

Chapter 15

Oscillations

First Edition

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Preface

Oscillation is the recurring motion of a system about an equilibrium configuration. It appears in structures, sensors, vehicles, clocks, molecules, electrical circuits and, in the next chapter, as the local motion that supports a mechanical wave. The recurring shape of a graph is not the deepest idea. The deeper idea is that a stable system can store energy in different forms and repeatedly exchange them while inertia carries it through equilibrium.

This chapter develops that idea in layers. We begin with the ideal linear oscillator and deliberately measure displacement from the *actual equilibrium position*. That choice makes a vertical spring look mathematically identical to a horizontal one after gravity has shifted its equilibrium. We then connect the differential equation to amplitude, phase, initial conditions and energy; use the projection of uniform circular motion as a representation of phase; develop simple, physical and torsional pendulums; and add damping and a sinusoidal driving force. The chapter then turns the same equations into tools for parameter identification, vibration interpretation and numerical checking.

The central modelling question is always: *what force or torque is restoring, what has been linearised, what variable is observed, and which input is being held fixed?* These questions matter especially near resonance. A force-driven oscillator does not have one universal “resonance frequency”: the peak of displacement and the peak of velocity are different statements about different observables.

The Guideline

Treat every oscillation problem first as a problem about equilibrium and model choice. Locate the equilibrium, define the displacement from it, identify the restoring and dissipative forces, state the approximation that makes the model linear, and only then solve the differential equation. A sinusoidal graph is a consequence of the model, not its definition.

Pedagogical boundary

The core chapter uses linear, single-degree-of-freedom models. Finite-amplitude nonlinear pendulum motion is deliberately not developed as a second advanced module. The selected advanced pathway is a symmetric pair of coupled linear oscillators, which prepares normal modes and the transition to waves without making advanced material a prerequisite for the core.

Conventions and notation

Displacement x is measured from the *actual stable equilibrium* unless another origin is explicitly stated. A spring stiffness is denoted by k_s so that it is not confused with a wave number in later chapters. The viscous damping coefficient is b , with

$$\beta = \frac{b}{2m}, \quad \zeta = \frac{\beta}{\omega_0},$$

and the undamped natural angular frequency is ω_0 . In an underdamped free response, $\omega_d = \sqrt{\omega_0^2 - \beta^2}$. A sinusoidal drive uses Ω ; these three frequencies are not interchangeable. The oscillation period is T and $f = 1/T$. The logarithmic decrement is ℓ , distinct from the driven phase lag δ ; the quality factor is $Q = m\omega_0/b$. Amplitude A is non-negative, while the phase constant ϕ is an angle in radians.

Symbol	Meaning	SI unit
$x, \dot{x}, \ddot{x}$	displacement from equilibrium, velocity, acceleration	m, m s ⁻¹ , m s ⁻²
m	oscillating mass	kg
k_s	linear spring stiffness	N m ⁻¹
A	displacement amplitude, $A \geq 0$	m
ϕ	phase constant	rad
ω_0	undamped natural angular frequency	rad s ⁻¹
ω_d	underdamped free angular frequency	rad s ⁻¹
Ω	imposed driving angular frequency	rad s ⁻¹
b	viscous damping coefficient in $-b\dot{x}$	kg s ⁻¹
$\beta = b/(2m)$	damping rate	s ⁻¹
$\zeta = \beta/\omega_0$	damping ratio	1
κ_θ	torsional stiffness	N m rad ⁻¹
I_O	moment of inertia about the actual pivot O	kg m ²

We use $g = 9.81 \text{ m s}^{-2}$ when a numerical value is required. Radians are dimensionless in SI but are written when they clarify angular meaning. Numerical examples retain guard digits internally and round final values to a precision justified by the stipulated inputs.

Note

A measured or synthetic data record is identified as such. Values introduced only to illustrate a model are *stipulated teaching parameters*; they are not claims about equipment or measurements at AGH University of Krakow.

Chapter 15

Oscillations

A mass hangs from a vertical spring. Before release, gravity stretches the spring. If the mass is pulled a little farther down and released, should gravity appear explicitly in the equation for every subsequent oscillation? The answer depends on the coordinate origin. The physical weight has not disappeared, but a wise coordinate choice can make its constant contribution cancel exactly against the static spring force. This simple observation will become a general method: first find equilibrium, then describe small motion about it.

Learning goals

After studying the core of this chapter, you should be able to:

- derive harmonic motion from a linear restoring force after choosing the correct equilibrium coordinate;
- determine amplitude and phase from position and velocity at a specified time;
- connect displacement, velocity, acceleration and energy without losing their phase relations;
- derive and assess the small-angle simple-pendulum approximation and compare it with physical and torsional pendulums;
- distinguish underdamped, critically damped and overdamped free responses;
- separate transient and steady-state motion in a driven oscillator and identify displacement and velocity response peaks for a fixed-force drive;
- infer frequency and damping from a stated low-amplitude decay model, including checks on sampling and peak selection;
- validate a numerical oscillator calculation against analytical limits and identify the physical input model being simulated.

Key message

A linear oscillator is organised by four linked ideas: a stable equilibrium, a restoring force proportional to displacement, inertia, and a phase that records where the system is in its cycle. Damping and forcing modify this structure; they do not remove the need to define the equilibrium and the observable precisely.

15.1 Stable equilibrium and harmonic motion

15.1.1 What makes an equilibrium oscillate?

A mass hanging at rest on a spring is already deformed by gravity. If it is pulled a little farther down and released, its motion is measured from the hanging position, not from the unstretched spring length. This distinction is the first modelling decision in an oscillator problem. A suitable origin simplifies the equation because it incorporates the constant forces that balance before motion begins.

An equilibrium configuration has zero net force. In one dimension, let q denote an arbitrary position coordinate and let q_{eq} satisfy $F(q_{\text{eq}}) = 0$. Equilibrium alone does not imply stability: a pencil balanced on its point also has a possible equilibrium. A stable configuration resists small disturbances. When displaced to either side, the system experiences a restoring tendency towards the equilibrium. Inertia then explains the overshoot. A mass approaching equilibrium has a nonzero velocity and therefore travels through the point where the restoring force vanishes. On the other side the force reverses, slows the mass, and eventually turns it back. Repeated exchange of stored energy and kinetic energy produces oscillation.

For a smooth force function, expand about the equilibrium. With $x = q - q_{\text{eq}}$,

$$F(q_{\text{eq}} + x) = F(q_{\text{eq}}) + F'(q_{\text{eq}})x + \mathcal{O}(x^2). \quad (15.1)$$

The prime means a derivative with respect to position. The constant term vanishes by the equilibrium condition. If $F'(q_{\text{eq}}) < 0$, positive x produces negative force and negative x produces positive force, to leading order. Define the local positive stiffness

$$k_s = -F'(q_{\text{eq}}) > 0, \quad F_x \simeq -k_s x. \quad (15.2)$$

Stiffness measures how rapidly the restoring force changes with displacement; its unit is N m^{-1} . A straight force–displacement graph describes an ideal linear spring. A curved graph can still have an approximately linear tangent near a suitable equilibrium. The allowable amplitude must then be checked, rather than assumed from the word “small.”

15.1.2 The harmonic equation and its two initial conditions

Newton’s second law for a constant mass gives

$$m\ddot{x} + k_s x = 0, \quad \ddot{x} + \omega_0^2 x = 0, \quad \boxed{\omega_0 = \sqrt{\frac{k_s}{m}}}. \quad (15.3)$$

An overdot means differentiation with respect to time, so $\dot{x}$ is velocity and $\ddot{x}$ is acceleration. This is a second-order differential equation: specifying acceleration as a function of position does not yet specify the whole motion. Both the initial position and the initial velocity are needed. For instance, a mass initially at equilibrium can remain there if its initial velocity is zero, or execute a finite oscillation if it is given a velocity.

The trial functions $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$ each have a second derivative equal to $-\omega_0^2$ times the original function. Their linear combination therefore satisfies the equation:

$$x(t) = C \cos(\omega_0 t) + D \sin(\omega_0 t). \quad (15.4)$$

Substitution at $t = 0$ gives $C = x_0$ and $D = v_0/\omega_0$. Thus an immediately useful initial-value solution is

$$\boxed{x(t) = x_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t)}. \quad (15.5)$$

The coefficient v_0/ω_0 has units of length, as it must. This form avoids a phase-quadrant decision when the goal is simply to predict the motion. In the next section the same solution is expressed through amplitude and phase, which are often more informative experimentally.

