phase or energy error.

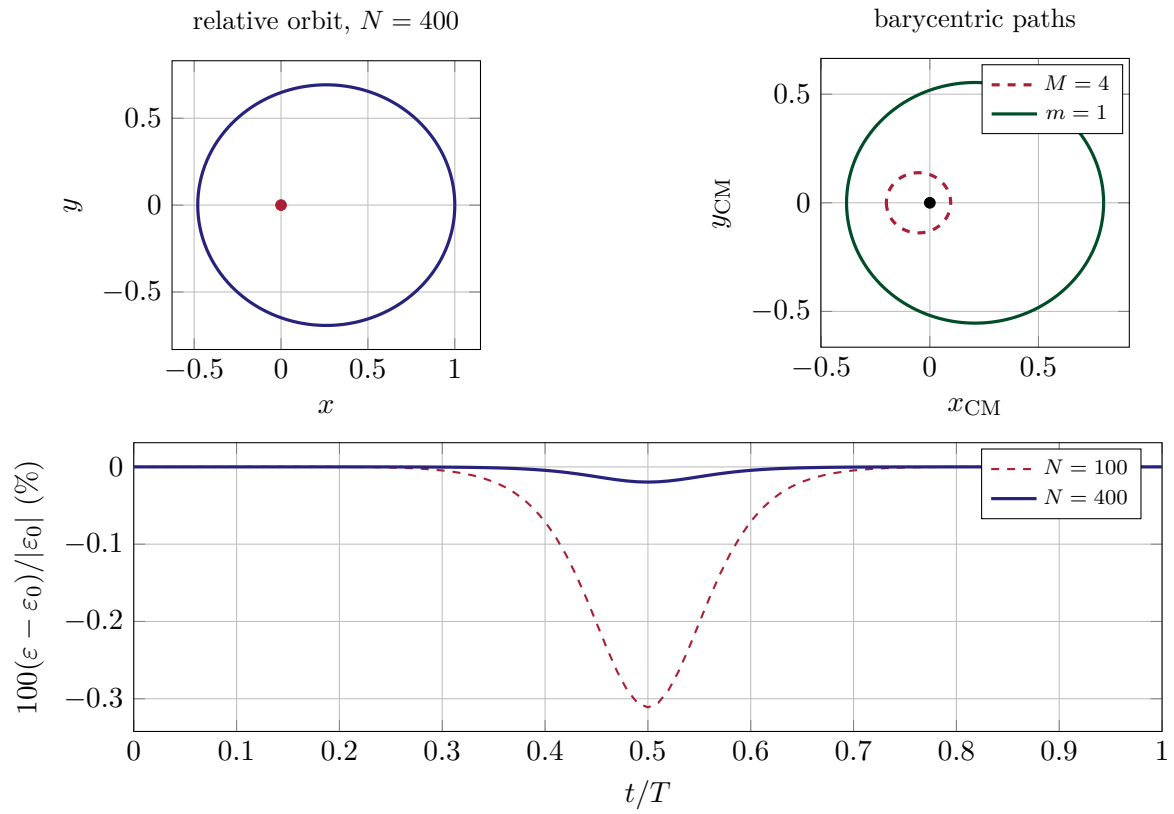


Figure 13.30: F13-07: numerical-model outputs for the bound two-body example. Top left: relative orbit. Top right: reconstructed barycentric paths. Bottom: relative energy error for two timesteps. Angular momentum remained within 4×10^{-15} relative error in the tested runs. All curves are generated from the stated velocity-Verlet model, not measured data.

13.10.4 A reproducible orbit check and an acceptance criterion

The numerical orbit above can be reproduced without astronomical data or a specialized dynamics package. The following standard-Python implementation integrates the stated relative problem using $G(M + m) = 5$, $\mathbf{r}_0 = (1, 0)$ and $\mathbf{v}_0 = (0, 1.8)$. Its normalization makes length and time dimensionless model units. It computes the analytic period from the initial energy, advances exactly N steps to that period, and records three diagnostics. No external files are read.

The algorithm is: evaluate acceleration at the old position; make a half velocity step; use that velocity for a full position step; reevaluate acceleration; complete the velocity step. Evaluate energy and angular momentum on the full-step state. Mixing a half-step velocity with a full-step position in a diagnostic would test different times and give a misleading conservation error.

```
from math import sqrt, pi

def norm(q):
    return sqrt(q[0]*q[0] + q[1]*q[1])

def acceleration(q, kappa):
    radius = norm(q)
    return [-kappa*x/radius**3 for x in q]

def orbit_check(N):
    kappa = 5.0
    r0 = [1.0, 0.0]
    r, v = r0[:], [0.0, 1.8]
    E0, L0 = -3.38, 1.8
    a = -kappa/(2.0*E0)
    period = 2.0*pi*sqrt(a**3/kappa)
    dt = period/N
    maximum_E, maximum_L = 0.0, 0.0
    for step in range(N):
        old_a = acceleration(r, kappa)
        half_v = [v[j]+0.5*dt*old_a[j]
                  for j in range(2)]
        r = [r[j]+dt*half_v[j] for j in range(2)]
        new_a = acceleration(r, kappa)
        v = [half_v[j]+0.5*dt*new_a[j]
              for j in range(2)]
        E = 0.5*sum(x*x for x in v)-kappa/norm(r)
        L = r[0]*v[1]-r[1]*v[0]
        maximum_E = max(maximum_E, abs(E-E0)/abs(E0))
        maximum_L = max(maximum_L, abs(L-L0)/abs(L0))
        closure = norm([r[j]-r0[j] for j in range(2)])
    return maximum_E, maximum_L, closure

for N in (100, 200, 400, 800):
    energy_error, angular_error, closure = orbit_check(N)
    print(N, energy_error, angular_error, closure)
```

This listing reproduces the reported convergence to the precision justified by floating-point arithmetic. Exact last digits can vary with implementation. The code intentionally stops short of an impact or close-encounter model: the tested trajectory has nonzero periapsis, and the point-force singularity is never crossed. Changing the initial state requires reassessing temporal resolution, not merely reusing the same N .

Energy error and orbital closure answer different questions. Energy fixes orbit size and, together with angular momentum, eccentricity. It does not specify where the body is on that orbit at a given time. A method can maintain a small bounded energy oscillation while accumulating a timing or phase error. Conversely, excellent angular-momentum conservation here follows from the central-force kick structure, and does not verify that the value of κ or the initial conditions were

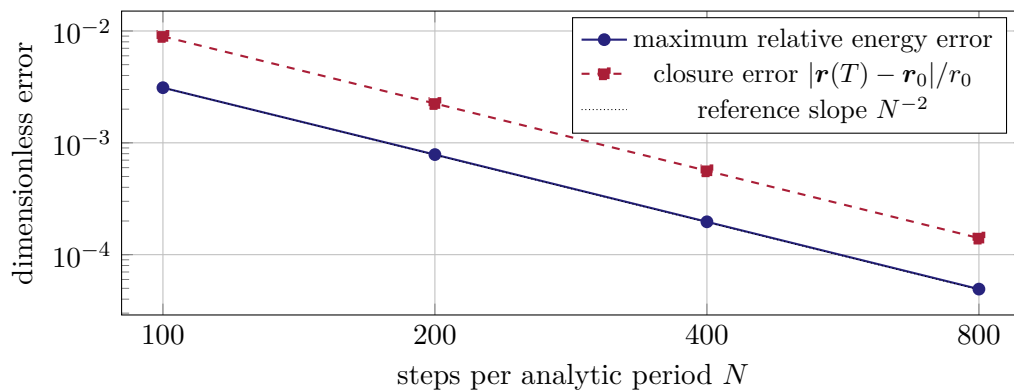


Figure 13.31: Independent velocity-Verlet refinement for the chapter’s normalized two-body orbit. The initial radius has magnitude $r_0 = 1$, so the listed closure distance also equals the dimensionless normalized closure error. Both diagnostics approach second-order behavior; near-roundoff angular-momentum error is omitted from this log plot because it follows a different error scale.

entered correctly. An acceptance criterion should therefore name more than one diagnostic.

Example 13.25: Choose a verified timestep for a stated tolerance

For the normalized numerical orbit, require maximum relative energy error below 10^{-4} and one-period closure distance below 5×10^{-4} length units. Among $N = 100, 200, 400, 800$, identify the smallest tested acceptable step count. Explain why angular-momentum conservation alone would give an incomplete answer.

Compare actual checks. At $N = 400$, the energy error is 1.9674×10^{-4} and closure distance is 5.6245×10^{-4} . Both fail the stated limits. At $N = 800$, the corresponding errors are 4.9214×10^{-5} and 1.4066×10^{-4} , so both pass. The coarser $N = 100, 200$ runs fail by larger amounts. Hence the smallest acceptable tested count is $N = 800$, with $\Delta t = T/800 = 0.00223429$ time units.

Interpret the evidence. Doubling N reduces each leading error by approximately four, providing evidence for the stated second-order method. Angular momentum is conserved near roundoff even for the coarse runs that fail the other limits, so that invariant alone cannot select an adequate timestep. A phase estimate near the initial apoapsis is closure distance divided by the apoapsis radius; since that radius is one here, the $N = 400$ result corresponds to about 5.62×10^{-4} rad. It is small but measurably larger than the permitted closure criterion.

