

Chapter 14

Fluid Statics and Dynamics

First Edition

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Physics I

Physics for Technical and Engineering Students -
Volume I

Mechanics, Fluids, Waves, and Thermodynamics

1 January 2026

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Preface

A fluid transmits forces through pressure, transports mass through moving surfaces, and dissipates mechanical energy through viscosity. These three ideas connect a reservoir wall, a floating instrument, a hydraulic lift and a narrow flow channel. They also show why choosing the right physical model matters as much as carrying out the algebra.

This chapter develops a controlled transition from the mechanics of discrete bodies to continuum fluid mechanics. Pressure is introduced as a scalar field whose forces depend on surface orientation. Static balance then leads to pressure measurement, submerged forces and buoyancy. Flow is treated through mass conservation and a carefully qualified mechanical-energy balance. Newtonian viscosity supplies the constitutive information needed to understand a real pressure drop even when the mean speed remains constant.

The core is calculus based, with the necessary force and integral arguments explained at their first use. Applications concern technical and engineering systems, including readers in Space Technologies who may need to rebuild an undergraduate foundation. The advanced section deepens the circular-tube model through shear-force balance, numerical integration, energy dissipation and the limits of continuum wall conditions. Capillarity is a compact retained extension with its contact angle explicitly measured through the liquid.

The numerical values describe stipulated teaching models unless a passage explicitly identifies a measurement. The laboratory data are synthetic and are provided so that inference and residual analysis can be studied without pretending an experiment has been performed. Every worked example, diagnostic and end problem has a reasoned answer or an explicit reference analysis in this chapter.

The Guideline

Begin by drawing the system or control volume. Name the pressure reference, fluid properties, relevant surfaces and sign conventions. State whether the flow is steady, incompressible, inviscid or fully developed. Then select a force, mass or energy balance and test the resulting answer against those assumptions.

Pedagogical boundary

General Navier–Stokes solution methods, turbulence closure, compressible-flow thermodynamics, detailed pump design and structural certification lie outside this chapter. An ideal result is a model prediction. Its interpretation requires a separate check of the physical assumptions.

Conventions and notation

The vertical coordinate z is positive upward. Depth h is positive downward from a named free surface. Numerical calculations use $g = 9.81 \text{ m s}^{-2}$. Pressure is a scalar; arrows represent forces or velocities, not a direction possessed by pressure itself. On a selected fluid volume with outward surface normal $\hat{n}$, a static pressure contribution is $d\mathbf{F} = -p\hat{n}dA$.

Symbol	Meaning	SI unit
$p, p_{\text{abs}}, p_{\text{gauge}}, p_{\text{ref}}$	pressure; absolute, gauge and reference values	Pa
ρ	mass density	kg m^{-3}
$\dot{V}, \dot{m}$	volume and mass flow rates	$\text{m}^3 \text{ s}^{-1}; \text{kg s}^{-1}$
$v, \bar{v}, v_{\text{max}}$	local speed, section mean speed, centreline maximum	m s^{-1}
R, D, L	tube inner radius, diameter $2R$, length	m
η_v, ν	dynamic viscosity; kinematic viscosity η_v/ρ	$\text{Pa s}; \text{m}^2 \text{ s}^{-1}$
γ_s, θ_c	surface tension; contact angle through liquid	$\text{N m}^{-1}; \text{rad}$
Re, Kn	Reynolds and Knudsen numbers	dimensionless
H_E, H_G, h_L	energy head, hydraulic-grade head, head loss	m
σ_x	standard uncertainty in quantity x	unit of x

A gauge pressure is a difference, $p_{\text{gauge}} = p_{\text{abs}} - p_{\text{ref}}$. Its reference is always named. Pressure differences can be evaluated with either absolute or gauge values when the same reference is used consistently. Absolute pressure is required when the comparison concerns vacuum or a stipulated vapor-pressure limit.

The symbol v is used in local streamline equations. For a section model, $\bar{v}$ is the area average of the normal speed. If a nonuniform velocity profile matters to energy or momentum transport, an explicit profile integral replaces an unqualified substitution of the mean. The tube radial coordinate is $r \geq 0$; a signed transverse cut uses y , with $-R \leq y \leq R$.

Density, viscosity and surface tension in numerical problems are stipulated inputs. They are not precision material-property claims. Angles written in degrees are evaluated with degree-aware trigonometric functions; calculus identities use radians. An instructional drawing may be schematic even when the accompanying parameters and graph are quantitative.

Chapter 14

Fluid Statics and Dynamics

Motivation and roadmap

A two-port pressure sensor reads zero. That observation establishes equality of its port pressures, but it does not establish vacuum. A narrow tube carries steady flow with the same mean speed at both ends, yet its pressure falls. The decrease may supply viscous dissipation rather than acceleration. A sealed body floats because its displaced fluid supports its weight, while another body of the same external shape sinks because its average density is larger. These examples demand different balances, even though all involve a fluid.

The progression is from local definitions to static force balance, then to transport and model testing. First identify what is being measured or balanced. Next write the governing relation and explain the role of each term. Finally test the answer through a sign, a limit, a conservation statement or an independently evaluated integral.

Learning goals

After studying the chapter you should be able to distinguish pressure from its resultant force; calculate hydrostatic pressures, gate forces and moments with both sides included; explain hydraulic force/displacement tradeoffs and buoyancy equilibrium; conserve mass with variable area, density and velocity profile; derive and qualify Bernoulli's equation; calculate Newtonian laminar tube flow and distinguish mean from maximum speed; interpret signed capillary rise; and separate numerical convergence from physical validation in a data or computational investigation.

14.1 The continuum model and pressure

A fluid yields to a sustained shear stress: its neighbouring layers can continue to move relative to one another rather than maintain a fixed shear deformation. Both liquids and gases are fluids. This statement does not mean that a moving fluid cannot exert tangential forces; viscosity, introduced later, describes precisely such forces. An ordinary fluid at rest has no viscous shear stress, so its mechanical stress is represented by a scalar pressure. We first develop this static description and then apply conservation laws to flow.

The same molecule-based substance can require different models in different devices. Water in a laboratory tank is well represented as a nearly incompressible continuum. A dilute gas passing through a very small opening may require molecular reasoning. The decision concerns the length and time scales of the process, not simply whether the substance is called a liquid or a gas. A model should retain the physical effects that materially change the requested result and state which effects it neglects.

14.1.1 Local fields and density

Instead of following every molecule, the continuum model assigns density, pressure and velocity to each macroscopic position. A local density is an average over a small sampling volume:

$$\rho = \frac{dm}{dV}. \quad (14.1)$$

Here the differential notation represents a continuum limit, not an instruction to shrink a real sample below molecular dimensions. The sample must contain enough molecules for fluctuations to average out while remaining small compared with the scale on which the macroscopic properties change. This separation of scales permits both a smooth field and meaningful spatial variation.

For a uniform body, integration gives $m = \rho V$. For a nonuniform body the correct relation is $m = \int_V \rho dV$. The average density m/V is still useful, but it need not equal the local density at any specified point. The average density of a sealed composite object includes its interior cavities and all of its contents when its external volume is used. Confusing the density of its solid material with the density of the complete object leads to wrong floating predictions.

Density has SI units kg m^{-3} . It depends in general on pressure, temperature and composition. Calling a liquid incompressible means that density changes are negligible for the accuracy and pressure range of the chosen problem. It is an approximation, not a claim that no pressure could ever compress a liquid. Treating a gas as incompressible can also be useful for a sufficiently small pressure change, but a large change in gas pressure or temperature generally requires a density model.

14.1.2 Pressure and the force on a surface

Consider a small surface belonging to a selected volume of fluid. Let its outward unit normal be $\hat{n}$. The surrounding fluid pushes inward on that surface, so

$$d\mathbf{F} = -p\hat{n} dA. \quad (14.2)$$

The negative sign refers to the normal of the selected body. For the opposite body on the other side of the same interface, the outward normal is reversed, and the interaction force is reversed. This is Newton's third law, not a change in the pressure.

Definition 14.1: Pressure

Static pressure is the isotropic normal stress in an ordinary fluid at rest. It is a scalar field. A surface element acquires a force direction through its normal, with $d\mathbf{F} = -p\hat{n} dA$. The pascal is $1 \text{ Pa} = 1 \text{ N m}^{-2}$.

Pressure at a point is independent of the orientation of a sufficiently small sensor. One way to understand this isotropy is to imagine a tiny triangular fluid wedge. Pressure forces on its differently oriented faces must balance in the shrinking-wedge limit. Face areas scale as length squared, while its weight scales as length cubed. Weight therefore becomes negligible compared with the face forces in that limit. Resolving the face forces then gives equal pressure on the faces. A finite fluid element can still have different pressures at its top and bottom because those faces occupy different positions.

Uniform pressure can produce large individual face forces and zero total force on a closed body. For a box, opposite equal-area faces cancel pairwise. For a curved body, the equivalent cancellation is $\int_S \hat{n} dA = \mathbf{0}$. A pressure gradient, rather than a large uniform pressure alone, gives the net pressure force on a closed fluid element. Pressure nevertheless matters to deformation: a uniformly compressed vessel can experience substantial material stress even though its total translational pressure force vanishes.

Figure 14.1 separates these ideas. The arrows belong to forces on a selected fluid element; the instrument compares two pressure values. Neither representation assigns a vector direction to pressure itself.

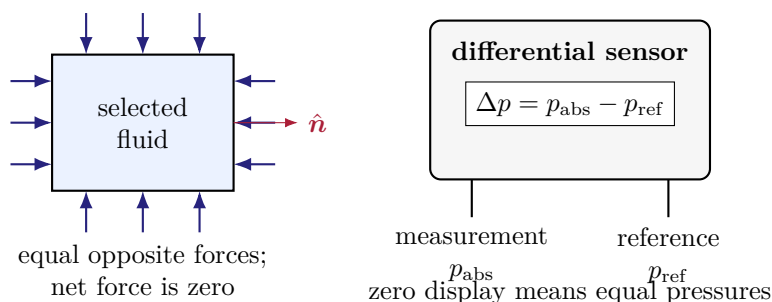


Figure 14.1: Uniform static pressure and a differential sensor. The equal opposite arrows on the selected element are qualitative force representations; the outward normal is shown separately. The instrument has two distinct ports and reports their pressure difference. A zero reading does not establish vacuum.

14.1.3 Absolute, gauge and differential readings

An absolute pressure uses vacuum as its zero. A differential pressure is the difference between two specified pressures. A gauge commonly compares a process pressure with the local atmosphere, but the actual instrument reference must be identified:

$$p_{\text{gauge}} = p_{\text{abs}} - p_{\text{ref}}. \quad (14.3)$$

The local atmospheric pressure is not always the standard atmosphere. A conventional tyre gauge usually senses pressure relative to its surroundings. A sealed-reference sensor or a process transmitter may use a different reference. The display alone does not reveal that choice.

Gauge pressure can be negative without the absolute pressure being negative. For example, 80 kPa absolute relative to 100 kPa ambient is -20 kPa gauge. A compressive absolute pressure remains positive in this example. A pump that reduces the pressure on one side of a piston does not create a special attractive pressure force: the larger pressure on the other side produces the net force.

For a flat surface with uniform pressure on each side, the net force is $(p_1 - p_2)A$, perpendicular to the surface. The absolute forces p_1A and p_2A can each be much larger than their difference. Using absolute pressure on one side and gauge pressure on the other mixes reference zeros and produces a fictitious extra force.

Example 14.1: E14-01: A referenced sensor

A differential sensor connects its measurement port to a vessel and its reference port to the local atmosphere. A separate barometer reads 101.3 kPa absolute; the differential display is +146.7 kPa. Find the vessel's absolute pressure, interpret a zero display, and find the display if the reference is changed to a sealed chamber at 85.0 kPa while the vessel pressure is unchanged. **Model and strategy.** The sensor measures a difference. Adding the named reference reconstructs the absolute pressure; the atmospheric value is supplied rather than assumed.

$$p_{\text{abs}} = p_{\text{gauge}} + p_{\text{ref}} = (146.7 + 101.3) \text{ kPa} = 248.0 \text{ kPa}.$$

A zero display means that both ports have equal pressure. With this atmospheric reference, that would mean 101.3 kPa absolute, so it would not be vacuum. After the reference changes, the reading is

$$248.0 \text{ kPa} - 85.0 \text{ kPa} = 163.0 \text{ kPa}.$$

The vessel pressure has not changed; the measured difference has. If this difference acts across

a uniform 20.0 cm^2 diaphragm, its net force magnitude is $163000(20.0 \times 10^{-4}) = 326 \text{ N}$, directed from the higher-pressure side toward the lower-pressure side. The area conversion is essential: $1 \text{ cm}^2 = 10^{-4} \text{ m}^2$.

Check. Adding 85.0 kPa to the new display recovers 248.0 kPa . The positive sign is consistent with the measurement port having the larger pressure. A sensor calibrated only for a smaller range could not be assumed to report this value faithfully; the calculation describes the ideal measurement relation.

Guided checks

Diagnostic X14-01. A student concludes that a zero gauge reading means no pressure. The repair is to name the reference: zero means equal pressures at the two ports. Only a reference known to be vacuum converts that equality into a zero absolute pressure. For an atmospheric reference, zero gauge pressure means atmospheric absolute pressure.

Caution

The shortcut $F = pA$ applies to uniform pressure on a plane surface. A nonuniform pressure requires integration; a curved surface also requires its changing normal. Always select the body first, then account for the pressure on every relevant side.

14.2 Hydrostatics and pressure measurement

Hydrostatics is mechanical equilibrium of a fluid in a specified body-force field. In the laboratory, gravity is usually nearly uniform and points downward. Resting does not mean that forces are absent: the pressure field supports the fluid's weight. The balance is local, so it can be applied without knowing the entire shape of a vessel.

14.2.1 Deriving the pressure gradient

Choose z upward and take a thin horizontal fluid slice with face area A and thickness dz . Its bottom and top lie at z and $z + dz$. In Figure 14.2, bottom pressure pushes upward, top pressure pushes downward, and weight points downward. Horizontal face forces cancel because the pressure has no horizontal gradient in this simple model. Vertical equilibrium is

$$p(z)A - p(z + dz)A - \rho g A dz = 0.$$

After division by $A dz$ and passage to the continuum limit,

$$\frac{dp}{dz} = -\rho g. \quad (14.4)$$

The units verify the relation: $\text{Pa/m} = \text{kg m}^{-2}\text{s}^{-2}$, the same as density times acceleration. The negative derivative means that pressure decreases on moving upward. If depth $h = z_0 - z$ is measured downward from a free surface, the derivative instead is $dp/dh = \rho g$.

For constant density and locally constant g , integrating between a known level z_0 and a second level z gives

$$p(z) = p_0 + \rho g(z_0 - z) = p_0 + \rho gh. \quad (14.5)$$

At a liquid free surface exposed to a uniform atmosphere, p_0 is approximately the local atmospheric pressure. A sealed tank can instead have a specified gas pressure above the liquid. This changes the pressure intercept, while the same liquid density determines the slope. Equal-depth pressure does not require equal liquid volume or equal vessel width.

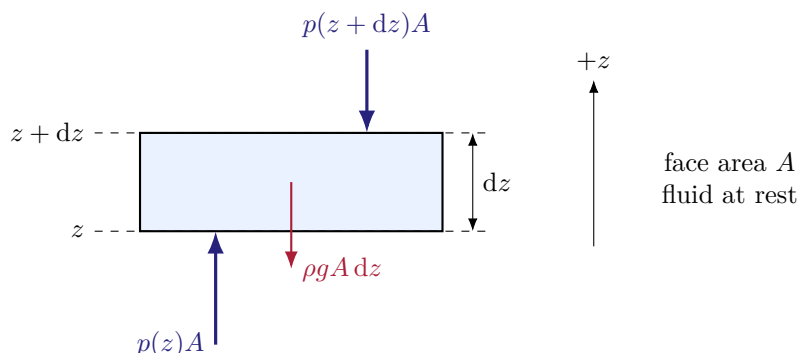


Figure 14.2: Force balance on a thin horizontal fluid slice. The lower face has the higher pressure; the small difference of the two pressure forces supports the weight. Arrow lengths illustrate their ordering and are not a numerical force scale.

The general relation retains variations of density or gravity:

$$p(z_2) - p(z_1) = - \int_{z_1}^{z_2} \rho(z)g(z) dz. \quad (14.6)$$

An atmospheric column cannot normally be extrapolated to arbitrary height using a single ground-level air density. Likewise, a deep liquid column may require compressibility if the requested accuracy is high enough. The constant-density law should be justified against the pressure range, not used as an identity for all fluids.

14.2.2 Connected vessels and layered liquids

At equal elevation within the same connected static liquid, pressure is equal when gravity is the only relevant body force. A horizontal pressure difference would accelerate the liquid and contradict static equilibrium. The connected-liquid restriction matters: separated vessels with different gas pressures can have different liquid pressures at the same elevation. At a broad flat liquid–liquid interface, pressure is continuous when capillary curvature effects are negligible. The pressure slope changes when density changes; pressure itself does not jump in this approximation.

Layered liquids are therefore handled one segment at a time. For upper density ρ_1 , thickness H_1 , lower density ρ_2 and depth h measured downward from the upper free surface,

$$p(h) - p_0 = \begin{cases} \rho_1 g h, & 0 \leq h \leq H_1, \\ \rho_1 g H_1 + \rho_2 g (h - H_1), & h > H_1. \end{cases} \quad (14.7)$$

A stable simple layering has the denser liquid below. Immiscibility and a sufficiently broad interface are part of this model. Figure 14.3 shows how to read the slopes and the continuous interface value rather than memorize a new formula for each arrangement.

Example 14.2: E14-S01: Reading a layered column

A vessel open to 100.0 kPa atmosphere contains 0.60 m of a liquid of density 800 kg m^{-3} above 0.90 m of water of density 1000 kg m^{-3} . Find the interface and bottom pressures, the force difference across a horizontal 0.0200 m^2 bottom cover exposed to atmosphere outside, and the lower-layer density inferred if a bottom sensor instead reports 13.80 kPa gauge.

System and assumptions. Both liquids are static, immiscible, and approximately incompressible. Gravity is 9.81 m s^{-2} . The cover is at a single depth. Its inside pressure is uniform, and its outside reference is the stated atmosphere.

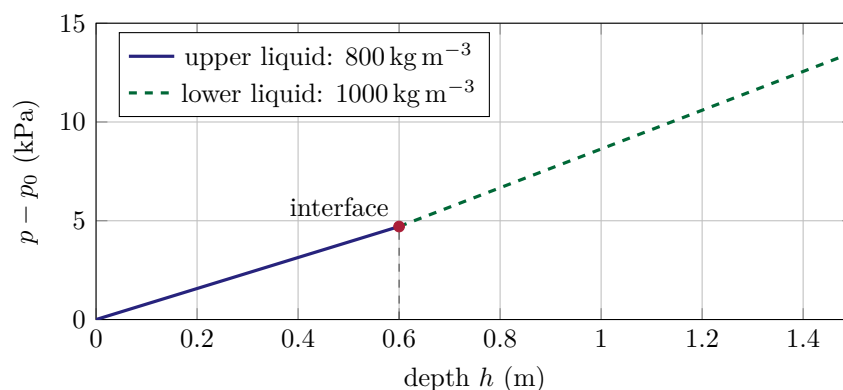


Figure 14.3: A continuous pressure-depth curve for two stipulated liquid layers. The interface lies at 0.60 m. Its pressure is the weight per area of the upper column; the steeper lower segment reflects the larger density. The horizontal axis is depth, not vertical position.

Forward calculation. Add the hydrostatic increments, beginning at the known free surface:

$$\begin{aligned} p_{\text{int}} - p_0 &= (800)(9.81)(0.60) = 4708.8 \text{ Pa}, \\ p_{\text{bottom}} - p_0 &= 4708.8 + (1000)(9.81)(0.90) = 13537.8 \text{ Pa}. \end{aligned}$$

Thus $p_{\text{int}} = 104.709 \text{ kPa}$ and $p_{\text{bottom}} = 113.538 \text{ kPa}$ absolute. The net bottom-cover force magnitude is

$$F = (13537.8)(0.0200) = 270.756 \text{ N} \simeq 271 \text{ N},$$

directed downward on the horizontal cover. Using the absolute inside pressure alone would incorrectly include the opposing atmospheric force.

Inverse calculation. Keep the upper-layer contribution and solve for the new lower-layer density:

$$\rho_2 = \frac{13800 - 4708.8}{(9.81)(0.90)} = 1029.70 \text{ kg m}^{-3}.$$

The sensor reading does not justify saying that the original stipulated density was wrong: it defines a different inferred case under the same model. Figure 14.3 corresponds to the forward case.

Checks. Each segment contributes positively downward. The denser lower liquid gives the larger slope. A relative 1% error in the small pressure difference assigned to a layer can produce a comparable density error; uncertainty in the upper contribution must also be included in a careful measurement.

The so-called hydrostatic paradox follows from this local law. Three differently shaped open vessels with the same bottom area, same liquid density and same depth have the same gauge force on their horizontal bottoms. They can contain different liquid weights. There is no force-balance contradiction: sloping side walls exert vertical force components on the fluid. The bottom force need equal the total fluid weight only when the other boundaries supply no net vertical component. This is another reason to select the complete system before comparing forces.

14.2.3 Manometers as pressure paths

A manometer converts a pressure difference into a hydrostatic height difference. The reliable method is to trace a path through the fluid, adding $\rho g \Delta h$ on descending and subtracting it on ascending. Equal-height points have equal pressure only when they belong to the same connected static fluid. The drawing must show an actual connected lower bend; two disconnected vertical columns do not

form a U-tube.

Figure 14.4 shows a gas-pressure measurement using a heavy manometer liquid. The left gas pressure depresses that liquid on the left. With negligible gas-density correction, pressures immediately above both liquid surfaces equal their port pressures. The right interface lies Δh higher than the left; descending through the heavy liquid from right to left gives

$$p_A - p_B = \rho_m g \Delta h. \quad (14.8)$$

If gas density cannot be neglected, the gas-column pressure changes must be included separately. Likewise, a differential manometer between liquid-filled pipes includes the density of the pipe liquid; the heavy liquid alone is not the whole pressure path.

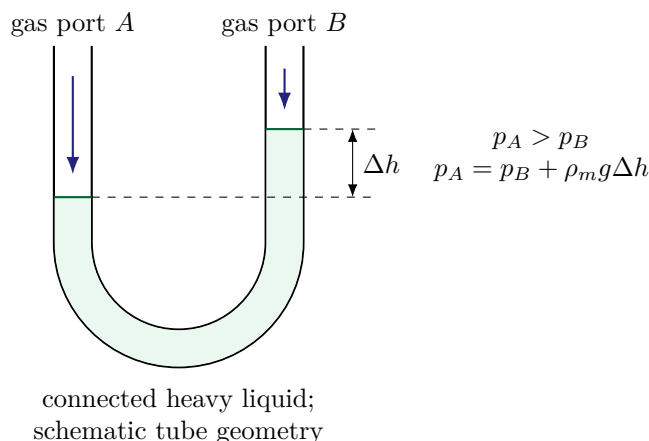


Figure 14.4: A true U-tube with a connected lower bend. The higher gas pressure at A depresses the left liquid level. The measured quantity is the vertical difference of the two interfaces, not the liquid length along the tube. Gas-column corrections are neglected here.

Example 14.3: E14-S02: A differential U-tube

Two water-filled pressure taps A and B are at the same elevation. A U-tube connects them through a heavy liquid of stipulated density 13600 kg m^{-3} . The heavy-liquid interface in the B limb is 0.180 m higher than in the A limb. Find $p_A - p_B$. Explain why $\rho_m g \Delta h$ alone is not correct in this water-filled case.

Strategy. Trace a hydrostatic path from one tap to the other, retaining both liquids. Let d_A and d_B be the depths of the interfaces below their equal-height taps. Geometry gives $d_A - d_B = \Delta h$.

At the left interface $p_L = p_A + \rho_w g d_A$. At the right interface $p_R = p_B + \rho_w g d_B$. The left interface is lower in the connected heavy liquid, so $p_L = p_R + \rho_m g \Delta h$. Combining these relations,

$$p_A - p_B = \rho_m g \Delta h - \rho_w g (d_A - d_B) = (\rho_m - \rho_w) g \Delta h.$$

For $\rho_w = 1000 \text{ kg m}^{-3}$,

$$p_A - p_B = (13600 - 1000)(9.81)(0.180) = 22249.08 \text{ Pa} = 22.25 \text{ kPa}.$$

The gas approximation would give 24.01 kPa and overestimate the difference by 1.77 kPa . It ignores the pressure gained on descending through the extra water length on the A side.

Sign and limit checks. The left interface is depressed, consistent with $p_A > p_B$. If the two densities were equal, the device would contain one continuous liquid and equal-height taps would have equal pressure, regardless of an arbitrarily named internal interface. The factor $\rho_m - \rho_w$ correctly vanishes in that limit.

14.2.4 Forces on submerged plane surfaces

For a plane gate, all pressure-force elements are parallel, so a scalar force integral is sufficient after the net pressure has been established:

$$F = \int_A \Delta p \, dA. \quad (14.9)$$

Here Δp is the pressure difference across the gate. A positive convention points from side 1 to side 2. If it varies in sign, both the force and moment must retain those signs. Common atmospheric pressure cancels only when it actually acts on both sides. Sealed gas pressure, unequal liquid levels, or a vacuum on one side must be included.

Take a vertical rectangular gate of width b and height H . Its top is h_0 below a liquid free surface. Let y run downward from the top, so a thin horizontal strip has area $b \, dy$. With atmosphere on both the free surface and the opposite gate face,

$$\Delta p(y) = \rho g(h_0 + y).$$

The resultant force and its moment about the top edge are

$$F = \rho g b \int_0^H (h_0 + y) \, dy = \rho g b \left(h_0 H + \frac{H^2}{2} \right), \quad (14.10)$$

$$M_{\text{top}} = \rho g b \int_0^H (h_0 + y)y \, dy = \rho g b \left(\frac{h_0 H^2}{2} + \frac{H^3}{3} \right). \quad (14.11)$$

The factor y in the second integral is the lever arm. This is why calculating total force alone cannot locate its line of action. The center of pressure is

$$y_{\text{cp}} = \frac{M_{\text{top}}}{F} = \frac{H}{2} + \frac{H^2}{12(h_0 + H/2)}. \quad (14.12)$$

The additional term is positive. Lower strips carry a larger fraction of the load than upper strips, so the resultant lies below the geometric centroid.

Figure 14.5 now gives the gate its own uncrowded representation. The pressure arrows grow with depth, while the resultant is placed at the moment-weighted depth. The companion manometer is separately resolved in Figure 14.4.

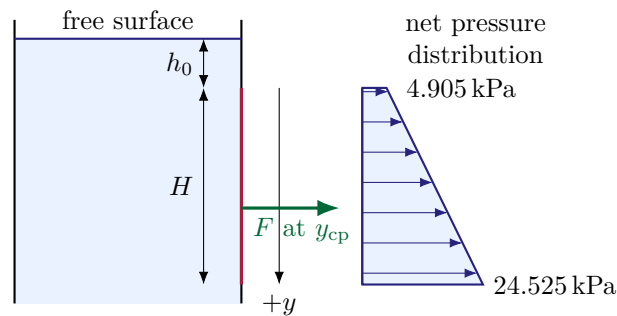


Figure 14.5: The vertical gate of Example E14-02 and its net pressure profile. The top lies 0.50 m below the surface and the gate height is 2.00 m. Geometry and profile endpoints are computed from those inputs; width into the page is 1.20 m. The green resultant acts 1.222 m below the top.

Example 14.4: E14-02: A submerged gate

A vertical rectangular gate is 1.20 m wide and 2.00 m high. Its top edge is 0.50 m below the surface of a liquid of density 1000 kg m^{-3} . Both the liquid surface and the other side of the gate are at the same atmospheric pressure. Find the net force, its moment about the top, and its line of action.

System and strategy. Select the gate. Subtract the two surface pressures before integrating. A horizontal strip at coordinate y carries $\rho g(h_0 + y)b dy$. The moment requires the same strip force multiplied by its lever arm y .

$$F = (1000)(9.81)(1.20) \left[(0.50)(2.00) + \frac{2.00^2}{2} \right] = 35316 \text{ N} = 35.316 \text{ kN},$$

$$M_{\text{top}} = (1000)(9.81)(1.20) \left[\frac{(0.50)(2.00)^2}{2} + \frac{2.00^3}{3} \right] = 43164 \text{ N m}.$$

Consequently,

$$y_{\text{cp}} = \frac{43164}{35316} = 1.2222 \text{ m},$$

or 1.7222 m below the liquid surface. The force points horizontally from the liquid toward the atmospheric side.

Independent force check. A linear profile has its average equal to its value at the centroid. The mean gauge pressure is $\rho g(0.50 + 1.00) = 14715 \text{ Pa}$. Multiplying by area 2.40 m^2 gives 35316 N again. This average-pressure shortcut establishes the force but does not establish the center of pressure.

Interpretation. The centroid is 1.00 m below the top. The deeper center of pressure is the correct point for replacing the distributed load by a single force in a rigid-body equilibrium calculation. A hinge or support must also supply the necessary force and torque balance; the hydrostatic calculation alone does not determine every support reaction.