Definition 15.1: Simple harmonic motion

Simple harmonic motion is motion satisfying $\ddot{x} = -\omega_0^2 x$, with constant $\omega_0 > 0$ and displacement measured from equilibrium. Acceleration is proportional to displacement and points in the opposite direction. A sinusoidal position is the resulting solution of this model.

A full cycle corresponds to an increase of 2π in the argument of the sine or cosine. Hence

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k_s}}, \quad f = \frac{1}{T} = \frac{\omega_0}{2\pi}. \quad (15.6)$$

Angular frequency is expressed in rad s^{-1} and frequency in hertz, meaning cycles per second. The radian is dimensionless in SI, but retaining “rad” makes the distinction between angular frequency and cycle frequency visible. Increasing mass reduces the frequency because a given restoring force produces less acceleration. Increasing stiffness raises it. Amplitude does not appear in the ideal formula because doubling the displacement also doubles the restoring force and the speed scale; the geometrically larger cycle is completed in the same time.

15.1.3 The vertical spring: natural length and actual equilibrium

Take downward as positive. Let y be extension measured from the natural length of a massless linear spring. The spring force is upwards when it is extended, so

$$m\ddot{y} = mg - k_sy, \quad y_{\text{eq}} = \frac{mg}{k_s}. \quad (15.7)$$

Now introduce $x = y - y_{\text{eq}}$. The reference point is fixed, so $\ddot{x} = \ddot{y}$. Substitution gives

$$m\ddot{x} = mg - k_s(x + y_{\text{eq}}) = -k_sx. \quad (15.8)$$

Gravity remains an actual force. Its constant contribution is cancelled by the part of spring force already present at static equilibrium. Figure 15.1 distinguishes the two lengths and the actual forces. The equilibrium extension marker joins the natural-length level to the equilibrium level; it is not a measure of the entire spring length.

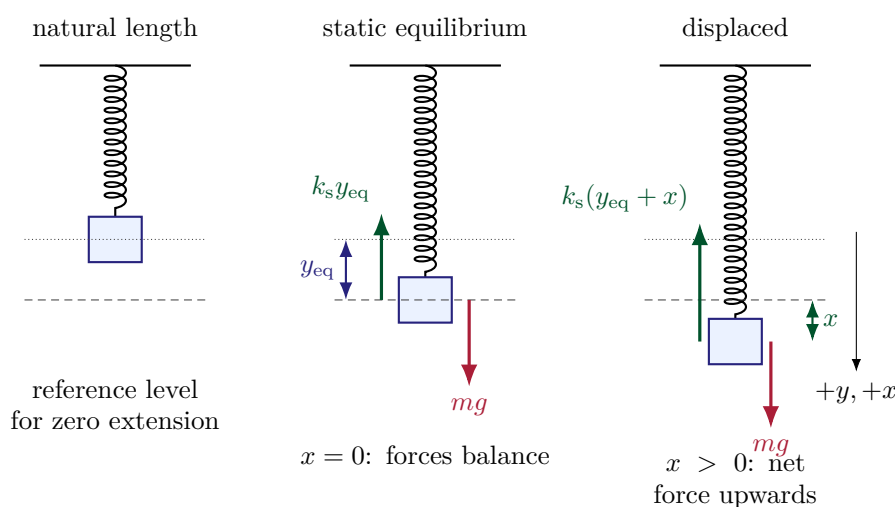


Figure 15.1: F15-01. Two coordinate origins for a vertical ideal spring. Dotted lines mark the mass-centre level for the natural length and dashed lines mark static equilibrium. The extension is $y = y_{\text{eq}} + x$. The equilibrium forces are equal; below equilibrium the upward spring force is larger. Geometry and force lengths are schematic, while their signs and relative magnitudes follow the linear model.

Example 15.1: Vertical spring with a shifted origin

A mass $m = 0.80 \text{ kg}$ hangs from an ideal vertical spring with $k_s = 50 \text{ N m}^{-1}$. It is displaced an additional 4.0 cm downwards from static equilibrium and released. Find the static extension, natural angular frequency, period, and motion with $t = 0$ at release. Give the result in both coordinate systems of Figure 15.1.

Model and strategy. Neglect spring mass, damping, and variation of g . Use $g = 9.81 \text{ m s}^{-2}$. Find static equilibrium first, then shift the coordinate and impose release from rest. The two coordinate descriptions must have the same acceleration.

$$y_{\text{eq}} = \frac{(0.80)(9.81)}{50} \text{ m} = 0.15696 \text{ m}, \quad \omega_0 = \sqrt{\frac{50}{0.80}} = 7.9057 \text{ rad s}^{-1},$$

$$T = \frac{2\pi}{7.9057} \text{ s} = 0.79477 \text{ s}.$$

With $x_0 = 0.040 \text{ m}$ and $v_0 = 0$, the two descriptions are

$$x(t) = 0.040 \cos(7.9057t) \text{ m}, \quad y(t) = [0.15696 + 0.040 \cos(7.9057t)] \text{ m},$$

where numerical t is in seconds. At release the net force is $-k_s x_0 = -2.00 \text{ N}$, giving $\ddot{x}(0) = -2.50 \text{ m s}^{-2}$. Independently, $mg - k_s y(0) = 7.848 - 9.848 = -2.00 \text{ N}$.

Interpretation and limits. At equilibrium the weight and spring force are both nonzero but cancel. Altering g changes the equilibrium extension while leaving $\sqrt{k_s/m}$ unchanged if the same mass and stiffness remain valid. This conclusion assumes the spring is still linear about its new extension. A nonlinear spring may have a different local stiffness at the new equilibrium.

Guided checks

Check 1. At $x = A > 0$ in SHM, what is the acceleration sign? **Answer.** It is negative, since $\ddot{x} = -\omega_0^2 A$: the mass accelerates towards equilibrium.

Check 2. Should a separate constant $+mg$ remain when a vertical-spring displacement is measured from static equilibrium? **Answer.** No. The equilibrium spring force already balances the weight; substitution cancels both constant terms.

Check 3. Does every stable equilibrium produce exact SHM? **Answer.** No. A nonzero positive local stiffness supports a leading harmonic approximation; finite displacements can sample nonlinear forces. Some stable minima even have zero quadratic curvature.

15.1.4 Potential curvature, stability, and the approximation range

For conservative motion, $F = -dU/dq$. At equilibrium $U'(q_{\text{eq}}) = 0$, and a Taylor expansion gives

$$U(q_{\text{eq}} + x) = U(q_{\text{eq}}) + \frac{1}{2}U''(q_{\text{eq}})x^2 + \mathcal{O}(x^3). \quad (15.9)$$

If $U''(q_{\text{eq}}) > 0$, the equilibrium is a strict local minimum and $k_s = U''(q_{\text{eq}})$ is positive. This is a sufficient stability condition, not a necessary one. The potential $U = \alpha x^4/4$ with $\alpha > 0$ also has a stable minimum at $x = 0$, although $U''(0) = 0$. Its force is $-\alpha x^3$ and there is no nonzero harmonic frequency obtained from linearisation. Conversely, $U'' < 0$ signals instability, while $U'' = 0$ requires examination of higher-order terms.

Figure 15.2 shows a harmonic potential and one with a positive quartic correction. Both have the same curvature at zero. Farther away, the additional potential rises more rapidly, and its slope produces a stronger restoring force. A measured period can then depend on amplitude. Stability and harmonicity are separate questions.

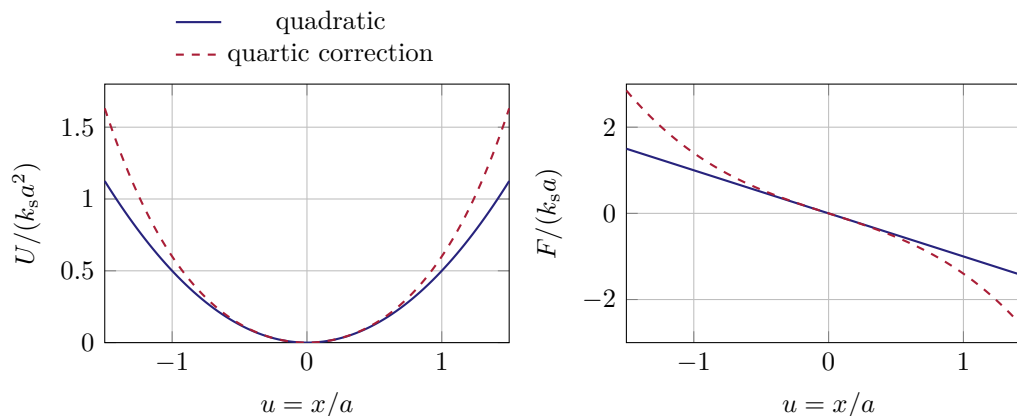


Figure 15.2: Local harmonic approximation and a hardening restoring force. With arbitrary length scale a , the compared potentials are $U/(k_s a^2) = u^2/2$ and $u^2/2 + 0.1u^4$. The force curves are their negative derivatives. They agree to first order at equilibrium and differ increasingly at larger displacement.