Scope of the pass. This test certifies these specified diagnostics for one period and this initial state. A longer integration, a more eccentric orbit or a close encounter needs its own refinement study. Conservation, convergence, correct physical inputs and a clearly stated duration are complementary requirements.

13.10.5 What the numerical diagnostics mean

The energy error of a time-reversible, second-order method can oscillate without a large secular drift over the simulated interval, while the phase still accumulates error. A conservation plot therefore complements rather than replaces trajectory comparison. Conversely, exact-looking angular-momentum conservation does not guarantee a correct gravitational parameter or correct initial conditions. Validation must test both the algorithm and the physical inputs.

The point-force singularity at $r = 0$ deserves special care. Inserting an arbitrary softening length changes the physical model. Such a modification can be useful in a different simulation context, but it must be declared and validated for that context rather than silently introduced to prevent numerical failure.

Take-home message

The advanced lesson is not that a computer “solves gravity.” It is that a well-posed model, a reproducible numerical method and physics-based diagnostics work together. The same discipline applies to many engineering simulations governed by conservation laws.

13.11 Chapter summary and working formula sheet**Summary**

Newtonian gravitation is an attractive inverse-square interaction. Superposition is vector addition. Outside a spherical source, $g = GM/r^2$ and the local constant- g model is only a small-height approximation. With $U(\infty) = 0$, the potential is $U = -GMm/r$ and escape corresponds to non-negative total energy. Gravity supplies the inward force of a circular orbit, giving $v_c \propto r^{-1/2}$ and $T \propto r^{3/2}$. Central-force angular momentum conservation gives Kepler’s equal-area law. Tides arise from field gradients after subtracting the acceleration of an extended body’s centre. Model limitations must be identified before numerical precision is interpreted as physical accuracy.

Formula summary**Point sources and spherical exterior fields**

$$\mathbf{g}(\mathbf{r}) = - \sum_i GM_i \frac{\mathbf{r} - \mathbf{r}_i}{|\mathbf{r} - \mathbf{r}_i|^3}, \quad \mathbf{F} = m\mathbf{g}, \quad g(r) = \frac{GM}{r^2} \quad (r \geq R).$$

Altitude variation

$$\frac{g(R+h)}{g(R)} = \left(1 + \frac{h}{R}\right)^{-2}.$$

Potential and escape, with $U(\infty) = 0$

$$U = -\frac{GMm}{r}, \quad E = \frac{1}{2}mv^2 - \frac{GMm}{r}, \quad v_{\text{esc}} = \sqrt{\frac{2GM}{r}}.$$

Circular test-particle orbit

$$v_c = \sqrt{\frac{GM}{r}}, \quad T_{\text{orb}} = 2\pi\sqrt{\frac{r^3}{GM}}, \quad E = -\frac{GMm}{2r}.$$

Kepler ellipse

$$\frac{dA}{dt} = \frac{L}{2m}, \quad r_p = a(1-e), \quad r_a = a(1+e), \quad T^2 = \frac{4\pi^2 a^3}{GM}.$$

First-order tidal field about $R\hat{\mathbf{i}}$

$$\Delta\mathbf{g} \simeq \frac{GM}{R^3}(2\delta x\hat{\mathbf{i}} - \delta y\hat{\mathbf{j}} - \delta z\hat{\mathbf{k}}), \quad |\delta| \ll R.$$

Before accepting a gravitational result, ask: Is r measured from the correct centre? Is altitude being confused with radius? Is the source model point-like or spherical where required? Is the potential zero stated? Is the system isolated for energy conservation? Does a test-mass approximation hide source recoil? Do numerical conservation errors shrink when the timestep is refined?

Take-home message

Gravity becomes a coherent subject when force, field, energy and orbit descriptions are treated as different views of the same model. The strongest solutions state the geometry and assumptions first, use conservation where appropriate, and test approximations numerically or dimensionlessly rather than by intuition alone.

13.12 Exercises and problems

Use $G = 6.67430 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$ and, where specified, the stipulated central gravitational parameter $\kappa = GM = 4.00 \times 10^{14} \text{ m}^3 \text{ s}^{-2}$. These are model data, not a claim that every example describes Earth. State geometry, directions, assumptions and the zero of potential. Neglect a probe's effect on its source except in the explicitly two-body problems. Every item has a complete solution in the next section; stable problem identifiers retain the original numbering.

A. Attraction, fields and superposition

P13-01. Distinguish a *source mass*, a *test mass*, gravitational force and gravitational field. Why can the field be discussed before choosing a particular test mass?

P13-02. Two positive point masses lie on the x axis at $x = -a$ and $x = +a$. At the origin, draw the two field contributions if the masses are equal. What changes if the right-hand mass is doubled?

P13-03. Diagnostic X13-01. Two positive masses are at $x = 0$ and $x = L$. Explain why solving only an equality of magnitudes can generate an algebraic candidate outside the interval that is not a physical zero-field point.

P13-04. Verify from dimensions that Gm_1m_2/r^2 has units of force. If every mass in a fixed geometry is multiplied by 3 and every distance by 2, by what factor does every gravitational-force vector change?

P13-05. A 2.0 kg test mass is at the origin. Sources of 5.0 kg and 8.0 kg are at (3.0, 0) m and (0, 4.0) m. Find the gravitational force vector on the test mass and its magnitude.

P13-06. Sources $16M_0$ and M_0 are separated by 10.0 m. Locate the zero-field point between them and explain why it lies close to the smaller source.

B. Spherical sources and local gravity

P13-07. At altitude $h = 0.10R$ above a spherical source, find $g(R+h)/g(R)$. By what percentage does the surface value overestimate the exact value if the exact value is the denominator?

P13-08. Diagnostic X13-02. Find the largest h/R for which the constant surface value overestimates the exact inverse-square value by less than 2.0%, using the exact value in the denominator.

P13-09. A spherical model body has $R = 3.40 \times 10^6 \text{ m}$ and surface gravitational acceleration 3.70 ms^{-2} . Infer its total mass and mean density. State why the mean density does not reveal a unique radial density profile.

P13-10. A probe moves from $r = R$ to $r = 2R$ outside a spherical source. Find the factor by which g changes. Why is calling $2R$ an “altitude of $2R$ ” incorrect?

P13-11. Starting from the exact ratio $g(R+h)/g(R) = (1+h/R)^{-2}$, derive the first-order fractional change $[g(R+h) - g(R)]/g(R) \simeq -2h/R$ for $h/R \ll 1$.

P13-12. A spherical body's radius is known but G is not independently available. A surface measurement gives g_s . Which combination of source properties is determined directly by $g_s R^2$? What additional information is required to infer M separately?

C. Potential, escape and circular orbits

P13-13. A 500 kg test body moves radially from $r_1 = 8.00 \times 10^6$ m to $r_2 = 1.20 \times 10^7$ m around a source with $GM = 4.00 \times 10^{14} \text{ m}^3 \text{ s}^{-2}$. Find ΔU .

P13-14. Diagnostic X13-03. For $GM = 4.00 \times 10^{14} \text{ m}^3 \text{ s}^{-2}$ and $r_0 = 2.00 \times 10^7$ m, find the escape speed. Explain why the result is independent of the test-body mass.

P13-15. A body is launched at $v = 0.70v_{\text{esc}}$ from radius r_0 . Express its specific mechanical energy ε in units of GM/r_0 and classify the state as bound or unbound.

P13-16. Two circular test-particle orbits about the same source have $r_2 = 9r_1$. Find v_2/v_1 , T_2/T_1 and $|E_2|/|E_1|$. For the energy ratio, use the same test-body mass in both states.

P13-17. For $GM = 4.00 \times 10^{14} \text{ m}^3 \text{ s}^{-2}$ and circular radius $r = 1.00 \times 10^7$ m, calculate v_c and T_{orb} .

P13-18. A 750 kg test mass changes from one ideal circular orbit at $r_1 = 8.00 \times 10^6$ m to another at $r_2 = 1.60 \times 10^7$ m about the stipulated source $GM = 4.00 \times 10^{14} \text{ m}^3 \text{ s}^{-2}$. Find the difference $E_2 - E_1$ between the two circular-state energies. Why is this not by itself the propulsive energy required for a real transfer?

P13-19. Diagnostic X13-04. Derive v_{esc}/v_c at the same radius and explain why a higher circular orbit has a longer period even though its orbital speed is smaller.

D. Kepler motion and tides

P13-20. An ellipse has $a = 1.00 \times 10^7$ m and $e = 0.20$. Find r_p , r_a and the ratio v_p/v_a .

P13-21. Two test-particle Kepler ellipses orbit the same source with semimajor axes $a_2 = 2a_1$. Find T_2/T_1 .

P13-22. Diagnostic X13-05. A student places eight points at equal 45° changes in true anomaly and says they are positions separated by equal times. Explain the error and give the conserved quantity that supplies the correct construction.