Guided checks

Diagnostic X14-02. Using centroid pressure times area gives the correct force for this plane linear-pressure case. Placing that force at the centroid gives the wrong moment. Force matching and moment matching are separate requirements, and the lower strips have the larger moment-weighted contribution.

14.2.5 Differential pressure on both sides

Both-side loading can change not only the magnitude but also the direction and position of the resultant. If the net profile on a rectangular gate is linear, write

$$\Delta p(y) = c + ky, \quad F = b \left(cH + \frac{kH^2}{2} \right), \quad M_{\text{top}} = b \left(\frac{cH^2}{2} + \frac{kH^3}{3} \right). \quad (14.13)$$

The constant c includes top-depth and gas-pressure differences, while k includes the difference of density slopes. One side's absolute hydrostatic law cannot be substituted for Δp . A uniform extra pressure changes force and moment, pulling the center of pressure toward the centroid as that uniform term dominates.

If Δp changes sign, the separate positive and negative portions form opposing loads. Their resultant need not lie within the gate. If $F = 0$ while $M \neq 0$, they produce a pure couple, and there is no finite center-of-pressure coordinate. Such a result is physically meaningful; dividing by zero or forcing the line of action onto the panel would hide the load structure.

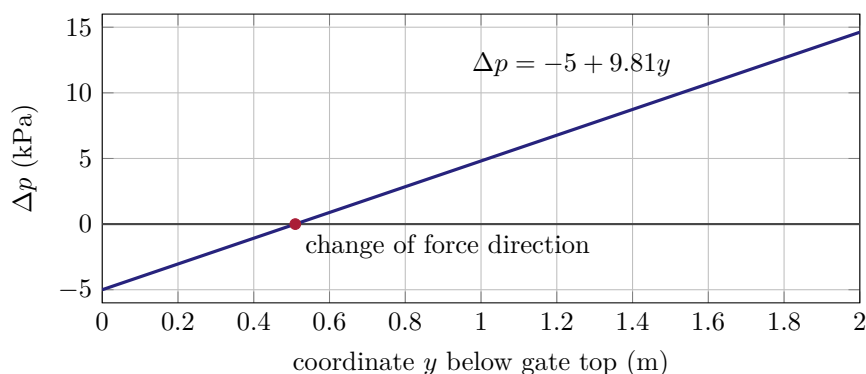


Figure 14.6: Signed pressure difference for Example E14-S03. The dry-side chamber adds a uniform 5.00 kPa opposing pressure. Above $y = 0.510$ m the net force reverses; below it the hydrostatic contribution wins. Force is the signed area times width, and moment is its depth-weighted integral.

Example 14.5: E14-S03: Opposing chamber pressure

A vertical gate of width $b = 1.20$ m and height $H = 2.00$ m has its top at the water surface. Water density is 1000 kg m^{-3} . Water-surface gas pressure is 100.0 kPa absolute. The dry chamber on the other side is uniformly at 105.0 kPa . Find the signed net force, its moment about the top, and the resultant location. Take positive force from water toward the dry chamber.

Pressure model. At depth y , water pressure is $100000 + 9810y$ in pascals. Subtracting the opposing 105000 Pa gives $\Delta p = -5000 + 9810y$. Figure 14.6 shows why the upper strip forces oppose the lower ones.

Force and moment. Using signed strip contributions,

$$F = 1.20 \left[(-5000)(2.00) + \frac{9810(2.00)^2}{2} \right] = 11544 \text{ N},$$

$$M_{\text{top}} = 1.20 \left[\frac{(-5000)(2.00)^2}{2} + \frac{9810(2.00)^3}{3} \right] = 19392 \text{ N m}.$$

The resultant points toward the dry chamber and acts at

$$y_{\text{cp}} = \frac{19392}{11544} = 1.6798 \text{ m}$$

below the top. The negative top load shifts it deeper than the $2H/3 = 1.333 \text{ m}$ location for water against equal gas pressure.

Checks and limiting case. The profile changes sign at $y = 5000/9810 = 0.5097 \text{ m}$. The mean differential pressure is 4810 Pa , giving $4810(2.40) = 11544 \text{ N}$. If the opposing gas-pressure excess were 9810 Pa , the mean net pressure would vanish: $F = 0$, while $M = 7848 \text{ N m}$ would remain. That loading is a couple, so no finite single-force location could reproduce it.

The location in the simplest one-sided case has useful limiting behaviour. At $h_0 = 0$, the triangular pressure profile gives $y_{\text{cp}} = 2H/3$. At $h_0 \gg H$, the pressure varies little relative to its large common value, and $y_{\text{cp}} \rightarrow H/2$. Figure 14.7 makes this approach visible without changing the gate geometry.

14.2.6 Inclined gates and vertical depth

Hydrostatic pressure depends on vertical depth, not on distance along a sloping surface. For a rectangular gate of length L down an incline at angle θ to the horizontal, let s measure distance down the gate from its top. A point has vertical depth $h = h_0 + s \sin \theta$; a strip has area $b ds$. The

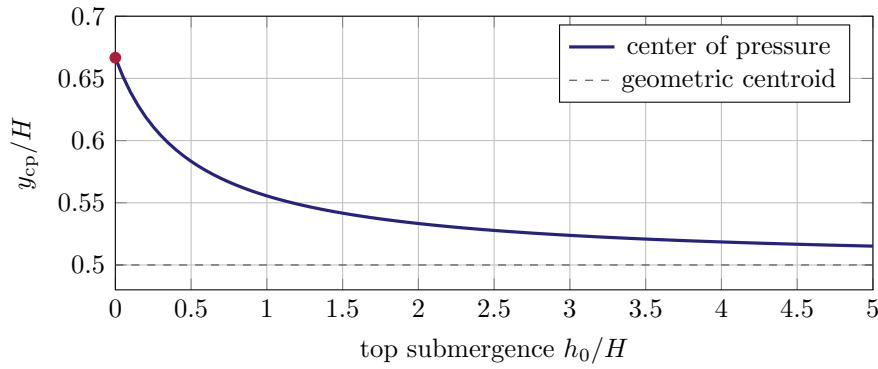


Figure 14.7: Center of pressure for a vertical rectangular gate against the same atmospheric reference. The curve is dimensionless and quantitative. Increasing top submergence makes the load more nearly uniform relative to its mean; its line of action approaches the centroid from below.

force elements remain normal to the gate. Their moment lever arm about the top edge is s , because force and gate direction are perpendicular:

$$F = \rho g b \left(h_0 L + \frac{L^2 \sin \theta}{2} \right), \quad (14.14)$$

$$M_{\text{top}} = \rho g b \left(\frac{h_0 L^2}{2} + \frac{L^3 \sin \theta}{3} \right). \quad (14.15)$$

These relations reduce to the vertical ones at $\theta = 90^\circ$. For a horizontal gate, $\theta = 0$, every strip has the same pressure and the resultant acts at its centroid. Figure 14.8 shows why replacing s by depth without the sine factor would overestimate the pressure slope.

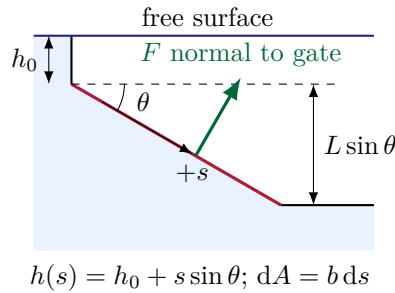


Figure 14.8: Pressure on an inclined plane gate separating liquid at lower left from atmosphere at upper right. The illustrated geometry uses $L = 2.00$ m, $h_0 = 0.40$ m and $\theta = 30^\circ$. The resultant is normal to the gate. Its shown outward direction corresponds to liquid lying on the lower-left side of the gate surface in this schematic.

Example 14.6: E14-S07: An inclined access panel

A rectangular submerged panel has width $b = 1.00$ m, length $L = 2.00$ m along its slope, and angle 30.0° to the horizontal. Its top is $h_0 = 0.40$ m below the free surface. Water density is 1000 kg m^{-3} , and equal atmosphere acts on both sides. Find the force magnitude and its location measured along the panel from the top.

Model and method. The panel is plane; pressure-force elements are parallel to its normal. Depth is $0.40 + s \sin 30^\circ$, not $0.40 + s$. Integrate force and moment using the slope coordinate:

$$F = 9810(1.00) \left[(0.40)(2.00) + \frac{(2.00)^2(0.5)}{2} \right] = 17658 \text{ N},$$

$$M_{\text{top}} = 9810(1.00) \left[\frac{(0.40)(2.00)^2}{2} + \frac{(2.00)^3(0.5)}{3} \right] = 20928 \text{ N m}.$$

Thus $s_{\text{cp}} = 20928/17658 = 1.1852 \text{ m}$. Its vertical depth is $0.40 + (1.1852)(0.5) = 0.9926 \text{ m}$. The centroid is at slope coordinate 1.00 m and depth 0.90 m.

Check. Centroid pressure $9810(0.90) = 8829 \text{ Pa}$ times area 2.00 m^2 gives 17658 N. The line of action lies deeper than the centroid as expected. The force magnitude is not a horizontal force magnitude; horizontal and vertical components require resolving the gate normal after the resultant has been found.

14.3 Pascal's principle and buoyancy

14.3.1 Pressure increments in a hydraulic system

An externally imposed pressure increment in a confined static fluid is transmitted throughout the connected fluid in the ideal incompressible model. This is Pascal's principle. It concerns an increment superposed on the hydrostatic field. It does not assert that total pressure is identical at different elevations: the background pressure gradient remains $dp/dz = -\rho g$.

For two frictionless pistons at the same elevation, with the same external atmospheric reference, negligible piston weights and face areas A_1 and A_2 , the load-induced pressure increment gives

$$\frac{F_1}{A_1} = \frac{F_2}{A_2} = \Delta p. \quad (14.16)$$

The forces in this equation are the corresponding net piston loads. If piston masses are not negligible, their weights belong in the piston force balances. If the external reference pressures differ, those differences also belong there. Naming the forces avoids silently treating an actuator load as the entire fluid force on a piston.

Incompressibility and a sealed fluid volume relate the piston displacements. For input motion inward by positive Δx_1 and output motion outward by positive Δx_2 ,

$$A_1 \Delta x_1 = A_2 \Delta x_2. \quad (14.17)$$

The products are volumes. Combining the force and displacement relations gives $F_1 \Delta x_1 = F_2 \Delta x_2$. A large output force comes with a correspondingly smaller output displacement. The hydraulic device conserves work in this ideal quasistatic equal-height limit; it does not supply free energy.

Figure 14.9 corrects the crucial geometric condition: the two active piston faces are at equal elevation. Their visible widths merely distinguish the piston sizes; the stated physical areas, rather than widths in a two-dimensional drawing, determine the ratio.

Real hydraulic systems lose mechanical energy through friction and fluid resistance. They also store energy through elastic deformation of hoses, compression of fluid and trapped gas, and motion of the load. The ideal static ratio remains a useful first model, but it should not be interpreted as an efficiency claim for a complete operating machine.

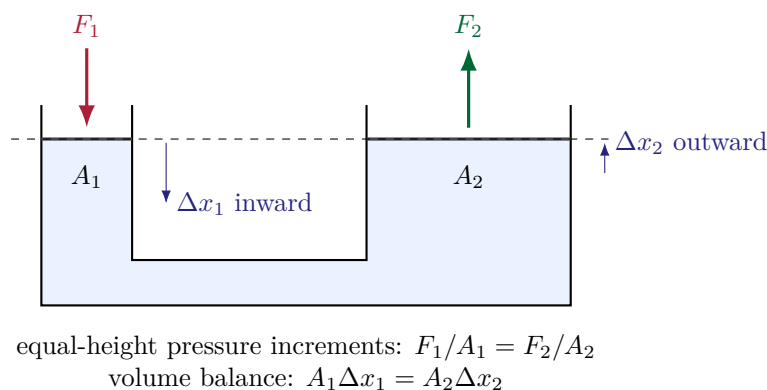


Figure 14.9: An ideal equal-elevation hydraulic pair. Piston face elevations are equal in the represented reference state; physical cross-sectional areas are specified in the calculation. Force arrows indicate directions and do not encode the force ratio. The floating-body content of the original combined figure is developed separately below.

Example 14.7: E14-03: Lift and floating block

Part A. An ideal equal-elevation hydraulic pair has $A_1 = 6.0 \text{ cm}^2$ and $A_2 = 240 \text{ cm}^2$. An input force of 180 N moves the small piston inward by 0.40 m. Find the output force, outward displacement and input/output work.

Model and reasoning. Friction, leakage, piston weights and density changes are neglected. The area ratio is 40, so pressure equality and volume balance give

$$F_2 = 40(180) = 7200 \text{ N} = 7.20 \text{ kN}, \quad \Delta x_2 = \frac{0.40}{40} = 0.010 \text{ m}.$$

The pressure increment is $180/(6.0 \times 10^{-4}) = 300 \text{ kPa}$. Input work is $180(0.40) = 72 \text{ J}$; output work is $7200(0.010) = 72 \text{ J}$. Equal work is the check that force multiplication is paid for by reduced displacement. These relations describe the ideal equal-height approximation; a significant height difference must be treated explicitly.

Part B. A sealed composite object has mass $m = 2.8 \text{ kg}$ and external volume $V = 4.0 \times 10^{-3} \text{ m}^3$. It floats in a liquid of stipulated density 1000 kg m^{-3} . Neglect capillary forces. Find its displaced volume and submerged fraction. Determine whether it can still float partially if its mass becomes 4.8 kg with the same external volume.

Force balance. For floating equilibrium, buoyancy equals weight. Therefore

$$V_{\text{displaced}} = \frac{m}{\rho_f} = 2.8 \times 10^{-3} \text{ m}^3, \quad \frac{V_{\text{displaced}}}{V} = 0.70.$$

The average object density is 700 kg m^{-3} , less than the liquid density, so partial floating is possible. In the heavier case, the required displaced volume is $4.8 \times 10^{-3} \text{ m}^3$, larger than the available external volume. Full immersion supplies only 39.24 N buoyancy, whereas the heavier weight is 47.09 N. Without another upward force, it cannot float.

Interpretation. The 70% result is a volume fraction. It is a height fraction only for a body with constant horizontal cross-sectional area. The heavier calculation fails the geometric feasibility check; it is not a meaningful “120% submerged” floating state.

14.3.2 Hydraulic pistons at different heights

Suppose the output face is above the input face by $\Delta z = z_2 - z_1$. Hydrostatic balance gives $p_2 = p_1 - \rho g \Delta z$. With equal outside pressure and net loads F_i , the static relation is

$$\frac{F_2}{A_2} = \frac{F_1}{A_1} - \rho g \Delta z. \quad (14.18)$$

The output pressure is smaller at the higher face. Pascal's principle still transmits an imposed increment throughout the fluid; it does not remove this elevation correction. During finite piston movement, the face elevations and hence Δz change, so a constant-force calculation must specify whether it is instantaneous or an approximation over a small stroke.

For an infinitesimal quasistatic motion transferring volume $dV = A_1 dx_1 = A_2 dx_2$, the pressure work difference is

$$F_1 dx_1 - F_2 dx_2 = (p_1 - p_2) dV = \rho g \Delta z dV.$$

This is the increase of the liquid's gravitational potential energy when volume is removed at level z_1 and added at level z_2 . The work difference does not indicate a failure of energy conservation; it identifies the stored energy omitted by a naive equal-height calculation.

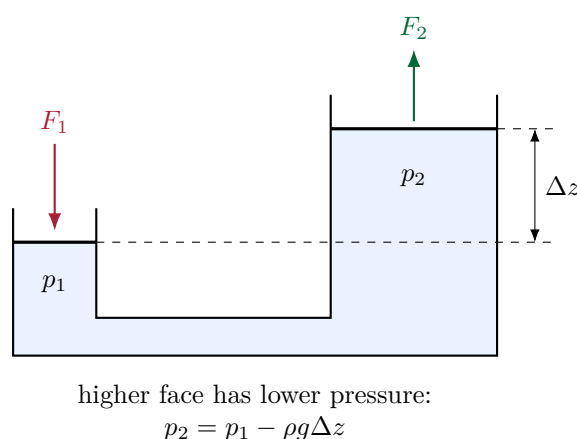


Figure 14.10: A hydraulic pair with the output piston above the input piston. The elevation difference is explicit. The piston widths and force-arrow lengths are schematic; the physical areas and pressure correction are supplied by the model.

Example 14.8: E14-S04: Elevation and hydraulic work

An ideal hydraulic device contains liquid of density 900 kg m^{-3} . At a reference instant, its output face is 1.50 m above its input face. Their areas are $A_1 = 5.00 \text{ cm}^2$ and $A_2 = 150 \text{ cm}^2$. Both outer faces are at the same atmospheric pressure, and piston weights are neglected. An input load of 150 N acts inward. Find the supported output load and account for the work in a small stroke transferring $1.00 \times 10^{-5} \text{ m}^3$.

Pressure balance. Input gauge pressure is $150 / (5.00 \times 10^{-4}) = 300000 \text{ Pa}$. The hydrostatic correction is $(900)(9.81)(1.50) = 13243.5 \text{ Pa}$. Hence

$$F_2 = (150 \times 10^{-4})(300000 - 13243.5) = 4301.35 \text{ N}.$$

Equal-height force multiplication would give 4500 N and overestimate the supported load by 198.65 N .

Small-stroke energy. Volume balance gives $dx_1 = 0.0200 \text{ m}$ inward and $dx_2 = 6.667 \times 10^{-4} \text{ m}$ outward. Evaluating work to first order at the reference pressures,

$$W_{\text{in}} \simeq 150(0.0200) = 3.000 \text{ J}, \quad W_{\text{out}} \simeq F_2 \frac{dV}{A_2} \simeq 2.86757 \text{ J}.$$

The difference 0.13244 J equals $\rho g \Delta z dV$, the first-order increase of the liquid's potential energy. **Finite-stroke check.** The face separation grows by $dV(1/A_1 + 1/A_2) = 0.020667 \text{ m}$ during this stroke. If the input force stays constant, integrating the changing pressure difference adds the second-order energy

$$\frac{1}{2} \rho g \left(\frac{1}{A_1} + \frac{1}{A_2} \right) (dV)^2 = 0.000912 \text{ J}.$$

The exact output work within this incompressible quasistatic model is therefore 2.86665 J. The difference from the first-order estimate is small because the stroke changes the elevation separation by only about 1.4%. For a large stroke, the varying geometry must be integrated rather than hidden in a constant-force approximation.

14.3.3 Buoyancy from a simple block

Consider a fully submerged rectangular body with horizontal top and bottom area A , vertical height H , and top at depth h . The fluid is at rest, uniform in density, and extends over all of the body's surface. Side forces cancel horizontally. Bottom pressure is $p_0 + \rho_f g(h + H)$, while top pressure is $p_0 + \rho_f g h$. Their vertical difference is

$$F_B = [p_{\text{bottom}} - p_{\text{top}}] A = \rho_f g H A = \rho_f g V.$$

The uniform pressure term cancels. Moving the body deeper increases both face pressures equally and leaves this net force unchanged in the constant-density model. Figure 14.11 separates the pressure forces from the object's weight; its motion depends on their resultant together with any support forces.

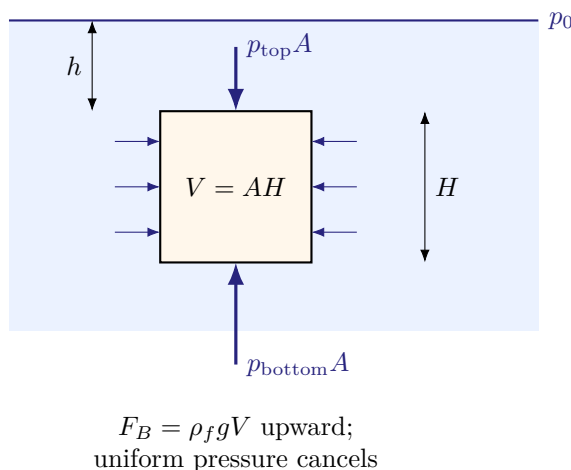


Figure 14.11: Hydrostatic origin of buoyancy for a rectangular immersed body. Opposite horizontal pressure forces cancel. The bottom pressure exceeds the top by $\rho_f g H$, giving an upward net force equal to the displaced fluid's weight. Individual arrows are qualitative and do not include the object's weight.

14.3.4 Why the result holds for arbitrary shape

The block calculation reveals the mechanism but does not restrict buoyancy to boxes. Imagine replacing an arbitrarily shaped immersed body with fluid occupying exactly the displaced region. That imaginary fluid is in equilibrium under its weight and the pressure forces exerted by the surrounding fluid. The surrounding pressure field at its boundary is the same as the pressure field around the actual body. Therefore its net pressure force is upward and equals the weight of that

displaced fluid. This replacement argument derives Archimedes' principle without requiring the body to have flat faces.

The same result follows directly from a surface integral. Let $\hat{n}$ point outward from the displaced region. The total hydrostatic force is

$$\mathbf{F}_B = - \int_S p \hat{n} dA.$$

Applying the divergence theorem component by component turns this into $-\int_V \nabla p dV$. Static balance is $\nabla p = \rho_f \mathbf{g}$, where $\mathbf{g} = -g\hat{z}$. Thus

$$\mathbf{F}_B = - \int_{V_{\text{displaced}}} \rho_f \mathbf{g} dV, \quad F_B = \rho_f g V_{\text{displaced}} \quad \text{for uniform density and gravity.} \quad (14.19)$$

This advanced-looking integral simply adds the forces on all boundary patches; the replacement argument supplies the same reasoning at the core level. If fluid density varies with position, the integral gives the weight of the actual displaced density distribution, rather than a single arbitrarily chosen density times volume.

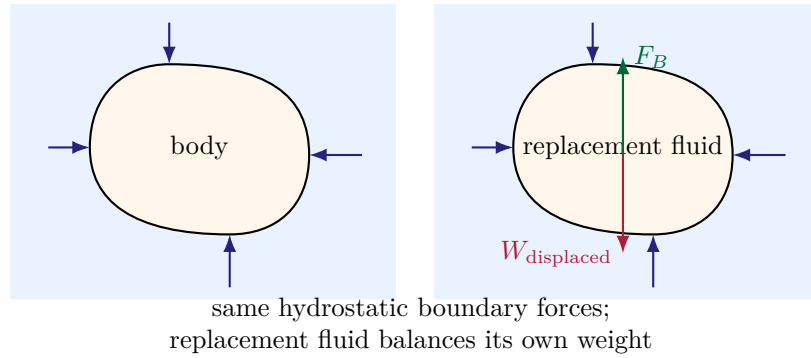


Figure 14.12: The replacement-fluid argument for an arbitrary submerged shape. The surrounding fluid applies the same boundary pressure to the actual body and the imagined displaced fluid. The imagined fluid is static, so its net pressure force balances its weight. Schematic geometry and arrow magnitudes are used.

For a partially immersed body, it is convenient to use pressures relative to the surrounding atmosphere. The exposed part then has approximately zero gauge pressure, while the wetted part supplies the hydrostatic contribution. The result is again the weight of the liquid displaced below its free surface, provided atmospheric density and surface-tension contributions are negligible. Neglecting air buoyancy is often excellent for dense objects in water, but it should be reconsidered when weighing low-density objects precisely.

A crucial boundary condition is fluid access. If an object seals against a tank floor, liquid may not act on its underside. The complete closed-surface pressure argument no longer applies unchanged, and suction or contact forces can matter. Saying that every object touching a liquid must feel $\rho_f g V$ upward ignores the missing lower-face pressure. Archimedes' formula requires the hydrostatic pressure forces represented by the displaced-fluid construction.

14.3.5 Floating, sinking and constrained immersion

Buoyancy is a force, not a statement that an object is in equilibrium. For an unsupported object at rest at a free surface, vertical force balance gives

$$\rho_f g V_{\text{displaced}} = mg, \quad \frac{V_{\text{displaced}}}{V_{\text{object}}} = \frac{\rho_{\text{object,avg}}}{\rho_f}. \quad (14.20)$$

Partial floating therefore requires $\rho_{\text{object,avg}} < \rho_f$ under these assumptions. A ship floats because its average density, including enclosed spaces, can be smaller than that of water; the density of the steel alone does not determine the vessel's average density.

A fully submerged rigid body in a uniform liquid has constant buoyancy as it changes depth, because its displaced volume and fluid density stay constant. If its weight exceeds buoyancy, it accelerates downward until other forces or boundaries intervene. If buoyancy exceeds weight, it accelerates upward until it reaches a different immersion state or another force balances it. Equality gives neutral translational buoyancy; it does not by itself establish rotational stability.

For a rigid upright body with constant horizontal area A , submerged volume is Ad , where d is draft. Adding mass Δm gives $\Delta d = \Delta m / (\rho_f A)$ until the geometry or a limiting freeboard condition changes. For a tapered body, the submerged volume must instead be integrated over its varying area. Shape affects draft, the position of the center of buoyancy, and rotational behaviour even though the force for a specified displaced volume is always given by Archimedes' relation.

The buoyant resultant acts through the centroid of the displaced fluid volume in uniform gravity and density, called the center of buoyancy. Weight acts through the object's center of mass. Their lines of action can create a torque when the body tilts. In the upright force balance in Figure 14.13, these lines coincide. Equal-length opposite force arrows indicate the actual floating equilibrium, rather than suggesting a net acceleration.

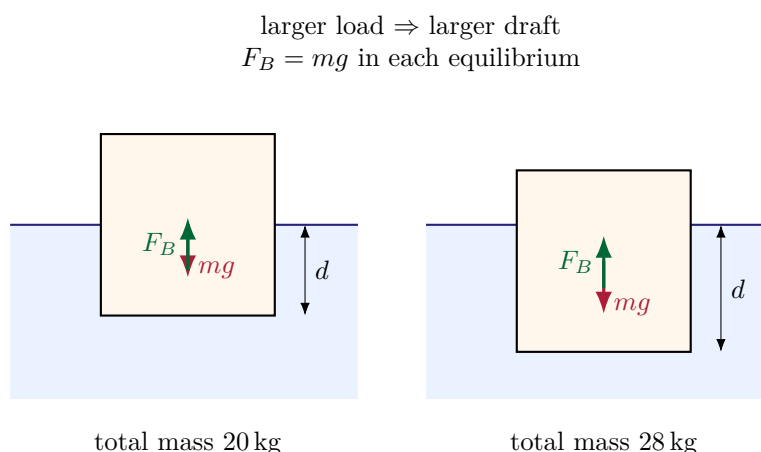


Figure 14.13: Two floating equilibria for the rectangular float of Example E14-S06. The draft rises from 0.100 m to 0.140 m after an 8.00 kg load is added. Within each panel, equal opposite arrows represent force balance; their magnitudes are schematic across panels. A volume fraction is a draft fraction here because the horizontal area is constant.

Example 14.9: E14-S06: Load and freeboard

A sealed rectangular float has dry mass 20.0 kg, vertical height $H = 0.200$ m and horizontal area $A = 0.200 \text{ m}^2$. It floats upright in water of density 1000 kg m^{-3} . Find its initial draft and freeboard, the draft after adding 8.00 kg, and the largest added mass consistent with at least 0.0500 m of freeboard. Assume the load preserves upright equilibrium and neglect air buoyancy and surface tension.

Model and strategy. The submerged volume is Ad . Floating force balance gives $\rho_f g Ad = mg$, so $d = m / (\rho_f A)$. Freeboard is $H - d$.

Initially,

$$d_0 = \frac{20.0}{(1000)(0.200)} = 0.100 \text{ m}, \quad H - d_0 = 0.100 \text{ m}.$$

With the load, total mass is 28.0 kg and $d_1 = 0.140$ m, leaving 0.0600 m freeboard. The permitted maximum draft is $0.200 - 0.0500 = 0.150$ m. Hence the largest total mass is $(1000)(0.200)(0.150) = 30.0$ kg, and the largest added mass is 10.0 kg.

Interpretation and check. The 8.00 kg load increases draft by 0.0400 m, consistent with $\Delta d = \Delta m / (\rho_f A)$. At full immersion the volume is 0.0400 m^3 and the maximum buoyancy

corresponds to 40.0 kg. That larger capacity does not satisfy the specified freeboard constraint. The result is a vertical-force capacity under the stipulated upright model, not a certification of tilt stability or behaviour in waves.

14.3.6 Apparent weight and instrument forces

A spring balance supports a submerged object through tension T . If the object is at rest, fully immersed, and not touching the vessel, its vertical force balance is $T + F_B - mg = 0$. The balance therefore reads

$$T = mg - \rho_f g V. \quad (14.21)$$

This reading is often called apparent weight, but the gravitational force mg has not decreased. The fluid supplies part of the upward support. If buoyancy exceeds weight, a string pulling upward cannot hold equilibrium with a negative tension; an additional downward constraint is needed.

The fluid simultaneously receives the downward reaction to buoyancy. If the object hangs from an external support over a vessel resting on a scale, the vessel scale reading increases by F_B when the object is immersed without contact. If instead the object and its support both rest on that same scale, buoyancy is an internal interaction for the combined system: the scale ultimately supports the total added object mass. Figure 14.14 distinguishes the separately selected bodies.