Example 15.2: Small oscillations about a biased nonlinear equilibrium

A 0.40 kg proof mass has force $F(q) = P - kq - \alpha q^3$, with $k = 40 \text{ N m}^{-1}$, $\alpha = 4000 \text{ N m}^{-3}$, and constant load $P = 0.832 \text{ N}$. Verify equilibrium at $q_{\text{eq}} = 0.020 \text{ m}$, obtain the local harmonic frequency, and assess the linear force approximation for $|x| \leq 2.0 \text{ mm}$.

Strategy. The constant load shifts equilibrium; the derivative of the complete force, rather than k alone, determines the local stiffness. At the stated position,

$$kq_{\text{eq}} + \alpha q_{\text{eq}}^3 = 0.800 + 0.032 = 0.832 \text{ N} = P.$$

Writing $q = q_{\text{eq}} + x$ and cancelling the constant terms gives

$$F = -(k + 3\alpha q_{\text{eq}}^2)x - 3\alpha q_{\text{eq}}x^2 - \alpha x^3.$$

Thus $k_s = 40 + 3(4000)(0.020)^2 = 44.8 \text{ N m}^{-1}$, and

$$\omega_0 = \sqrt{44.8/0.40} = 10.583 \text{ rad s}^{-1}, \quad T = 0.59371 \text{ s}.$$

For nonzero x , the nonlinear remainder divided by $k_s|x|$ is bounded by

$$\frac{3\alpha|q_{\text{eq}}|A + \alpha A^2}{k_s} = \frac{0.480 + 0.016}{44.8} = 0.0111.$$

The linear restoring force is therefore accurate to a little over one percent relative to its magnitude throughout the stated interval. At $x = 0$ both forces vanish and the limiting relative-error bound remains meaningful.

Check and interpretation. The local frequency exceeds $\sqrt{k/m} = 10.0 \text{ rad s}^{-1}$. Constant loading can therefore change frequency through a changed equilibrium stiffness, even though it does not do so for a truly linear spring. The force-error bound is not itself a rigorous bound on period error; it tests the first modelling step.

15.1.5 Inferring stiffness and effective moving mass

The mass in $m\ddot{x} + k_s x = 0$ is the inertia associated with the displacement coordinate. A sensor, fixture, and carriage moving together contribute to it. Ignoring those parts can make a period-based stiffness estimate disagree with a static force–extension measurement. A spring’s own distributed

inertia can also matter, but it requires a separate model and is excluded from the massless-spring approximation used here.

A useful inverse relation follows by squaring the period formula. If an added mass m_a moves with an unknown carriage mass m_c , then

$$T^2 = \frac{4\pi^2}{k_s} m_a + \frac{4\pi^2 m_c}{k_s}. \quad (15.10)$$

A straight T^2 versus m_a graph has slope $4\pi^2/k_s$ and intercept $4\pi^2 m_c/k_s$. The graph in Figure 15.3 makes the slope and intercept serve different purposes. A nonzero intercept is evidence for an effective mass in the stated model, rather than a reason to force the fit through the origin.

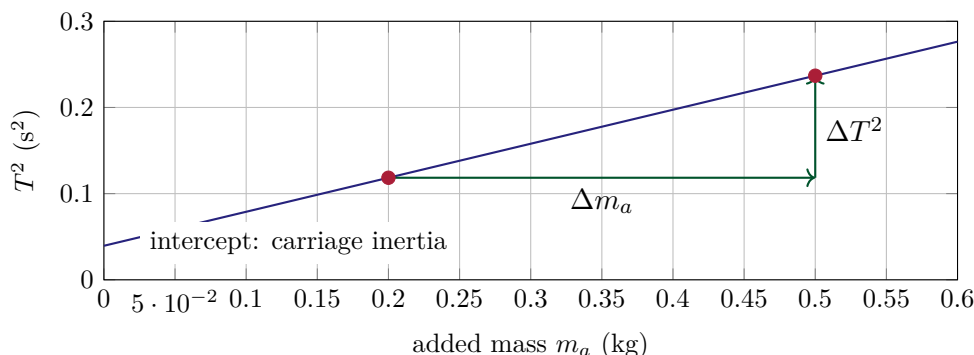


Figure 15.3: Synthetic period calibration for a massless linear spring with $k_s = 100 \text{ N m}^{-1}$ and carriage mass $m_c = 0.100 \text{ kg}$. The marked points are the two inputs used in Example 15.3; the line is the exact stipulated model, not a fit to experimental measurements.

Example 15.3: Infer stiffness and carriage mass from two periods

An ideal spring supports a moving carriage and an added mass. With added masses $m_1 = 0.200 \text{ kg}$ and $m_2 = 0.500 \text{ kg}$, the synthetic periods are $T_1 = 0.344144 \text{ s}$ and $T_2 = 0.486693 \text{ s}$. Determine the stiffness and the unknown carriage mass. Explain why one period would be insufficient.

Model. All attached parts move rigidly together, the spring is massless and linear, and damping is negligible. Subtracting the two squared-period equations eliminates the unknown carriage mass:

$$T_2^2 - T_1^2 = \frac{4\pi^2}{k_s}(m_2 - m_1), \quad k_s = \frac{4\pi^2(m_2 - m_1)}{T_2^2 - T_1^2}.$$

Here $T_1^2 = 0.118435 \text{ s}^2$ and $T_2^2 = 0.236870 \text{ s}^2$, so $k_s = 100.0 \text{ N m}^{-1}$ to input rounding. Then

$$m_c = \frac{k_s T_1^2}{4\pi^2} - m_1 = 0.1000 \text{ kg}.$$

The same calculation using the second period returns 0.1000 kg , an independent consistency check within this model.

Interpretation. One period determines only the ratio $(m_c + m_a)/k_s$, leaving two unknown parameters. The second mass changes inertia without changing stiffness and separates them. In an actual experiment, several mass values permit a residual check: systematic curvature can signal changing stiffness or an inadequate effective-mass model. The synthetic values establish the method but do not demonstrate an actual instrument calibration.

The harmonic equation now supplies a frequency and predicts motion once two initial conditions are specified. Amplitude, phase, and energy describe the same prediction in forms better suited to comparing different initial states.

15.2 Phase, initial conditions and energy

15.2.1 Amplitude and phase from a complete initial state

The sine–cosine solution can be combined into one cosine:

$$x(t) = A \cos(\omega_0 t + \phi), \quad A \geq 0. \quad (15.11)$$

Amplitude is the greatest magnitude of displacement from equilibrium. The phase angle $\omega_0 t + \phi$ identifies where the system lies in its cycle; the phase constant ϕ specifies that position at the chosen time origin. Amplitude is a physical scale of the motion, whereas phase also depends on the convention used to label time. A sine convention is equally valid if its phase is transformed consistently.

Angles in the trigonometric functions are in radians. Differentiation gives

$$v(t) = -A\omega_0 \sin(\omega_0 t + \phi), \quad (15.12)$$

$$a(t) = -A\omega_0^2 \cos(\omega_0 t + \phi) = -\omega_0^2 x(t). \quad (15.13)$$

Velocity is displaced by a quarter cycle from position, while acceleration has the opposite sign to position. Their amplitudes have different units and magnitudes: A , $A\omega_0$, and $A\omega_0^2$. Superposing numerical x , v , and a without normalization would therefore be misleading. Figure 15.4 uses explicit normalizations to compare phase while retaining correct derivative relations.

At $t = 0$,

$$x_0 = A \cos \phi, \quad v_0 = -A\omega_0 \sin \phi. \quad (15.14)$$

Squaring and adding removes the phase:

$$A = \sqrt{x_0^2 + (v_0/\omega_0)^2}. \quad (15.15)$$

For $A > 0$, a quadrant-aware phase is

$$\phi = \text{atan2}(-v_0/\omega_0, x_0) \pmod{2\pi}. \quad (15.16)$$

The two arguments are the sine-like and cosine-like coordinates of one vector, with the same units. A one-argument inverse tangent yields the same result for two opposite quadrants and therefore loses essential information. If $x_0 = v_0 = 0$, then $A = 0$ and the solution is $x(t) = 0$; its phase is undefined because every phase produces the same zero motion. Dividing by A in that case would introduce a spurious mathematical singularity.