P13-23. In the first-order tidal model, take $GM = 4.00 \times 10^{14} \text{ m}^3 \text{ s}^{-2}$, $R = 1.00 \times 10^8$ m and a radial offset $\delta x = 2.00 \times 10^5$ m. Find the magnitude of the radial differential acceleration.

P13-24. For the same source and separation as [P13-23](#), use a transverse offset $\delta y = 2.00 \times 10^5$ m. Find the transverse differential acceleration vector and state its physical direction.

P13-25. Compare the exact near-side differential acceleration at $R = 1.00 \times 10^8$ m with near-side distance $d = 1.00 \times 10^6$ m (so $\delta x = -d$) to the first-order tidal approximation for $GM = 4.00 \times 10^{14} \text{ m}^3 \text{ s}^{-2}$. Find the fractional difference relative to the exact differential acceleration.

E. Modelling and data interpretation

P13-26. Diagnostic X13-06/X13-08. For each case, identify the first model assumption you would investigate rather than blindly adding more significant figures: (a) a low-altitude satellite in a residual atmosphere; (b) two bodies of comparable mass; (c) an orbit near a strongly nonspherical rotating source; (d) a hypothetical motion with $v = 0.5c$.

P13-27. A synthetic circular-orbit record gives $r = 1.00 \times 10^7$ m and $T = 1.000 \times 10^4$ s. Infer GM . State what the orbit does and does not determine about G and M individually.

P13-28. In $\kappa = 4\pi^2 r^3 / T^2$, suppose independent relative standard uncertainties are $\sigma_r/r = 0.0010$ and $\sigma_T/T = 0.0020$. Estimate σ_{GM}/GM using first-order propagation. Which contribution is larger?

P13-29. A simulation of a nominally circular orbit shows total energy drifting monotonically by 2% per orbit, while halving the timestep reduces the drift to about 0.5%. What does this pattern suggest? Name two further checks before interpreting the drift as physical escape.

F. Advanced challenges

P13-30. For masses $M = 3m$, express the reduced mass μ in terms of m and identify the mass appearing in the relative equation $\ddot{\mathbf{r}} = -G(M + m)\mathbf{r}/r^3$.

P13-31. With normalized $G(M + m) = 5$ and specific angular momentum $\ell = 2$, find the radius of the circular minimum of V_{eff} . Verify the sign of the second derivative there.

P13-32. Diagnostic X13-07. A second-order orbit integrator has maximum relative energy error 3.2×10^{-3} at timestep Δt in its asymptotic convergence regime. Predict the errors at $\Delta t/2$ and $\Delta t/4$. Explain why a factor-of-four trend is stronger evidence than one visually smooth trajectory.

P13-33. Two bodies with masses $M = 9m$ have separation 10.0 m. Locate each body relative to their common centre of mass. If the relative orbit is an ellipse, what geometric scaling relates each barycentric orbit to the relative orbit?

G. Further engineering and analytical problems

P13-34. A shell and an interior cavity. A thin, uniform spherical shell has mass $M = 6.00 \times 10^{22}$ kg and radius $R = 2.00 \times 10^6$ m. Choose $\Phi(\infty) = 0$. (a) Find the field and potential at $r = 0.40R$ and $r = 2R$. (b) Calculate the work done by gravity when a 100 kg probe moves between two points inside the cavity. (c) Explain why zero field inside does not mean zero potential there.

P13-35. Potential inside a solid sphere. A nonrotating uniform sphere has radius $R = 6.00 \times 10^6$ m and $GM = \kappa$. (a) Obtain $\Phi(r)$ for $0 \leq r \leq R$ by integrating its interior field and matching the exterior potential. (b) Find $\Phi(0)$, $\Phi(R)$ and the inward field magnitude at $r = R/2$. (c) Find the escape speed from the centre in the idealized model of a probe moving through a frictionless narrow tunnel without materially changing the density. Check continuity at the surface.

P13-36. Gravitational and apparent weight. A spherical model planet has $R = 6.00 \times 10^6$ m, $GM = \kappa$, and rotation period $T_{\text{rot}} = 1.00 \times 10^4$ s. A 1000 kg object rests on a scale at its equator. (a) Find the gravitational acceleration and gravitational force. (b) Find the scale reading in newtons and the apparent weight per unit mass. (c) Find the rotation period at which an unsupported equatorial object would just have the circular speed. Ignore deformation and atmospheric forces.

P13-37. A zero field with a nonzero potential. Two equal point sources of mass M occupy $(\pm a, 0, 0)$. (a) Find $\mathbf{g}$ and Φ at the origin with zero potential at infinity. (b) Obtain the first-order field near the origin for a small displacement $x\hat{i}$, then for $y\hat{j}$. (c) Decide whether a freely moving test particle placed at the origin is stable against arbitrary small perturbations. Work to first order in $|x|/a$ or $|y|/a$.

P13-38. Reading a potential graph. For a point source define $q = r/R$ and $f(q) = \Phi/(\kappa/R)$. Figure 13.32 gives the graph for $1 \leq q \leq 4$. (a) Read $f(1)$, $f(2)$ and $f(4)$. (b) From its slope determine the sign and magnitude of the normalized radial field $g_r/(\kappa/R^2)$ at $q = 2$. (c) For $m = 200 \text{ kg}$ and $R = 5.00 \times 10^6 \text{ m}$, calculate the work done by gravity in a slow outward move from $q = 1$ to $q = 2$. Explain why a positive slope is compatible with an inward gravitational field.

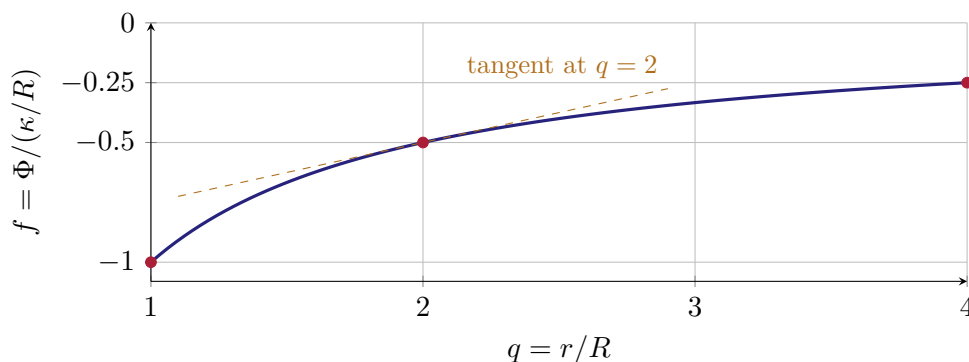


Figure 13.32: The potential increases outward toward zero. The dashed tangent is drawn at $q = 2$; the radial field is the negative gradient. The displayed interval is exterior to the chosen radius R .

P13-39. Maximum axial field of a ring. A thin uniform ring has radius $a = 1.00 \times 10^7 \text{ m}$ and total gravitational parameter $GM = \kappa$. On its symmetry axis let $z > 0$. (a) Starting from $\Phi(z) = -\kappa/\sqrt{a^2 + z^2}$, obtain $g_z(z)$. (b) Find where the field magnitude is largest and calculate that maximum. (c) Check its limiting behavior at the centre and at large z ; explain why the ring cannot be replaced by a central point mass near its centre.

P13-40. Vis viva at both apses. An ideal test particle follows an ellipse with $a = 1.00 \times 10^7 \text{ m}$, $e = 0.20$, and central parameter κ . (a) Calculate the speeds at periapsis and apoapsis using vis viva. (b) Find the specific orbital energy and period. (c) Verify angular momentum and energy independently from the two apsidal states.

P13-41. A bound trajectory that intersects its source. A spherical source has $GM = \kappa$ and physical radius $R = 6.00 \times 10^6 \text{ m}$. At $r_0 = 1.00 \times 10^7 \text{ m}$ a probe has zero radial velocity and tangential speed $v_0 = 0.80\sqrt{\kappa/r_0}$. (a) Find its specific energy and classify it as bound or unbound in the exterior point-source model. (b) Determine the semimajor axis, eccentricity and the other apsidal radius of the corresponding mathematical ellipse. (c) Explain whether this is a physically sustainable exterior orbit about the stated body. Ignore rotation and drag before surface contact.

P13-42. An ideal Hohmann transfer. Two coplanar prograde circular orbits about $GM = \kappa$ have radii $r_1 = 8.00 \times 10^6 \text{ m}$ and $r_2 = 1.60 \times 10^7 \text{ m}$. Assume instantaneous tangential impulses and a coast on an ellipse whose apses touch the two circles. (a) Find the transfer semimajor axis and speeds immediately after the first burn and before the second. (b) Find the two positive speed increments and their sum. (c) Calculate the coasting time. Distinguish this ideal Δv requirement from a spacecraft fuel requirement.

P13-43. A tidal tolerance for a formation. Two freely falling probes have small instantaneous radial separation $L = 100$ m at source-centre radius r around $GM = \kappa$. (a) Find the smallest r for which the first-order radial differential acceleration is no more than $1.00 \times 10^{-5} \text{ m s}^{-2}$. (b) Evaluate L/r at the boundary to test the linearization. (c) Would a transverse separation of the same length produce stretching or compression, and what is its differential acceleration magnitude there? This is an instantaneous tidal specification, not a passive station-keeping guarantee.