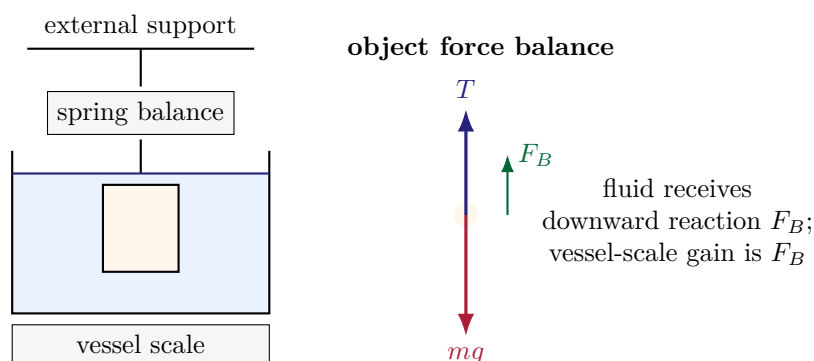


Figure 14.14: An object suspended from an external support and immersed without touching its vessel. The spring balance reads $mg - F_B$, while the vessel scale gains F_B through the force exerted on the liquid. The right panel is a free-body diagram of the object; arrows indicate forces, with schematic lengths.

Example 14.10: E14-S05: Density by apparent weighing

A compact solid object has mass 1.560 kg and reads 13.3416 N on a spring balance when fully immersed in water of stipulated density 1000 kg m^{-3} . The support is external to the water vessel, and the object does not touch the vessel. Neglect air buoyancy. Find the object's volume and average density, the vessel-scale increase, and the spring-balance reading in a liquid of density 800 kg m^{-3} .

Chosen system. On the object, weight acts downward; tension and buoyancy act upward. Its gravitational weight is $mg = (1.560)(9.81) = 15.3036 \text{ N}$. Thus water buoyancy is

$$F_B = mg - T = 15.3036 - 13.3416 = 1.9620 \text{ N}.$$

Volume follows by dividing by $\rho_w g$:

$$V = \frac{1.9620}{(1000)(9.81)} = 2.000 \times 10^{-4} \text{ m}^3 = 200.0 \text{ cm}^3, \quad \rho_o = \frac{1.560}{2.000 \times 10^{-4}} = 7800 \text{ kg m}^{-3}.$$

The vessel scale gains 1.9620 N because the object exerts that downward reaction on the fluid.

Expressed as an equivalent mass display, the gain is $F_B/g = 0.2000$ kg; this does not mean that the object has transferred mass to the vessel.

In the lighter liquid, buoyancy is $(800)(9.81)(2.000 \times 10^{-4}) = 1.5696$ N, so the new spring reading is $15.3036 - 1.5696 = 13.7340$ N. Less dense liquid supplies less support and the balance reads more.

Checks. The inferred object density exceeds both fluid densities, consistent with positive upward tension. The inferred volume uses the difference of two force readings, so measurement errors in that difference can dominate the density uncertainty. If a 0.010 N uncertainty were the only uncertainty in buoyancy, the relative volume uncertainty would be about $0.010/1.962 = 0.51\%$. A trapped bubble or contact with the tank would violate the assumed force model and could create a larger systematic error.

Guided checks

Diagnostic X14-03. A press that multiplies force by 50 reduces ideal output displacement by the same factor for equal-height incompressible motion, so it does not multiply work. A floating calculation with $V_{\text{displaced}}/V_{\text{object}} = 1.18$ fails the available-volume constraint; it predicts that partial floating is impossible. For a submerged balance measurement, distinguish gravitational weight mg from the support-force reading $mg - F_B$.

The three hydrostatic tools now have a common origin. The pressure gradient follows from local force balance; hydraulic transmission follows by comparing pressures in a connected liquid; buoyancy follows by integrating those pressure forces over an immersed boundary. None requires a separate source of mechanical energy. Their limits are also connected: unknown pressure references, inaccessible wetted surfaces, density changes, surface tension or additional body forces require the chosen model to be revised before its formulas are applied.

14.4 Mass conservation in flow

A hydrostatic problem concerns a fluid that remains at rest. A transport problem asks how much fluid crosses a chosen surface, how that rate changes, and which pressure forces can sustain the motion. The distinction between a moving parcel and a fixed observation region is especially useful. We may follow the same material particles, as in Newton's laws, or keep a *control volume* fixed in space and count what enters and leaves. Mass conservation applies to both descriptions, but its mathematical bookkeeping looks different.

14.4.1 From local velocity to a measured flow rate

Suppose a small area element dA has a unit normal $\mathbf{n}$ pointing in the chosen positive direction. During time dt , fluid moving with velocity $\mathbf{v}$ sweeps a signed volume $(\mathbf{v} \cdot \mathbf{n})dA dt$ through it. Only the normal velocity component contributes: a velocity tangent to the surface carries no fluid through it. Integration across the entire section gives

$$\dot{V} = \int_A \mathbf{v} \cdot \mathbf{n} dA, \quad \dot{m} = \int_A \rho \mathbf{v} \cdot \mathbf{n} dA. \quad (14.22)$$

The dimensions are respectively volume per time and mass per time. A section can even contain simultaneous forward and reverse flow; the signed integral then gives the net transport, not the sum of the magnitudes.

Define the mean normal velocity by

$$\bar{v} = \frac{1}{A} \int_A \mathbf{v} \cdot \mathbf{n} dA. \quad (14.23)$$

Consequently,

$$\dot{V} = A\bar{v}, \quad (14.24)$$

and, if density is uniform across that section,

$$\dot{m} = \rho A\bar{v}. \quad (14.25)$$

The first formula is exact for the defined area mean; it does not require a flat velocity profile. The second additionally requires that density can be taken outside the integral. If both density and velocity vary across a gas stream, an ordinary velocity mean and an ordinary density mean do not in general multiply to the correct mass flux. The correlation between the two distributions matters.

A flow meter normally reports a transport rate or a section mean. A small velocity probe reports something closer to a local speed. These measurements are interchangeable only after the profile is known. Figure 14.15 shows why even a perfectly smooth profile can give a large difference between the centreline and mean speeds. The area element in a circular section is an annulus, $dA = 2\pi r dr$; a narrow strip near the centre represents less area than an equally wide annulus near the wall.

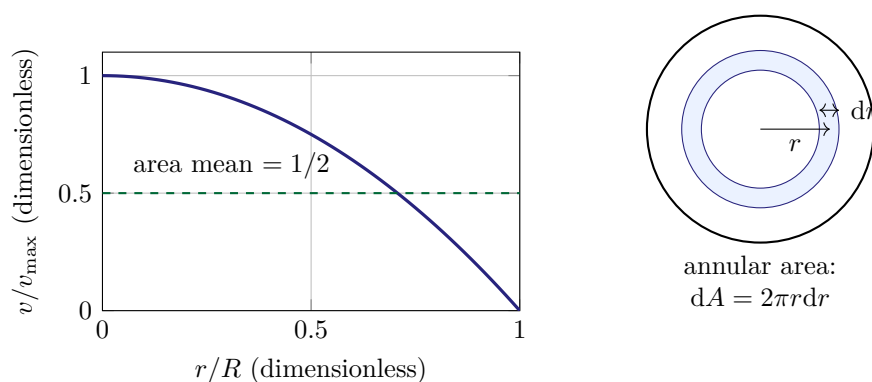


Figure 14.15: Area averaging of a nonuniform circular-tube velocity profile. The curve is quantitative; the annulus is schematic. The mean is obtained with annular area weights, rather than by taking an unweighted average of radial positions.

Example 14.11: E14-F01: Recovering transport from a local profile

A circular tube has radius $R = 2.00$ mm. An independently established axisymmetric velocity profile is $v(r) = v_0[1 - (r/R)^2]$, with $v_0 = 0.200$ m s⁻¹. The density is uniform, $\rho = 1000$ kg m⁻³. Find (a) the volume rate, (b) the mass rate, and (c) the error in treating the centreline speed as the mean.

Model and strategy. The section is normal to the velocity. Integrate the observed profile; its origin need not yet be known. The annular weighting in Figure 14.15 is essential.

$$\dot{V} = 2\pi v_0 \int_0^R \left(1 - \frac{r^2}{R^2}\right) r dr = 2\pi v_0 \left(\frac{R^2}{2} - \frac{R^2}{4}\right) = \frac{\pi R^2 v_0}{2}.$$

Thus $\bar{v} = v_0/2 = 0.100$ m s⁻¹, and

$$\dot{V} = 1.257 \times 10^{-6} \text{ m}^3 \text{ s}^{-1} = 1.257 \text{ mL s}^{-1}, \quad \dot{m} = 1.257 \times 10^{-3} \text{ kg s}^{-1}.$$

Using Av_0 instead gives twice the true rate: the relative overestimate is $(2\dot{V} - \dot{V})/\dot{V} = 100\%$.

Check and interpretation. The mean lies between the wall speed, zero, and the central speed. The result is smaller than the flat-profile rate because most of the section moves more slowly than the centre. The integral supplies transport; it does not by itself establish whether the stipulated profile is dynamically possible for a given pressure gradient.

14.4.2 A control-volume balance and its steady limit

For a stationary observation volume, let M_{CV} be the mass presently inside. Conservation states

$$\frac{dM_{CV}}{dt} = \sum_{in} \dot{m} - \sum_{out} \dot{m}, \quad M_{CV} = \int_{CV} \rho dV. \quad (14.26)$$

Here inlet and outlet rates are quoted as positive magnitudes in their respective directions. An equivalent compact formula uses outward normals on every boundary and a single negative surface integral. The sign is physical: a net outward rate decreases the stored mass.

Steady means that the velocity, density and other fields at each fixed position do not change with time. For a fixed control volume in steady flow, the accumulation term vanishes. In a streamtube, its side boundary is tangent to the velocity, so there is no flux through the sides. One inlet and one outlet then give

$$\rho_1 A_1 \bar{v}_1 = \rho_2 A_2 \bar{v}_2. \quad (14.27)$$

If density has the same value at both sections,

$$A_1 \bar{v}_1 = A_2 \bar{v}_2. \quad (14.28)$$

The familiar smaller-area/higher-speed statement is therefore a conditional consequence, not the definition of mass conservation. A changing density, a branch, a leak or temporary storage changes the equation needed to connect the speeds.

A *streamline* is tangent to the instantaneous velocity field. A *pathline* is the history of one material particle. In steady flow, following a particle traces the same curve as a streamline. In unsteady flow these curves generally differ. Steadiness also does not mean laminar flow: a fluctuating turbulent velocity field may have a statistically steady average. Conversely, a laminar flow can change with time, such as the smooth start-up of liquid in a tube.

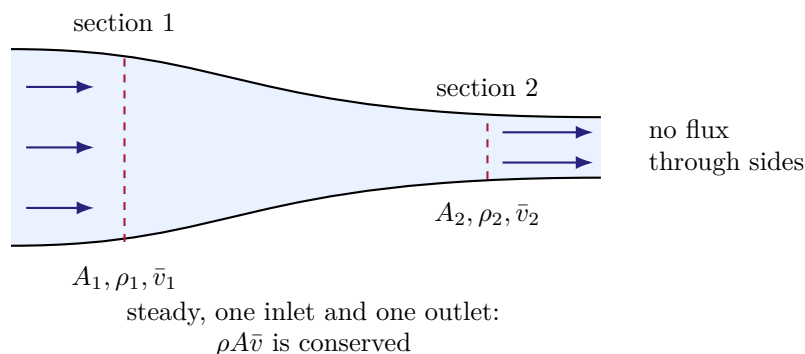


Figure 14.16: A variable-area streamtube. The chosen control sections and zero side flux make the mass balance explicit. Cross-sectional areas refer to physical sections, not heights measured from the drawing.

Example 14.12: E14-04: A converging flow passage

A steady passage with one inlet and one outlet carries $\dot{m} = 3.0 \text{ kg s}^{-1}$. At section 1, $A_1 = 0.015 \text{ m}^2$ and $\rho_1 = 1000 \text{ kg m}^{-3}$. At section 2, $A_2 = 0.0050 \text{ m}^2$. Find the inlet mean speed and the outlet mean speed for (a) equal density and (b) stipulated $\rho_2 = 800 \text{ kg m}^{-3}$. Explain what can be concluded if the outlet density is unknown.

Strategy and assumptions. Use mass continuity with uniform density on each control section and no storage or side branch. The inlet speed is

$$\bar{v}_1 = \frac{\dot{m}}{\rho_1 A_1} = \frac{3.0}{(1000)(0.015)} = 0.200 \text{ m s}^{-1}.$$

For (a), unchanged density gives

$$\bar{v}_2 = \frac{3.0}{(1000)(0.0050)} = 0.600 \text{ m s}^{-1}.$$

For (b), the same mass rate at lower density requires

$$\bar{v}_2 = \frac{3.0}{(800)(0.0050)} = 0.750 \text{ m s}^{-1}.$$

Both cases carry 3.0 kg s^{-1} , but their outlet volume rates are respectively 3.0 L s^{-1} and 3.75 L s^{-1} .

Underdetermination and check. If ρ_2 is unknown, one equation contains two unknowns, ρ_2 and $\bar{v}_2$. A density measurement or an additional material/thermodynamic relation is required. The stipulated lower density is a mathematical case; it is not a claim that ordinary water undergoes that large density change in this apparatus. Substitution of each result into $\rho A \bar{v}$ recovers the specified mass rate.

Guided checks

Diagnostic X14-04. A nozzle area decreases by a factor of three, and a student triples the speed. Is area information sufficient? **Reasoned answer:** only when the same volume rate crosses both sections. In the present steady one-stream model, unchanged density supplies that condition. If density changes, conserve $\rho A \bar{v}$; if mass accumulates or branches off, include those terms first.

Storage is particularly easy to overlook when a reservoir is filling. Figure 14.17 separates the transport observations from the height change. The free surface is moving, so the whole apparatus is not a steady flow system even when the inlet and outlet rates are individually constant.

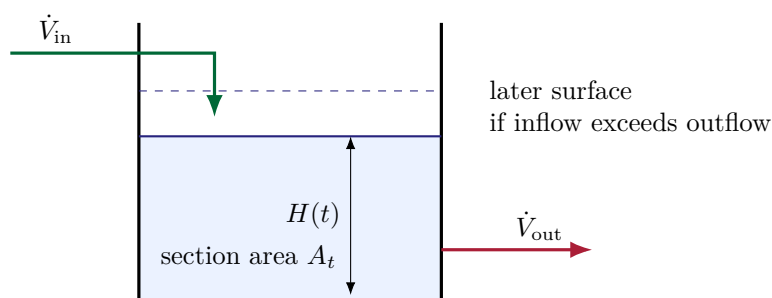


Figure 14.17: A filling control volume with a moving liquid level. The drawing is schematic. A constant supply rate does not eliminate accumulation; the difference between inlet and outlet volume rates sets the height increase.

Example 14.13: E14-F02: A tank with unequal transport rates

A vertical tank of constant cross-sectional area $A_t = 0.120 \text{ m}^2$ receives an incompressible liquid at 5.00 L s^{-1} and discharges it at 3.00 L s^{-1} . The density is 1000 kg m^{-3} . Find (a) the stored-mass increase per second, (b) the level increase in 30.0 s , and (c) the inlet pipe's mean speed if its area is $2.00 \times 10^{-3} \text{ m}^2$.

Balance. With rates measured relative to the fixed tank, $M = \rho A_t H$, so

$$\frac{dM}{dt} = \rho(\dot{V}_{\text{in}} - \dot{V}_{\text{out}}) = (1000)(0.00500 - 0.00300) = 2.00 \text{ kg s}^{-1}.$$

For constant area and density,

$$\frac{dH}{dt} = \frac{0.00200}{0.120} = 0.0167 \text{ m s}^{-1}, \quad \Delta H = 0.500 \text{ m}.$$

The pipe speed is independently determined by its inlet rate:

$$\bar{v}_{\text{in}} = \frac{0.00500}{2.00 \times 10^{-3}} = 2.50 \text{ m s}^{-1}.$$

Check. The added volume is 0.0600 m^3 , equal to $A_t \Delta H$, and the added mass is 60.0 kg . Equating inlet and outlet rates would incorrectly force the level to remain fixed. The answer assumes that the specified discharge is maintained as the level changes; a passive outlet driven only by liquid height would generally have a changing rate.

14.5 Bernoulli's equation and its limits

Continuity constrains transport but does not determine the pressure difference that drives it. Mechanical energy supplies a second relation under an ideal model. The pressure in this relation is *static pressure*, the isotropic stress exerted by the fluid locally. The kinetic term is energy associated with ordered fluid motion. It is not an additional isotropic pressure that should be added to every hydrostatic force calculation.

14.5.1 Deriving the equation from pressure work

Take a steady, incompressible flow through a narrow streamtube, with negligible viscosity and no pump or turbine between two sections. During a short time, a volume $\Delta V = A_1 \Delta s_1 = A_2 \Delta s_2$ enters at one end and the same volume leaves at the other. The two transported masses are equal, $\Delta m = \rho \Delta V$. Fluid already between the sections has the same energy distribution before and after the displacement because the flow is steady.

The upstream fluid pushes the selected material forward with pressure work $+p_1 A_1 \Delta s_1 = p_1 \Delta V$. The selected material pushes against the downstream pressure, giving work $-p_2 A_2 \Delta s_2 = -p_2 \Delta V$. Thus the *net* pressure work is $(p_1 - p_2) \Delta V$. The gravitational potential-energy change of the transported mass is $\rho g \Delta V (z_2 - z_1)$, where z is measured upward from one common reference. The kinetic-energy change is $\frac{1}{2} \rho \Delta V (v_2^2 - v_1^2)$. Including gravity in potential energy, the work–energy theorem becomes

$$(p_1 - p_2) \Delta V = \frac{1}{2} \rho \Delta V (v_2^2 - v_1^2) + \rho g \Delta V (z_2 - z_1).$$

Divide by the nonzero transported volume and rearrange:

$$p_1 + \frac{1}{2} \rho v_1^2 + \rho g z_1 = p_2 + \frac{1}{2} \rho v_2^2 + \rho g z_2. \quad (14.29)$$

This derivation explains the pressure terms' signs: inlet pressure supplies energy while outlet pressure opposes the displacement. It also explains why pressure, kinetic energy and gravitational energy appear in the same balance.

The same result can be derived locally by resolving Newton's second law along a streamline. For a constant-density inviscid fluid, $dp + \rho v dv + \rho g dz = 0$; integration reproduces Eq. (14.29). This differential statement makes the dynamical mechanism clear. A pressure gradient and gravity jointly

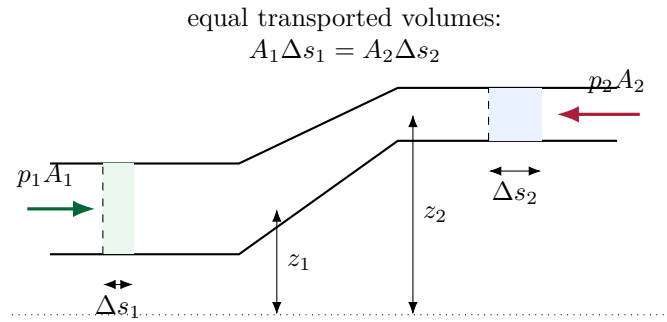


Figure 14.18: Pressure work in the Bernoulli derivation. Inlet and outlet pressure forces act into the selected material from opposite ends. The swept slices have equal physical volume; widths, heights and force-arrow lengths are schematic. The elevation reference is common to both sections.

accelerate the fluid; continuity then constrains which speed field fits the passage. One should not imagine that fluid somehow “knows” its speed and creates a pressure change without forces.

The result is commonly written

$$p + \frac{1}{2}\rho v^2 + \rho g z = \text{constant along a streamline.} \quad (14.30)$$

Different streamlines need not share the same constant. A shared reservoir or an independently justified irrotational region can make comparison across streamlines possible. Irrotational flow means zero vorticity, $\nabla \times \mathbf{v} = 0$, in the region considered; it does not mean that the entire fluid has zero angular momentum about an arbitrary origin.

14.5.2 Pressure head, velocity head and elevation head

Every term in Eq. (14.29) is an energy per unit volume and has units Pa. Dividing by ρg gives the energy per unit weight, conventionally expressed as a length:

$$\frac{p}{\rho g} + \frac{v^2}{2g} + z = H. \quad (14.31)$$

The terms are pressure head, velocity head and elevation head. The *energy grade line* has elevation H above the chosen datum. The *hydraulic grade line* has elevation $z + p/(\rho g)$, lower by the velocity head. In a slow liquid connected to a vertical piezometer open to the same pressure reference, the hydraulic grade line corresponds approximately to the equilibrium level in that tube. The energy grade line includes the flow’s kinetic energy and generally is not an ordinary static liquid level.

Only pressure *differences* occur between two Bernoulli states of one constant-density fluid. Subtracting one common reference pressure therefore leaves the energy equation unchanged. This permits gauge pressures in a comparison, provided that both are referred to the same reference. A cavitation or gas-state calculation instead needs absolute pressure. Similarly, moving the elevation datum shifts both total heads by the same constant without changing any prediction.

Method

Before using Bernoulli, define the two points and ask whether the flow is steady, density is approximately constant, viscous losses are negligible, and the points share the appropriate streamline constant. Identify any pump, turbine or other mechanical-energy transfer. Finally, check the speed definition: the streamline formula uses local speed. A section-mean energy equation needs an appropriate profile correction when kinetic-energy flux is nonuniform; that distinction is developed in Section 14.9.

14.5.3 A horizontal contraction and the area-ratio convention

A smooth horizontal contraction offers a clean ideal application. Let the two physical cross-sectional areas be A_1 and A_2 , with $A_2 < A_1$, and approximate the profiles by uniform speeds at the chosen sections. Continuity gives $v_1 = (A_2/A_1)v_2$. Define the *area ratio*

$$\beta_A = \frac{A_2}{A_1}.$$

When a bare β is used in the area-ratio formula, it means β_A . Bernoulli at equal elevation then yields

$$\Delta p = p_1 - p_2 = \frac{1}{2}\rho v_2^2(1 - \beta_A^2), \quad (14.32)$$

so

$$\dot{V} = A_2 \sqrt{\frac{2\Delta p}{\rho(1 - \beta_A^2)}}. \quad (14.33)$$

For circular sections one can instead define the *diameter ratio* $\beta_D = D_2/D_1$. Since area is proportional to diameter squared, $\beta_A = \beta_D^2$ and the denominator is $1 - \beta_D^4$. Confusing the two definitions produces an incorrect meter formula even though every term still has the correct dimensions.

Pressure taps should sense static pressure without confronting the stream with an opening that brings it to rest. The ideal equation estimates the transport corresponding to a measured static-pressure difference. A real differential-pressure meter needs a discharge coefficient based on its geometry and operating regime. Such a coefficient is an experimentally supported correction, not permission to ignore the underlying assumptions.

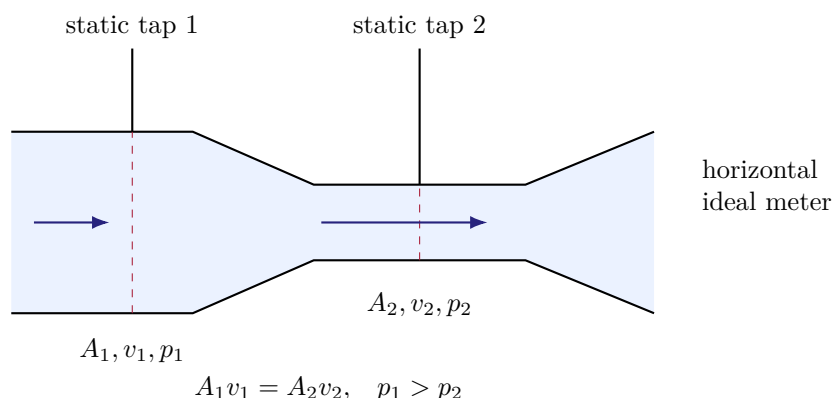


Figure 14.19: A horizontal differential-pressure meter, with a smooth contraction. Static taps compare the two pressures. The passage is schematic: area ratios are supplied by the physical dimensions, rather than inferred from this side view.

Example 14.14: E14-05: An ideal differential-pressure flow meter

A horizontal meter carries liquid of density $\rho = 1000 \text{ kg m}^{-3}$. Its areas are $A_1 = 4.0 \times 10^{-3} \text{ m}^2$ and $A_2 = 1.0 \times 10^{-3} \text{ m}^2$. The measured static-pressure difference is $p_1 - p_2 = 18.0 \text{ kPa}$. Estimate (a) both section speeds and (b) the ideal volume rate, and explain the model limitation.

Model. Assume steady incompressible flow, equal section elevations, negligible losses and approximately uniform profiles. The area ratio is $\beta_A = 0.25$, so continuity gives $v_2 = 4v_1$. Substitution into Bernoulli gives

$$18.0 \times 10^3 = \frac{1}{2}(1000)(16 - 1)v_1^2.$$

The positive-speed branch appropriate to the stated flow direction is

$$v_1 = 1.549 \text{ m s}^{-1}, \quad v_2 = 6.197 \text{ m s}^{-1},$$

followed by

$$\dot{V} = A_1 v_1 = 6.20 \times 10^{-3} \text{ m}^3 \text{ s}^{-1} = 6.20 \text{ L s}^{-1}.$$

Checks. Both $A_1 v_1$ and $A_2 v_2$ give the same rate. The dynamic terms are 1.200 kPa and 19.200 kPa; their difference recovers 18.0 kPa. The smaller section has the greater speed and lower static pressure, as required by this horizontal ideal model.

Interpretation. The number is an ideal estimate. A real device's tapping geometry, viscosity and loss affect calibration. A pressure difference alone does not certify that the section profiles are uniform or that the ideal coefficient is unity.

The speeds in this example give a particularly useful head diagram. Specify $z = 1.00 \text{ m}$ and an ideal total head $H = 10.00 \text{ m}$, with a common gauge-pressure reference. The inlet velocity head is 0.1223 m and the throat velocity head is 1.9572 m. The hydraulic grade line therefore falls from 9.8777 m to 8.0428 m as the kinetic contribution grows; total head stays constant. These are quantitative heights in Figure 14.20, not decorative bars.

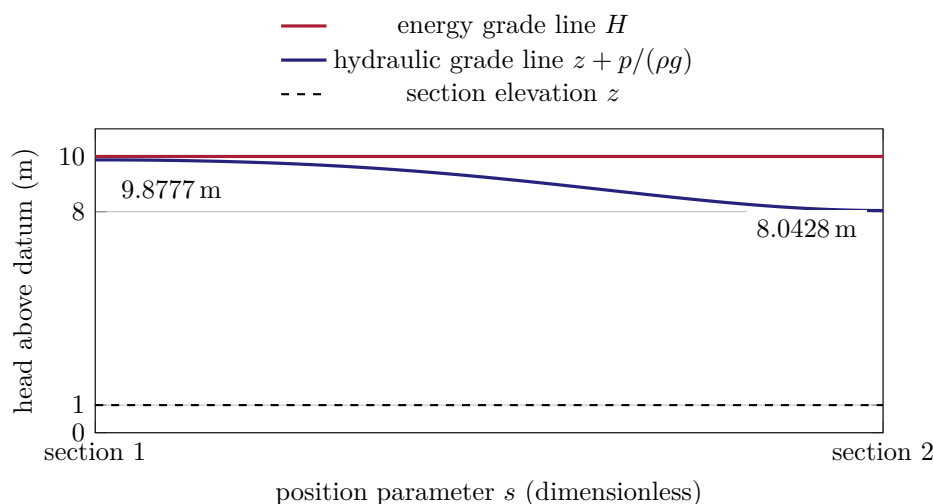


Figure 14.20: An energy-consistent head representation for E14-05, with stipulated $z = 1 \text{ m}$ and $H = 10 \text{ m}$. The end values are quantitative. A prescribed smooth speed interpolation joins the two sections; it does not constitute a solution for the detailed contraction geometry. The vertical gap between the two grade lines is $v^2/(2g)$.

Guided checks

Diagnostic X14-05. Two equal-area horizontal sections have a nonzero measured pressure drop. Can Eq. (14.33) explain it? **Reasoned answer:** its denominator tends to zero because equal areas require equal speeds in this model; Bernoulli then requires equal static pressures. The nonzero measured difference indicates loss, external work, inconsistent pressure references or another failure of the assumed measurement model. Division by zero is a warning about applicability.

14.5.4 Elevation changes and a pressure requirement

If the sections differ in elevation, the pressure difference includes a gravitational term:

$$p_1 - p_2 = \frac{1}{2} \rho (v_2^2 - v_1^2) + \rho g (z_2 - z_1). \quad (14.34)$$

A rising stream needs energy for increased gravitational potential. Its static pressure can fall even while its speed falls. A descending stream can speed up while its pressure increases if the gravitational

contribution is large enough. Consequently, “faster means lower pressure” is a description of a particular horizontal lossless comparison, rather than a universal law connecting any two fluid points.