Example 15.4: Recover amplitude and phase from an initial state

For the spring oscillator of Example 15.1, suppose that $x_0 = +2.5 \text{ cm}$ and $v_0 = -0.20 \text{ m s}^{-1}$. Find amplitude and phase in the cosine convention, then determine the next zero crossing and the first two extrema after $t = 0$.

Strategy. Use both initial data. The frequency is unchanged because the same ideal mass and spring are used. With $\omega_0 = 7.905694 \text{ rad s}^{-1}$,

$$A = \sqrt{0.025^2 + (-0.20/7.905694)^2} \text{ m} = 0.0355668 \text{ m}.$$

Since $x_0 > 0$ and $v_0 < 0$, both $\cos \phi$ and $\sin \phi$ are positive. Thus

$$\phi = \text{atan2}(0.0252982, 0.025) = 0.791327 \text{ rad}, \quad x(t) = 0.0355668 \cos(7.905694t + 0.791327) \text{ m}.$$

The next zero satisfies $\omega_0 t + \phi = \pi/2$, giving $t = 0.098596 \text{ s}$. At that event $v = -A\omega_0 = -0.281 \text{ m s}^{-1}$, so the mass crosses towards negative displacement. Extrema satisfy $v = 0$. The

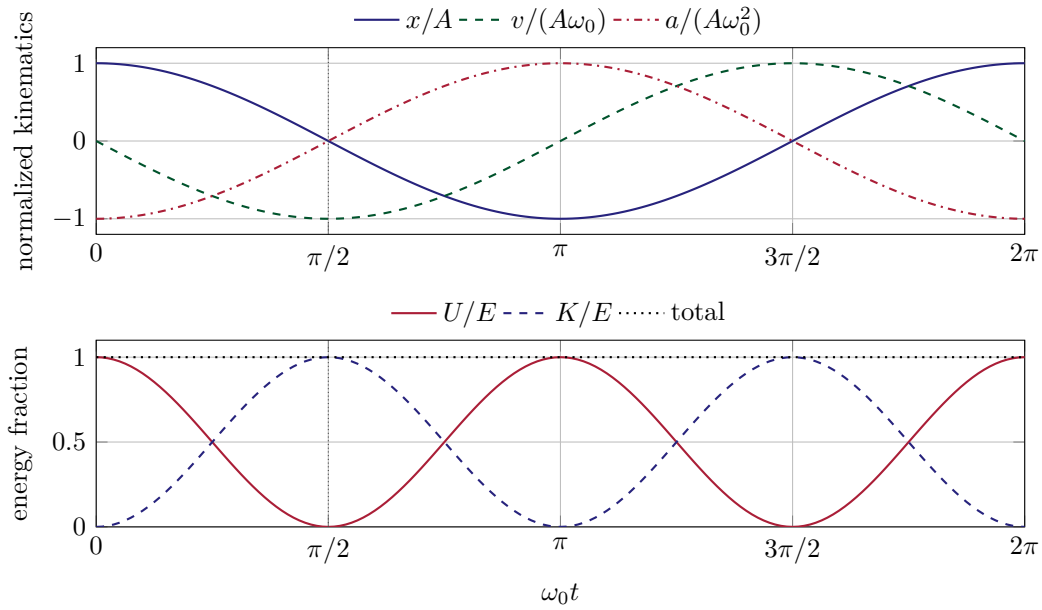


Figure 15.4: F15-02. Linked kinematics and energy for $x = A \cos \omega_0 t$. At the dotted event line $\omega_0 t = \pi/2$, position and acceleration vanish, velocity is most negative, and kinetic energy is maximal. The lower curves exchange energy twice during each displacement cycle. All plotted vertical quantities are dimensionless.

next two phases are π and 2π :

$$t_{\min} = \frac{\pi - \phi}{\omega_0} = 0.297288 \text{ s}, \quad t_{\max} = \frac{2\pi - \phi}{\omega_0} = 0.694671 \text{ s}.$$

Their displacements are $-A$ and $+A$, respectively.

Checks. Substitution at zero returns both original initial conditions. The difference between the two extremum times is $T/2 = 0.39738 \text{ s}$, as expected. Position alone would have admitted both a mass approaching equilibrium and one moving away from it; the velocity resolves the ambiguity.

15.2.2 Energy exchange and the phase portrait

For a mass and ideal spring, with potential energy zero at equilibrium,

$$K = \frac{1}{2} m \dot{x}^2, \quad U = \frac{1}{2} k_s x^2. \quad (15.17)$$

Differentiate their sum rather than assuming conservation from a graph:

$$\frac{dE}{dt} = m \dot{x} \ddot{x} + k_s x \dot{x} = \dot{x} (m \ddot{x} + k_s x) = 0.$$

The equation of motion therefore implies constant mechanical energy. At a turning point $|x| = A$, velocity is zero and

$$E = \frac{1}{2} k_s A^2 = \frac{1}{2} m \omega_0^2 A^2. \quad (15.18)$$

Potential energy is largest at the turning points, not at equilibrium. At equilibrium the mass has its largest speed and zero potential energy with this reference choice. It does not stop there simply because acceleration is momentarily zero.

Substituting the sinusoidal solution gives $U/E = \cos^2(\omega_0 t + \phi)$ and $K/E = \sin^2(\omega_0 t + \phi)$. Each squared function repeats after a phase increase of π , half a displacement cycle. This is the origin

of the doubled energy-oscillation frequency in Figure 15.4. Both energies are nonnegative; energy exchange is not represented by treating the negative parts of a displacement curve as negative energy.

Eliminating time gives

$$\boxed{v^2 = \omega_0^2(A^2 - x^2)}, \quad \frac{x^2}{A^2} + \frac{v^2}{A^2\omega_0^2} = 1. \quad (15.19)$$

A plot of velocity against displacement is a phase portrait. It is an ellipse with dimensional axes, or a circle when both axes are normalized as in Figure 15.5. At a given non-extremal position there are two possible velocities. The upper half of the portrait represents motion towards increasing x , and the lower half motion towards decreasing x . The arrows follow the equations $\dot{x} = v$ and $\dot{v} = -\omega_0^2 x$.

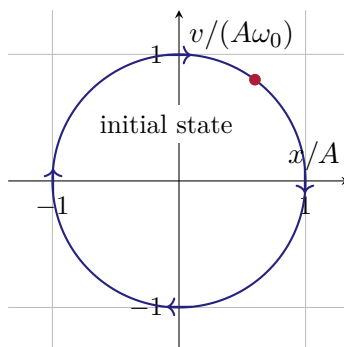


Figure 15.5: Normalized phase portrait of a nonzero ideal harmonic oscillation. Motion follows the circle clockwise for axes x/A and $v/(A\omega_0)$. The marked state has positive displacement and positive velocity, so it moves towards the positive turning point. A dimensional x - v plot is generally an ellipse.

For a vertical spring, the conserved quadratic potential about equilibrium already combines spring and gravitational energies. Indeed, $k_s y^2/2 - mgy = k_s(y - y_{\text{eq}})^2/2 - k_s y_{\text{eq}}^2/2$. The last term is a constant and does not affect dynamics. Counting $k_s x^2/2$ and then adding a new gravitational term $-mgx$ would double-count part of the same physical potential.

Example 15.5: Use energy and phase to locate a specified event

An ideal oscillator has $m = 0.250 \text{ kg}$, $k_s = 100 \text{ N m}^{-1}$, and amplitude $A = 0.0400 \text{ m}$. It is released from the positive turning point at $t = 0$. Find its total energy, the speed when $x = +0.0240 \text{ m}$, the first time it reaches that position, and its signed displacement and distance travelled by half a period.

Strategy. Energy gives speed without solving for time; phase then selects the first event. The release conditions give $x = A \cos \omega_0 t$ with $\omega_0 = \sqrt{100/0.250} = 20.0 \text{ rad s}^{-1}$.

$$E = \frac{1}{2}(100)(0.0400)^2 = 0.0800 \text{ J}, \quad U(0.0240) = 0.0288 \text{ J}.$$

Consequently $K = 0.0512 \text{ J}$ and

$$|v| = \sqrt{2K/m} = 0.640 \text{ m s}^{-1}.$$

The first encounter is on the decreasing-position branch, so $v = -0.640 \text{ m s}^{-1}$. Its time is

$$t = \frac{\cos^{-1}(0.0240/0.0400)}{20.0} = 0.0463648 \text{ s}.$$

By half a period the mass has moved monotonically from $+A$ to $-A$: displacement is $-2A = -0.0800 \text{ m}$ and distance is $2A = 0.0800 \text{ m}$. Returning to the same position after a whole period would give zero displacement but distance $4A$.