P13-44. Fitting synthetic Kepler data. Set $r_* = 1.00 \times 10^7$ m, $x = (r/r_*)^3$, and $y = T^2/(1.00 \times 10^8 \text{ s}^2)$. Four independent synthetic records are

x	1	8	27	64
y	0.99	8.03	27.00	63.98

Treat x as exact and assign each y a known independent standard uncertainty $\sigma_y = 0.050$. (a) Fit $y = bx$ by minimizing $\chi^2 = \sum_i (y_i - bx_i)^2 / \sigma_y^2$. (b) Infer GM and its standard uncertainty from the fit. (c) Give the residuals and χ^2 with its degrees of freedom. State whether this result proves the gravity model or identifies G and M separately. The data and uncertainty are stipulated, not observational claims.

P13-45. Correlation in an inferred gravitational parameter. A circular-orbit estimate uses $\kappa = 4\pi^2 r^3 / T^2$. The relative standard uncertainties are $u_r = 0.0010$ and $u_T = 0.0020$, with correlation coefficient $\rho_{rT} = +0.75$. (a) Derive and calculate the first-order relative variance of κ . Compare it with the independent-error result. (b) If ten orbital records share the same fractional radius-calibration uncertainty 0.0010, explain why that calibration contribution cannot be divided by $\sqrt{10}$. (c) State what additional evidence is required before quoting a smaller final uncertainty from averaging.

13.13 Complete reasoned solutions

The statements are deliberately varied: some require a vector, some a ratio, some a model decision. A valid solution gives the requested result and explains its interpretation. A few calculations display guard digits to make independent checks reproducible; those digits do not imply accuracy beyond the stipulated inputs. Potential and energy use zero at infinite separation unless a problem explicitly uses normalized units.

A. Attraction, fields and superposition

Solution P13-01. A source mass belongs to the matter distribution producing the field. A test mass is sufficiently small that its effect on that prescribed distribution can be neglected. The force $\mathbf{F}$ is the interaction experienced by a specified body, measured in newtons; the gravitational field $\mathbf{g}$ is force per unit test mass, measured in $\text{N kg}^{-1} = \text{m s}^{-2}$. For a point source at $\mathbf{r}_s$,

$$\mathbf{g}(\mathbf{r}) = -GM \frac{\mathbf{r} - \mathbf{r}_s}{|\mathbf{r} - \mathbf{r}_s|^3}, \quad \mathbf{F} = m\mathbf{g}.$$

The source and field point determine $\mathbf{g}$ before m is chosen. Doubling a test mass doubles its force but leaves its acceleration and the prescribed field unchanged. In a comparable-mass system both bodies move, so the fixed-source simplification requires revision.

Solution P13-02. Let each original source have mass M . At the origin the left source produces $-GM\hat{\mathbf{i}}/a^2$ and the right produces $+GM\hat{\mathbf{i}}/a^2$. Equal opposite vectors cancel. If only the right mass becomes $2M$, its contribution doubles while its distance remains a , giving

$$\mathbf{g}(0) = \left(-\frac{GM}{a^2} + \frac{2GM}{a^2} \right) \hat{\mathbf{i}} = \frac{GM}{a^2} \hat{\mathbf{i}}.$$

The arrow points toward the more massive source. Attraction determines each arrow toward its own source, rather than imposing the same algebraic sign on both contributions. Figure 13.33 separates contributions from their sum.

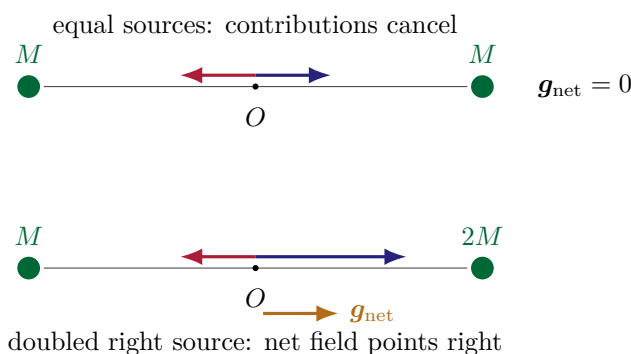


Figure 13.33: Field contributions at the midpoint. Red and blue arrows represent the left and right sources; orange represents their sum in the lower case. The source separation is the same in both cases. Arrow lengths show the 1 : 1 and 1 : 2 ratios.

Solution P13-03. For $x < 0$, both source masses lie to the right, so both fields point in $+x$. For $x > L$, both lie to the left, so both fields point in $-x$. Neither exterior region admits cancellation for positive masses. Only $0 < x < L$ puts the sources on opposite sides of the field point. Equating magnitudes there is legitimate because the directions are already opposite. Squaring an equation, or writing distances without first fixing the region, can discard sign information and generate another algebraic candidate. Substitution into the original signed field equation is therefore essential: equality of two positive magnitudes is a necessary cancellation condition only after opposite directions have been established.

Solution P13-04. The unit calculation is

$$[G] \frac{[m_1][m_2]}{[r]^2} = (\text{m}^3 \text{kg}^{-1} \text{s}^{-2}) \frac{\text{kg}^2}{\text{m}^2} = \text{kg m s}^{-2} = \text{N}.$$

Scaling every mass by 3 multiplies each source–target product by 9; scaling every distance by 2 divides each inverse-square factor by 4. Thus every pair force and the resultant force scale by 9/4. This statement assumes a geometrically similar configuration, including the target mass: all unit direction vectors remain unchanged. A field instead scales by 3/4, because its definition divides out the target mass. The distinction provides a useful check against confusing force with acceleration.

Solution P13-05. Each force points from the origin toward its own source. The sources lie on mutually perpendicular positive axes, so

$$F_x = G \frac{(2.0)(5.0)}{3.0^2} = 7.41589 \times 10^{-11} \text{ N}, \quad F_y = G \frac{(2.0)(8.0)}{4.0^2} = 6.67430 \times 10^{-11} \text{ N}.$$

Their sum is $\mathbf{F} = (7.42\hat{\mathbf{i}} + 6.67\hat{\mathbf{j}}) \times 10^{-11} \text{ N}$. Orthogonality gives $|\mathbf{F}| = \sqrt{F_x^2 + F_y^2} = 9.98 \times 10^{-11} \text{ N}$. Its direction is $\tan^{-1}(F_y/F_x) = 42.0^\circ$ above $+x$. The magnitude exceeds either component but is less than their scalar sum, as required for perpendicular vectors. Dividing by 2.0 kg yields the field if desired; adding magnitudes directly would overestimate the force.

Solution P13-06. Place $16M_0$ at $x = 0$ and M_0 at $x = 10.0 \text{ m}$. Between them the two fields are opposite. With $0 < x < 10.0 \text{ m}$,

$$\frac{16GM_0}{x^2} = \frac{GM_0}{(10.0 - x)^2} \Rightarrow \frac{4}{x} = \frac{1}{10.0 - x}.$$

Both denominators are positive in the selected interval. Hence $4(10.0 - x) = x$ and $x = 8.00 \text{ m}$: the smaller source is 2.00 m away. The distance ratio is 4 : 1, the square root of the mass ratio 16 : 1. Being closer to the smaller source amplifies its inverse-square field enough to match the larger one. Checking both contributions gives $16/8^2 = 1/2^2 = 1/4$ in common units.

B. Spherical sources and local gravity

Solution P13-07. Outside a spherical source, $g(r) = GM/r^2$. At $h = 0.10R$ the radius is $1.10R$, giving

$$\frac{g(R+h)}{g(R)} = \frac{1}{1.10^2} = 0.826446.$$

The requested error uses the exact altitude value as denominator:

$$\frac{g_s - g}{g} = \frac{g_s}{g} - 1 = 1.10^2 - 1 = 0.210 = 21.0\%.$$

Using g_s in the denominator would instead give 17.36%, which answers a different question. The constant surface value overestimates because the actual field weakens outward. The ratio cancels both G and the source mass; it depends only on the relative altitude under the stated spherical model.

Solution P13-08. Let $u = h/R \geq 0$. The exact-denominator error is $(1+u)^2 - 1$. Imposing the strict tolerance gives

$$(1+u)^2 - 1 < 0.020 \Rightarrow u < \sqrt{1.020} - 1 = 9.95049 \times 10^{-3}.$$

Thus the upper boundary is about 0.995% of the source radius. Because the wording is “less than,” the boundary itself gives exactly 2.0% and is excluded; mathematically there is a supremum rather than an attained largest altitude. To first order the error is $2u$, predicting $u < 0.010$, close to but slightly less restrictive than the exact inequality. Stating the denominator and strictness avoids two common ambiguity sources.