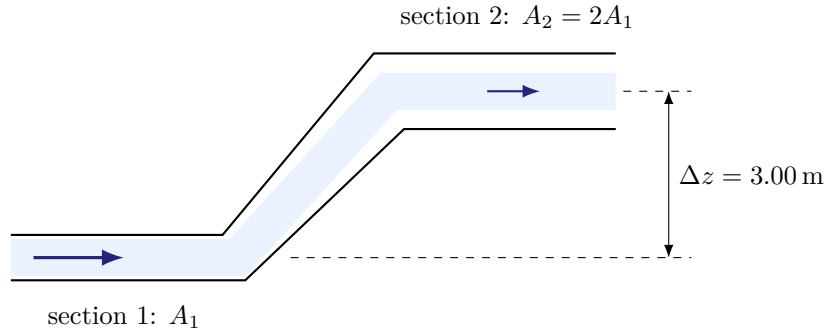


Figure 14.21: A rising passage that broadens, used in E14-F03. The ideal mean speed decreases as the physical area doubles, while the elevation rises. The geometry is schematic; the supplied elevation and area data govern the calculation.

Example 14.15: E14-F03: Slower flow can also have lower pressure

An ideal liquid line carries $\dot{V} = 4.00 \times 10^{-3} \text{ m}^3 \text{ s}^{-1}$ with $\rho = 1000 \text{ kg m}^{-3}$. It rises by 3.00 m while broadening from $A_1 = 2.00 \times 10^{-3} \text{ m}^2$ to $A_2 = 4.00 \times 10^{-3} \text{ m}^2$. The inlet gauge pressure is 80.0 kPa. Determine (a) both speeds, (b) outlet gauge pressure and (c) whether greater speed identifies the lower-pressure point.

Model and calculation. Use steady, incompressible, lossless flow with uniform section speeds and no pump between the sections. Continuity gives $v_1 = 2.00 \text{ m s}^{-1}$ and $v_2 = 1.00 \text{ m s}^{-1}$. Bernoulli gives

$$p_2 = p_1 + \frac{1}{2}\rho(v_1^2 - v_2^2) - \rho g \Delta z = 80000 + 1500 - 29430 = 52070 \text{ Pa}.$$

Thus $p_2 = 52.1 \text{ kPa}$ gauge. Pressure *falls* while speed *falls*; the elevation change costs more energy than the speed reduction releases.

Energy check. Taking $z_1 = 0$, the inlet energy density is $80000 + 2000 = 82000 \text{ J m}^{-3}$. The outlet value is $52070 + 500 + 29430 = 82000 \text{ J m}^{-3}$. Both are equal. A real line with positive viscous loss would require a greater inlet pressure to achieve the same outlet condition and flow rate.

14.5.5 Stagnation pressure and a Pitot measurement

A fluid parcel brought to rest without appreciable loss has a stagnation point. At the same elevation as the nearby moving fluid, Bernoulli gives

$$p_0 - p = \frac{1}{2}\rho v^2, \quad v = \sqrt{\frac{2(p_0 - p)}{\rho}}. \quad (14.35)$$

Here p_0 is the stagnation pressure and p the static pressure. A Pitot opening faces upstream and samples a stagnation condition, whereas a static tap is arranged to avoid stopping the incident stream. The pressure difference is called dynamic pressure in this ideal relation; it is not the pressure exerted on every oriented surface by a moving fluid.

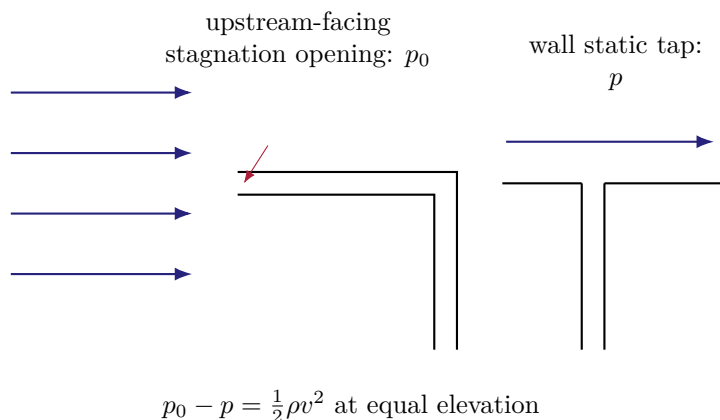


Figure 14.22: The sensing distinction in a Pitot/static measurement. The incident flow and apparatus are schematic. The upstream-facing opening brings fluid locally to rest; the flush wall tap lies parallel to the local flow and measures static pressure. Both sensing openings are shown at the same elevation. Connecting tubing is shown only to identify the two pressure signals.

Example 14.16: E14-F04: Speed from a stagnation-pressure difference

A Pitot/static measurement in an approximately incompressible gas gives $p_0 - p = 240 \text{ Pa}$ at one location. Use stipulated density $\rho = 1.20 \text{ kg m}^{-3}$ and a stipulated sound speed $c = 340 \text{ m s}^{-1}$. Find the local speed and Mach number, then explain whether the result directly gives a pipe's mean speed.

Model. Assume that the two sensing positions have equal elevation, that losses in bringing fluid to rest are negligible, and that the density changes are small enough for the incompressible relation. Then

$$v = \sqrt{\frac{2(240)}{1.20}} = 20.0 \text{ m s}^{-1}, \quad \text{Ma} = v/c = 0.0588.$$

The low Mach number supports the small-compressibility approximation; it does not check alignment, calibration or probe disturbance.

Local-versus-mean answer. The instrument samples a local speed. In a pipe, one must measure or otherwise establish the velocity profile and integrate across the section to infer $\dot{V}$. A centreline Pitot reading in the parabolic flow of Figure 14.15 would be twice the mean. Substitution gives $\frac{1}{2}(1.20)(20.0)^2 = 240 \text{ Pa}$, which checks the pressure conversion.

14.5.6 Efflux from a tank and the role of its area

Choose the free surface of an open tank as point 1 and a free jet just outside an outlet as point 2. Both are exposed to the same ambient pressure, so their pressures cancel. If the outlet is h below the surface and the tank's area A_t is much greater than the outlet area a , continuity makes the surface speed small compared with the jet speed. Neglecting that speed and losses gives Torricelli's result,

$$v \simeq \sqrt{2gh}. \quad (14.36)$$

The pressure at the fast exposed jet is ambient, rather than zero absolute pressure. Its kinetic energy comes principally from the height lost by the liquid.

A finite-area correction is instructive. With approximately uniform speeds, continuity gives surface speed magnitude $v_s = (a/A_t)v$. Bernoulli gives

$$v = \sqrt{\frac{2gh}{1 - (a/A_t)^2}}. \quad (14.37)$$

This is a quasi-steady estimate: at each height the apparatus is treated as if its instantaneous flow could be represented by the steady formula. It assumes that the shape of the reservoir and outlet permits that approximation. The formula's divergence as $a \rightarrow A_t$ is not a real unlimited speed; a slowly changing, large-reservoir model has ceased to apply.

For a small outlet, the liquid level obeys $A_t dh/dt = -a\sqrt{2gh}$. Separating variables and integrating from h_0 to $h(t)$ gives

$$\sqrt{h(t)} = \sqrt{h_0} - \frac{a}{A_t} \sqrt{\frac{g}{2}} t, \quad t_{\text{empty}} = \frac{A_t}{a} \sqrt{\frac{2h_0}{g}}. \quad (14.38)$$

The mathematical model predicts a linearly decreasing square root of height, while height itself decreases quadratically. Very near emptying, surface shape, outlet geometry and unsteady effects undermine the simple description. The ideal time is consequently an estimate for the stated model.

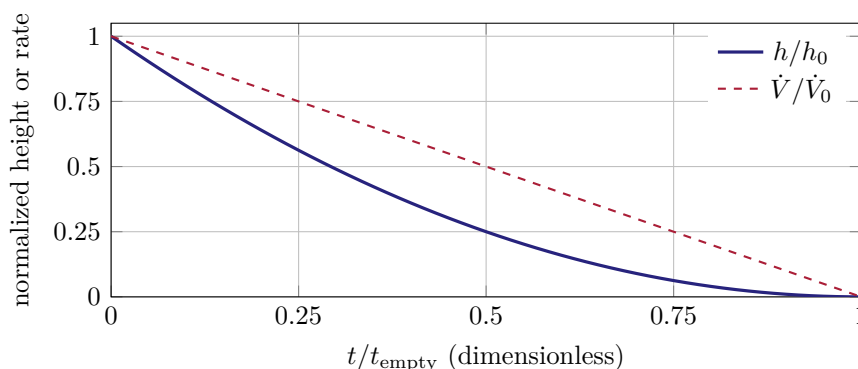


Figure 14.23: Ideal draining predictions for a constant-area tank with a small outlet. These analytical curves describe the quasi-steady lossless model, not measured data. At half the predicted emptying time, the height has fallen to one quarter, while the volume rate has fallen to one half of its initial value.

Example 14.17: E14-F05: Estimating an ideal drain time

An open tank has constant area $A_t = 0.200 \text{ m}^2$ and initial depth $h_0 = 0.800 \text{ m}$ above a small outlet of area $a = 1.00 \times 10^{-4} \text{ m}^2$. Neglect losses and assume quasi-steady efflux. Find (a) the initial jet speed and volume rate, (b) the ideal emptying time, and (c) the height halfway through that time.

Choice of approximation. The area ratio is $a/A_t = 5.00 \times 10^{-4}$; its square is 2.50×10^{-7} . Neglecting free-surface kinetic energy therefore has a tiny effect within this ideal model. Initially,

$$v_0 = \sqrt{2(9.81)(0.800)} = 3.962 \text{ m s}^{-1}, \quad \dot{V}_0 = av_0 = 3.962 \times 10^{-4} \text{ m}^3 \text{ s}^{-1}.$$

Integrating the changing-rate balance, rather than dividing the total volume by its initial rate, gives

$$t_{\text{empty}} = \frac{0.200}{1.00 \times 10^{-4}} \sqrt{\frac{2(0.800)}{9.81}} = 807.7 \text{ s} = 13.46 \text{ min}.$$

At $t = t_{\text{empty}}/2$, $\sqrt{h} = \sqrt{h_0}/2$, so $h = h_0/4 = 0.200 \text{ m}$.

Check. The initial volume is $A_t h_0 = 0.160 \text{ m}^3$. Dividing by $\dot{V}_0$ gives 403.9 s , exactly half the ideal time because the rate declines linearly to zero. The time integral of that rate is the triangle area $\dot{V}_0 t_{\text{empty}}/2 = 0.160 \text{ m}^3$. A real contracted jet and viscous loss commonly reduce discharge relative to this estimate.

14.5.7 Absolute pressure and the upper limit of a siphon

Bernoulli can predict a pressure too low to sustain the assumed single-phase liquid. For an ideal siphon fed by a large open reservoir, let the crest lie h_c above the reservoir surface and have speed v_c . Then

$$p_{c,\text{abs}} = p_{\text{atm}} - \rho g h_c - \frac{1}{2} \rho v_c^2. \quad (14.39)$$

If the calculated absolute pressure approaches the liquid's vapor pressure, vapor cavities can form and the single-phase model becomes unreliable. Dissolved gas, nucleation and losses may matter before a simple vapor-pressure boundary is reached. Gauge pressure can be negative without being physically impossible; absolute pressure is the relevant quantity for this check.

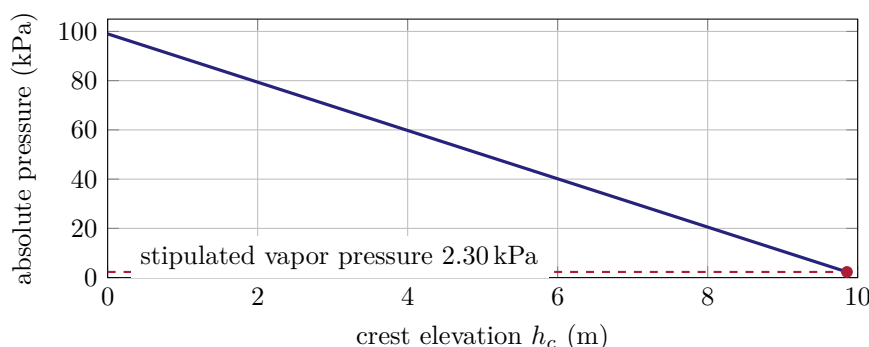


Figure 14.24: Ideal siphon-crest pressure at stipulated constant speed 2.00 m s^{-1} , atmospheric pressure 101 kPa , liquid density 1000 kg m^{-3} and vapor pressure 2.30 kPa . The liquid-model line ends at the calculated vapor-pressure boundary; it is not extended as a physically valid negative-pressure liquid prediction.

Example 14.18: E14-F06: A siphon pressure-consistency bound

Use stipulated $p_{\text{atm}} = 101 \text{ kPa}$, $p_{\text{vap}} = 2.30 \text{ kPa}$ and $\rho = 1000 \text{ kg m}^{-3}$. A lossless siphon has crest speed $v_c = 2.00 \text{ m s}^{-1}$. Determine (a) the crest absolute and gauge pressures at $h_c = 6.00 \text{ m}$ and (b) the largest crest height for which the ideal pressure remains at least p_{vap} . Explain the meaning of this bound.

Pressure calculation. Taking the reservoir surface as the elevation datum,

$$p_{c,\text{abs}} = 101000 - (1000)(9.81)(6.00) - \frac{1}{2}(1000)(2.00)^2 = 40140 \text{ Pa}.$$

Thus the absolute pressure is 40.1 kPa and the gauge pressure is -60.9 kPa . The negative gauge value is consistent with a positive absolute pressure well above the stipulated vapor pressure.

Boundary. Setting $p_{c,\text{abs}} = p_{\text{vap}}$ gives

$$h_{c,\text{max}} = \frac{p_{\text{atm}} - p_{\text{vap}}}{\rho g} - \frac{v_c^2}{2g} = \frac{101000 - 2300 - 2000}{9810} = 9.86 \text{ m}.$$

This is a model-consistency upper bound, not a guaranteed working height. Additional losses lower the crest pressure for a given elevation and speed; gas release and imperfect priming can also disrupt operation. Substituting the unrounded boundary into Eq. (14.39) returns 2.30 kPa , the endpoint shown in Figure 14.24.

Caution

A correct Bernoulli algebraic result can fail physically. Check the inferred absolute pressure, the assumed density, the origin of the kinetic-energy profile, and any omitted loss or work input. In a long constant-area pipe, an observed pressure decrease usually does not imply an increase of the incompressible mean speed. The mechanical energy may instead be converted irreversibly into internal energy.

14.6 Viscosity, pressure loss and flow regime

The ideal model leaves out a material property that explains many everyday transport observations. Honey and water do not flow equally readily through the same narrow passage under the same pressure difference. More fundamentally, a fluid can resist *shear while moving*, although it cannot sustain an arbitrary static shear stress as an ordinary elastic solid does. Dynamic viscosity describes that dissipative response for a Newtonian fluid.

14.6.1 Newtonian shear and the no-slip boundary condition

Consider parallel layers with velocity in the x direction, changing with coordinate y . Define τ_{xy} as the x component of traction acting on a surface whose outward normal points in the positive y direction. A Newtonian fluid obeys

$$\tau_{xy} = \eta_v \frac{dv_x}{dy}. \quad (14.40)$$

A positive velocity gradient corresponds to a positive x traction on that positive- y face; the traction on the opposite face has the opposite sign. Dynamic viscosity η_v is positive and has units Pa s. The gradient has units s^{-1} , so their product has the required stress units. Viscosity depends on the material and its state, especially temperature. Numerical properties in this chapter are stipulated teaching values, rather than material specifications.

Figure 14.25 makes the constitutive relation tangible. For two large plates separated by distance b , the lower stationary and the upper moving steadily at U , a simple pressure-gradient-free model gives $v_x = Uy/b$. Its shear-stress magnitude is $\eta_v U/b$. The applied force on upper-plate area A_p must be $F = \eta_v A_p U/b$, and the mechanical input power is $FU = \eta_v A_p U^2/b$. The force scales linearly with speed while the dissipated power scales quadratically. This is a particular laminar shear model, not a general drag law for an object in a turbulent stream.

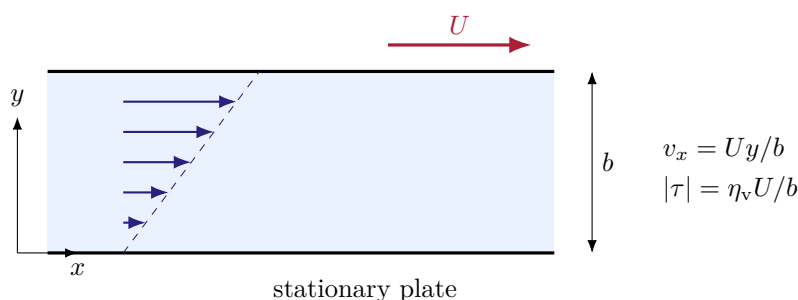


Figure 14.25: Simple Newtonian shear between parallel plates. The linear velocity profile satisfies no slip at both walls. Arrow lengths are proportional to local speed on this schematic scale. The moving upper plate receives a backward viscous traction and must be driven forward to maintain its speed.

The *no-slip condition* says that the fluid velocity at a solid boundary equals the boundary's velocity. At a stationary impermeable wall, both tangential velocity and normal penetration vanish. No slip does not say that the entire fluid must be at rest; it creates a velocity variation near a moving stream's wall. It is a successful continuum boundary condition at ordinary engineering scales,

although rarefied gases and specialised micro- or nanoscale interfaces can require a slip model. That limitation is a reason to check scales rather than to abandon a useful model in ordinary tubes.

Newtonian viscosity is also a material assumption. Suspensions, polymer solutions and other complex fluids may have stress that depends nonlinearly on the shear rate or on prior deformation. One cannot assign a single constant η_v and apply the following tube formula without checking that this model is appropriate.

14.6.2 How a pressure gradient balances shear in a tube

Take a horizontal straight circular tube of radius R and length L . Assume steady incompressible flow, uniform viscosity, axial symmetry and a fully developed velocity field $v(r)$ independent of axial position. Fully developed means that the *shape* and magnitude of the velocity profile no longer change downstream; it does not mean that pressure stops changing. Let $\Delta p = p_{\text{in}} - p_{\text{out}} > 0$.

A coaxial cylindrical fluid core of radius r and length L receives net forward pressure force $\Delta p \pi r^2$. The surrounding fluid exerts a backward shear force of magnitude $|\tau(r)| 2\pi r L$. The core has no net axial acceleration in this developed steady state, so the magnitudes balance:

$$|\tau(r)| = \frac{\Delta p}{2L} r.$$

Since speed decreases outward, $dv/dr < 0$ and Newtonian stress gives

$$-\eta_v \frac{dv}{dr} = \frac{\Delta p}{2L} r.$$

Integrate from radial position r to the wall and use $v(R) = 0$:

$$v(r) = \frac{\Delta p}{4\eta_v L} (R^2 - r^2), \quad 0 \leq r \leq R. \quad (14.41)$$

The centreline speed is therefore

$$v_{\text{max}} = \frac{\Delta p R^2}{4\eta_v L}. \quad (14.42)$$

The zero slope at $r = 0$ respects axial symmetry; the zero wall speed satisfies no slip. The core-force argument independently identifies the linear shear-stress distribution, which checks the derivative of the parabolic profile.

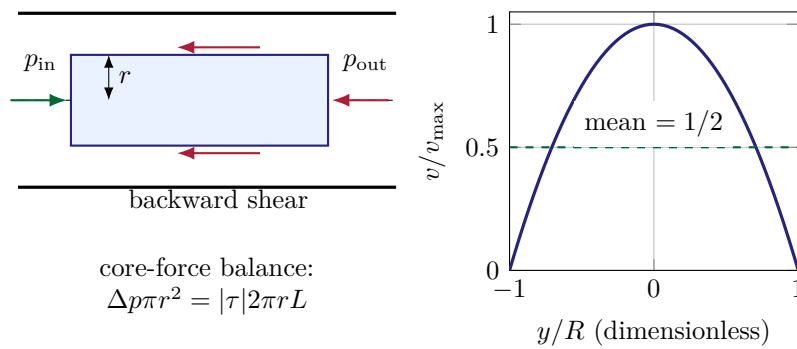


Figure 14.26: The horizontal laminar-tube model. Pressure force on a cylindrical core balances viscous shear. The right panel uses signed transverse coordinate y/R , while physical radius is nonnegative. The area mean is half the centreline speed.

14.6.3 Poiseuille transport, resistance and power

Integrate the profile over annuli, rather than over a diameter:

$$\begin{aligned}\dot{V} &= \int_0^R v(r) 2\pi r \, dr = \frac{\pi \Delta p}{2\eta_v L} \int_0^R (R^2 r - r^3) \, dr \\ &= \frac{\pi R^4 \Delta p}{8\eta_v L}.\end{aligned}\tag{14.43}$$

Dividing by physical area yields

$$\bar{v} = \frac{\dot{V}}{\pi R^2} = \frac{\Delta p R^2}{8\eta_v L} = \frac{v_{\max}}{2}.\tag{14.44}$$

This factor of two is specifically an *area-mean* property of the parabolic circular-tube profile. A planar channel has a different area weighting, and a turbulent profile need not have the same ratio.

Writing $\Delta p = \mathcal{R}_h \dot{V}$ defines the hydraulic resistance

$$\mathcal{R}_h = \frac{8\eta_v L}{\pi R^4}, \quad P_{\text{diss}} = \Delta p \dot{V}.\tag{14.45}$$

The product pressure times volume per time has units J s^{-1} and equals the mechanical energy converted by viscosity per second in the fully developed horizontal section. The mean speed at its inlet and outlet is the same. The pressure loss does not create an increase of mean kinetic energy; it supplies dissipation. At constant flow, increasing length or viscosity increases the required pressure difference. At constant pressure, it decreases transport.

The R^4 dependence is unusually sensitive. A 10% larger radius gives $1.1^4 = 1.4641$ times the rate when the other inputs remain fixed. A 10% smaller radius gives $0.9^4 = 0.6561$ times the rate. The effects are asymmetric because the dependence is nonlinear. Radius errors, deposits and small bore changes can therefore dominate narrow-tube transport. This inference is useful only while the flow remains Newtonian, laminar and fully developed.

14.6.4 Reynolds number and a regime audit

Compare an inertial scale $\rho \bar{v}^2 / D$ with a viscous scale $\eta_v \bar{v} / D^2$. Their ratio is

$$\text{Re} = \frac{\rho \bar{v} D}{\eta_v}, \quad D = 2R.\tag{14.46}$$

The quantity is dimensionless. Small Reynolds number supports strong viscous influence; increasing it allows inertial effects and instability to become more important. Ordinary circular-pipe flow is often treated as laminar below values around 2000, with an intermediate transition region and commonly turbulent behaviour at several thousand and above. These are guidance ranges, not universal boundary conditions: inlet disturbances, roughness, development length and the manner of forcing affect transition.

A self-consistency check should follow, rather than precede, the calculation when the speed is initially unknown. Assume a laminar model, compute $\bar{v}$, then evaluate Re. If that predicted Reynolds number is far outside the credible laminar range, reject the laminar result as a description of the apparatus. Substituting an unknown turbulent viscosity into Poiseuille's formula would conceal the missing physics rather than solve it.

The additional fully developed assumption concerns inlet development. Fluid entering a tube generally adjusts from the inlet profile toward the downstream developed profile. The required development distance depends on the inlet condition and on the tolerated departure from the final profile. A low Reynolds number alone does not prove that a short tube reaches a fully developed

profile; that assumption needs separate geometrical or experimental support. The horizontal derivation also excludes the axial component of gravity; an inclined tube requires the corresponding elevation contribution in its driving potential.

Example 14.19: E14-06: A viscous tube and a consistency audit

A Newtonian liquid has stipulated $\rho = 1000 \text{ kg m}^{-3}$ and $\eta_v = 1.00 \times 10^{-3} \text{ Pa s}$. A horizontal straight tube has radius $R = 0.50 \text{ mm}$ and length $L = 2.0 \text{ m}$ under pressure drop $\Delta p = 5.0 \text{ kPa}$. Assume fully developed laminar flow and find (a) transport and both velocity scales, (b) Reynolds number, and (c) dissipated power. Audit the assumed regime and the effect of a 10% radius increase.

Transport. Converting every length to metres,

$$\dot{V} = \frac{\pi(0.50 \times 10^{-3})^4(5.0 \times 10^3)}{8(1.00 \times 10^{-3})(2.0)} = 6.14 \times 10^{-8} \text{ m}^3 \text{ s}^{-1}.$$

Therefore

$$\bar{v} = \frac{\dot{V}}{\pi R^2} = 0.0781 \text{ m s}^{-1}, \quad v_{\max} = 2\bar{v} = 0.156 \text{ m s}^{-1}.$$

Regime. With $D = 1.00 \text{ mm}$,

$$\text{Re} = \frac{(1000)(0.078125)(1.00 \times 10^{-3})}{1.00 \times 10^{-3}} = 78.1.$$

This low Reynolds number supports the laminar assumption. The tube length is 2000 diameters, but a developed inlet region remains a distinct modeling assumption; a length-to-diameter ratio by itself is not a proof. The calculation also does not check bore shape, property variation or experimental contamination.

Power and sensitivity.

$$P_{\text{diss}} = (5000)(6.1359 \times 10^{-8}) = 3.07 \times 10^{-4} \text{ W}.$$

At fixed $\Delta p, L, \eta_v$, increasing R by 10% multiplies $\dot{V}$ by $1.1^4 = 1.4641$, while $\bar{v}$ increases by 1.1^2 and Re by 1.1^3 . The revised Reynolds number is approximately 104, still low. The scaling is thus self-consistent for this modest change.

Independent force check. The wall shear predicted by the core balance is $|\tau_w| = \Delta p R / (2L) = 0.625 \text{ Pa}$. Multiplying by wall area $2\pi RL$ gives the same force as $\Delta p \pi R^2$. The derivative of the velocity profile at R reproduces this shear.

The quantitative pressure graph in Figure 14.27 uses the same example. An arbitrarily specified outlet gauge pressure of zero fixes the otherwise irrelevant pressure datum. The pressure decreases linearly, whereas the developed profile and its mean remain unchanged. The second panel gives a different view of the radius sensitivity: normalized rate is a fourth-power function, while normalized Reynolds number at fixed pressure drop is a third-power function. Increasing radius can eventually undermine the laminar premise, so the graph is shown only over a modest interval whose Reynolds numbers remain small for this example.

Guided checks

Diagnostic X14-06. A student substitutes centreline speed into a Reynolds number defined with mean speed. What changes? **Reasoned answer:** the definition has been changed without acknowledgment. For this circular parabolic profile, $v_{\max} = 2\bar{v}$, so the numerical value doubles. State the length and velocity scales whenever quoting a Reynolds number; comparisons to a regime range require consistent definitions.

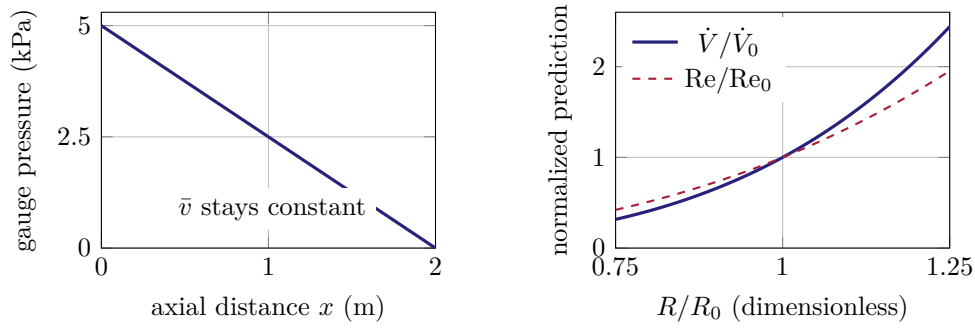


Figure 14.27: Pressure loss and radius sensitivity for E14-06. Left: a constant-speed developed flow loses pressure rather than gaining kinetic energy. Right: at fixed pressure drop, transport and Reynolds number respond differently to radius. The curves are analytical predictions within the stated laminar model.

Key message

A fully developed horizontal constant-area flow can have constant mean speed and a falling static pressure. Its pressure work is dissipated by viscosity. Continuity remains valid, but the lossless Bernoulli relation does not describe the pressure change along the tube.

14.7 Surface tension and capillarity

At a liquid interface, surface tension γ_s describes an energetic cost per area and a force per contact-line length, with equivalent units J m^{-2} and N m^{-1} . It acts tangentially to the interface. A narrow tube demonstrates how this interfacial effect can compete with gravity. Let θ_c be the equilibrium contact angle *measured through the liquid*. A wetting meniscus has liquid rising against the tube walls, producing a concave-up cross-sectional shape; a non-wetting meniscus has a convex dome-shaped meniscus and a depressed level. Figure 14.28 illustrates these geometrical distinctions.

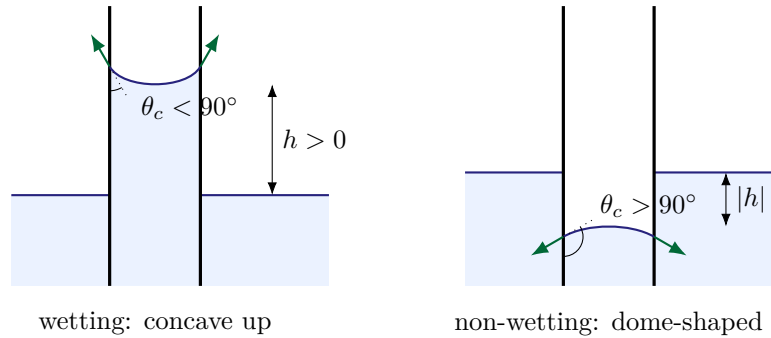


Figure 14.28: Rise and depression geometries. Contact angles are measured through the liquid, between the downward wall direction and the tangent into the meniscus. The outward tangent arrows at the two contacts have upward components for wetting and downward components for non-wetting. Meniscus shapes and heights are schematic; liquid shading inside each tube ends at its own interface, rather than extending across at the reservoir level.