Check. Differentiating the position gives $v = -0.800 \sin(0.927295) = -0.640 \text{ m s}^{-1}$ at the calculated event. The position is inside $[-A, A]$, so the energy-derived speed is real. The two methods determine different aspects of the same event and agree on its velocity.

Figure 15.6 shows why positions beyond $|x| = A$ are inaccessible for the energy in this example. The gap between the horizontal total-energy line and the potential curve is kinetic energy. The same graph also explains why energy alone cannot choose the sign of velocity: it gives the square of velocity, and initial conditions select which branch is occupied.

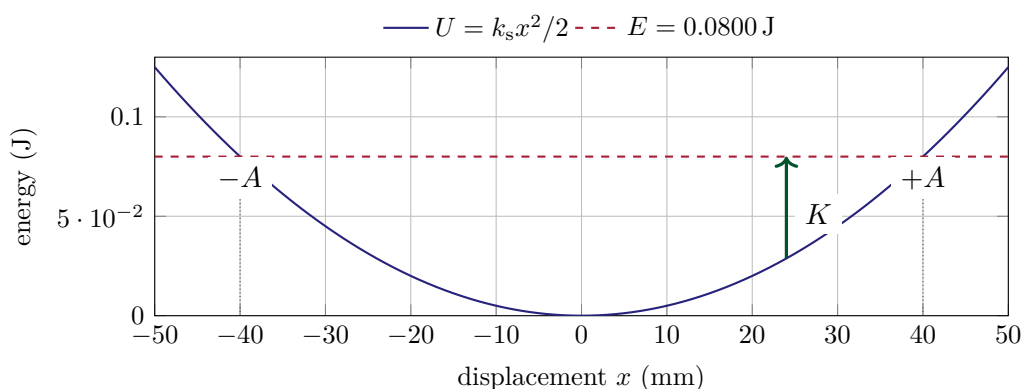


Figure 15.6: Energy and allowed motion for Example 15.5. The intersections $U = E$ occur at $x = \pm 40 \text{ mm}$ and are turning points. At $x = 24 \text{ mm}$, the marked vertical gap is $K = 0.0512 \text{ J}$. The potential outside the turning points exceeds the available energy.

15.2.3 Inferring a state from uncertain measurements

An initial state obtained from measurements has uncertainty, which carries into amplitude and phase. For independent small standard uncertainties σ_x and σ_v , treating ω_0 as known, first-order propagation gives

$$\sigma_A^2 \simeq \left(\frac{x_0}{A} \sigma_x \right)^2 + \left(\frac{v_0}{\omega_0^2 A} \sigma_v \right)^2, \quad (15.20)$$

$$\sigma_\phi^2 \simeq \left(\frac{v_0}{\omega_0 A^2} \sigma_x \right)^2 + \left(\frac{x_0}{\omega_0 A^2} \sigma_v \right)^2. \quad (15.21)$$

These formulas follow by differentiating amplitude and the quadrant-aware phase locally and adding the variances of independent contributions. They are local approximations, not universal rules for large uncertainties. Correlated position and velocity estimates require covariance terms. Near $A = 0$, phase cannot be reliably inferred because a very small noise vector can point in any direction. A phase uncertainty crossing 0 and 2π must be interpreted as an angular interval, not as two physically distant states.

Example 15.6: Amplitude and phase from uncertain initial data

For $\omega_0 = 10.0 \text{ rad s}^{-1}$ known exactly in the stated model, measurements give $x_0 = 0.0300 \text{ m}$ with standard uncertainty 0.0010 m and $v_0 = 0.400 \text{ m s}^{-1}$ with standard uncertainty 0.020 m s^{-1} . The errors are independent. Find the inferred amplitude and cosine phase, with first-order standard uncertainties.

State and quadrant. The scaled velocity is $v_0/\omega_0 = 0.0400 \text{ m}$, so $A = \sqrt{0.0300^2 + 0.0400^2} = 0.0500 \text{ m}$. Both initial position and velocity are positive. Therefore $\cos \phi > 0$ and $\sin \phi < 0$,

locating phase in quadrant IV:

$$\phi = 2\pi - \tan^{-1}(4/3) = 5.35589 \text{ rad.}$$

Using Eqs. (15.20)–(15.21),

$$\sigma_A = \sqrt{(0.6 \times 0.0010)^2 + (0.08 \times 0.020)^2} = 0.00171 \text{ m,}$$

$$\sigma_\phi = \sqrt{(16 \times 0.0010)^2 + (1.2 \times 0.020)^2} = 0.0288 \text{ rad.}$$

Thus $A = (0.0500 \pm 0.0017) \text{ m}$ and $\phi = (5.356 \pm 0.029) \text{ rad}$, where the $\pm$ entries denote standard uncertainties under the stated assumptions.

Interpretation and check. The normalized state is $(x/A, v/(A\omega_0)) = (0.6, 0.8)$, the marked point in Figure 15.5. Substituting the inferred phase gives positive v_0 , as required. Velocity uncertainty contributes more to amplitude uncertainty here. The inferred amplitude is well above its uncertainty, so the local phase description is meaningful; the same procedure would be unreliable for a nearly motionless oscillator.

Method

Reconstruct harmonic motion by locating equilibrium and establishing ω_0 , then using both x_0 and v_0 . Determine the phase quadrant from their signs and check both reconstructed initial values. Use energy for a speed at a specified position, and phase for a specified event time. Neither a speed magnitude nor a position measurement alone identifies the complete state.

The time curves, energy equation, and phase portrait are mutually consistent descriptions. A circular construction makes their phase relations especially easy to visualize, provided its auxiliary coordinates are kept distinct from the physical motion.

15.3 Circular-motion representation

15.3.1 A geometric representation of the sinusoid

Consider an auxiliary point rotating counterclockwise at constant angular speed ω_0 on a circle of radius A . Its angle measured from the positive horizontal axis is $\theta = \omega_0 t + \phi$. Its horizontal coordinate is

$$x = A \cos \theta = A \cos(\omega_0 t + \phi). \quad (15.22)$$

The auxiliary vertical coordinate is $y_{\text{aux}} = A \sin \theta$. Differentiating the horizontal projection gives $v = -\omega_0 y_{\text{aux}}$. Thus the upper half of the geometric circle corresponds to negative oscillator velocity; it must not be confused with the upper half of the x – v phase portrait, which corresponds to positive velocity. The two circles use different vertical coordinates.

Figure 15.7 uses a single phase angle to locate a point on the circle and its corresponding displacement on a cosine curve. The projection makes the double-valued nature of phase at a specified position visible: most vertical projection lines intersect the circle twice, with opposite signs of the auxiliary vertical coordinate and therefore opposite velocity signs.

Example 15.7: Read oscillator phase from a projection

An oscillator has $A = 30 \text{ mm}$ and $\omega_0 = 10.0 \text{ rad s}^{-1}$. At $t = 0$ its auxiliary phase radius points at $2\pi/3$. Find its position and velocity, compare with the state at phase $4\pi/3$, and explain

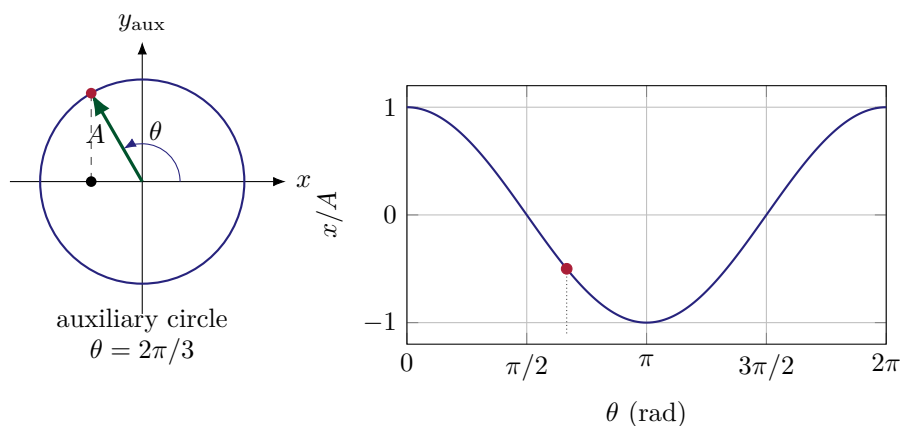


Figure 15.7: F15-03. Projection of a uniformly rotating auxiliary radius and its corresponding cosine curve. The marked phase $2\pi/3$ gives $x/A = -1/2$. Since $y_{\text{aux}} > 0$, the oscillator velocity is negative. The circle describes phase geometrically; it is not the spatial trajectory of the spring mass.

what position alone does not determine.