Solution P13-09. The surface relation is $g_s = GM/R^2$, so

$$M = \frac{g_s R^2}{G} = \frac{3.70(3.40 \times 10^6)^2}{6.67430 \times 10^{-11}} = 6.41 \times 10^{23} \text{ kg}.$$

Divide by the spherical volume, rather than by the surface area:

$$\bar{\rho} = \frac{3M}{4\pi R^3} = 3.89 \times 10^3 \text{ kg m}^{-3}.$$

The units are mass per volume. For a general spherical density profile, $M = 4\pi \int_0^R \rho(r)r^2 dr$. Many different functions have the same weighted integral. Therefore the mean density does not establish uniform density or determine an interior field profile. Exterior gravity measures total mass under spherical symmetry, not its unique radial arrangement.

Solution P13-10. The radius doubles, so the inverse-square law gives

$$\frac{g(2R)}{g(R)} = \left(\frac{R}{2R}\right)^2 = \frac{1}{4}.$$

It is the source-centre distance r that enters the gravitational formula. Altitude is $h = r - R$, measured from the surface. At $r = 2R$ it is $h = R$. Conversely an altitude $2R$ would mean $r = 3R$ and a field $g_s/9$. The distinction is physical, not merely notation: using an altitude directly as r changes the distance to the source and can produce a large error.

Solution P13-11. Write $u = h/R$. A Taylor expansion about $u = 0$ gives

$$(1 + u)^{-2} = 1 - 2u + 3u^2 + O(u^3).$$

Subtracting the surface ratio 1 yields $[g(R+h) - g(R)]/g(R) = -2h/R + O((h/R)^2)$. The first-order coefficient can also be checked from $g'(R) = -2GM/R^3 = -2g_s/R$. For positive altitude the change is negative, consistent with weaker attraction. The approximation requires $|h|/R \ll 1$ and the omitted quadratic term is positive. It describes an exterior spherical-source field; extending it deep into a material body would require the interior density model.

Solution P13-12. Multiplying the measured acceleration by the squared radius gives

$$g_s R^2 = GM = \kappa,$$

with units $\text{m}^3 \text{s}^{-2}$. The observation directly determines the gravitational parameter. It does not separately measure a mass in kilograms if G is unknown. For any proposed positive G' , a mass $M' = \kappa/G'$ gives the same spherical surface acceleration. An independent determination of G , or independent information fixing M in the adopted mass standard, breaks this degeneracy. Increasing the precision of the same surface measurement does not remove it.

C. Potential, escape and circular orbits

Solution P13-13. With infinity as the zero, $U(r) = -\kappa m/r$. The difference is

$$\Delta U = \kappa m \left(\frac{1}{r_1} - \frac{1}{r_2} \right) = (4.00 \times 10^{14})(500) \left(\frac{1}{8.00 \times 10^6} - \frac{1}{1.20 \times 10^7} \right) = +8.33 \times 10^9 \text{ J}.$$

The units of $\kappa m/r$ reduce to joules. Separating an attracted pair raises its potential energy toward zero, explaining the positive result. Gravity itself does work $W_g = -\Delta U = -8.33 \times 10^9 \text{ J}$ during the outward displacement. A slow lift by an external agent would supply the positive work if kinetic energy stayed constant. A constant- g formula is inappropriate for this large radial change.

Solution P13-14. At the threshold of escape, the body reaches infinity with zero remaining speed. Conservation of mechanical energy gives

$$\frac{1}{2}mv_{\text{esc}}^2 - \frac{\kappa m}{r_0} = 0, \quad v_{\text{esc}} = \sqrt{\frac{2\kappa}{r_0}} = 6.32 \times 10^3 \text{ m s}^{-1}.$$

The test-body mass cancels because inertial kinetic energy and gravitational potential energy are both proportional to it. This is the minimum speed in the ideal fixed-source, gravity-only exterior after launch; atmosphere, collision geometry or thrust scheduling can change mission requirements. The dimensions $\kappa/r_0 = \text{m}^2 \text{s}^{-2}$ check the square root. The result is a speed threshold, not a statement that any arbitrary launch direction avoids impact.

Solution P13-15. Divide mechanical energy by the test-body mass. Since $v_{\text{esc}}^2 = 2\kappa/r_0$,

$$\varepsilon = \frac{1}{2}(0.70)^2 v_{\text{esc}}^2 - \frac{\kappa}{r_0} = (0.49 - 1) \frac{\kappa}{r_0} = -0.51 \frac{\kappa}{r_0}.$$

The negative specific energy means that the isolated Kepler motion is bound. Its magnitude is less than the initial binding potential because the launch supplied kinetic energy. This classification does not guarantee a safe orbit: zero angular momentum can give radial return, and an ellipse may intersect the body. The scalar speed and energy alone classify boundness; direction and source size are needed to decide the detailed physical trajectory.

Solution P13-16. Circular balance gives $v_c \propto r^{-1/2}$, the period is $T \propto r^{3/2}$, and the circular total energy is $E = -\kappa m/(2r)$. Therefore

$$\frac{v_2}{v_1} = 9^{-1/2} = \frac{1}{3}, \quad \frac{T_2}{T_1} = 9^{3/2} = 27, \quad \frac{|E_2|}{|E_1|} = \frac{1}{9}.$$

The last ratio assumes the same test-body mass, now stated explicitly in the problem. For unequal masses it would be $(m_2/m_1)/9$, whereas the specific-energy ratio remains $1/9$. All three are dimensionless. The outer orbit is slower, takes much longer to traverse, and is less tightly bound.

Solution P13-17. Gravity supplies the inward centripetal acceleration, so $v_c^2/r = \kappa/r^2$. Thus

$$v_c = \sqrt{\frac{4.00 \times 10^{14}}{1.00 \times 10^7}} = 6.32456 \times 10^3 \text{ m s}^{-1} = 6.32 \text{ km s}^{-1}.$$

One circumference at constant speed takes

$$T = \frac{2\pi r}{v_c} = 9.93459 \times 10^3 \text{ s} = 2.76 \text{ h}.$$

The independent expression $2\pi\sqrt{r^3/\kappa}$ gives the same period and confirms dimensions of time. The input is already a radius measured from the source centre, so no surface-radius correction is needed. There is no second outward physical force balancing gravity in the inertial description.

Solution P13-18. Each circular state has $E = -\kappa m/(2r)$. Hence

$$E_2 - E_1 = \frac{\kappa m}{2} \left(\frac{1}{r_1} - \frac{1}{r_2} \right) = \frac{(4.00 \times 10^{14})(750)}{2} \left(\frac{1}{8.00 \times 10^6} - \frac{1}{1.60 \times 10^7} \right) = 9.38 \times 10^9 \text{ J}.$$

The outer circular state is less negative in energy, so the difference is positive. Its potential increase is partly offset by a decrease in circular kinetic energy. This endpoint difference is the net change of spacecraft mechanical energy in a stipulated fixed-mass model. Fuel consumption also depends on exhaust speed, changing spacecraft mass, burn directions and trajectory; propulsion energy includes exhaust and losses. The endpoint result alone cannot determine those quantities.

Solution P13-19. At the same radius the escape and circular speeds are $\sqrt{2\kappa/r}$ and $\sqrt{\kappa/r}$, giving $v_{\text{esc}}/v_c = \sqrt{2}$. The circular speed decreases with radius as $r^{-1/2}$, but the circumference increases as r . Their ratio gives

$$T = \frac{2\pi r}{v_c} \propto r^{3/2}.$$

Both a longer path and a smaller speed increase the period. Escape requires zero total energy, while circular motion has negative total energy $-\kappa m/(2r)$. The factor $\sqrt{2}$ follows from that different energy condition; an escape-speed trajectory at a given point is not another circular orbit there.

D. Kepler motion and tides

Solution P13-20. The source occupies an ellipse's focus. Apsidal distances are

$$r_p = a(1 - e) = 8.00 \times 10^6 \text{ m}, \quad r_a = a(1 + e) = 1.20 \times 10^7 \text{ m}.$$

At both apses the velocity is tangential, so conservation of specific angular momentum gives $r_p v_p = r_a v_a$. Therefore $v_p/v_a = r_a/r_p = 1.50$. The result is independent of κ because it uses the fixed orbital geometry and central-force conservation. The closer apsis has the greater speed. Equal speeds at both apses would violate angular momentum unless the two radii were equal, which is precisely the circular case $e = 0$.

Solution P13-21. For test-particle Kepler orbits about the same source,

$$T = 2\pi\sqrt{a^3/\kappa}, \quad \frac{T_2}{T_1} = \left(\frac{a_2}{a_1}\right)^{3/2} = 2^{3/2} = 2\sqrt{2} = 2.83.$$

Eccentricity does not appear: ellipses with the same semimajor axis and central parameter have the same period even though their instantaneous speeds differ. The comparison requires the same κ . In a comparable-mass two-body problem the relevant parameter is $G(M + m)$, so different companion masses would require corresponding corrections rather than uncritical reuse of this ratio.

Solution P13-22. The areal rate about the attracting focus is constant:

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{\ell}{2}.$$

For a small fixed angular interval, the swept area is approximately $r^2\Delta\theta/2$. It is larger near apoapsis, where r is larger, so it takes longer there. Equal true-anomaly steps are therefore not equal-time steps on an eccentric ellipse. Correct equal-time positions divide the swept area into equal parts about the focus; equivalently one may advance mean anomaly uniformly and solve Kepler's equation. Equal angles do give equal times for a circular orbit, because its radius stays constant.