Around the circular contact line of tube radius R , the vertical surface-tension resultant is

$$F_{\gamma,z} = 2\pi R \gamma_s \cos \theta_c. \quad (14.47)$$

For a narrow tube in a large reservoir, neglect the meniscus-volume correction and balance this resultant with the weight difference associated with a column height h , positive upward:

$$2\pi R \gamma_s \cos \theta_c = \rho g \pi R^2 h.$$

Consequently,

$$h = \frac{2\gamma_s \cos \theta_c}{\rho g R}. \quad (14.48)$$

A negative h is depression, rather than negative liquid mass. In that case the signed balance describes a deficit relative to the reservoir-level reference column. The inverse-radius dependence explains the growing importance of capillarity in narrow passages. A macroscopic contact angle, fixed surface tension and continuum liquid are still assumptions; contamination, hysteresis and very small scales may invalidate them.

Example 14.20: E14-07: Rise or depression in a narrow tube

Use stipulated $\gamma_s = 0.072 \text{ N m}^{-1}$, $\rho = 1000 \text{ kg m}^{-3}$ and $R = 0.50 \text{ mm}$. Find the ideal capillary height for (a) $\theta_c = 30^\circ$ and (b) $\theta_c = 120^\circ$.

For the wetting case,

$$h = \frac{2(0.072) \cos 30^\circ}{(1000)(9.81)(0.50 \times 10^{-3})} = 0.0254 \text{ m} = 25.4 \text{ mm}.$$

For the non-wetting case, $\cos 120^\circ = -1/2$, so

$$h = -0.0147 \text{ m} = -14.7 \text{ mm}.$$

The signs match the two geometries in Figure 14.28. Dimensions reduce to length, and halving R doubles either magnitude. These teaching values do not assert universal liquid–glass contact properties; the model neglects meniscus volume, evaporation and contact-angle hysteresis.

Guided checks

Diagnostic X14-07. Does replacing $\cos \theta_c$ by its absolute value improve the formula?
Reasoned answer: it erases the distinction between rise and depression. Keep the signed cosine and define upward height as positive. If only a magnitude is requested, take the absolute value after determining the physical sign.

14.8 Engineering, computation and measurement connections

The useful outcome of a fluid calculation is a statement about an observable under a specified model. Pressure-port readings, collected volume, piston displacement and body immersion are observables. Density, viscosity and wall conditions link them through equations. A calculation becomes testable when the observations and the linking assumptions are separated.

14.8.1 A pressure reference is part of the apparatus

A differential sensor requires two connections even when only one is drawn prominently in an apparatus sketch. Record the reference port and its elevation, the liquid filling any connecting lines, and the zero procedure. A line containing liquid contributes a hydrostatic correction between a pressure tap and a remote sensor. Trapped gas can introduce a compliant volume and delay the reading. A gauge calibrated to atmosphere cannot be interpreted as an absolute sensor without an independently specified atmospheric reference.

For a proposed hydrostatic investigation, measure pressure difference at several depths in a stationary liquid. Plot the difference against depth rather than silently converting a sensor display into a force. A straight-line slope estimates ρg only if the density is approximately constant and the reference is stable. An intercept can indicate zero offset, an incorrectly located height origin or a real pressure

difference at the chosen origin. Repeating readings at increasing and decreasing depth helps reveal hysteresis or drift.

A pressure-depth experiment and a tube-flow experiment test different physics. The first can operate at rest without a viscosity model. The second infers a constitutive parameter from moving fluid and needs independent checks of the flow profile, radius and temperature. Agreement with hydrostatics therefore cannot certify a Poiseuille flow model.

14.8.2 A synthetic viscosity investigation

Consider a proposed experiment with a horizontal circular tube whose stipulated inner radius is $R = 0.500$ mm and length $L = 2.00$ m. Collect the effluent for a measured time at each imposed pressure difference. Divide collected volume by elapsed time to obtain $\dot{V}$. Record fluid temperature and repeated trials. The data below are *synthetic teaching data*; they were not collected from an apparatus.

Δp (kPa)	0	1	2	3	4	5
$\dot{V}$ (mL min ⁻¹)	0.04	0.77	1.47	2.27	2.94	3.75

The ideal laminar model predicts a line through the origin, $\dot{V} = S\Delta p$, with $S = \pi R^4/(8\eta_v L)$. Fitting an intercept initially is useful because forcing the line through zero can hide a sensor offset. A nonzero fitted intercept does not automatically identify its physical cause; one should investigate pressure zero, collection bias, leakage and timing procedures.

Figure 14.29 separates agreement with a line from the pattern of the remaining differences. Both graphs use the same six data values. The residual is observed flow minus fitted flow. A systematic curved pattern would challenge the assumed linear relation, while an apparent straight line across a narrow interval might still conceal changing viscosity or tube dimensions.

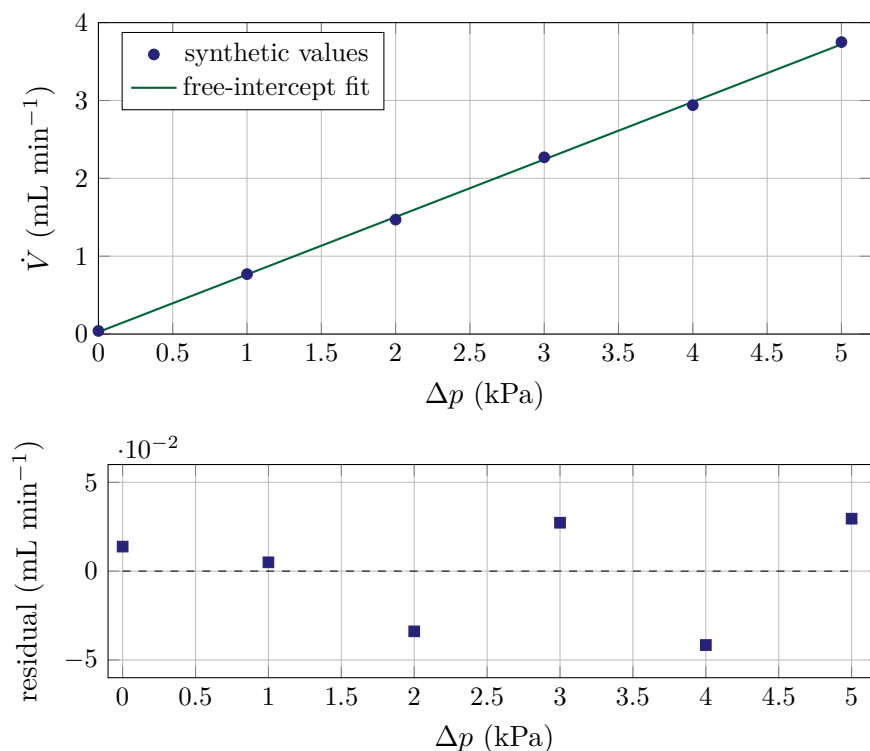


Figure 14.29: Synthetic flow data and residuals for the same fitted model. The residuals change sign without a clear monotonic trend in this small sample. This supports studying the linear model over the specified range; it does not identify viscosity independently of geometry or establish a fully developed profile.

Example 14.21: Viscosity from synthetic data

Statement. Using the six values above and $R = 0.500$ mm, $L = 2.00$ m, fit $y = mx + b$ with $x = \Delta p$ in kPa and $y = \dot{V}$ in mL min⁻¹. Infer viscosity from the slope and estimate the statistical slope uncertainty if independent equal-variance errors in y dominate and the x values and geometry are treated as exact.

Strategy and fit. Minimize the sum of squared residuals. With $\bar{x}$ and $\bar{y}$ the arithmetic means, differentiating the sum with respect to m and b gives

$$m = \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sum_i (x_i - \bar{x})^2}, \quad b = \bar{y} - m\bar{x}.$$

Here $\bar{x} = 2.5$, $\bar{y} = 1.87333$, $\sum (x_i - \bar{x})^2 = 17.5$, and $\sum (x_i - \bar{x})(y_i - \bar{y}) = 12.9300$. Thus

$$m = 0.738857 \text{ mL min}^{-1} \text{ kPa}^{-1}, \quad b = 0.0261905 \text{ mL min}^{-1}.$$

Convert the slope before using SI geometry:

$$S = m \frac{10^{-6}}{60 \times 10^3} = 1.23143 \times 10^{-11} \text{ m}^3 \text{ s}^{-1} \text{ Pa}^{-1}.$$

Therefore

$$\eta_v = \frac{\pi R^4}{8LS} = 9.96554 \times 10^{-4} \text{ Pa s}.$$

The fitted parameter is conditional on the stipulated radius and laminar developed-flow model.

Statistical estimate. For $n = 6$ values and two fitted parameters, the residual standard deviation is

$$s_y = \sqrt{\frac{\sum_i [y_i - (mx_i + b)]^2}{n - 2}} = 0.0343165 \text{ mL min}^{-1}.$$

Consequently $\sigma_m = s_y / \sqrt{17.5} = 0.00820320 \text{ mL min}^{-1} \text{ kPa}^{-1}$. Since viscosity is proportional to m^{-1} , first-order propagation gives $\sigma_{\eta_v} / \eta_v = \sigma_m / m = 1.11\%$, or $\sigma_{\eta_v} = 1.11 \times 10^{-5} \text{ Pa s}$. The intercept uncertainty from the same model is $0.0248364 \text{ mL min}^{-1}$, so this small sample does not establish a significant physical offset.

Checks and limitations. The fitted line's residuals sum to zero, and their scalar product with $x_i - \bar{x}$ is zero, as required by the least-squares normal equations. Neither a small residual nor this statistical uncertainty includes an error in the radius, viscosity variation with temperature or an inadequate wall model. A reported viscosity should include those contributions when they are known.

14.8.3 Radius sensitivity and uncertainty

The fourth power of radius makes bore measurement unusually consequential. Logarithmic differentiation of the laminar formula, holding the model assumptions fixed, gives

$$\frac{d\dot{V}}{\dot{V}} = 4 \frac{dR}{R} + \frac{d\Delta p}{\Delta p} - \frac{d\eta_v}{\eta_v} - \frac{dL}{L}.$$

This is a signed first-order sensitivity relation, not yet an uncertainty formula. For independent standard uncertainties, variances add:

$$\left(\frac{\sigma_{\dot{V}}}{\dot{V}}\right)^2 \simeq \left(4 \frac{\sigma_R}{R}\right)^2 + \left(\frac{\sigma_{\Delta p}}{\Delta p}\right)^2 + \left(\frac{\sigma_{\eta_v}}{\eta_v}\right)^2 + \left(\frac{\sigma_L}{L}\right)^2. \quad (14.49)$$

If measurements share a calibration source, covariance terms must be retained. For example, a common length calibration can correlate radius and tube length. Because their sensitivity signs

differ, the associated covariance contribution can reduce or increase the combined variance according to its sign. Independence must be justified by the measurement procedure, rather than inferred from different column headings.

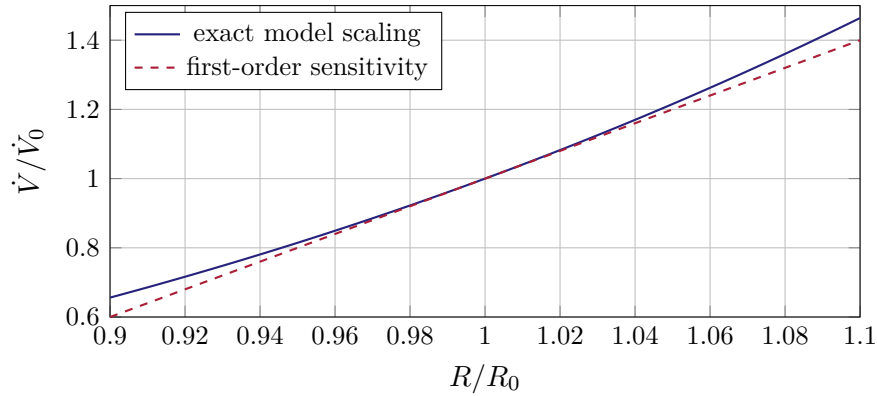


Figure 14.30: Radius sensitivity at fixed pressure difference, length and viscosity. The tangent correctly describes small changes but increasingly misses the fourth-power curvature as the radius change grows. Both curves are dimensionless quantitative predictions.

14.8.4 One available head, two limiting models

A reservoir free surface above an outlet supplies gravitational energy per unit weight. An ideal short outlet mainly converts that head into kinetic energy. A long narrow tube mainly dissipates it. Treating the entire hydrostatic head as a Poiseuille pressure drop is a useful approximation when the outlet kinetic-energy flux is small; it should not be represented as exact for an accelerating fluid leaving a reservoir.

For a steady incompressible section, kinetic-energy transport is weighted by v^3 , because faster fluid crosses the section more rapidly and carries more kinetic energy per mass. Define

$$\alpha = \frac{1}{A\bar{v}^3} \int_A v^3 dA.$$

Then the transported kinetic-energy head is $\alpha\bar{v}^2/(2g)$. A uniform profile has $\alpha = 1$. The parabolic circular-tube profile has $\alpha = 2$, derived in the advanced section. With that profile at an atmospheric outlet, negligible reservoir speed, and entrance/minor losses deliberately omitted, a reduced head balance is

$$h = \frac{8\eta_v L}{\rho g R^2} \bar{v} + \frac{\alpha\bar{v}^2}{2g}. \quad (14.50)$$

The first term is the developed tube friction loss; the second is the outgoing kinetic-energy flux. This mixed model still approximates inlet development and outlet geometry. It improves energy bookkeeping without becoming a complete piping-system design.

Example 14.22: A low-head narrow tube

Statement. A large reservoir supplies $h = 0.500$ m of head to a tube with $R = 0.750$ mm, $L = 2.00$ m, $\rho = 1000$ kg m⁻³ and $\eta_v = 1.00 \times 10^{-3}$ Pa s. Compare the purely resistive approximation with Eq. (14.50) using $\alpha = 2$. Neglect entrance and minor losses and assess the scale of the kinetic correction.

Approximation. Assigning the full head to viscous pressure drop gives $\Delta p = \rho gh = 4905$ Pa and

$$\bar{v}_0 = \frac{\rho gh R^2}{8\eta_v L} = 0.172441 \text{ m s}^{-1}, \quad \dot{V}_0 = 3.04729 \times 10^{-7} \text{ m}^3 \text{ s}^{-1}.$$

Head-consistent correction. Write $a = 8\eta_v L / (\rho g R^2) = 2.89954 \text{ s}$ and $b = \alpha / (2g) = 0.101937 \text{ s}^2 \text{ m}^{-1}$. The positive solution of $b\bar{v}^2 + a\bar{v} - h = 0$, in a form avoiding subtractive cancellation, is

$$\bar{v} = \frac{2h}{a + \sqrt{a^2 + 4bh}} = 0.171408 \text{ m s}^{-1}.$$

It gives $\dot{V} = 3.02904 \times 10^{-7} \text{ m}^3 \text{ s}^{-1}$ and $\text{Re} = 257.113$. The friction head is $a\bar{v} = 0.497005 \text{ m}$ and the outgoing kinetic head is $b\bar{v}^2 = 0.00299499 \text{ m}$, which sum to 0.500000 m .

Interpretation. The purely resistive approximation overestimates the mean speed by 0.603% in this reduced model. The lossless efflux value $\sqrt{2gh} = 3.13209 \text{ m s}^{-1}$ describes a different limiting geometry and is much larger. The laminar Reynolds number supports the selected tube regime, but does not verify the omitted inlet loss or the developed-profile outlet assumption.

14.8.5 A reproducible computation and its assessment

For positive pressure drop, the following optional Python listing evaluates the baseline tube model. Required results and plotting data are already embedded in this chapter; software is not needed to compile or understand it.

```
from math import pi

def laminar_tube(rho, eta, radius, length, delta_p):
    if min(rho, eta, radius, length) <= 0:
        raise ValueError("material and geometry inputs must be positive")
    qdot = pi*radius**4*delta_p/(8*eta*length)
    vmean = qdot/(pi*radius**2)
    reynolds = rho*abs(vmean)*(2*radius)/eta
    return qdot, vmean, reynolds

print(laminar_tube(1000, 1e-3, 0.50e-3, 2.0, 5000))
# (6.13592315e-8, 0.078125, 78.125)
```

The signed pressure difference determines flow direction. The Reynolds number above uses the magnitude of the mean velocity as its characteristic speed. A zero pressure difference should give zero flow; reversing the pressure difference should reverse the flow without changing this Reynolds magnitude. These are useful tests because they examine limiting behavior and convention consistency rather than merely duplicating one positive-pressure calculation.

An adequate experimental report identifies the pressure zero, inner-bore measurement, temperature, collection and timing method, repeated trials and uncertainty sources. Its reference analysis should include a regime estimate, residual plot and a discussion of whether entrance length or wall behavior could bias viscosity inference. A result is defensible when the assumptions and limitations explain what was measured, not when a software output prints many digits. Problem P14-33 supplies a complete reference plan and evaluation criteria.

14.9 Advanced Topics for Technical and Engineering Students

This section derives the pressure-driven circular-tube profile, checks its transported energy and dissipation, and then separates numerical integration from physical validation. The prerequisites are force balance, ordinary derivatives and one-dimensional integrals. The final discussion of slip and rarefaction is optional enrichment; it introduces limits of the core model without requiring a general fluid partial differential equation.

14.9.1 Shear-force derivation of the parabolic profile

Consider a straight horizontal circular tube of radius R . A coaxial cylindrical fluid core of radius $r \leq R$ and length L is selected as the system. Assume steady, incompressible, axisymmetric, fully developed flow of a Newtonian fluid. Fully developed means that the axial velocity profile does not change with axial position. The core therefore has no net axial acceleration caused by downstream profile development. Constant viscosity and no slip are separate assumptions, not consequences of steady motion.

Pressure at the upstream end pushes in the positive axial direction; downstream pressure pushes in the negative direction. Their resultant is

$$F_p = \Delta p \pi r^2, \quad \Delta p = p_{\text{in}} - p_{\text{out}} > 0. \quad (14.51)$$

The surrounding fluid acts on the cylindrical side through resisting shear. Write $\tau(r)$ for the positive magnitude of the backward shear stress on this selected core. Its force magnitude is

$$F_\tau = \tau(r) 2\pi r L. \quad (14.52)$$

Axial force balance requires $F_p - F_\tau = 0$. Hence

$$\tau(r) = \frac{\Delta p}{2L} r. \quad (14.53)$$

The stress vanishes at the centreline and increases linearly to the wall. A larger core receives pressure driving proportional to its end area, r^2 , but resisting shear acts over a side area proportional to r ; their balance therefore requires stress proportional to r .

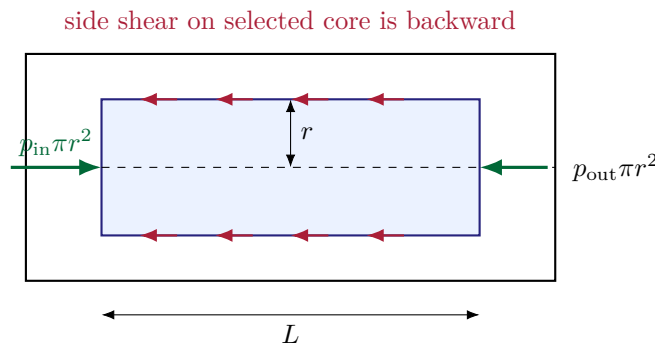


Figure 14.31: Selected fluid-core forces in a horizontal developed-flow model. Pressure acts inward on the two end faces. The net pressure driving is balanced by shear on the cylindrical side. Arrow lengths are schematic; the signed force balance determines the result.

Velocity decreases as radial distance increases, so the chosen positive resisting-stress magnitude satisfies

$$\tau = -\eta_v \frac{dv}{dr}. \quad (14.54)$$

The minus sign follows from the negative radial velocity gradient, not from a negative viscosity. Combining the two relations and integrating gives

$$\frac{dv}{dr} = -\frac{\Delta p}{2\eta_v L} r, \quad v(r) = -\frac{\Delta p}{4\eta_v L} r^2 + C.$$

No slip at the stationary wall, $v(R) = 0$, fixes the integration constant. Thus

$$v(r) = \frac{\Delta p}{4\eta_v L}(R^2 - r^2). \quad (14.55)$$

The modelling sequence is force balance, Newtonian constitutive relation, then wall boundary condition. Changing the wall condition changes the constant, while changing the constitutive law may change the shape itself. Continuity alone supplies neither result.

14.9.2 Annular integration and the correct section average

A narrow annulus has area $dA = 2\pi r dr$. Therefore

$$\dot{V} = 2\pi \int_0^R v(r) r dr \quad (14.56)$$

$$= \frac{\pi \Delta p}{2\eta_v L} \int_0^R (R^2 r - r^3) dr = \frac{\pi R^4 \Delta p}{8\eta_v L}. \quad (14.57)$$

It follows that $\bar{v} = \dot{V}/(\pi R^2) = v_{\max}/2$. Averaging velocities at equal radial intervals without their annular areas gives a different answer. Outer intervals contain more area than equally wide inner intervals. That geometric weighting is why an unweighted mean along a diameter cannot replace the section average.

Let $s = r/R$ and $u = v/v_{\max} = 1 - s^2$. The normalized flow is

$$q = \frac{\dot{V}}{v_{\max} R^2} = 2\pi \int_0^1 (1 - s^2) s ds = \frac{\pi}{2}. \quad (14.58)$$

The unitless radial measure contains the factor s . One can check the average independently by setting $w = s^2$: since $2s ds = dw$, the normalized area average becomes $\int_0^1 (1 - w) dw = 1/2$.

14.9.3 Transported kinetic energy and viscous dissipation

The kinetic-energy flux through a section is $\frac{1}{2}\rho \int_A v^3 dA$. For the parabolic profile,

$$\int_A v^3 dA = 2\pi R^2 v_{\max}^3 \int_0^1 (1 - s^2)^3 s ds = \frac{\pi R^2 v_{\max}^3}{4}.$$

Since $\bar{v} = v_{\max}/2$, division by $A\bar{v}^3$ gives $\alpha = 2$. In the same way, the momentum-flux correction is $\int_A v^2 dA/(A\bar{v}^2) = 4/3$. These factors belong to different balances. Neither changes mass conservation, for which $\int_A v dA = A\bar{v}$ by definition.

In a fully developed constant-area horizontal tube, the velocity profile and therefore its transported kinetic-energy flux are the same at both ends. The pressure work rate $\Delta p \dot{V}$ is instead converted into internal energy through viscosity. Locally, the mechanical dissipation per unit volume for this simple shear field is $\eta_v (dv/dr)^2$. Integrating over the tube gives

$$P_{\text{diss}} = 2\pi L \int_0^R \eta_v \left(\frac{dv}{dr} \right)^2 r dr \quad (14.59)$$

$$= \frac{\pi \Delta p^2}{2\eta_v L} \int_0^R r^3 dr = \frac{\pi \Delta p^2 R^4}{8\eta_v L} = \Delta p \dot{V}. \quad (14.60)$$

This equality is an energy check derived independently of the section-flow integral. The cumulative fraction of dissipation inside radius r is $(r/R)^4$. The shear gradient is largest near the wall, so the outer region contributes disproportionately.

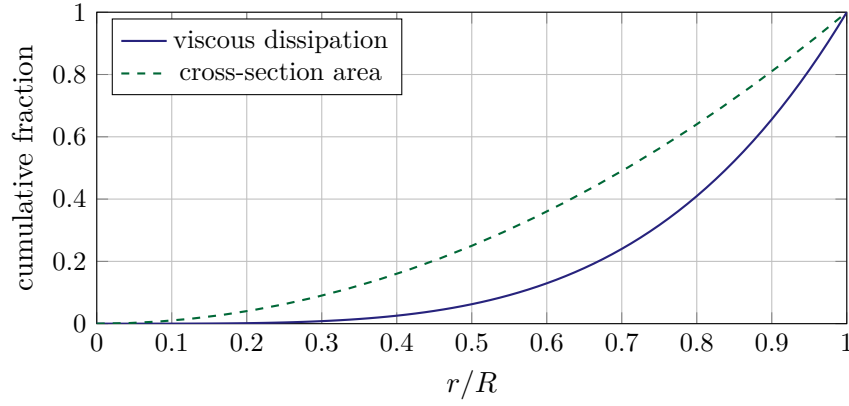


Figure 14.32: Cumulative radial contributions in developed laminar tube flow. The central half-radius region contains 25% of the area but only 6.25% of the viscous dissipation. The remaining 93.75% occurs in the outer annulus in this Newtonian model.

Example 14.23: Pressure work and dissipation

Statement. For the baseline tube with $R = 0.500$ mm, $L = 2.00$ m, $\eta_v = 1.00 \times 10^{-3}$ Pa s and $\Delta p = 5.00$ kPa, calculate the wall shear stress and viscous power. Explain why no kinetic-energy increase is required between developed sections.

Calculation. The wall stress is $\tau_w = \Delta p R / (2L) = 0.625$ Pa. With $\dot{V} = 6.13592315 \times 10^{-8} \text{ m}^3 \text{ s}^{-1}$, pressure power is

$$P = \Delta p \dot{V} = 3.06796158 \times 10^{-4} \text{ W} = 0.306796 \text{ mW}.$$

The integral in Eq. (14.60) gives the same value from the squared shear gradient, confirming the energy balance. Since the developed profile is unchanged, both ends transport the same kinetic energy per time. The pressure energy is dissipated inside the tube. This does not mean total energy is destroyed; its mechanical part becomes internal energy, with any resulting heat transfer depending on the thermal boundary conditions.

14.9.4 Numerical quadrature of a prescribed profile

Partition $0 \leq s \leq 1$ into N equal radial intervals. At the midpoint $s_{j+1/2} = (j + 1/2)/N$, evaluate $u = 1 - s^2$ and multiply by the exact annulus area. The normalized midpoint estimate is

$$q_N = \pi \sum_{j=0}^{N-1} \left[1 - s_{j+1/2}^2 \right] \left(s_{j+1}^2 - s_j^2 \right), \quad s_j = j/N. \quad (14.61)$$

Because $s_{j+1}^2 - s_j^2 = 2s_{j+1/2}/N$, this is also midpoint quadrature for $2\pi(s - s^3)$. Its exact discrete sum is

$$q_N = \frac{\pi}{2} + \frac{\pi}{4N^2}, \quad \frac{q_N - q}{q} = \frac{1}{2N^2}.$$

One can obtain the expression by expanding the midpoint powers and summing j , j^2 and j^3 . It shows that the approximation is an overestimate and predicts the sign and order before a graph is drawn.

N	q_N	relative error (%)	refinement ratio
2	1.767145868	12.5000	—
4	1.619883712	3.12500	4
8	1.583068173	0.781250	4
16	1.573864288	0.195313	4
32	1.571563317	0.0488281	4
64	1.570988074	0.0122070	4

The exact result is $q = 1.570796327$. Doubling N divides the error by four, so the numerical order is two. The evidence concerns integration of a smooth prescribed function. It cannot establish that a real inlet profile has become parabolic or that its wall satisfies no slip.

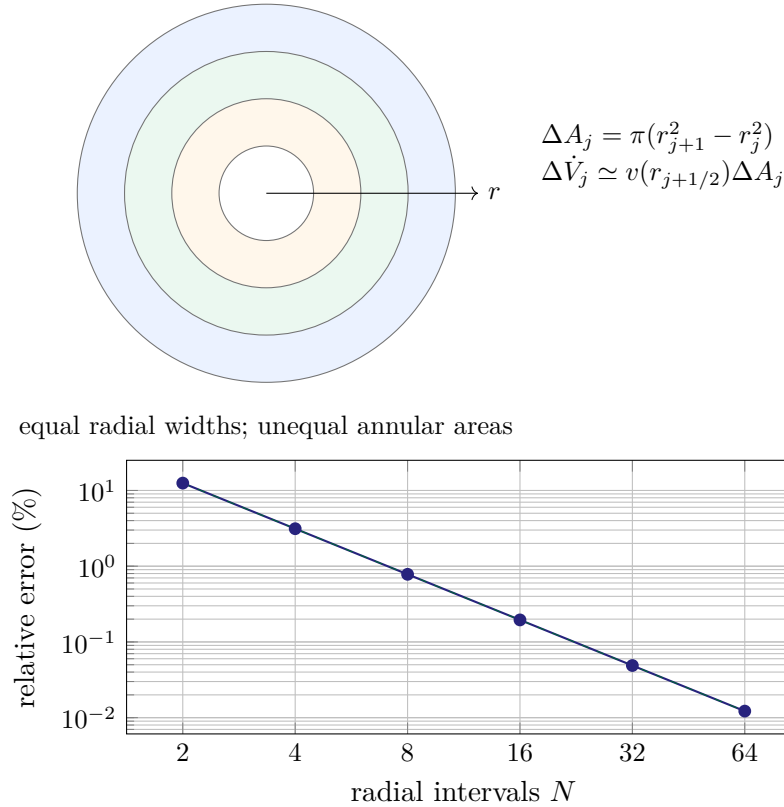


Figure 14.33: Annular midpoint integration and its convergence. The upper construction shows why equal radial intervals require different area weights. The points below are discrete sums; the dashed curve is the independently derived error $50/N^2$ in percent. Their agreement tests the quadrature, while the physical profile remains an assumption.