Projection and derivative. For the first state,

$$x(0) = 0.030 \cos(2\pi/3) = -0.0150 \text{ m},$$

$$v(0) = -(0.030)(10.0) \sin(2\pi/3) = -0.2598 \text{ m s}^{-1}.$$

The mass is on the negative side and moves farther negative. At $4\pi/3$, the same cosine gives $x = -0.0150 \text{ m}$ but the sine is negative, so $v = +0.2598 \text{ m s}^{-1}$: the mass moves towards equilibrium.

Check and conclusion. Both states obey $x^2 + (v/\omega_0)^2 = A^2$. Their energies are equal and their velocity directions opposite. A position observation therefore loses the direction of motion and cannot determine phase uniquely. A second observation supplying velocity, or a time-resolved sequence resolving direction, restores that information.

15.3.2 Changing the time origin and distinguishing real trajectories

Suppose a new time coordinate is $t' = t - t_s$, with zero at a later event. Then $t = t' + t_s$ and

$$x = A \cos[\omega_0 t' + (\phi + \omega_0 t_s)].$$

The new phase constant is $\phi' = \phi + \omega_0 t_s$ modulo 2π , while amplitude and frequency are unchanged. Figure 15.8 compares two phase constants for the same amplitude and frequency. Increasing phase shifts the cosine curve to earlier times; a phase difference $\Delta\phi$ corresponds to a time shift $\Delta\phi/\omega_0$. This relation is useful when aligning two records acquired with different trigger times.

Caution

The phase circle is a representation of the solution, not an additional physical force model. The spring mass has a one-dimensional trajectory. No physical centripetal force should be added to its free-body diagram merely because an auxiliary point moves on a circle.

A real pendulum does move along a circular arc, and therefore has a radial constraint force as well as a tangential restoring force. Its dynamics must be derived from those real forces or their torque, rather than inferred from the auxiliary phase-circle construction.

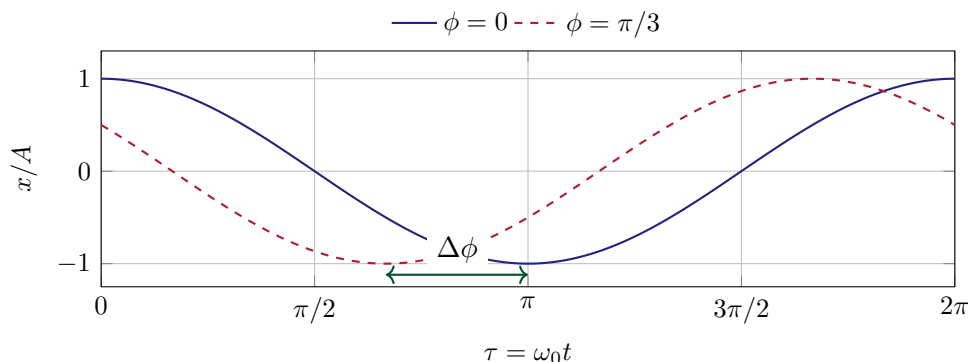


Figure 15.8: Equal-amplitude oscillations differing by phase $\pi/3$. The dashed curve reaches its negative extremum earlier by $\pi/(3\omega_0)$. A change of time origin changes phase convention; it does not change the underlying natural frequency.

15.4 Pendulums and torsional oscillation

15.4.1 Simple pendulum: exact equation and harmonic limit

A simple pendulum consists of a point-like bob of mass M at distance L from a fixed pivot, attached by a massless inextensible connector. Let θ be its signed angle from downward vertical, with positive θ to the right. The connector tension points towards the pivot and contributes no tangential force. Weight has tangential component $-Mg \sin \theta$. Tangential acceleration is $L\ddot{\theta}$, so

$$ML\ddot{\theta} = -Mg \sin \theta, \quad \boxed{\ddot{\theta} + \frac{g}{L} \sin \theta = 0}. \quad (15.23)$$

Equivalently, torque about the pivot is $-MgL \sin \theta$ and the point-mass inertia is ML^2 . Both approaches give the same equation. The mass cancels because gravitational restoring torque and inertial resistance are both proportional to M . This cancellation does not extend to air resistance or a connector with appreciable distributed mass.

Figure 15.9 compares the geometries of three angular oscillators. For the extended body, the pivot-to-centre-of-mass distance is d , whereas inertia depends on the entire mass distribution. The torsional oscillator has an elastic restoring torque rather than a gravitational one.

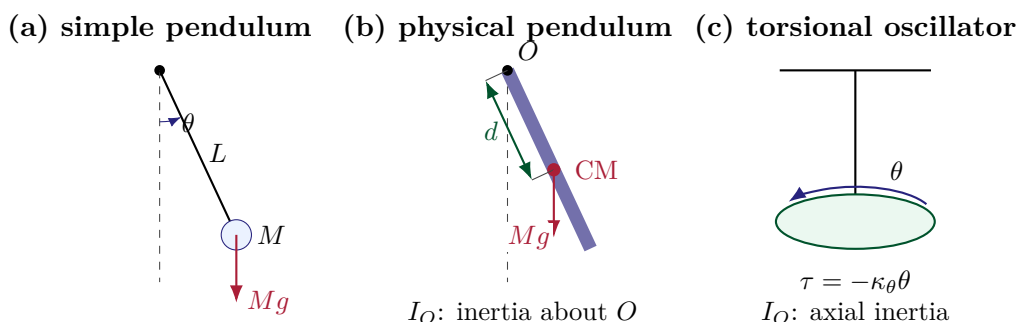


Figure 15.9: F15-04. Three schematic angular oscillators. In the physical pendulum, the dimension line for d is parallel to, and has endpoints corresponding exactly to, the pivot–CM segment. Simple and physical pendulums are harmonic only in the small-angle limit; linear torsion produces harmonic motion within its elastic torque range.

The sine prevents the exact simple-pendulum equation from being harmonic. Its expansion, valid for angles in radians, is

$$\sin \theta = \theta - \frac{\theta^3}{6} + \mathcal{O}(\theta^5).$$

Keeping the linear term gives

$$\ddot{\theta} + \frac{g}{L}\theta = 0, \quad \boxed{\omega_0 = \sqrt{g/L}}, \quad \boxed{T = 2\pi\sqrt{L/g}}. \quad (15.24)$$

Arc displacement $s = L\theta$ obeys the same harmonic equation in this limit. The small parameter is the angular amplitude in radians, not the arc length alone: a given arc displacement can be a small angle for a long pendulum and a large angle for a short one.

15.4.2 Actual forces, radial acceleration, and tension

The real bob follows a circular arc even when its angular history is approximately sinusoidal. Its radial acceleration points towards the pivot and has magnitude v^2/L , where $v = L|\dot{\theta}|$. Let $\mathcal{T}$ denote connector tension to distinguish it from period. Resolving real forces in the inward radial direction gives

$$\mathcal{T} - Mg \cos \theta = M \frac{v^2}{L}, \quad \mathcal{T} = Mg \cos \theta + ML\dot{\theta}^2. \quad (15.25)$$

Tension supplies part of the net radial force and gravity supplies the rest. A separate force labelled “centripetal force” is unnecessary. Figure 15.10 isolates the bob’s actual forces, then shows the gravitational torque lever arm in a separate panel. Resolving components is a calculation performed on these forces, not a reason to add the components as extra forces.

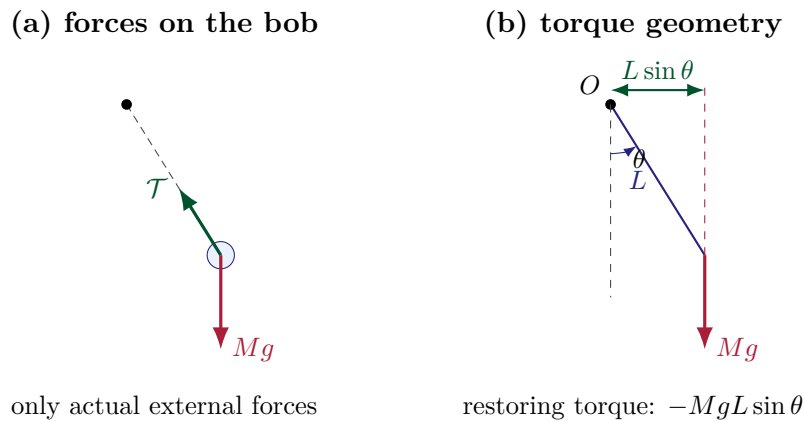


Figure 15.10: Force and torque descriptions of the same simple pendulum. Tension points towards the pivot and weight is vertical. The perpendicular distance from the pivot to the weight’s line of action is $L \sin \theta$, giving a torque towards downward equilibrium. Lengths and force arrows are schematic.