Solution P13-23. Choose $+x$ radially outward from the source through the formation centre. The central field is $g_x = -\kappa/R^2$ and its radial derivative is $+2\kappa/R^3$. Thus the linear differential field at a positive radial offset is

$$\Delta g_x \simeq \frac{2\kappa\delta x}{R^3} = \frac{2(4.00 \times 10^{14})(2.00 \times 10^5)}{(1.00 \times 10^8)^3} = +1.60 \times 10^{-4} \text{ m s}^{-2}.$$

The requested magnitude is $1.60 \times 10^{-4} \text{ m s}^{-2}$. The outward sign denotes weaker inward attraction at the outer probe, measured relative to the accelerating centre. It does not mean gravity repels that probe. The offset ratio 0.002 supports first-order treatment; the absolute central acceleration is 0.0400 m s^{-2} .

Solution P13-24. At a small transverse displacement δy from the source–centre axis, the source direction has a transverse component pointing back toward that axis. Linearization gives

$$\Delta g_y = -\frac{\kappa \delta y}{R^3} = -\frac{(4.00 \times 10^{14})(2.00 \times 10^5)}{(1.00 \times 10^8)^3} = -8.00 \times 10^{-5} \text{ m s}^{-2}.$$

Thus $\Delta \mathbf{g} = -8.00 \times 10^{-5} \hat{\mathbf{j}} \text{ m s}^{-2}$ to first order for the stipulated positive offset. The sign describes transverse compression: probes above and below the axis accelerate toward it relative to the centre. The transverse coefficient is half the radial stretching coefficient in magnitude. A change in the radial component caused by this transverse offset first appears at second order.

Solution P13-25. Use the outward x direction of Solution P13-23. The near-side probe has coordinate offset $\delta x = -d$, with $d = 1.00 \times 10^6 \text{ m}$. Its exact differential acceleration is

$$\Delta g_x^{\text{exact}} = -\kappa \left[\frac{1}{(R-d)^2} - \frac{1}{R^2} \right] = -8.12162 \times 10^{-4} \text{ m s}^{-2}.$$

The linear estimate is $\Delta g_x^{(1)} = -2\kappa d/R^3 = -8.00 \times 10^{-4} \text{ m s}^{-2}$. Taking the requested relative difference in magnitude gives

$$\frac{|\Delta g_x^{\text{exact}} - \Delta g_x^{(1)}|}{|\Delta g_x^{\text{exact}}|} = 0.0149749 = 1.50\%.$$

The exact attraction is slightly stronger than predicted by the linear model. The separation ratio $d/R = 0.01$ is small, but a differential quantity's relative truncation error is of first order in that ratio; it need not equal the much smaller error relative to the central acceleration.

E. Modelling and data interpretation

Solution P13-26. (a) Residual atmosphere introduces drag, so the purely conservative two-body force model should be checked first. Its importance depends on density, speed, area-to-mass ratio and duration. (b) Comparable masses invalidate a fixed source: use relative motion with $G(M+m)$ and track the common centre of mass. (c) A nonspherical rotating source can have multipole gravity and a time-dependent orientation; investigate the spherical-source approximation and distinguish inertial motion from effects of a rotating coordinate frame. (d) At $v = 0.5c$, Newtonian low-speed dynamics is inadequate; relativistic kinematics and the gravitational regime must be assessed. The first issue is the model, not the number of printed decimals. These observations identify assumptions to investigate; they do not prescribe one universal correction without further physical information.

Solution P13-27. Circular dynamics gives $T^2 = 4\pi^2 r^3/\kappa$. Solving for the parameter,

$$\kappa = \frac{4\pi^2(1.00 \times 10^7)^3}{(1.000 \times 10^4)^2} = 3.94784 \times 10^{14} \text{ m}^3 \text{ s}^{-2}.$$

The radius and period are a synthetic record, so this value is their inferred model parameter; it need not equal the 4.00×10^{14} used in other examples. Only the product GM is observable from this orbit within the circular test-particle model. To infer a separate source mass in kilograms requires an independent G . A separate G measurement cannot be obtained merely by making this same orbital record more precise.

Solution P13-28. Logarithmic differentiation of $\kappa = 4\pi^2 r^3 T^{-2}$ gives $d\kappa/\kappa = 3 dr/r - 2 dT/T$. For independent standard uncertainties, the variance adds in quadrature:

$$\frac{\sigma_\kappa}{\kappa} \simeq \sqrt{(3 \times 0.0010)^2 + (2 \times 0.0020)^2} = 0.0050 = 0.50\%.$$

The radius term is 0.0030 and period term 0.0040, so the period contributes more variance: 16 versus 9 in common units of 10^{-6} . The minus sign in the sensitivity to T does not subtract independent variances. A correlation would introduce a signed covariance term, addressed in P13-45. This propagation assumes sufficiently small errors and a valid circular-orbit model.

Solution P13-29. A reduction from 2% to 0.5% on halving the timestep is a factor of four, consistent with leading error proportional to Δt^2 . It suggests numerical truncation error rather than evidence for physical energy growth in an isolated conservative orbit. Check at least one more refinement and verify the expected convergence of both energy and orbital phase against an analytic circular solution. Also monitor angular momentum and confirm that the code includes only the stipulated central force. A monotonic drift is particularly worth investigating if a symplectic method should normally produce bounded energy fluctuations. Apparent escape should not be interpreted physically until it persists under justified refinement and a specified nonconservative model.

F. Advanced challenges

Solution P13-30. The reduced mass is

$$\mu = \frac{Mm}{M+m} = \frac{3m \cdot m}{4m} = \frac{3}{4}m.$$

This is the mass multiplying the relative kinetic energy, not the mass to insert alone into the relative acceleration. Subtracting the two bodies' acceleration equations gives $\ddot{\mathbf{r}} = -G(M+m)\mathbf{r}/r^3$, so here $M+m = 4m$ is the mass in the gravitational parameter. The force in the reduced-coordinate equation may be written $\mu\ddot{\mathbf{r}} = -GMm\mathbf{r}/r^3$, consistently using $\mu(M+m) = Mm$. Distinguishing these roles prevents a spurious factor of μ in the orbital period.

Solution P13-31. In the specified normalized units the effective potential per unit reduced mass is $V_{\text{eff}} = \ell^2/(2r^2) - \kappa_{\text{rel}}/r$, with $\kappa_{\text{rel}} = 5$ and $\ell = 2$. Its first derivative vanishes when

$$-\frac{\ell^2}{r^3} + \frac{\kappa_{\text{rel}}}{r^2} = 0, \quad r_c = \frac{\ell^2}{\kappa_{\text{rel}}} = 0.800.$$

The curvature is $V''_{\text{eff}} = 3\ell^2/r^4 - 2\kappa_{\text{rel}}/r^3$. Using $\ell^2 = \kappa_{\text{rel}}r_c$ yields $V''_{\text{eff}}(r_c) = \kappa_{\text{rel}}/r_c^3 = 9.765625 > 0$. Thus the stationary point is a radial minimum at fixed angular momentum. These numbers are normalized quantities, not SI metres or SI potential values unless an additional normalization is supplied.

Solution P13-32. In the stated asymptotic regime, a second-order error has leading form $e = C(\Delta t)^2$. Therefore

$$e(\Delta t/2) = \frac{3.2 \times 10^{-3}}{4} = 8.0 \times 10^{-4}, \quad e(\Delta t/4) = \frac{3.2 \times 10^{-3}}{16} = 2.0 \times 10^{-4}.$$

The errors must be measured consistently over the same physical interval and with the same norm. A repeated factor of four checks a quantitative prediction of the method. A visually smooth trajectory says little about its accumulated phase, energy or apsidal error. Refinement does not prove every implementation detail correct, but it is stronger evidence than appearance and can expose failure to reach the asymptotic regime.

Solution P13-33. Let r_M and r_m be positive distances from the common centre of mass. The bodies are on opposite sides, so $r_M + r_m = 10.0$ m and $9mr_M = mr_m$. Hence $r_m = 9r_M$, giving $r_M = 1.00$ m and $r_m = 9.00$ m. More generally, with relative vector $\mathbf{r} = \mathbf{r}_m - \mathbf{r}_M$ and centre of mass at the origin,

$$\mathbf{r}_M = -\frac{m}{M+m}\mathbf{r} = -0.1\mathbf{r}, \quad \mathbf{r}_m = \frac{M}{M+m}\mathbf{r} = 0.9\mathbf{r}.$$

Thus each barycentric ellipse is a scaled copy of the relative ellipse with the same eccentricity. Its semimajor axis is respectively $0.1a$ or $0.9a$, and the heavy body's position vector is opposite the relative vector at every instant. This describes simultaneous positions, not bodies independently moving with unrelated orbital phases.