Example 14.24: E14-08: Integrate a known profile

Statement. For $u(s) = 1 - s^2$, estimate $q = \dot{V}/(v_{\max} R^2)$ by annular midpoint quadrature with $N = 4, 8, 16$ and compare with $\pi/2$. How many radial intervals guarantee a relative quadrature error below 0.1% for this particular profile?

Solution. The discrete sum gives $q_4 = 1.619883712$, $q_8 = 1.583068173$ and $q_{16} = 1.573864288$. Against $q = 1.570796327$, the respective relative errors are 3.125%, 0.78125% and 0.1953125%. Their ratio is four on each doubling, as predicted by $1/(2N^2)$.

To require error below 0.001, solve $1/(2N^2) < 0.001$, or $N > \sqrt{500} = 22.3607$. The smallest integer choice is $N = 23$, giving 0.094518%; $N = 22$ gives 0.103306% and fails the requested bound. This result uses the exact error formula for the specified profile and should not be transferred unchanged to another profile or quadrature rule.

Interpretation. The numerical model approaches the analytic integral. Near a real inlet, profile development can still invalidate the assumed parabola. Convergence therefore answers a mathematical question, while physical validation requires geometry, pressure, viscosity, regime and wall-condition evidence.

Guided checks

Diagnostic X14-08. Six stable numerical digits do not establish the physical validity of Poiseuille flow. They can establish accurate integration of the prescribed profile. Physical validation separately requires evidence for laminar developed flow, Newtonian behavior, geometry and boundary conditions. Problem P14-28 develops the convergence interpretation.

14.9.5 Continuum breakdown and wall-condition limits

A continuum field describes an average over molecular scales while resolving macroscopic variation. For a gas, the Knudsen number compares molecular mean free path λ with a relevant channel scale L_c :

$$\text{Kn} = \lambda/L_c. \quad (14.62)$$

For stipulated $\lambda = 70 \text{ nm}$, the values are $\text{Kn} = 1.4 \times 10^{-3}$ at $L_c = 50 \mu\text{m}$ and $\text{Kn} = 0.14$ at $L_c = 0.50 \mu\text{m}$. The second channel has much weaker separation between molecular and device scales. Ordinary no-slip continuum modelling then requires reconsideration. These are scale comparisons, not universal thresholds for every gas observable.

Even when a continuum description is useful in the bulk, the wall condition can be refined. As an illustrative mathematical boundary model, let a stationary wall permit a slip length $\ell_s \geq 0$, so that $v(R) = \ell_s[-v'(R)]$. Keep the same Newtonian shear-force balance. Integrating and imposing this boundary condition yields

$$v(r) = \frac{\Delta p}{4\eta_v L}(R^2 - r^2 + 2\ell_s R), \quad \dot{V}_{\text{slip}} = \dot{V}_{\text{no-slip}} \left(1 + \frac{4\ell_s}{R}\right).$$

This example is a prescribed boundary model, not a determination of ℓ_s for a material. It recovers no slip as $\ell_s \rightarrow 0$ and demonstrates why high numerical accuracy cannot compensate for an inappropriate boundary condition.

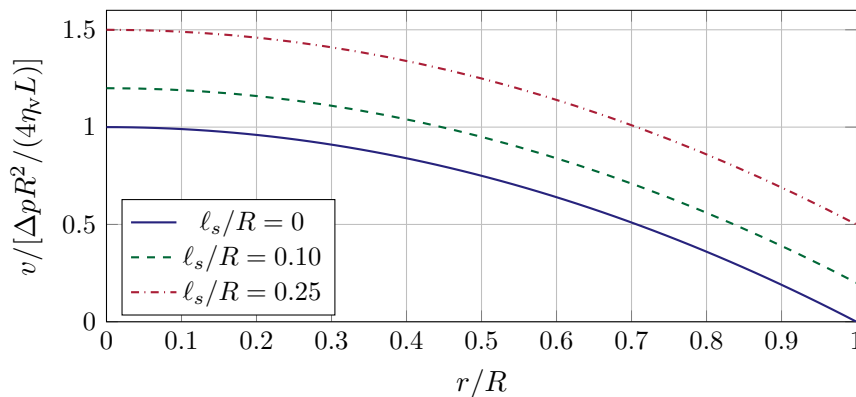


Figure 14.34: A prescribed slip-length extension at the same pressure gradient, radius and viscosity. Curves differ by a constant offset; the wall speed is nonzero when the imposed slip length is positive. The plot illustrates a boundary-model consequence and supplies no empirical slip-length values.

Example 14.25: A prescribed wall refinement

Statement. Within the slip-length model above, take $\ell_s/R = 0.10$ at fixed pressure gradient and geometry. Find the wall speed relative to the no-slip centreline scale, the flow-rate ratio and the actual centreline-to-mean-speed ratio.

Solution. Let $v_0 = \Delta p R^2 / (4\eta_v L)$ denote the no-slip centreline scale. The boundary speed is $v(R) = 2(\ell_s/R)v_0 = 0.20v_0$, while the centreline speed is $1.20v_0$. Annular integration gives $\bar{v} = (1/2 + 2\ell_s/R)v_0 = 0.70v_0$, so $\dot{V}_{\text{slip}}/\dot{V}_{\text{no-slip}} = 1.40$ and $v(0)/\bar{v} = 12/7 \simeq 1.71429$. The familiar factor of two is therefore a consequence of the no-slip parabolic model, not a universal property of all tube flows. The shear gradient remains unchanged at the fixed pressure gradient, consistently with the unchanged core-force balance.

Take-home message

Two different questions must be answered: does the numerical calculation converge to the stated mathematical model, and does that model adequately describe the chosen experiment? Accurate quadrature, a low Reynolds number and a small residual each supply useful evidence, but they address different parts of that assessment.

14.10 Chapter summary and working formula sheet

Pressure is a scalar field whose local force depends on surface orientation. Static force balance gives the pressure gradient; integrating pressure gives both a resultant and its moment. Buoyancy follows from the complete pressure-force balance around displaced fluid. Ideal hydraulics conserves volume and work while trading force for displacement.

For flow, mass conservation and energy balance provide different information. Continuity fixes mass transport. Bernoulli relates pressure, local kinetic energy and elevation along a qualified ideal streamline. Viscosity connects stress to velocity gradient and supplies a dissipation mechanism. In a developed tube, pressure can decrease without increasing mean speed. Surface tension acts at an interface and capillary height retains the sign of its contact-angle factor.

Formula summary**Pressure and static balance**

$$p_{\text{gauge}} = p_{\text{abs}} - p_{\text{ref}}, \quad d\mathbf{F} = -p\hat{\mathbf{n}}dA, \quad \frac{dp}{dz} = -\rho g.$$

For constant density, $p_2 - p_1 = -\rho g(z_2 - z_1)$. A plane gate requires $F = \int_A \Delta p dA$ and a moment integral to locate the resultant.

Hydraulics and buoyancy

$$F_1/A_1 = F_2/A_2, \quad A_1 \Delta x_1 = A_2 \Delta x_2, \quad F_B = \rho_f g V_{\text{displaced}}.$$

The hydraulic force relation assumes equal piston elevations and properly subtracted surrounding pressure. Partial floating requires $m/\rho_f < V_{\text{object}}$, with no additional vertical force in the simple model.

Mass and ideal streamline energy

$$\dot{m} = \rho A \bar{v}, \quad \dot{V} = A \bar{v}, \quad p + \frac{1}{2} \rho v^2 + \rho g z = \text{constant along a streamline.}$$

Steady incompressible single-stream flow has the same $\dot{V}$ at each section. Bernoulli further requires negligible viscosity, no added shaft work and the stated streamline conditions.

Horizontal Newtonian developed laminar tube

$$v(r) = \frac{\Delta p}{4\eta_v L}(R^2 - r^2), \quad \dot{V} = \frac{\pi R^4 \Delta p}{8\eta_v L}, \quad v_{\max} = 2\bar{v}.$$

$$\text{Re} = \frac{\rho \bar{v} D}{\eta_v}, \quad P_{\text{diss}} = \Delta p \dot{V}, \quad \tau_w = \frac{\Delta p R}{2L}.$$

A signed flow calculation can use signed Δp and mean axial velocity; the Reynolds characteristic speed is their magnitude. The energy-flux correction is $\alpha = 2$ for this profile.

Interface and scale limits

$$h = \frac{2\gamma_s \cos \theta_c}{\rho g R}, \quad \text{Kn} = \lambda/L_c.$$

The contact angle is measured through liquid; meniscus-volume corrections and dynamic wetting are omitted from the capillary-column estimate.

Before accepting a result, inspect its pressure reference, area units, velocity definition, force directions, energy terms, regime and limiting behavior. A complete answer explains why the selected law applies and what would cause it to cease applying. The next chapter, Oscillations, returns to time-dependent motion. Its core requires the earlier mechanics of forces and energy rather than the optional slip or numerical details developed here.

14.11 Exercises and problems

Use $g = 9.81 \text{ m s}^{-2}$. Fluid properties are stipulated inputs unless identified as measurements. For each calculation state the physical model, show the governing relation, retain intermediate digits and give units. Full solutions to all tasks and subparts follow in Section 14.12. The diagnostic tags X14-01–X14-08 link the core misconceptions to corresponding practice tasks; the diagnostic explanations in the exposition remain valid independently.

Problems 1–10 develop pressure, force and equilibrium; 11–19 develop conservation and ideal flow; 20–27 address constitutive models and interfaces; 28–33 require advanced or experimental reasoning. Problems 34–43 deliberately change the representation or combine principles. The later tasks identify any additional assumptions needed; they are not prerequisites for the core material.

A. Pressure, hydrostatics and measurement

P14-01. [X14-01] A pressure gauge connected to a vessel reads 0 kPa while its reference port is open to an atmosphere at 99.5 kPa absolute. A student concludes that the vessel is evacuated. Diagnose the error and state the vessel pressure implied by the ideal gauge reading.

P14-02. A cube of side 0.20 m is surrounded by a static fluid at uniform pressure 300 kPa. Find the pressure-force magnitude on one face. What is the vector sum of pressure forces over all six faces if gravity is neglected and the pressure is truly uniform?

P14-03. A liquid of density 1000 kg m^{-3} has a free surface at 101.0 kPa absolute. Find the gauge and absolute pressure 6.50 m below the surface, assuming constant density.

P14-04. Two gas spaces are connected by a U-tube manometer containing a liquid of density 920 kg m^{-3} . The liquid level on side A is 0.180 m lower than on side B. Neglect gas-density effects. Find $p_A - p_B$ and explain the sign.

P14-05. [X14-02] A vertical rectangular gate 0.80 m wide and 1.50 m high has its upper edge at a liquid free surface. A student uses the pressure at mid-depth, obtains the correct total force, and places the resultant at mid-depth. Find the total force and the correct center-of-pressure depth for $\rho = 1000 \text{ kg m}^{-3}$, assuming equal atmospheric pressure on the dry side and the liquid surface.

P14-06. A vertical rectangular panel is 2.0 m wide and extends from the free surface to depth 3.0 m in a liquid of density 1000 kg m^{-3} . Find the net hydrostatic force and its depth below the free surface when equal atmosphere acts on the dry side.

B. Hydraulics and buoyancy

P14-07. [X14-03] An ideal hydraulic press has output-piston area 30 times its input area. A 150 N input force acts while the input piston moves 0.300 m. Find the ideal output force, output displacement and input/output work. Explain why “30 times the force” does not mean “30 times the energy.”

P14-08. A uniform sealed object of average density 780 kg m^{-3} floats in a liquid of density 1000 kg m^{-3} . What fraction of its volume is submerged if surface-tension forces are negligible?

P14-09. A fully submerged object has volume $2.00 \times 10^{-3} \text{ m}^3$ and mass 3.00 kg. It is held at rest by a vertical force while immersed in a liquid of density 1000 kg m^{-3} . Find the buoyant force and the required holding-force magnitude and direction.

P14-10. A sealed object has external volume $1.5 \times 10^{-3} \text{ m}^3$ and mass 1.8 kg. Can it float partially in a liquid of density 1000 kg m^{-3} ? If not, find its downward-minus-upward force imbalance when fully submerged, neglecting drag and surface tension.

C. Mass conservation and flow kinematics

P14-11. [X14-04] A steady gas flow has $\dot{m} = 4.0 \text{ kg s}^{-1}$. At section 1, $A_1 = 0.020 \text{ m}^2$ and $\rho_1 = 1.20 \text{ kg m}^{-3}$. At section 2, $A_2 = 0.010 \text{ m}^2$ and $\rho_2 = 0.80 \text{ kg m}^{-3}$. Find the mean speeds. Explain why $A_1 v_1 = A_2 v_2$ would be wrong here and comment on whether the resulting speeds suggest that a constant-density gas model would be questionable.

P14-12. Water-like liquid with stipulated $\rho = 997 \text{ kg m}^{-3}$ passes through a section of area $3.00 \times 10^{-4} \text{ m}^2$ at mean speed 0.750 m s^{-1} . Find $\dot{V}$ and $\dot{m}$.

P14-13. An incompressible nozzle contracts from 20 cm^2 to 5.0 cm^2 . The upstream mean speed is 1.20 m s^{-1} . Find the downstream mean speed.

P14-14. Explain the difference between a streamline and a pathline. Give a condition under which they coincide and a reason they can differ.

D. Bernoulli reasoning and ideal flow

P14-15. [X14-05] A horizontal pipe has equal area at sections 1 and 2, equal density, and steady flow, but a measured pressure drop of 12 kPa. A student inserts the equal speeds into lossless Bernoulli and obtains $p_1 = p_2$. What does the disagreement diagnose? List at least two physical effects or modelling changes that could account for it.

P14-16. A large open tank discharges through a small outlet 1.80 m below the free surface. Estimate the ideal outlet speed when losses and free-surface speed are neglected.

P14-17. A horizontal ideal meter has $A_1 = 6.0 \times 10^{-3} \text{ m}^2$, $A_2 = 3.0 \times 10^{-3} \text{ m}^2$, $\rho = 1000 \text{ kg m}^{-3}$ and $p_1 - p_2 = 7.50 \text{ kPa}$. Find v_1 , v_2 and $\dot{V}$.

P14-18. An ideal incompressible flow has the same cross-section area and speed at two points. Point 2 is 8.0 m higher than point 1. If $p_1 = 250 \text{ kPa}$ and $\rho = 1000 \text{ kg m}^{-3}$, find p_2 . What energy conversion does the result represent?

P14-19. A student applies Bernoulli between the upstream and downstream sides of a running pump but includes no pump-work term. Explain why this is not the Bernoulli model developed in the chapter, even if density is constant and the flow is steady.

E. Viscosity, pressure loss and Reynolds number

P14-20. [X14-06] In fully developed laminar circular-tube flow, a student uses the centerline speed in a Reynolds number formula defined with the mean speed. By what factor does the reported Re differ from the mean-speed definition? Explain the source of the factor.

P14-21. In the Poiseuille model, a tube radius decreases by 5.0% while Δp , L and η_v are unchanged. By what factor does $\dot{V}$ change? Give the percentage decrease.

P14-22. A Newtonian liquid has $\rho = 900 \text{ kg m}^{-3}$, $\eta_v = 1.50 \times 10^{-3} \text{ Pa s}$ and flows through a straight horizontal circular tube with $R = 0.80 \text{ mm}$ and $L = 1.50 \text{ m}$ under $\Delta p = 3.00 \text{ kPa}$. Assuming fully developed laminar flow, find $\dot{V}$, $\bar{v}$ and Re.

P14-23. A constant-area horizontal pipe carries steady incompressible flow at essentially the same mean speed at two stations but has a pressure decrease of 20 kPa. For $\rho = 1000 \text{ kg m}^{-3}$, express this pressure decrease as an equivalent head $\Delta p/(\rho g)$. Why should it not be interpreted as a kinetic-energy increase if the speeds are equal?

P14-24. For an internal-flow calculation, $\rho = 850 \text{ kg m}^{-3}$, $D = 0.020 \text{ m}$, $\bar{v} = 0.30 \text{ m s}^{-1}$ and $\eta_v = 0.050 \text{ Pa s}$. Find Re. What can and cannot be concluded from this one number?

F. Surface tension and capillarity

P14-25. [X14-07] A capillary calculation gives $\theta_c = 110^\circ$. A student replaces $\cos \theta_c$ by $|\cos \theta_c|$ because “height must be positive.” Explain the error and state whether the liquid level rises or falls relative to the reservoir in the sign convention of Eq. (14.48).

P14-26. For fixed γ_s , ρ , g and θ_c , how does the capillary height change if the tube radius is halved? What physical limitation should be remembered before extrapolating this scaling to arbitrarily small radii?

P14-27. A liquid has stipulated $\gamma_s = 0.060 \text{ N m}^{-1}$, $\rho = 800 \text{ kg m}^{-3}$ and $\theta_c = 60^\circ$. Find the ideal rise in tubes of radius 0.40 mm and 0.80 mm. Verify the inverse-radius scaling.

G. Advanced synthesis, verification and model limits

P14-28. [X14-08] The annular midpoint calculation of the normalized parabolic profile gives relative errors 3.125%, 0.78125% and 0.19531% for $N = 4, 8, 16$. Determine the observed error-reduction factor on each refinement and infer the numerical order. Explain why this does not validate no slip or fully developed flow in a real tube.

P14-29. Starting with a cylindrical fluid core of radius r in a straight horizontal tube, derive $\tau(r) = \Delta p r / (2L)$ from axial force balance. Then use $\tau = -\eta_v dv/dr$ and $v(R) = 0$ to recover the parabolic velocity profile.

P14-30. A gas has stipulated molecular mean free path $\lambda = 70 \text{ nm}$. Calculate Kn for characteristic channel scales $50 \mu\text{m}$ and $0.50 \mu\text{m}$. Which case is more likely to require reconsidering the continuum/no-slip model, and why?

P14-31. In the Poiseuille model, independent standard uncertainties are $\sigma_R/R = 1.0\%$, $\sigma_{\Delta p}/\Delta p = 2.0\%$, $\sigma_{\eta_v}/\eta_v = 1.5\%$ and $\sigma_L/L = 0.5\%$. Using first-order independent uncertainty propagation, estimate $\sigma_{\dot{V}}/\dot{V}$. Which input dominates?

P14-32. A large reservoir free surface is 0.50 m above the atmospheric outlet of a long circular tube with $R = 0.75 \text{ mm}$ and $L = 2.0 \text{ m}$. A liquid has $\rho = 1000 \text{ kg m}^{-3}$ and $\eta_v = 1.00 \times 10^{-3} \text{ Pa s}$. The tube is full, the reservoir speed is negligible, and the flow is quasi-steady and laminar. (a) As an approximation, assign the available head entirely to fully developed viscous resistance: $\Delta p \simeq \rho gh$. Find $\dot{V}$, $\bar{v}$ and Re. (b) Compare with the lossless small-orifice estimate $\sqrt{2gh}$ and explain the different assumptions. (c) For a developed parabolic outlet profile, check the neglected outlet kinetic head $2\bar{v}^2/(2g)$ as a fraction of h . State why the tube result is approximate even when laminar flow is plausible; neglect entrance and other minor losses in this check.

P14-33. Design a measurement plan to test the radius-fourth-power dependence of laminar tube flow. Specify the directly measured quantities, at least two systematic effects, a regime check and one graphical test. Do not assume that a straight line through a transformed plot by itself proves the model.

H. Graphs, diagrams and combined applications

P14-34. A pressure graph with two liquid layers. Figure 14.35 gives the *gauge* pressure at depth h below a free surface at 97.0 kPa absolute. Both layers are static, immiscible and of constant density. (a) Infer each density from the graph slope, explaining why the pressure remains continuous at the interface. (b) Find the absolute pressure at $h = 2.50 \text{ m}$. (c) A sensor at $h = 3.00 \text{ m}$ is referenced to a sealed gas at 102.0 kPa absolute. Predict its signed differential reading. (d) Sketch the corresponding absolute-pressure graph and identify what changes and what remains unchanged.

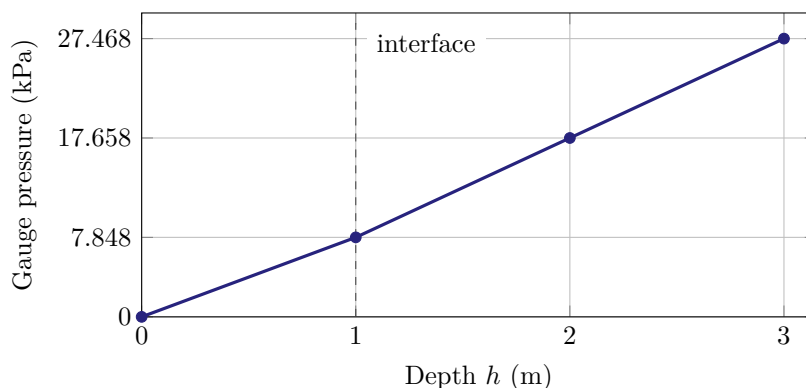


Figure 14.35: Stipulated two-layer hydrostatic graph for P14-34. The free-surface pressure defines the zero of this gauge graph. The change of slope encodes the density change; it is not a pressure jump.

P14-35. A gate with pressure on both sides. A vertical rectangular gate has width $b = 1.20 \text{ m}$ and height $H = 2.00 \text{ m}$; its top is $h_0 = 0.50 \text{ m}$ below a liquid free surface. The liquid has density 1000 kg m^{-3} . The gas above it is at $p_{\text{atm}} + 12.0 \text{ kPa}$, while a uniform gas on the dry

side is at $p_{\text{atm}} + 5.00 \text{ kPa}$ (Figure 14.36). Take y downward from the gate top. (a) Construct $\Delta p(y) = p_{\text{wet}}(y) - p_{\text{dry}}$, separating the constant and hydrostatic parts. (b) Calculate the net force and its moment about the top edge. (c) Locate the resultant and explain why using only liquid gauge pressure relative to the atmosphere would be wrong. The pressure-force resultant is required, not the hinge reaction or actuator force.

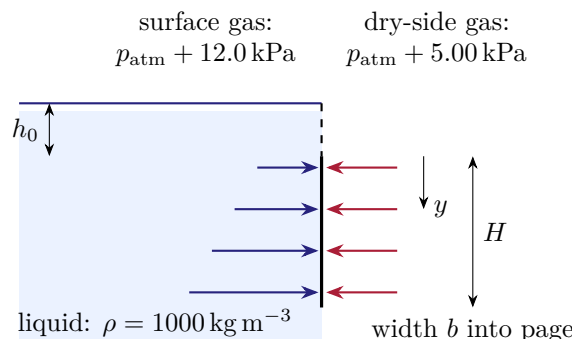


Figure 14.36: Both pressure fields in P14-35 act normal to the gate, in opposite directions. Arrow lengths are schematic; the force and moment use the stated pressure fields. The uniform dry-side pressure must be subtracted before locating the net resultant.

P14-36. Equilibrium at an oil–water interface. A sealed uniform cylinder of area $A = 0.0100 \text{ m}^2$, height $H = 0.200 \text{ m}$ and mass $m = 1.80 \text{ kg}$ rests vertically across an interface between oil ($\rho_o = 800 \text{ kg m}^{-3}$) and water ($\rho_w = 1000 \text{ kg m}^{-3}$), as in Figure 14.37. The oil layer is thick enough that the cylinder remains wholly immersed in the two liquids. Neglect surface tension. (a) Derive and find the length x in water. (b) Draw the vertical force diagram and calculate each force at equilibrium. (c) For a small upward displacement ϵ , while both ends remain in their original liquids, find the change in net upward force and decide whether vertical equilibrium is stable. Evaluate it for $\epsilon = 0.0100 \text{ m}$.

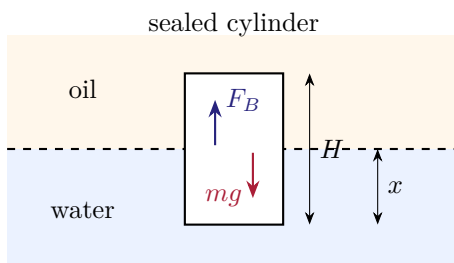


Figure 14.37: The interface geometry in P14-36. The buoyant force includes the weights of both displaced liquids. The shown immersion split is schematic; the equilibrium value must follow from force balance. The two force arrows mark directions, not force magnitudes.

P14-37. Reading a velocity profile quantitatively. Figure 14.38 describes fully developed flow of a Newtonian liquid in a straight horizontal circular tube: $R = 1.00 \text{ mm}$, $L = 0.500 \text{ m}$, $v_{\text{max}} = 0.400 \text{ m s}^{-1}$, $\rho = 1000 \text{ kg m}^{-3}$ and $\eta_v = 1.00 \times 10^{-3} \text{ Pa s}$. (a) Use the area element $2\pi r dr$ to derive the mean speed and flow rate. (b) Find the pressure drop and wall shear-stress magnitude consistent with this graph. (c) Find the fraction of the total flow carried within $r \leq R/2$; compare it with that region's area fraction. (d) Calculate the mean-speed Reynolds number and distinguish the laminar-regime check from verification of the no-slip boundary condition.

P14-38. A linear fit that fails at larger pressure drops. The following *synthetic teaching data* describe a straight horizontal tube of measured inner radius $R = 0.500 \text{ mm}$ and length $L = 0.250 \text{ m}$. Its liquid has density 1000 kg m^{-3} ; viscosity is to be inferred. The reference line in Figure 14.39 uses only the two lowest-pressure points. No experimental uncertainties are supplied.

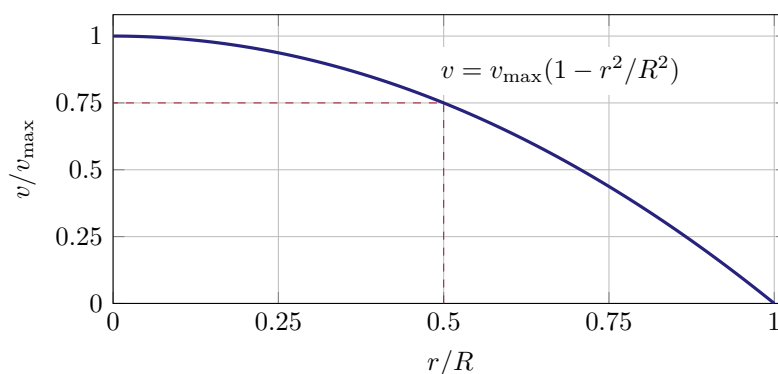


Figure 14.38: The normalized model profile for P14-37. A graph against radius cannot be averaged with equal radial weights to obtain a cross-section mean; annular area increases with radius. The central half-radius region provides a quantitative test of that distinction.

Δp (kPa)	1	2	3	4	5
$\dot{V}$ (mL s ⁻¹)	0.092	0.184	0.274	0.360	0.438

(a) Obtain the low-pressure conductance $G = \dot{V}/\Delta p$ in SI units, and infer an effective viscosity using the developed laminar model. (b) Calculate residuals relative to that reference line and the largest relative discrepancy. (c) Compute the Reynolds number at the largest measured flow, using the inferred viscosity. (d) Give two possible physical or measurement explanations for the residual pattern, and specify an additional measurement that would help distinguish them. Explain why the supplied data do not permit a statistical rejection threshold.

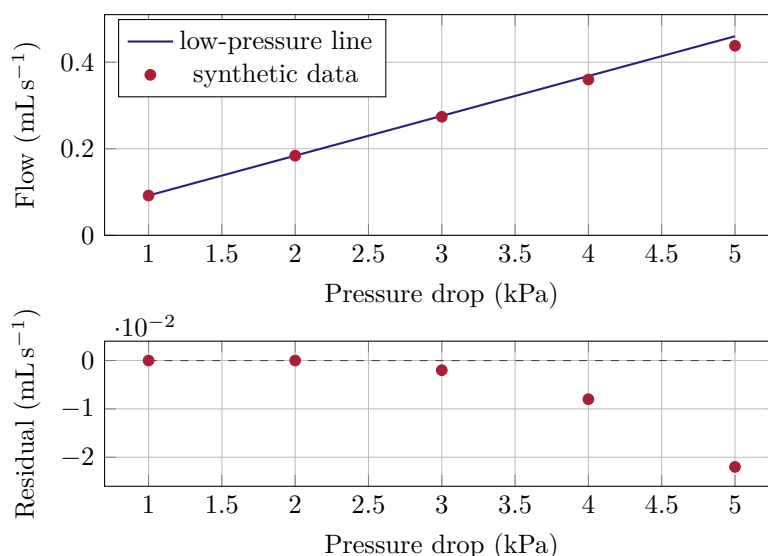


Figure 14.39: The shared data and residual model for P14-38. Residual means observed minus reference flow. The increasingly negative residual is small on the main flow graph but prominent on its own scale. These stipulated numbers are not evidence from a performed experiment.