If the bob is released from rest at angle θ_0 , exact energy conservation yields

$$v^2 = 2gL(\cos \theta - \cos \theta_0).$$

At the bottom, $\mathcal{T} = Mg[1 + 2(1 - \cos \theta_0)]$, which exceeds the static weight for nonzero amplitude. At a turning point its speed is zero but tangential acceleration is nonzero. These checks help distinguish an angular turning point from a state of force equilibrium.

15.4.3 How accurate is the small-angle model?

A statement such as “the small-angle error is below one percent” is incomplete unless it identifies the quantity and denominator. Approximating the exact restoring factor $\sin \theta$ by θ overestimates it. For $0 < |\theta| < \pi$, the error relative to the exact factor is

$$\epsilon_F = \frac{|\theta| - \sin |\theta|}{\sin |\theta|} = \frac{|\theta|}{\sin |\theta|} - 1. \quad (15.26)$$

An alternative denominator $|\theta|$ gives a slightly different numerical error. Neither is the same as period error, because a period depends on the whole trajectory and its speed, rather than only the instantaneous force at maximum displacement.

The first finite-amplitude period correction, stated here to test the harmonic approximation, is

$$T \simeq T_0 \left(1 + \frac{\theta_0^2}{16} + \frac{11\theta_0^4}{3072} \right), \quad T_0 = 2\pi\sqrt{L/g}. \quad (15.27)$$

Its expansion parameter is θ_0 in radians. The correction is positive: larger swings have longer periods. An exact quadrature check can be obtained from energy by substituting $\sin(\theta/2) = \sin(\theta_0/2) \sin \psi$:

$$\frac{T}{T_0} = \frac{2}{\pi} \int_0^{\pi/2} \frac{d\psi}{\sqrt{1 - \sin^2(\theta_0/2) \sin^2 \psi}}. \quad (15.28)$$

The substitution removes the turning-point singularity of the direct time integral. Evaluation of this bounded integrand supplies the checked inline period data in Figure 15.11. Understanding or using the core harmonic formula does not require a special-function solution.

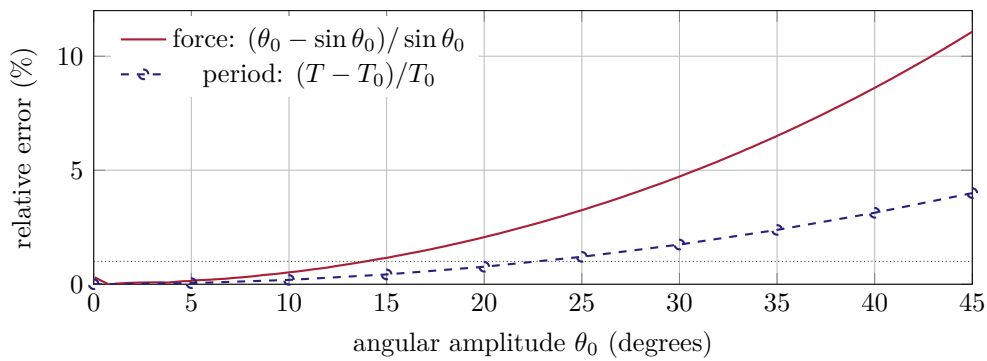


Figure 15.11: Two distinct tests of the simple-pendulum approximation. Force error uses the exact sine as denominator. Period error uses the small-angle period T_0 as denominator and is evaluated from the energy quadrature. In particular, an angle limit that is adequate for timing can fail a stricter instantaneous-force criterion.

At 10° , force error is about 0.51% while period error is about 0.19%. At 15° , they are about 1.15% and 0.43%, respectively. A one-percent force criterion therefore allows an angle of only about 14° , whereas a one-percent period criterion permits a substantially larger amplitude. These differences are physically useful: an experiment intended to determine a period needs a timing-error criterion, while an acceleration prediction may need the force criterion.

15.4.4 Physical pendulum: inertia and lever arm

A physical pendulum is a rigid body swinging about a fixed horizontal axis. Its centre of mass is a distance $d > 0$ below the pivot in the stable orientation, and its moment of inertia about the actual pivot is I_O . The weight can be represented at the centre of mass for calculating gravitational torque in a uniform field. Thus

$$I_O \ddot{\theta} = -Mgd \sin \theta, \quad \omega_0 = \sqrt{\frac{Mgd}{I_O}}, \quad T = 2\pi \sqrt{\frac{I_O}{Mgd}} \quad (15.29)$$

for the final two relations in the small-angle limit. The connector or bearing reaction at the pivot exerts no torque about that pivot for an ideal frictionless constraint. The period depends on both inertia and centre-of-mass distance. Using the total body length in place of d would generally give a wrong restoring torque.

A point mass gives $I_O = ML^2$ and $d = L$, recovering the simple-pendulum result. A uniform rod of length L pivoted at one end has $I_O = ML^2/3$ and $d = L/2$, giving $T = 2\pi\sqrt{2L/(3g)}$. Its equivalent simple-pendulum length is $I_O/(Md) = 2L/3$, which is shorter than the rod. Equal masses do not imply equal periods if their distributions differ.

Example 15.8: Compare ideal pendulum models

Compare (a) a simple pendulum of length 0.70 m, (b) a uniform rod of length 0.80 m pivoted at one end, and (c) a torsional oscillator with $I_O = 0.012 \text{ kg m}^2$ and $\kappa_\theta = 0.20 \text{ N m rad}^{-1}$. Find each ideal period. For the simple pendulum, assess a 10° angular amplitude using the period correction.

Models and calculation. The first two use gravity with small angular displacements; the third assumes a linear restoring torque. For the simple pendulum,

$$T_0 = 2\pi\sqrt{0.70/9.81} = 1.67840 \text{ s.}$$

With $\theta_0 = 10\pi/180 = 0.174533 \text{ rad}$, Eq. (15.27) gives $T = 1.68160 \text{ s}$, about 0.1907% longer. This is a model-validity correction, not a replacement for stating the harmonic assumption. For the rod, inertia and lever arm must refer to the same pivot:

$$T_0 = 2\pi\sqrt{\frac{M(0.80)^2/3}{Mg(0.80/2)}} = 2\pi\sqrt{\frac{2(0.80)}{3(9.81)}} = 1.46502 \text{ s.}$$

For torsion,

$$T = 2\pi\sqrt{I_O/\kappa_\theta} = 2\pi\sqrt{0.012/0.20} = 1.53906 \text{ s.}$$

Interpretation. The equations have the same harmonic structure but their stiffnesses are different: MgL , Mgd , and κ_θ are angular restoring coefficients. A numerical similarity between periods does not justify interchanging their physical models.

15.4.5 Choosing a pivot: an inertia–torque tradeoff

Moving a pivot closer to the centre of mass reduces its inertia contribution Md^2 , but also reduces the gravitational restoring torque. The parallel-axis theorem gives $I_O = I_C + Md^2$, where I_C is inertia about a parallel axis through the centre of mass. Introduce the radius of gyration $r_g = \sqrt{I_C/M}$, so

$$T^2 = \frac{4\pi^2}{g} \left(d + \frac{r_g^2}{d} \right). \quad (15.30)$$

For positive d , differentiating the bracket yields $1 - r_g^2/d^2$. It vanishes at $d = r_g$, and the second derivative $2r_g^2/d^3$ is positive. The shortest available period therefore occurs at $d = r_g$ if that pivot can be realized in the body. Figure 15.12 shows both competing terms and the resulting minimum. The divergence as $d \rightarrow 0$ has a clear interpretation: gravity cannot restore a rigid body whose pivot passes exactly through its centre of mass. A distant pivot has strong torque but still larger inertia, making its period long again. The optimization therefore expresses a physical competition rather than an algebraic trick.