G. Further engineering and analytical solutions

Solution P13-34. (a) The shell theorem gives zero field at every interior cavity point and a constant potential fixed by matching at the surface:

$$\mathbf{g}(0.40R) = 0, \quad \Phi(0.40R) = -\frac{GM}{R} = -2.00229 \times 10^6 \text{ J kg}^{-1}.$$

At $r = 2R$, the shell acts externally as a point mass:

$$\mathbf{g}(2R) = -\frac{GM}{4R^2} \hat{\mathbf{r}} = -0.250286 \hat{\mathbf{r}} \text{ m s}^{-2}, \quad \Phi(2R) = -\frac{GM}{2R} = -1.00115 \times 10^6 \text{ J kg}^{-1}.$$

(b) Both cavity points have the same potential, so $W_g = -m(\Phi_f - \Phi_i) = 0$, regardless of the path inside the cavity. (c) A field is a spatial derivative of potential: $\mathbf{g} = -\nabla\Phi$. A constant need not be zero. Choosing infinity as the zero fixes the negative cavity constant; changing the convention would shift every potential by the same constant without altering any force or work.

Solution P13-35. (a) Uniform density gives enclosed mass $M(r) = M(r/R)^3$ and interior radial field $g_r = -\kappa r/R^3$. Since $d\Phi/dr = -g_r$, integrate from the surface, where $\Phi(R) = -\kappa/R$:

$$\Phi(r) - \Phi(R) = \int_R^r \frac{\kappa s}{R^3} ds = \frac{\kappa(r^2 - R^2)}{2R^3}, \quad \Phi(r) = -\frac{\kappa}{2R} \left(3 - \frac{r^2}{R^2} \right).$$

(b) Consequently $\Phi(0) = -1.00 \times 10^8 \text{ J kg}^{-1}$, $\Phi(R) = -6.67 \times 10^7 \text{ J kg}^{-1}$, and $|g_r(R/2)| = \kappa/(2R^2) = 5.56 \text{ m s}^{-2}$. The potential is finite and lowest at the centre, while the field vanishes there. (c) Escape from the centre requires $v_{\text{esc}}(0) = \sqrt{-2\Phi(0)} = \sqrt{3\kappa/R} = 1.41 \times 10^4 \text{ m s}^{-1}$. At R , both interior and exterior expressions give $\Phi = -\kappa/R$ and $g_r = -\kappa/R^2$. The matched curves in Figure 13.34 reflect that continuity. The tunnel is a stated idealization: real material interactions would defeat this conservative-motion calculation.

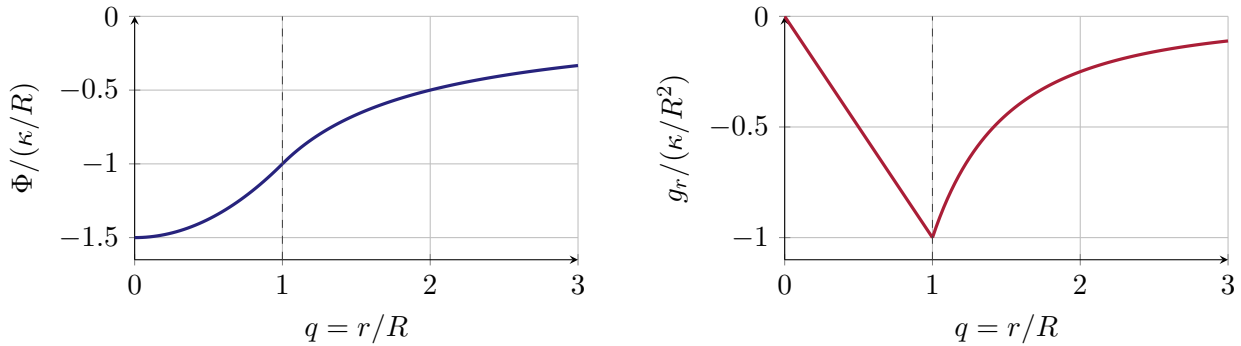


Figure 13.34: Uniform solid-sphere potential and signed radial field, matched at $q = 1$. The field is the negative potential slope. Its magnitude rises linearly inside and decays inversely as the squared radius outside; the interior potential is a parabola, not the point-source singular expression.

Solution P13-36. (a) The gravitational acceleration is $g = \kappa/R^2 = 11.1 \text{ m s}^{-2}$, giving inward gravitational force $mg = 1.11 \times 10^4 \text{ N}$. (b) In an inertial frame an object resting on the rotating equator moves in a circle. Gravity acts inward and the scale's normal force N outward, so $mg - N = m\omega^2 R$ with $\omega = 2\pi/T_{\text{rot}}$. Thus

$$\omega^2 R = 2.36871 \text{ m s}^{-2}, \quad \frac{N}{m} = g - \omega^2 R = 8.74241 \text{ m s}^{-2}, \quad N = 8.74 \times 10^3 \text{ N}.$$

The scale reads support force, smaller than gravitational force. (c) At zero support, $\omega^2 R = \kappa/R^2$, yielding $T_{\text{crit}} = 2\pi\sqrt{R^3/\kappa} = 4.62 \times 10^3 \text{ s}$. Faster rotation would require an inward attachment force in this rigid spherical idealization. Deformation of a real rotating planet is deliberately excluded; a centrifugal term is a coordinate-frame representation, not an additional inertial-frame interaction.

Solution P13-37. (a) Equal source fields cancel at the origin, while potentials add as scalars:

$$\mathbf{g}(0) = 0, \quad \Phi(0) = -\frac{2GM}{a}.$$

(b) Along the line joining sources,

$$g_x(x) = GM \left[\frac{1}{(a-x)^2} - \frac{1}{(a+x)^2} \right] \simeq \frac{4GM}{a^3} x.$$

A positive displacement receives a positive acceleration, away from the midpoint. Along the perpendicular y axis the horizontal fields cancel, whereas

$$g_y(y) = -\frac{2GM y}{(a^2 + y^2)^{3/2}} \simeq -\frac{2GM}{a^3} y.$$

That component is restoring. (c) The origin is a saddle equilibrium: transverse displacements restore, but longitudinal displacements grow. Stability against arbitrary perturbations therefore fails. Zero field identifies an equilibrium at rest; it does not by itself establish a potential minimum or a stable trap. The same transverse conclusion holds for the z direction by symmetry.

Solution P13-38. (a) The exterior curve is $f(q) = -1/q$, giving $f(1) = -1$, $f(2) = -1/2$ and $f(4) = -1/4$. (b) Because $\Phi = (\kappa/R)f(r/R)$, the chain rule gives

$$\frac{g_r}{\kappa/R^2} = -f'(q) = -\frac{1}{q^2}; \quad \left. \frac{g_r}{\kappa/R^2} \right|_{q=2} = -\frac{1}{4}.$$

The graph slopes upward but force points down the potential gradient, hence inward. (c) The outward move raises the potential by $\Delta\Phi = (\kappa/R)(-1/2 + 1) = \kappa/(2R) = 4.00 \times 10^7 \text{ J kg}^{-1}$. Gravity therefore does $W_g = -m\Delta\Phi = -8.00 \times 10^9 \text{ J}$. Slow motion only makes an external agent's compensating work easy to identify; the gravitational work itself depends on the endpoints, not on how slowly the probe moves.

Solution P13-39. (a) Differentiate the stated potential:

$$g_z = -\frac{d\Phi}{dz} = -\frac{\kappa z}{(a^2 + z^2)^{3/2}}.$$

(b) For $z > 0$, its magnitude is $h(z) = \kappa z(a^2 + z^2)^{-3/2}$. The derivative is $h'(z) = \kappa(a^2 - 2z^2)/(a^2 + z^2)^{5/2}$, positive before and negative after $z = a/\sqrt{2}$. Thus

$$z_{\max} = 7.07 \times 10^6 \text{ m}, \quad h_{\max} = \frac{2\kappa}{3\sqrt{3}a^2} = 1.54 \text{ m s}^{-2}.$$

(c) Near the centre, $g_z \simeq -\kappa z/a^3$ and vanishes at $z = 0$. Far away, $g_z \simeq -\kappa/z^2$. The remote point-mass approximation works because the ring's size becomes negligible, but it predicts a divergent central field if used near $z = 0$. The actual symmetry cancellation and finite potential forbid that replacement there.

Solution P13-40. (a) The apses are $r_p = 8.00 \times 10^6 \text{ m}$ and $r_a = 1.20 \times 10^7 \text{ m}$. Vis viva gives

$$v_p = \sqrt{\kappa(2/r_p - 1/a)} = 7.75 \times 10^3 \text{ m s}^{-1}, \quad v_a = \sqrt{\kappa(2/r_a - 1/a)} = 5.16 \times 10^3 \text{ m s}^{-1}.$$

(b) The specific energy is $\varepsilon = -\kappa/(2a) = -2.00 \times 10^7 \text{ J kg}^{-1}$ and $T = 2\pi\sqrt{a^3/\kappa} = 9.93 \times 10^3 \text{ s}$. (c) Using guard digits gives $r_p v_p = r_a v_a = 6.19677 \times 10^{10} \text{ m}^2 \text{ s}^{-1}$. Independently, $v_p^2/2 - \kappa/r_p = 3.00 \times 10^7 - 5.00 \times 10^7 = -2.00 \times 10^7$, while $v_a^2/2 - \kappa/r_a = 1.33333 \times 10^7 - 3.33333 \times 10^7 = -2.00 \times 10^7 \text{ J kg}^{-1}$. The two checks test both the radius-speed assignment and the shared orbit energy.