P14-39. Reading a pump-and-loss energy diagram. Water of density 1000 kg m^{-3} flows steadily at $\dot{V} = 2.00 \times 10^{-3} \text{ m}^3 \text{ s}^{-1}$ through a constant-diameter pipe, $D = 0.0400 \text{ m}$. Station A is at $z_A = 0$ and has gauge pressure 20.0 kPa . A pump adds head $h_p = 5.00 \text{ m}$, and the route from A to station B loses head $h_L = 2.00 \text{ m}$. Station B is at $z_B = 4.00 \text{ m}$. Use the one-dimensional energy model with equal section speed and kinetic correction $\alpha = 1$ at A and B. (a) Find the speed, velocity head and total mechanical head at A. (b) Use Figure 14.40 and the energy equation to find B's gauge pressure. (c) Find the power delivered to the fluid by the pump; find the electrical input if the overall electrical-to-fluid efficiency is 0.650 . (d) Identify the errors in calling the entire pressure

difference “friction loss” or applying lossless Bernoulli across the pump.

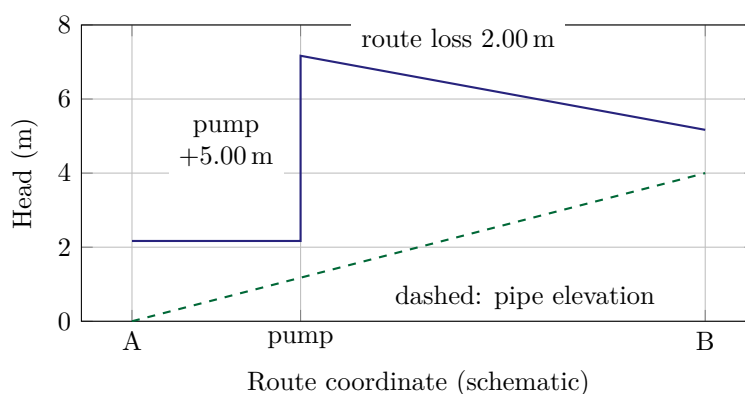


Figure 14.40: Energy bookkeeping for P14-39. The solid line is $p_{\text{gauge}}/(\rho g) + v^2/(2g) + z$, with an idealized jump across the pump. The route coordinate and linear decline are schematic; only the stipulated net head changes are used. The elevation line joins the known station elevations and does not specify detailed pipe geometry.

P14-40. A signed capillary-height graph. The equilibrium model in Figure 14.41 has $R = 0.500 \text{ mm}$, $\rho = 1000 \text{ kg m}^{-3}$, and $\gamma_s = 0.0720 \text{ N m}^{-1}$. The contact angle θ_c is measured through the liquid; $h > 0$ is above the reservoir. (a) Read the sign and calculate the height at 60° , 90° and 120° . (b) If the radius doubles, describe the change to the entire graph. (c) Explain why a measured height near $\theta_c = 90^\circ$ is a poor way to infer surface tension when contact-angle uncertainty is appreciable. (d) Calculate $\rho g R^2 / \gamma_s$ and explain its role as a scale comparison rather than a full validation of the meniscus model.

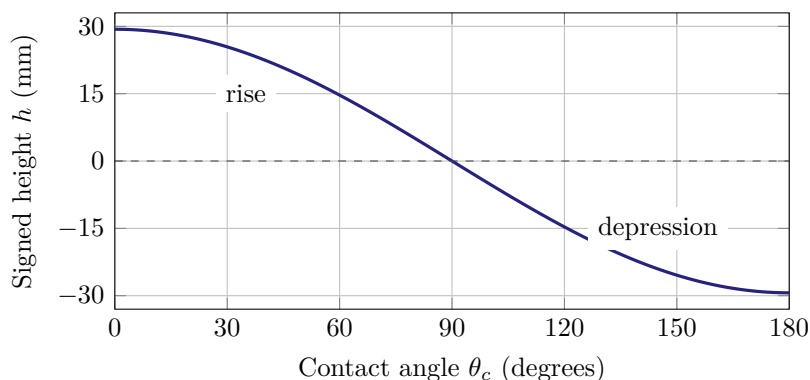


Figure 14.41: The conditional equilibrium model for P14-40. Rise and depression belong to one signed function. Changing the radius rescales the vertical axis; it does not shift the zero at 90° . Continuum, equilibrium contact angle and negligible extra forces remain assumptions.

P14-41. Constructing different streamline and pathline curves. A prescribed two-dimensional velocity field is spatially uniform but changes with time: $u = U = 1.00 \text{ m s}^{-1}$ and $v = at$ with $a = 0.400 \text{ m s}^{-2}$. A tracer begins at the origin at $t = 0$. (a) Find its pathline in parametric form and eliminate time. (b) At $t = 2.00 \text{ s}$, find the streamline through the tracer’s current position. (c) Sketch both curves for $0 \leq x \leq 3.00 \text{ m}$, marking the tracer and the common tangent there. Explain why different curves can have the same local tangent at that instant. This is a prescribed kinematic field; do not infer that it solves a particular force-driven fluid experiment.

P14-42. Finite free-surface speed in an ideal tank model. A tank of constant horizontal area $A_T = 0.0200 \text{ m}^2$ drains through an outlet of area $a = 2.00 \times 10^{-3} \text{ m}^2$. At one instant the free surface is $h = 0.800 \text{ m}$ above the outlet. Both locations are at atmospheric pressure. Adopt a quasi-steady,

incompressible, lossless model with uniform section speeds. (a) Include the free-surface speed v_s using continuity and derive the outlet speed v_e . (b) Calculate v_e , $\dot{V}$ and dh/dt at that instant. (c) Compare with neglecting v_s and give the relative outlet-speed error. (d) Identify two physical effects that could make this ideal result inaccurate and explain why the quasi-steady assumption is separate from the area-ratio approximation.

P14-43. An exponential draining approximation. A reservoir of constant area $A_T = 0.0100 \text{ m}^2$ feeds a long, full, straight circular tube: $R = 0.500 \text{ mm}$, $L = 1.00 \text{ m}$. Its free surface is height h above the atmospheric outlet. Take $\rho = 1000 \text{ kg m}^{-3}$ and $\eta_v = 1.00 \times 10^{-3} \text{ Pa s}$. Adopt quasi-steady developed laminar flow, neglect reservoir speed, outlet kinetic head, entrance losses and variable capillary backpressure, and set $\Delta p \simeq \rho gh$. (a) Derive $h(t)$ for $h(0) = 0.400 \text{ m}$; identify the time constant. (b) Find the time to reach 0.100 m . (c) Check the initial mean speed, Reynolds number and neglected outlet kinetic head using $\alpha = 2$. (d) Compare the viscous diffusion time $\rho R^2/\eta_v$ with the draining time constant and explain what that comparison supports, and what it cannot establish. (e) Give the expected straight-line graph for testing this model from height data and explain what curved residuals might indicate. This is an idealized proposed analysis, not a performed experiment.

14.12 Complete reasoned solutions

Every end problem has a solution below. For numerical work, additional digits show arithmetic checks; the precision of a real observation would be set by its uncertainty. Model plausibility and numerical consistency are separate checks. A conceptual or design question is assessed by its reasoning, so its solution provides the necessary distinctions or a defensible exemplar.

A. Pressure, hydrostatics and measurement

Solution P14-01: the pressure reference

For Problem P14-01, an ideal gauge measures the pressure difference between its vessel and reference ports:

$$p_{\text{gauge}} = p_{\text{abs}} - p_{\text{ref}}.$$

Thus $p_{\text{gauge}} = 0$ with $p_{\text{ref}} = 99.5 \text{ kPa}$ implies $p_{\text{abs}} = 99.5 \text{ kPa}$. The vessel and atmosphere are at equal pressure. A pressure difference can vanish while each pressure is appreciable; it does not mean that molecular pressure forces vanish. Evacuation would require an absolute reading approaching zero, or a zero differential reading against a known vacuum reference. The mistake is therefore a reference error, not an arithmetic error. The identity checks the answer directly: subtracting 99.5 kPa from the inferred vessel pressure gives the stated zero reading.

Solution P14-02: force versus resultant

The pressure is uniform over each face of the cube in Problem P14-02. The face area is $A = (0.20)^2 = 0.0400 \text{ m}^2$, so the inward normal force magnitude on one face is

$$F = pA = (300 \times 10^3)(0.0400) = 1.20 \times 10^4 \text{ N} = 12.0 \text{ kN}.$$

Each face has an opposite face with the same area, the same pressure and an opposite outward normal. Therefore the six inward pressure forces cancel in three equal-and-opposite pairs and their vector sum is zero. Adding their magnitudes would give a scalar sum, not the resultant force. In continuum notation $\mathbf{F} = -p \oint \mathbf{n} dA = 0$ for constant p . Gravity has deliberately been excluded; including it in a static fluid would introduce a pressure gradient and remove the assumption of uniform pressure.

Solution P14-03: hydrostatic gauge and absolute pressure

Take depth h positive downward from the surface. Static force balance and constant density give $dp/dh = \rho g$. Integrating with $p(0) = 101.0 \text{ kPa}$ gives

$$p(h) = 101.0 \text{ kPa} + \rho gh.$$

At $h = 6.50 \text{ m}$, the pressure increment is

$$\rho gh = (1000)(9.81)(6.50) = 63765 \text{ Pa} = 63.765 \text{ kPa}.$$

The surface-referenced gauge pressure is 63.8 kPa , and the absolute pressure is $164.765 \text{ kPa} \simeq 164.8 \text{ kPa}$. The dimensional check is $\text{kg m}^{-3} \text{ m s}^{-2} \text{ m} = \text{Pa}$. The larger pressure at greater depth is consistent with the weight of the overlying fluid; the surface pressure must be added only for the absolute result.

Solution P14-04: the sign in a manometer

Neglecting gas density makes the pressure at each gas–liquid interface equal to its corresponding gas-space pressure. Start at the higher interface on side B and move downward by 0.180 m through the manometer liquid. Pressure increases by $\rho g \Delta h$. At that lower elevation the connected static liquid has the same pressure across both limbs, so

$$p_A = p_B + \rho g \Delta h, \quad p_A - p_B = (920)(9.81)(0.180) = 1624.536 \text{ Pa}.$$

Thus $p_A - p_B = +1.62 \text{ kPa}$. The gas on A depresses its interface and is at the higher pressure. Swapping A and B would reverse the sign while leaving the magnitude unchanged. A useful independent sign check is the zero-height limit: equal interfaces would imply equal gas pressures under the stated negligible-gas-density model.

Solution P14-05: the gate centroid is not its centre of pressure

Equal atmosphere on the surface and dry side cancels from the net pressure in Problem P14-05. A strip of width b at depth h therefore experiences

$$dF = \rho gh b dh.$$

For $H = 1.50 \text{ m}$, $b = 0.80 \text{ m}$,

$$F = \rho gb \int_0^H h dh = \frac{\rho gb H^2}{2} = 8829 \text{ N} \simeq 8.83 \text{ kN}.$$

The moment about the upper edge is $M = \rho gb \int_0^H h^2 dh = \rho gb H^3/3$. Requiring the resultant to have the same moment gives

$$h_{\text{cp}} = \frac{M}{F} = \frac{2H}{3} = 1.00 \text{ m}.$$

Using midpoint pressure reproduces the force because pressure varies linearly on this rectangle. It does not reproduce the moment if the force is also placed at $H/2 = 0.750 \text{ m}$. Deeper strips have both larger force and larger lever arm, placing the resultant below the centroid. The computed depth lies inside the gate, as required for this everywhere-positive load.

Solution P14-06: a larger submerged panel

The panel starts at the free surface, and the equal external atmosphere removes the common constant pressure. The net pressure is $\Delta p(h) = \rho gh$ for $0 \leq h \leq 3.0 \text{ m}$. With $b = 2.0 \text{ m}$,

$$F = \int_0^H \rho gh b dh = \frac{(1000)(9.81)(2.0)(3.0)^2}{2} = 88290 \text{ N} = 88.3 \text{ kN}.$$

Its line of action is obtained from $Fh_{cp} = \int h dF$:

$$h_{cp} = \frac{\rho g b H^3 / 3}{\rho g b H^2 / 2} = \frac{2H}{3} = 2.00 \text{ m.}$$

The force is normal to the panel and points from the wet side toward the dry side. No assumption about a hinge or support is needed to locate the pressure resultant; calculating support reactions would require those additional geometric constraints. Both the force scaling $F \propto H^2$ and the deeper-than-centroid location follow from the triangular pressure distribution.

B. Hydraulics and buoyancy

Solution P14-07: force multiplication conserves ideal work

Use an ideal incompressible hydraulic system with equal relevant piston elevations, negligible piston weight and friction, and the same external reference pressure. Pascal transmission gives $F_i/A_i = F_o/A_o$. Thus

$$F_o = 30F_i = 30(150) = 4500 \text{ N.}$$

Conservation of displaced volume requires $A_i x_i = A_o x_o$, so

$$x_o = \frac{x_i}{30} = \frac{0.300}{30} = 0.0100 \text{ m.}$$

The input work is $W_i = F_i x_i = 150(0.300) = 45.0 \text{ J}$; the output work is $W_o = F_o x_o = 4500(0.0100) = 45.0 \text{ J}$. The force rises by the area ratio while displacement falls by its reciprocal. A claim of thirty times the energy implicitly keeps the output displacement unchanged and violates fluid-volume conservation. Real friction and other losses would reduce useful output work and require additional input, rather than generating energy.

Solution P14-08: submerged fraction

For static floating equilibrium, neglecting interfacial forces and air buoyancy, the upward buoyant force equals the object's weight:

$$\rho_f g V_{\text{sub}} = \rho_o g V.$$

Cancel g and solve for the submerged-volume fraction:

$$f = \frac{V_{\text{sub}}}{V} = \frac{\rho_o}{\rho_f} = \frac{780}{1000} = 0.780.$$

Hence 78.0% of the external volume is submerged. This statement refers to volume, not necessarily height: height fraction equals volume fraction only for a shape with constant horizontal cross-section. The value lies between zero and one, so partial floating is possible under the stipulated model. Because the object is sealed, the liquid displaces its external volume rather than entering it; its average density includes the complete mass distributed over that external volume.

Solution P14-09: holding a heavier submerged object

Choose upward positive. The pressure resultant is the weight of displaced liquid:

$$F_B = \rho_f g V = (1000)(9.81)(2.00 \times 10^{-3}) = 19.62 \text{ N.}$$

The downward weight is $mg = (3.00)(9.81) = 29.43 \text{ N}$. For the object to remain at rest, the holding force satisfies $F_H + F_B - mg = 0$. Therefore

$$F_H = mg - F_B = 9.81 \text{ N}$$

upward. An upward support is needed because the object's average density is $m/V = 1500 \text{ kg m}^{-3}$, exceeding the liquid density. Applying a downward holding force would make the downward imbalance larger. The answer assumes the holding mechanism does not materially change displaced volume or add buoyant forces to the chosen object; drag is absent at rest.

Solution P14-10: recognizing an impossible floating solution

Floating would require $V_{\text{sub}} = m/\rho_f = (1.8)/(1000) = 1.8 \times 10^{-3} \text{ m}^3$. The object can displace at most its external volume $V = 1.5 \times 10^{-3} \text{ m}^3$. Thus the required fraction would be 1.20, which cannot describe partial floating. Fully submerged, the downward-minus-upward force imbalance is

$$mg - F_B = (m - \rho_f V)g = (1.8 - 1.5)(9.81) = 2.943 \text{ N}.$$

The direction is downward, so the object accelerates downward if released while drag is initially negligible. The failed floating calculation is useful information: it diagnoses insufficient available displaced volume. It should not be reported as “120 percent submerged.” At later times drag could balance this imbalance at a terminal speed, but determining that speed requires a drag model not supplied here.

C. Mass conservation and flow kinematics**Solution P14-11: mass flow is conserved when density changes**

For steady one-dimensional flow without side branches or mass sources, the same mass rate crosses each section. With uniform stipulated section density, $\dot{m} = \rho A \bar{v}$. Thus

$$v_1 = \frac{4.0}{(1.20)(0.020)} = 166.7 \text{ m s}^{-1}, \quad v_2 = \frac{4.0}{(0.80)(0.010)} = 500 \text{ m s}^{-1}.$$

The volume rates are $A_1 v_1 = 3.333 \text{ m}^3 \text{ s}^{-1}$ and $A_2 v_2 = 5.00 \text{ m}^3 \text{ s}^{-1}$; they differ because volume is not conserved for this density-changing gas stream. Each volume rate times its density gives the same 4.0 kg s^{-1} . The stipulated density change itself rules out a constant-density model between the sections. The high speeds also motivate checking compressibility, but a Mach number cannot be assigned without a sound speed or gas state. Mass continuity remains valid even when incompressible Bernoulli would be inappropriate.

Solution P14-12: volume and mass rates

The section mean speed is defined so that $\dot{V} = A \bar{v}$. For Problem P14-12,

$$\dot{V} = (3.00 \times 10^{-4})(0.750) = 2.25 \times 10^{-4} \text{ m}^3 \text{ s}^{-1}.$$

With the stipulated density uniform at the section,

$$\dot{m} = \rho \dot{V} = (997)(2.25 \times 10^{-4}) = 0.224325 \text{ kg s}^{-1} \simeq 0.224 \text{ kg s}^{-1}.$$

The units distinguish transport of volume from transport of mass. A quick reverse check divides the mass rate by 997 kg m^{-3} and recovers the computed volume rate. A nonuniform velocity profile would not change the formula when $\bar{v}$ is the cross-section mean; substituting a centreline speed in its place would generally give the wrong flow rate.

Solution P14-13: contracting an incompressible nozzle

Incompressibility, steady single-stream flow and no leakage give $A_1 v_1 = A_2 v_2$. Hence

$$v_2 = \frac{A_1}{A_2} v_1 = \frac{20 \text{ cm}^2}{5.0 \text{ cm}^2} (1.20 \text{ m s}^{-1}) = 4.80 \text{ m s}^{-1}.$$

Because both areas use the same units, their ratio is dimensionless; conversion to square metres is unnecessary for this ratio. The factor of four is a consequence of maintaining the same volume rate through one quarter of the area. Continuity alone determines the speed ratio but does not specify the pressure difference or determine whether an actual nozzle has negligible losses. Those require an energy equation and a model for its terms.

Solution P14-14: instantaneous direction versus a particle history

A streamline is a curve whose tangent at every point follows the velocity field at one selected instant. A pathline is the trajectory of one labelled fluid particle as time passes. Streamlines use a frozen-time field; pathlines sample the evolving field along a particle's motion. They coincide geometrically in a steady field, wherever the velocity direction and the particle trajectory are well defined. They can differ in an unsteady field even though the pathline tangent at a particle's current position still equals its instantaneous velocity. For example, a spatially uniform flow whose transverse component grows with time has straight streamlines at each instant but curved particle trajectories. Problem P14-41 develops that example without relying on a memorized sketch. A photograph of short velocity arrows therefore does not by itself show complete particle histories.

D. Bernoulli reasoning and ideal flow**Solution P14-15: a failed lossless model**

Continuity gives equal section mean speeds because area and density are unchanged and the flow is steady. Under the student's additional lossless, horizontal, no-shaft-work assumptions, the pressure terms would consequently be equal. The measured difference demonstrates that this *complete set* of idealizations cannot account for the observation; it does not invalidate mass conservation. Viscous wall resistance or localized separation/mixing losses can remove mechanical energy at equal speeds. A turbine or other work extraction could also lower pressure. For real measurements, incorrect elevations, taps that do not measure the intended static pressure, or unrecognized side flow could change the model or the data interpretation. If the stated geometry and measurements are reliable and losses alone explain the difference, their equivalent head is

$$h_L = \frac{12000}{(1000)(9.81)} = 1.223 \text{ m}$$

for a water-density illustration. This last number is illustrative because density was not numerically supplied in the problem; the diagnostic conclusion does not depend on that choice.

Solution P14-16: the large-tank efflux limit

Choose point 1 at the free surface and point 2 at the atmospheric outlet. Both pressures are atmospheric; the large surface has negligible speed. Under steady or quasi-steady, incompressible, lossless flow along a streamline, Bernoulli gives $gh = v_2^2/2$. Therefore

$$v_2 = \sqrt{2gh} = \sqrt{2(9.81)(1.80)} = 5.9427 \text{ m s}^{-1} \simeq 5.94 \text{ m s}^{-1}.$$

The available gravitational energy per unit mass converts into kinetic energy per unit mass. A check squares the answer: $v_2^2/(2g) = 1.80 \text{ m}$. An actual small opening can have contraction and dissipation, so this is an ideal speed estimate. A long narrow tube with the same elevation head would require a viscous-flow model and generally produce a lower speed; the ideal-orifice formula is not a universal law for every outlet geometry.

Solution P14-17: continuity coupled to Bernoulli

The horizontal ideal meter is steady and incompressible, and losses are neglected. Continuity gives $v_2 = (A_1/A_2)v_1 = 2v_1$. Bernoulli then requires

$$p_1 - p_2 = \frac{\rho}{2}(v_2^2 - v_1^2) = \frac{3\rho}{2}v_1^2.$$

Thus

$$v_1 = \sqrt{\frac{2(7500)}{3(1000)}} = \sqrt{5} = 2.2361 \text{ m s}^{-1}, \quad v_2 = 4.4721 \text{ m s}^{-1}.$$

The common volume rate is

$$\dot{V} = A_1 v_1 = (6.0 \times 10^{-3})(2.2361) = 1.34164 \times 10^{-2} \text{ m}^3 \text{ s}^{-1}.$$

The downstream product $A_2 v_2$ gives the same result, while substitution into $\rho(v_2^2 - v_1^2)/2$ returns 7.50 kPa. The lower pressure occurs in the smaller, faster section in this ideal horizontal model. A measured device with appreciable loss requires calibration or a loss term; continuity remains applicable but the inferred ideal rate could be biased.

Solution P14-18: pressure energy and elevation

With the same speed at both points, the kinetic terms cancel. Ideal Bernoulli gives

$$p_1 + \rho g z_1 = p_2 + \rho g z_2, \quad p_2 = p_1 - \rho g(z_2 - z_1).$$

The elevation rise is 8.0 m, so

$$p_2 = 250000 - (1000)(9.81)(8.0) = 171520 \text{ Pa} \simeq 171.5 \text{ kPa}.$$

Use the same pressure reference for both stations; if the supplied pressure is gauge, the result is gauge, and if it is absolute, the result is absolute. The lost pressure energy per unit volume is $\rho g \Delta z$, equal to the gained gravitational potential energy per unit volume. The pressure drop need not indicate friction: here it occurs in an explicitly lossless model with equal speeds because the liquid climbs.

Solution P14-19: work added by a pump

A running pump transfers shaft work to the fluid. The Bernoulli equation developed for no shaft work does not include that transfer, so applying it across the pump omits a physically essential term. For a steady incompressible one-dimensional energy model, the appropriate bookkeeping is

$$\frac{p_1}{\rho g} + \frac{\alpha_1 \bar{v}_1^2}{2g} + z_1 + h_p = \frac{p_2}{\rho g} + \frac{\alpha_2 \bar{v}_2^2}{2g} + z_2 + h_L,$$

with pump head h_p and nonnegative dissipative loss h_L . The correction factors account for section velocity profiles when using means. Density constancy and steadiness do not force h_p to vanish. At equal elevation and speed, an operating pump can raise downstream pressure; lossless Bernoulli without a pump term would falsely forbid that result. Pump head must be specified, measured or inferred using a pump model.

E. Viscosity, pressure loss and Reynolds number

Solution P14-20: the velocity scale in Reynolds number

For the developed circular-tube profile $v(r) = v_{\max}(1 - r^2/R^2)$, averaging with annular area gives $\bar{v} = v_{\max}/2$. The prescribed Reynolds number is $\text{Re} = \rho \bar{v} D / \eta_v$. Substituting the centreline speed instead gives

$$\text{Re}_{\text{mistaken}} = \frac{\rho v_{\max} D}{\eta_v} = 2 \frac{\rho \bar{v} D}{\eta_v} = 2\text{Re}.$$

The reported number is therefore too large by a factor of two. This factor follows from the particular parabolic profile; it should not be transplanted to arbitrary velocity distributions. A definition based explicitly on centreline speed could be used consistently, but its numerical thresholds and comparison values would have to use that same definition. Mixing definitions silently creates the error.

Solution P14-21: finite radius changes are not linear

At fixed pressure drop, length and viscosity, the developed laminar model gives $\dot{V} \propto R^4$. A five percent radius reduction means $R_{\text{new}} = 0.95R_{\text{old}}$. Consequently

$$\frac{\dot{V}_{\text{new}}}{\dot{V}_{\text{old}}} = (0.95)^4 = 0.81450625,$$

so the flow decreases by $100(1 - 0.81450625) = 18.5494\% \simeq 18.5\%$. First-order differentiation would predict approximately $4 \times 5\% = 20\%$, which is a small-change estimate rather than the exact finite-change result. The fourth-power dependence explains why accurate inner-radius measurement is critical. The calculation also holds the physical flow regime fixed; a geometry change that altered entrance effects or the validity of developed flow would need a more complete model.

Solution P14-22: a laminar-tube calculation with a regime check

Use a Newtonian liquid, straight horizontal circular tube, no slip, and fully developed laminar flow. Convert $R = 0.80 \text{ mm} = 8.00 \times 10^{-4} \text{ m}$. Then

$$\dot{V} = \frac{\pi R^4 \Delta p}{8\eta_v L} = \frac{\pi(8.00 \times 10^{-4})^4(3000)}{8(1.50 \times 10^{-3})(1.50)} = 2.14466 \times 10^{-7} \text{ m}^3 \text{ s}^{-1}.$$

The section mean and diameter-based Reynolds number are

$$\bar{v} = \frac{\dot{V}}{\pi R^2} = 0.106667 \text{ m s}^{-1}, \quad \text{Re} = \frac{(900)(0.106667)(1.60 \times 10^{-3})}{1.50 \times 10^{-3}} = 102.4.$$

This is consistent with laminar flow in an ordinary tube. It does not establish an inertia-free creeping-flow limit, since the specified Reynolds number exceeds unity, and it does not itself prove a developed entrance profile. A separate analytic check uses $\bar{v} = R^2 \Delta p / (8\eta_v L)$ and reproduces the mean speed. The force-balance/profile derivation is given in Solution P14-29.

Solution P14-23: head loss without acceleration

The equivalent pressure head is the pressure difference divided by weight density:

$$h_{\Delta p} = \frac{\Delta p}{\rho g} = \frac{20000}{(1000)(9.81)} = 2.03874 \text{ m} \simeq 2.04 \text{ m}.$$

In a horizontal, equal-area, steady incompressible stream, the section mean speeds are equal. Their kinetic-energy terms therefore have no change. If there is no work transfer and the section energy representation is otherwise appropriate, the pressure-head decrease represents dissipated mechanical energy. More generally it can signal another omitted energy transfer or an inadequacy of the section model. It cannot be assigned to a kinetic-energy increase while maintaining equal speeds. “Head” here is an energy-per-unit-weight representation; the liquid need not visibly descend through this height in the actual horizontal pipe.

Solution P14-24: a Reynolds number is not a full model

From the stated diameter and mean-speed definition,

$$\text{Re} = \frac{\rho \bar{v} D}{\eta_v} = \frac{(850)(0.30)(0.020)}{0.050} = 102.$$

For ordinary internal flow in a circular tube, this supports a laminar-regime assumption. It does *not* imply that inertial terms are negligible everywhere: a creeping-flow approximation would require a suitably small Reynolds number and the relevant geometric/time scales, whereas 102 exceeds unity. Nor does the number establish that the velocity profile has finished developing, the wall is no slip, the liquid is Newtonian, or the density and viscosity are actually constant. Geometry, disturbances and boundary conditions remain relevant. The useful inference is therefore limited: turbulent pipe flow is not expected solely from this modest mean-speed Reynolds number, but Poiseuille applicability needs the additional assumptions stated with that model.

F. Surface tension and capillarity

Solution P14-25: negative height is meaningful

With angle measured through the liquid and height positive upward, the conditional capillary model is

$$h = \frac{2\gamma_s \cos \theta_c}{\rho g R}.$$

All quantities outside the cosine are positive. Since $\cos 110^\circ = -0.3420 < 0$, the height is negative: the tube liquid level is depressed below the reservoir. Replacing cosine by its magnitude changes the predicted direction of the vertical interfacial force and therefore changes the physics. It is not a harmless formatting convention. A positive magnitude could be reported separately as the *depth of depression*, but then its name and direction must be stated. Keeping the signed height makes the rise-to-depression transition at 90° continuous and preserves the distinction between wetting and nonwetting behavior in the declared model.

Solution P14-26: inverse-radius scaling and its boundary

At fixed $\gamma_s, \rho, g, \theta_c$, write $h = C/R$, with the signed constant $C = 2\gamma_s \cos \theta_c / (\rho g)$. Halving R gives $h_{\text{new}} = 2h_{\text{old}}$: the magnitude doubles while the sign stays the same. If $\theta_c = 90^\circ$, both heights are zero within this ideal model. Extending the result indefinitely toward microscopic radii is unjustified. Molecular-scale confinement can undermine continuum fluid properties, a macroscopic contact-angle description and the assumption of a simple equilibrium interface. Contact-angle hysteresis, contamination, trapped gas and an available finite liquid volume can matter at larger scales as well. The scaling is a consequence of the specified capillary model; it does not prove those assumptions remain valid after changing the tube.