Solution P13-41. (a) Here $v_0 = 5.06 \times 10^3 \text{ m s}^{-1}$ and

$$\varepsilon = \left(\frac{0.80^2}{2} - 1 \right) \frac{\kappa}{r_0} = -0.68 \frac{\kappa}{r_0} = -2.72 \times 10^7 \text{ J kg}^{-1} < 0.$$

(b) The mathematical Kepler ellipse has $a = -\kappa/(2\varepsilon) = r_0/1.36 = 7.35 \times 10^6 \text{ m}$. Since v_0 is below the local circular speed and the radial velocity is zero, the present position is apoapsis. Thus $r_a = r_0 = a(1+e)$, giving $e = 0.36$ and $r_p = a(1-e) = 4.71 \times 10^6 \text{ m}$. One may check $\ell^2 = \kappa a(1-e^2)$ from $\ell = r_0 v_0$. (c) The calculated periapsis lies inside $R = 6.00 \times 10^6 \text{ m}$. The real probe reaches the surface before completing that ellipse. Negative energy establishes boundness; it does not establish a sustainable collision-free exterior orbit. Continued motion would require a different interior or contact model.

Solution P13-42. (a) The tangent transfer has $a_t = (r_1 + r_2)/2 = 1.20 \times 10^7 \text{ m}$. Its periapsis and apoapsis speeds are

$$v_{t,p} = \sqrt{\kappa(2/r_1 - 1/a_t)} = 8.16497 \text{ km s}^{-1}, \quad v_{t,a} = \sqrt{\kappa(2/r_2 - 1/a_t)} = 4.08248 \text{ km s}^{-1}.$$

(b) The initial and final circular speeds are $v_{c,1} = 7.07107$ and $v_{c,2} = 5.00000 \text{ km s}^{-1}$. Both burns are prograde:

$$\Delta v_1 = v_{t,p} - v_{c,1} = 1.09390 \text{ km s}^{-1}, \quad \Delta v_2 = v_{c,2} - v_{t,a} = 0.917517 \text{ km s}^{-1},$$

giving total $\Delta v = 2.01142 \text{ km s}^{-1}$. The second positive burn circularizes an ellipse whose speed at apoapsis is below the local circular speed. (c) The coast lasts half the transfer period: $t_t = \pi \sqrt{a_t^3/\kappa} = 6.52968 \times 10^3 \text{ s} = 1.81 \text{ h}$. Figure 13.35 checks that both circles and the transfer ellipse touch at the burn locations. Fuel requires a propulsion model and spacecraft masses, for example a specified effective exhaust speed in the ideal rocket equation. Geometry and Δv alone do not supply them.

both impulses increase the prograde speed

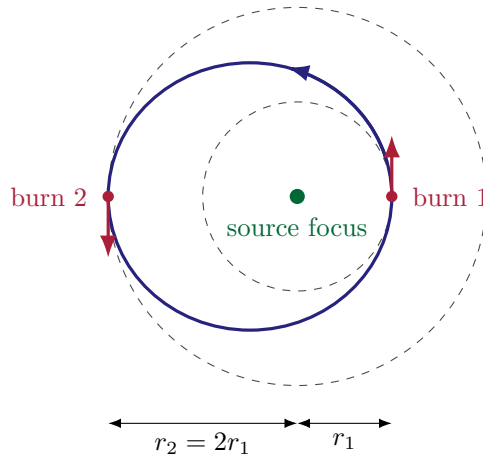


Figure 13.35: Ideal Hohmann geometry for the stipulated 2 : 1 radius ratio. The source is a focus, not the transfer ellipse's centre. Dashed circles are the endpoint orbits; the upper half of the solid ellipse is the coast from burn 1 to burn 2. Red arrows show local prograde directions, without encoding burn magnitudes.

Solution P13-43. (a) The first-order radial difference is $2\kappa L/r^3$. Applying the stipulated limit a_{lim} gives

$$r \geq \left(\frac{2\kappa L}{a_{\text{lim}}} \right)^{1/3} = \left[\frac{2(4.00 \times 10^{14})(100)}{1.00 \times 10^{-5}} \right]^{1/3} = 2.00 \times 10^7 \text{ m}.$$

(b) At this boundary $L/r = 5.00 \times 10^{-6}$, so neglected nonlinear terms are very small compared with the first-order tide. (c) A transverse displacement has coefficient $-\kappa/r^3$, hence magnitude $\kappa L/r^3 = 5.00 \times 10^{-6} \text{ m s}^{-2}$, directed toward the source–centre axis. It is compressive, unlike radial stretching. This instantaneous acceleration limit does not guarantee fixed separation throughout an orbit: initial relative velocity and subsequent coupled formation dynamics also matter.

Solution P13-44. (a) Differentiating χ^2 with respect to b yields $\sum_i x_i(y_i - bx_i) = 0$, so

$$b = \frac{\sum_i x_i y_i}{\sum_i x_i^2} = \frac{4888.95}{4890} = 0.999785276.$$

Known equal independent uncertainties give $\sigma_b = \sigma_y / \sqrt{\sum_i x_i^2} = 0.000715016$. (b) In physical units the period relation implies $b(1.00 \times 10^8 \text{ s}^2) = 4\pi^2 r_*^3 / \kappa$, thus

$$\kappa = \frac{4\pi^2 r_*^3}{b(1.00 \times 10^8 \text{ s}^2)} = 3.94869 \times 10^{14} \text{ m}^3 \text{ s}^{-2}, \quad \frac{\sigma_\kappa}{\kappa} = \frac{\sigma_b}{b} = 7.15169 \times 10^{-4}.$$

Therefore $\sigma_\kappa = 2.82 \times 10^{11} \text{ m}^3 \text{ s}^{-2}$. (c) The dimensionless residuals $y_i - bx_i$ are

$$(-0.0097853, \quad 0.0317178, \quad 0.00579755, \quad -0.00625767),$$

giving $\chi^2 = 0.469816$ with $4 - 1 = 3$ degrees of freedom. The small residuals show compatibility with this stipulated model and uncertainty assignment, not proof of gravitation. Since uncertainties were supplied as known, they should not be automatically rescaled downward by the small reduced χ^2 . The fit still measures the product GM , not the two factors separately.

Solution P13-45. (a) For small fractional fluctuations, $\delta\kappa/\kappa \simeq 3\delta r/r - 2\delta T/T$. Covariance propagation then gives

$$u_\kappa^2 = 9u_r^2 + 4u_T^2 - 12\rho_{rT}u_r u_T = 9 \times 10^{-6} + 16 \times 10^{-6} - 18 \times 10^{-6} = 7 \times 10^{-6}.$$

Hence $u_\kappa = 0.00264575 = 0.265\%$, versus 0.500% for independent errors. A positive correlation reduces this variance because the radius and period sensitivities have opposite signs. (b) Let the same fractional calibration error c affect every radius. Every inferred parameter then contains the same first-order term $3c$, so averaging ten records leaves that common contribution unchanged at 0.0030 relative standard uncertainty. Only independent fluctuations receive the familiar averaging reduction. (c) A smaller combined uncertainty requires a covariance model, evidence for which components are independent, calibration information, and a check of model errors such as noncircular motion or drag. The correlation supplied in part (a) must not be transferred to unrelated records without evidence.

13.14 Bridge to Chapter 14

Gravitation introduced a field that varies with position and a potential energy defined from a conservative force. The next part of the book turns from point particles and orbital motion toward matter distributed continuously in space. There the same habits remain essential: define the system, identify the relevant field or stress, track geometry and units, and state when a continuum approximation replaces a microscopic description.

The main gravitational ideas remain available as useful scales. Pressure inside a fluid, buoyant forces and planetary fluid configurations all involve the interaction between distributed matter and gravitational acceleration. When the spatial region is small enough, a nearly uniform $\mathbf{g}$ may be adequate; when it is not, the inverse-square variation developed here supplies the correction.

References

- [1] D. Halliday, R. Resnick, and J. Walker, *Fundamentals of Physics*, 12th ed., Volume 1, Wiley (2022), Chapter 13, especially Sections 13.1–13.8.
- [2] R. A. Serway and J. W. Jewett, Jr., *Physics for Scientists and Engineers with Modern Physics*, 9th ed., Brooks/Cole (2014), Chapter 13, especially Sections 13.1–13.6.
- [3] S. J. Ling, J. Sanny, and W. Moebs, *University Physics, Volume 1*, OpenStax, Rice University, Chapter 13, Sections 13.1–13.7.
- [4] National Institute of Standards and Technology, *CODATA Value: Newtonian constant of gravitation*, 2022 CODATA recommended value, $G = 6.67430(15) \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$; <https://physics.nist.gov/cuu/pdf/all.pdf>.
- [5] I. Neutelings, *Gravitational force and field*, TikZ.net, https://tikz.net/dynamics_gravity/.
- [6] A. Tsagkaropoulos, *Elliptical Orbit*, TikZ.net, <https://tikz.net/elliptical-orbit/>.