Solution P14-27: calculating two rises

The contact angle is 60° , so $\cos \theta_c = 1/2$ and the interfacial force is upward. At $R_1 = 0.40$ mm,

$$h_1 = \frac{2(0.060)(0.5)}{(800)(9.81)(0.40 \times 10^{-3})} = 0.0191131 \text{ m} = 19.1 \text{ mm}.$$

At $R_2 = 0.80$ mm,

$$h_2 = \frac{2(0.060)(0.5)}{(800)(9.81)(0.80 \times 10^{-3})} = 0.00955657 \text{ m} = 9.56 \text{ mm}.$$

The ratio $h_1/h_2 = 2$ exactly follows from $R_2/R_1 = 2$. The numerator has units N m^{-1} and the denominator has units N m^{-2} , so their ratio is a length. These are positive rises relative to the reservoir, using a contact angle through the liquid and the stipulated constant fluid properties. Uncertainty in those properties or angle has not been supplied and should not be invented.

G. Advanced synthesis, verification and model limits

Solution P14-28: convergence of a numerical integral

The error-reduction factors are

$$\frac{3.125}{0.78125} = 4, \quad \frac{0.78125}{0.19531} = 4.00005 \text{ approximately.}$$

The slight second-ratio difference is explained by rounding the last supplied error. If $E_N \simeq CN^{-p}$, doubling N gives $E_N/E_{2N} \simeq 2^p$. Therefore

$$p \simeq \frac{\ln(E_N/E_{2N})}{\ln 2} = 2.$$

The sequence supports second-order convergence for the annular midpoint integration of the prescribed smooth profile. It verifies a numerical method approaching that mathematical integral. It does not establish that a real fluid satisfies no slip, is Newtonian, is fully developed or remains laminar. A precisely integrated wrong physical profile is still a wrong physical model. The complete statement must identify the calculation verified, the convergence evidence and the physical assumptions that remain to be tested.

Solution P14-29: deriving the profile from a fluid core

Take a stationary cylindrical control surface around an axial fluid core of radius r and length L in fully developed steady flow through a horizontal tube. The velocity field has no axial development, so there is no net change of axial momentum flux through the end planes. The net pressure force is $\Delta p \pi r^2$. The surrounding fluid exerts resisting axial shear of magnitude $\tau(r)$ on the lateral area $2\pi r L$. Axial balance gives

$$\Delta p \pi r^2 = \tau(r) 2\pi r L, \quad \tau(r) = \frac{\Delta p}{2L} r.$$

For a Newtonian liquid whose speed decreases outward, $\tau = -\eta_v dv/dr$. Consequently

$$\frac{dv}{dr} = -\frac{\Delta p}{2\eta_v L} r, \quad v(r) = -\frac{\Delta p}{4\eta_v L} r^2 + C.$$

No slip at $r = R$ fixes $C = \Delta p R^2 / (4\eta_v L)$, yielding

$$v(r) = \frac{\Delta p}{4\eta_v L} (R^2 - r^2).$$

This satisfies $v(R) = 0$, centreline symmetry $v'(0) = 0$, and positive speed for positive driving pressure. Shear is zero at the axis and largest at the wall. Integrating $2\pi \int_0^R v r dr$ recovers $\dot{V} = \pi R^4 \Delta p / (8\eta_v L)$. All conclusions are conditional on the developed, Newtonian, no-slip model.

Solution P14-30: molecular and channel scales

Use $\text{Kn} = \lambda / L_c$ with both lengths in metres. For the larger channel,

$$\text{Kn}_1 = \frac{70 \times 10^{-9}}{50 \times 10^{-6}} = 1.4 \times 10^{-3}.$$

For the smaller channel,

$$\text{Kn}_2 = \frac{70 \times 10^{-9}}{0.50 \times 10^{-6}} = 0.14.$$

The second channel is only about seven mean free paths wide on the stipulated characteristic scale, so separation between molecular and device scales is much weaker. Wall slip and departures from a continuum local-equilibrium description deserve closer attention. The smaller Kn in the first case makes the usual continuum/no-slip approximation more plausible, but neither number alone certifies accuracy for every observable or boundary condition. The distinction concerns length-scale separation, whereas Reynolds number concerns an inertial-to-viscous comparison; one cannot replace the other.

Solution P14-31: independent uncertainty propagation

For $\dot{V} = C R^4 \Delta p \eta_v^{-1} L^{-1}$, logarithmic differentiation identifies relative sensitivity coefficients 4, 1, -1 , -1 . Under the stated independent, small standard-uncertainty approximation,

$$\left(\frac{\sigma_{\dot{V}}}{\dot{V}} \right)^2 \simeq \left(4 \frac{\sigma_R}{R} \right)^2 + \left(\frac{\sigma_{\Delta p}}{\Delta p} \right)^2 + \left(\frac{\sigma_{\eta_v}}{\eta_v} \right)^2 + \left(\frac{\sigma_L}{L} \right)^2.$$

Substitution gives

$$\frac{\sigma_{\dot{V}}}{\dot{V}} = \sqrt{0.040^2 + 0.020^2 + 0.015^2 + 0.005^2} = 0.0474342 \simeq 4.74\%.$$

The radius sensitivity contributes 4.0% before quadrature and accounts for about 71.1% of this variance budget, so it dominates. The negative sensitivities do not subtract uncertainties: their squares are positive for independent inputs. Correlated errors would introduce covariance terms. Model-form error, such as a failure of developed flow, is outside this supplied input-uncertainty budget and must not be silently included in the quoted 4.74%.

Solution P14-32: checking the available-head approximation

(a) The available gravitational head corresponds to $\rho gh = (1000)(9.81)(0.50) = 4905$ Pa. Assigning it entirely to developed viscous resistance, as the problem explicitly requests, gives

$$\dot{V} \simeq \frac{\pi(0.75 \times 10^{-3})^4(4905)}{8(1.00 \times 10^{-3})(2.0)} = 3.04729 \times 10^{-7} \text{ m}^3 \text{ s}^{-1}.$$

Then $\bar{v} = \dot{V}/(\pi R^2) = 0.172441 \text{ m s}^{-1}$ and $\text{Re} = \rho \bar{v}(2R)/\eta_v = 258.662$. This supports the laminar-regime choice for ordinary tube flow but does not itself certify full development or the entrance-loss approximation.

(b) The lossless small-orifice speed is $\sqrt{2gh} = 3.13209 \text{ m s}^{-1}$, about 18.16 times the viscous-tube mean. The orifice model assigns the available head to outlet kinetic energy; the present approximation assigns nearly all of it to tube resistance. They describe different physical limits. Applying the orifice result to this long narrow tube would neglect the dominant resistance.

(c) A developed parabolic outlet has kinetic-energy correction $\alpha = 2$. Its omitted kinetic head, using the approximate mean, is

$$h_K = \frac{2\bar{v}^2}{2g} = \frac{(0.172441)^2}{9.81} = 0.00303120 \text{ m}, \quad \frac{h_K}{h} = 0.00606239 = 0.606\%.$$

Thus assigning all available head to viscous loss is a good small-kinetic-head approximation for these inputs, within a model neglecting entrance and other losses. It is not an exact pressure boundary condition for flowing efflux. An improved balance includes both the viscous head and $\alpha \bar{v}^2/(2g)$; the relevant approximation error has now been quantified rather than hidden.

Solution P14-33: an exemplar radius-scaling investigation

A defensible proposed study uses several straight, circular tubes of the same accurately measured length and different measured *inner* radii. Use one Newtonian liquid, control temperature, measure the pressure difference between declared pressure-tap locations, and measure flow by timed collection of mass or volume. Repeat each setting, and record whether fluid properties are measured or taken from a stated source. Include a zero-flow/zero-pressure instrument check.

For each tube compute $\bar{v} = \dot{V}/(\pi R^2)$ and $\text{Re} = \rho \bar{v}(2R)/\eta_v$. Arrange enough inlet distance or assess entrance development separately; a low Reynolds number alone is insufficient. Keep pressure drop, viscosity and length fixed or normalize by their measured values. Plot $\dot{V}$ against R^4 ; the ideal slope is $\pi \Delta p/(8\eta_v L)$ and the intercept should be consistent with zero within the measurement model. A complementary log plot has expected slope four when the other factors are fixed.

Systematic effects include radius calibration or noncircular bore, temperature-dependent viscosity, end losses, bubbles, leaks and pressure-tap offsets. Radius error is especially amplified by the fourth power. Examine residuals against radius, pressure and temperature, and repeat selected tubes at several pressure drops. A good assessment rewards clear control variables, traceable radius measurements, regime/development checks, uncertainty treatment and explicit criteria for model agreement. A straight transformed plot supports the tested dependence over that range; it does not uniquely prove the complete flow model.

H. Graphs, diagrams and combined applications

Solution P14-34: slopes, offsets and sensor references

(a) Depth is positive downward, so hydrostatic slope is $dp/dh = \rho g$. The first slope is $7.848 \text{ kPa m}^{-1} = 7848 \text{ Pa m}^{-1}$, giving $\rho_1 = 7848/9.81 = 800 \text{ kg m}^{-3}$. The second slope is $(27.468 - 7.848)/2 = 9.810 \text{ kPa m}^{-1}$, giving $\rho_2 = 1000 \text{ kg m}^{-3}$. Across a flat static liquid interface without a capillary pressure jump, pressure is continuous even when its depth derivative changes. An arbitrary jump would violate the interface's normal force balance.

(b) The gauge pressure at $h = 2.50 \text{ m}$ is the integrated weight-density contribution of one metre of the upper liquid and 1.50 m of the lower:

$$p_g = (800)(9.81)(1.00) + (1000)(9.81)(1.50) = 22563 \text{ Pa}.$$

Add the stated absolute surface pressure to obtain $p_{\text{abs}} = 97000 + 22563 = 119563 \text{ Pa} = 119.563 \text{ kPa}$.

(c) At 3.00 m , the absolute pressure is $97.000 + 27.468 = 124.468 \text{ kPa}$. The sensor subtracts its own reference, not the liquid surface pressure, so its differential reading is $124.468 - 102.000 = +22.468 \text{ kPa}$.

(d) Shift the entire gauge graph upward by 97.0 kPa . The absolute graph passes through $(0, 97.0)$, $(1, 104.848)$ and $(3, 124.468)$ in metres and kilopascals. The interface depth and both slopes remain unchanged. Changing a pressure reference changes offsets, not the densities inferred from slopes.

Solution P14-35: a trapezoidal net gate load

(a) The wet and dry pressures are

$$p_{\text{wet}} = p_{\text{atm}} + 12000 + \rho g(h_0 + y), \quad p_{\text{dry}} = p_{\text{atm}} + 5000,$$

so the common atmosphere cancels but the two gas overpressures do not. The net field is

$$\Delta p(y) = 7000 + 9810(0.50 + y) = 11905 + 9810y \quad \text{Pa},$$

where y is in metres. It is positive across the gate and consists of a uniform part plus a triangular hydrostatic increment.

(b) Let $a = 11905 \text{ Pa}$ and $k = 9810 \text{ Pa m}^{-1}$. The force and top-edge moment are

$$F = b \int_0^H (a + ky) dy = b \left(aH + \frac{kH^2}{2} \right) = 52116 \text{ N},$$

$$M_{\text{top}} = b \int_0^H y(a + ky) dy = b \left(\frac{aH^2}{2} + \frac{kH^3}{3} \right) = 59964 \text{ N m}.$$

These correspond to 52.116 kN and 59.964 kN m . Independently, split the load into a rectangle of force baH acting at $H/2$ and a triangle of force $b k H^2/2$ acting at $2H/3$; summing their moments reproduces the integral.

(c) The resultant lies at $y_{\text{cp}} = M/F = 1.150587 \text{ m}$ below the top, hence $h_{\text{cp}} = h_0 + y_{\text{cp}} = 1.650587 \text{ m}$ below the liquid surface. It lies between $H/2$ and $2H/3$ as expected for this positive rectangle-plus-triangle load. Using wet-side pressure relative only to atmosphere would retain an unwanted 5.00 kPa dry-side contribution and overestimate the net force. Hinge and actuator forces are separate equilibrium quantities requiring their own support geometry.

Solution P14-36: displaced weight from both liquids

(a) The displaced water and oil volumes are Ax and $A(H - x)$. Their combined buoyant force is

$$F_B = gA[\rho_w x + \rho_o(H - x)].$$

Equilibrium with weight mg therefore gives

$$x = \frac{m/A - \rho_o H}{\rho_w - \rho_o} = \frac{180 - 160}{200} = 0.100 \text{ m}.$$

This lies between zero and $H = 0.200 \text{ m}$, so the assumed interface-spanning configuration is self-consistent. The average object density is $m/(AH) = 900 \text{ kg m}^{-3}$, between those of the two liquids.

(b) The vertical diagram contains upward F_B and downward mg , with no holding force. The displaced water mass is 1.000 kg , giving a buoyant contribution of 9.810 N . The displaced oil mass is 0.800 kg , giving 7.848 N . Their sum is 17.658 N , equal to $mg = 1.80(9.81) = 17.658 \text{ N}$. Interfacial forces were explicitly neglected.

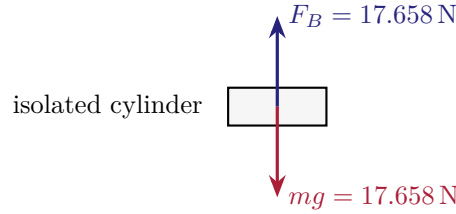


Figure 14.42: The equilibrium force diagram answering P14-36(b). Both displaced-liquid contributions are combined into F_B ; adding them again as separate external forces would count buoyancy twice. Equal arrow lengths encode equal force magnitudes.

(c) An upward displacement ϵ reduces x by ϵ and substitutes displaced oil for displaced water. The change in net upward force is

$$\delta F_{\text{up}} = -gA(\rho_w - \rho_o)\epsilon = -19.62 \text{ N m}^{-1}\epsilon.$$

For $\epsilon = 0.0100 \text{ m}$, it is -0.1962 N , a downward restoring force. A downward displacement gives an upward restoring force. Vertical equilibrium is stable within the interval where both cylinder ends remain in their original liquids; this argument does not establish rotational stability.

Solution P14-37: area weighting matters

(a) Annuli of radius r have area $2\pi r dr$. Thus

$$\begin{aligned} \dot{V} &= 2\pi v_{\text{max}} \int_0^R \left(1 - \frac{r^2}{R^2}\right) r dr = \frac{\pi R^2 v_{\text{max}}}{2}, \\ \bar{v} &= \frac{\dot{V}}{\pi R^2} = \frac{v_{\text{max}}}{2} = 0.200 \text{ m s}^{-1}. \end{aligned}$$

With $R = 1.00 \times 10^{-3} \text{ m}$, $\dot{V} = 6.28319 \times 10^{-7} \text{ m}^3 \text{ s}^{-1}$. Taking an unweighted radial average would give a different value because it ignores the greater area of outer annuli.

(b) The parabolic model has $v_{\text{max}} = \Delta p R^2 / (4\eta_v L)$. Hence $\Delta p = 4\eta_v L v_{\text{max}} / R^2 = 800 \text{ Pa}$. Differentiating the graph gives $dv/dr = -2v_{\text{max}} r / R^2$. The resisting wall shear magnitude is $\tau_w = \eta_v |v'(R)| = 2\eta_v v_{\text{max}} / R = 0.800 \text{ Pa}$. The independent balance $\tau_w = \Delta p R / (2L)$ returns the same result.

(c) Integrating only to $r = aR$ and dividing by the full flow gives $\dot{V}(aR) / \dot{V}(R) = 2a^2 - a^4$. At $a = 1/2$, the fraction is $7/16 = 0.4375$, while the area fraction is $a^2 = 0.25$. The core carries 43.75% of flow through 25% of the area because its speed is higher.

(d) $\text{Re} = 1000(0.200)(0.00200)/0.00100 = 400$. This supports the laminar-regime assumption for ordinary tube flow. The plotted zero wall speed is an imposed boundary condition of the model, not a consequence measured by this Reynolds-number calculation. Testing no slip requires boundary-sensitive evidence, and testing full development requires suitable inlet conditions or profile measurements.

Solution P14-38: fitting, residuals and restrained inference

(a) The first two points have the same flow-to-pressure ratio, so the stated origin-passing reference line has

$$G = 0.0920 \text{ mL s}^{-1} \text{ kPa}^{-1} = 9.20 \times 10^{-11} \text{ m}^3 \text{ s}^{-1} \text{ Pa}^{-1}.$$

Using $G = \pi R^4 / (8\eta_v L)$ with the measured tube geometry gives

$$\eta_{\text{eff}} = \frac{\pi(0.500 \times 10^{-3})^4}{8(0.250)(9.20 \times 10^{-11})} = 1.06712 \times 10^{-3} \text{ Pa s}.$$

It is an *effective* viscosity inferred within the developed-flow model. A biased radius or an unmodelled resistance could produce the same inferred value.

(b) Define residual as $r_i = \dot{V}_i - G\Delta p_i$. In increasing pressure order the residuals are

$$(0, 0, -0.002, -0.008, -0.022) \text{ mL s}^{-1}.$$

At 5 kPa the reference predicts 0.460 mL s^{-1} . Its relative residual is

$$\frac{-0.022}{0.460} = -0.0478261,$$

corresponding to -4.78% . The flow is increasingly below the calibrated linear line; inspecting the residual plot makes this trend clearer than inspecting only the main flow curve.

(c) The largest observed mean speed and its Reynolds number are

$$\bar{v} = \frac{0.438 \times 10^{-6}}{\pi(0.500 \times 10^{-3})^2} = 0.557679 \text{ m s}^{-1}, \quad \text{Re} = \frac{\rho\bar{v}(2R)}{\eta_{\text{eff}}} = 522.6.$$

This remains consistent with ordinary laminar tube flow and does not by itself justify a claim of transition to turbulence.

(d) A growing localized inertial/entrance loss can make the total required pressure nonlinear in flow; viscosity drift with temperature or pressure-sensor nonlinearity could also generate a trend. Measure temperature at every setting and repeat after returning to the low-pressure settings to test drift. Separate pressure taps across a long developed segment from the entrance/outlet to test end resistance. Calibrate the pressure sensor independently if nonlinearity is suspected. No uncertainties or repeated observations are supplied, so there is no defensible statistical rejection threshold. The synthetic pattern diagnoses a model/data issue that needs investigation; it does not identify a unique physical cause.

Solution P14-39: pressure change, elevation and shaft work

(a) The section area is $\pi D^2/4$, so

$$v = \frac{0.00200}{\pi(0.0400)^2/4} = 1.59155 \text{ m s}^{-1}, \quad \frac{v^2}{2g} = 0.129104 \text{ m}.$$

At A, $H_A = 20000/(1000 \times 9.81) + 0.129104 + 0 = 2.167840 \text{ m}$. The total head uses the declared section model with $\alpha = 1$.

(b) The energy line rises by the pump head and drops by the route loss: $H_B = H_A + 5.00 - 2.00 = 5.167840 \text{ m}$. At B subtract elevation 4.00 m and the unchanged velocity head. Thus $p_B/(\rho g) = 1.038736 \text{ m}$ and

$$p_B = 10190 \text{ Pa} = 10.19 \text{ kPa} \quad \text{gauge.}$$

Directly canceling equal velocity terms gives $p_B = p_A + \rho g(h_p - h_L - z_B + z_A)$ and reproduces the answer. Both checks keep the same pressure reference.

(c) The power supplied to the fluid is $P_f = \rho g \dot{V} h_p = (1000)(9.81)(0.00200)(5.00) = 98.1 \text{ W}$. With the stated overall electrical-to-fluid efficiency, $P_e = P_f/0.650 = 150.923 \text{ W} \simeq 151 \text{ W}$.

(d) The pressure difference combines a pump gain, dissipative loss and elevation change. It is therefore not equal to the friction loss alone. Applying lossless Bernoulli across the running pump would omit shaft work; applying it along the full route would also omit the declared loss. A head diagram is useful precisely because it distinguishes these separate contributions to the same final pressure.

Solution P14-40: interpreting the cosine graph

(a) The amplitude is $2\gamma_s/(\rho g R) = 0.0293578 \text{ m} = 29.3578 \text{ mm}$. Thus

$$h(60^\circ) = +14.6789 \text{ mm}, \quad h(90^\circ) = 0, \quad h(120^\circ) = -14.6789 \text{ mm}.$$

The signs mean rise, no signed displacement in the ideal model, and depression, respectively. The corresponding negative cosine must not be replaced by its magnitude.

(b) Doubling R halves every graph height, including negative values. Its zero remains at 90° because the radius rescales the amplitude and does not change the cosine factor.

(c) Inverting the model gives $\gamma_s = h \rho g R / (2 \cos \theta_c)$. Near 90° , the cosine and measured height are both small. A modest absolute angle error can cause a large fractional error in $\cos \theta_c$, so inferred surface tension becomes poorly conditioned. Linear sensitivity contains $\tan \theta_c \delta \theta_c$ with the angle uncertainty in radians, which grows strongly near 90° . At exactly 90° , the zero-height result contains no information about surface tension within this model.

(d) The scale ratio is

$$\frac{\rho g R^2}{\gamma_s} = \frac{(1000)(9.81)(0.500 \times 10^{-3})^2}{0.0720} = 0.0340625.$$

It compares gravitational and interfacial effects over the radius scale. Its small value supports the relevance of capillary forces at that scale, but does not establish a constant equilibrium contact angle, absence of contamination or applicability of a macroscopic interface model. Those require separate information.

Solution P14-41: distinct curves with the same local tangent

(a) Integrate the prescribed particle equations with the stated initial position:

$$\frac{dx}{dt} = U, \quad \frac{dy}{dt} = at \quad \Rightarrow \quad x = Ut, \quad y = \frac{at^2}{2}.$$

Eliminating t gives the pathline $y = ax^2/(2U^2)$. With the supplied numbers, it is the upward-opening parabola $y = (0.200 \text{ m}^{-1})x^2$.

(b) At $t = 2.00 \text{ s}$, the particle is at $(x, y) = (2.00, 0.800) \text{ m}$. Freeze the time in the field. Streamlines then satisfy $dy/dx = v/u = at/U = 0.800$, a constant. The particular streamline through the particle is

$$y - 0.800 \text{ m} = 0.800(x - 2.00 \text{ m}), \quad \text{or} \quad y = 0.800x - 0.800 \text{ m}.$$

(c) Sketch the parabola from the origin and the straight line with vertical intercept -0.800 m on axes covering the requested range. Mark their common point $(2.00, 0.800)$ m. The pathline slope there is $dy/dx = ax/U^2 = 0.800$, equal to the streamline slope. This shared tangent expresses the instantaneous velocity at the particle, but away from that point the pathline records different times while the streamline uses only the field at 2.00 s. A correct sketch therefore shows a tangency, not identical curves or an arbitrary-angle crossing.

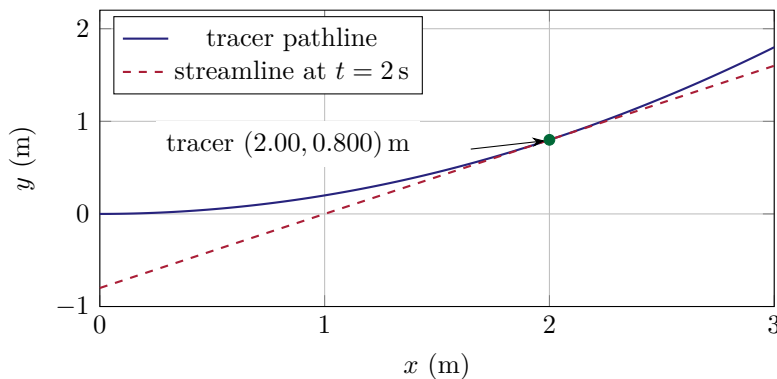


Figure 14.43: The requested sketch for P14-41(c). The particle history is a parabola; the instantaneous streamline through its current position is the tangent line there. Agreement of a local tangent does not make the two entire curves identical.

Solution P14-42: allowing the free surface to move

(a) Let both v_s and v_e denote positive speed magnitudes. Incompressible continuity gives $A_T v_s = a v_e$, so $v_s = \beta v_e$ with $\beta = a/A_T$. The stipulated quasi-steady Bernoulli model between surface and outlet gives

$$\frac{v_s^2}{2} + gh = \frac{v_e^2}{2} \Rightarrow v_e = \sqrt{\frac{2gh}{1 - \beta^2}}.$$

The condition $\beta < 1$ is necessary for this reduction. A tank with an outlet comparable to its surface area would also undermine the simple reservoir-flow idealization.

(b) Here $\beta = 0.100$. The outlet speed is

$$v_e = \sqrt{\frac{2(9.81)(0.800)}{0.990}} = 3.98178 \text{ m s}^{-1},$$

and the surface speed is $v_s = 0.398178 \text{ m s}^{-1}$. The volume rate is $\dot{V} = a v_e = 7.96355 \times 10^{-3} \text{ m}^3 \text{ s}^{-1}$. Since the reservoir volume decreases,

$$\frac{dh}{dt} = -\frac{\dot{V}}{A_T} = -0.398178 \text{ m s}^{-1}.$$

(c) Neglecting v_s gives $v_{e,0} = \sqrt{2gh} = 3.96182 \text{ m s}^{-1}$. Its fractional underestimate relative to the retained-surface-speed result is $1 - v_{e,0}/v_e = 1 - \sqrt{1 - \beta^2} = 0.00501256$, or 0.501%. Equivalently, the retained result is 0.504% above the simplified estimate; the denominators differ. Including initial kinetic energy at the moving surface is consistent with this ideal energy balance.

(d) Viscous losses and outlet contraction can change the speed/flow prediction. Nonuniform flow near the outlet can also invalidate uniform section speeds. Quasi-steady modeling means the field readjusts rapidly compared with the changing water level; a small area ratio only makes the surface-speed term small. The two are distinct assumptions, so one does not prove the other.

Solution P14-43: separating draining and adjustment times

(a) Under the declared head-to-viscous-pressure approximation,

$$\dot{V} = \frac{\pi R^4 \rho g h}{8 \eta_v L}, \quad A_T \frac{dh}{dt} = -\dot{V}.$$

Define $C = \pi R^4 \rho g / (8 \eta_v L A_T)$ and $\tau = 1/C$. Then $dh/dt = -Ch$, giving

$$h(t) = h_0 e^{-t/\tau}, \quad C = 2.40774 \times 10^{-5} \text{ s}^{-1}, \quad \tau = 41532.8 \text{ s} = 11.537 \text{ h}.$$

The exponential follows because the pressure-driven flow is proportional to the remaining head; it is not the lossless-orifice draining law.

(b) Set $h/h_0 = 0.100/0.400 = 1/4$. The time is

$$t = \tau \ln(h_0/h) = 41532.8 \ln 4 = 57576.7 \text{ s} = 15.994 \text{ h}.$$

(c) The largest mean speed occurs initially:

$$\bar{v}_0 = \frac{R^2 \rho g h_0}{8 \eta_v L} = 0.122625 \text{ m s}^{-1}, \quad \text{Re}_0 = 122.625.$$

For a parabolic outlet, $h_{K,0} = 2\bar{v}_0^2/(2g) = 0.00153281 \text{ m}$, which is 0.3832% of h_0 . This is a small outlet-kinetic-head correction for the stipulated model. As h decreases, $\bar{v} \propto h$, so $h_K/h \propto h$ and that fraction becomes smaller. Entrance losses or variable capillary backpressure were excluded explicitly; this estimate does not bound them.

(d) The viscous diffusion scale is $t_\nu = \rho R^2/\eta_v = 0.250 \text{ s}$, only 6.02×10^{-6} of τ . Rapid momentum diffusion across the radius compared with reservoir draining supports quasi-steady profile adjustment. It does not independently prove correct inlet development, a constant viscosity, the no-slip condition or negligible capillary effects.

(e) The predicted graph is $\ln[h(t)/h_0] = -t/\tau$, a straight line through the origin of slope $-2.40774 \times 10^{-5} \text{ s}^{-1}$. Compare actual height measurements with the untransformed exponential as well, using their measurement uncertainties. Curved residuals could indicate changing viscosity, an appreciable outlet energy term, entrance resistance, capillary backpressure, or a nonconstant reservoir area. A height-sensor offset can also create curvature in the log plot. Repeated data, temperature monitoring and independent geometry/pressure checks would help distinguish physical deviations from measurement bias.

14.13 Bridge to Chapter 15

The mechanical principles developed in this chapter provide the foundation for understanding oscillatory motion. When a system is displaced from a stable equilibrium position, restoring forces may drive it back toward equilibrium, while inertia carries it beyond that position, producing repeated motion. Chapter 15 develops this behavior systematically through simple harmonic motion, energy exchange, damping, driven oscillations and resonance. The present chapter supplies the mechanical foundations, force analysis and energy principles needed to understand how oscillations arise and how their characteristic frequencies and amplitudes are determined.

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