Example 15.9: Find the fastest pivot for a uniform rod

A uniform slender rod has length $L = 0.900 \text{ m}$. A frictionless horizontal pivot can be placed anywhere along the rod above its centre of mass. At small angular amplitude, find the pivot position giving the shortest period and compare that period with an end pivot. Assume the rod is rigid and $g = 9.81 \text{ m s}^{-2}$.

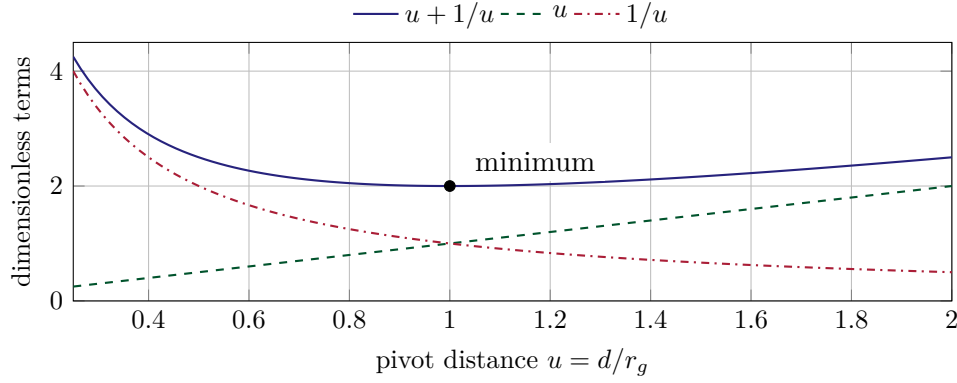


Figure 15.12: Physical-pendulum pivot optimization. The solid curve is $gT^2/(4\pi^2 r_g) = u + 1/u$. Its two contributions balance at $d = r_g$. The plotted domain excludes $d = 0$, where gravitational torque vanishes and the pendulum period formula diverges.

Strategy and geometry. For a uniform rod, $I_C = ML^2/12$, giving $r_g = L/\sqrt{12} = 0.259808$ m. The optimum pivot distance is $d = r_g$, which is less than $L/2$ and therefore lies on the rod. Measured from its upper end, the pivot is at

$$\frac{L}{2} - d = 0.450000 - 0.259808 = 0.190192 \text{ m.}$$

At that position,

$$T_{\min} = 2\pi\sqrt{\frac{2r_g}{g}} = 1.44606 \text{ s.}$$

For the end pivot, $d = L/2$ and

$$T_{\text{end}} = 2\pi\sqrt{\frac{2L}{3g}} = 1.55389 \text{ s.}$$

The minimum is about 6.94% shorter.

Check and interpretation. The optimum lies in the permitted interval $0 < d \leq 0.450$ m. Using the same inertia about the centre for both cases is essential; moving the pivot changes I_O . A pivot arbitrarily close to the centre would be slower, despite reducing inertia, because restoring torque approaches zero. A finite bearing size or structural constraint can restrict the available pivots; this ideal result supplies a reference rather than a complete bearing design.

15.4.6 Torsional oscillators and calibration by added inertia

A disk or rigid platform suspended from a thin wire can twist about its suspension axis. Within a linear elastic range, the restoring torque is $\tau = -\kappa_\theta\theta$, where $\kappa_\theta > 0$ is torsional stiffness. Its unit is conventionally N m rad^{-1} , distinguishing angular deformation from linear extension. Rotational dynamics gives

$$I_O\ddot{\theta} + \kappa_\theta\theta = 0, \quad \boxed{\omega_0 = \sqrt{\frac{\kappa_\theta}{I_O}}}, \quad \boxed{T = 2\pi\sqrt{\frac{I_O}{\kappa_\theta}}}. \quad (15.31)$$

The inertia must refer to the suspension axis. Gravity can establish vertical support without supplying the twisting torque, so the torsional period has no explicit g in this model. Rotational energy takes the analogous form $E = I_O\dot{\theta}^2/2 + \kappa_\theta\theta^2/2$. Larger inertia lengthens the period, while larger torsional stiffness shortens it.

A useful measurement adds an object with known axial inertia without changing the suspension. Squared periods before and after addition obey $T_1^2 = 4\pi^2 I_1 / \kappa_\theta$ and $T_2^2 = 4\pi^2 (I_1 + \Delta I) / \kappa_\theta$. Their difference isolates stiffness. This avoids needing the unknown platform inertia beforehand, just as changing mass isolated stiffness in the translational calibration.

Example 15.10: Calibrate a torsional platform with a known ring

An ideal torsional platform has period $T_1 = 0.800$ s. Adding a concentric rigid ring with known axial inertia $\Delta I = 0.00600$ kg m² changes the period to $T_2 = 1.200$ s. Find the torsional stiffness, the platform's original inertia, and its original angular frequency. State the essential experimental assumption.

Method. The added ring must rotate with the platform and leave the wire's torque law unchanged. Subtracting squared-period equations gives

$$\kappa_\theta = \frac{4\pi^2 \Delta I}{T_2^2 - T_1^2} = \frac{4\pi^2 (0.00600)}{1.440 - 0.640} = 0.296088 \text{ N m rad}^{-1}.$$

Then

$$I_1 = \frac{\kappa_\theta T_1^2}{4\pi^2} = 0.004800 \text{ kg m}^2, \quad \omega_0 = \frac{2\pi}{T_1} = 7.85398 \text{ rad s}^{-1}.$$

Check. The inertia ratio $(I_1 + \Delta I)/I_1 = 2.25$ equals $(T_2/T_1)^2$. Adding positive inertia has increased period, as required. If the ring slips, is off-axis, or changes the wire stiffness through a load-dependent material response, these equations no longer isolate the stated parameters. A real implementation should check repeatability and low-amplitude behaviour rather than treat two timings as proof of the model.

Method

For angular oscillation, identify the actual pivot or suspension axis, determine its inertia, and derive the restoring torque. For gravitational pendulums, use the pivot-to-CM distance and linearize only after writing the sine torque. For torsion, state the range in which torque is proportional to angle. An angular amplitude in radians supplies the small-angle criterion.

All four sections so far describe conservative oscillators. Real motion generally loses mechanical energy through drag, internal friction, or a damper. The next step is to add a specified dissipative force, retain the same equilibrium coordinate, and determine how this changes the time history and its interpretation.

15.5 Damped motion

15.5.1 Choosing a dissipative model

An instrument mount, a spring-supported payload and a laboratory oscillator all exchange energy with their surroundings. Some of that energy becomes thermal energy in a fluid or material. A useful first model represents this loss by a force proportional to the velocity of the moving body relative to a stationary support:

$$F_d = -b\dot{x}, \quad b > 0. \quad (15.32)$$

Here b has units $\text{N s m}^{-1} = \text{kg s}^{-1}$. The minus sign means that the damping force opposes the velocity; it does not necessarily oppose the displacement. A mass to the right of equilibrium and moving to the left has a rightward damping force but a leftward spring force. Drawing both forces in the same direction without checking the velocity would misrepresent the system.

The dashpot in Fig. 15.13 is an ideal element implementing Eq. (15.32). A real damper need not

obey this law at all speeds. Sliding friction is approximately independent of speed in some regimes; fluid drag can become proportional to the square of speed; and internal material loss can depend on frequency. The linear model is valuable because it is tractable, testable and often adequate for sufficiently small vibrations. Its parameters must describe the range being studied. “Damped” by itself does not specify a force law.

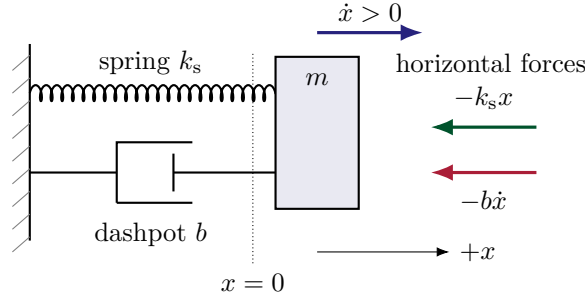


Figure 15.13: Linear spring–dashpot model, shown schematically with $x > 0$ and $\dot{x} > 0$. Both horizontal force components point left for this state. The spring force changes sign when x changes sign; the damping force changes sign when $\dot{x}$ changes sign. Vertical support and weight balance and are omitted from this horizontal component diagram.

With displacement measured from the equilibrium position, Newton’s law gives

$$m\ddot{x} + b\dot{x} + k_s x = 0 \quad \Longleftrightarrow \quad \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0, \quad \beta = \frac{b}{2m}, \quad \omega_0 = \sqrt{\frac{k_s}{m}}. \quad (15.33)$$

The natural angular frequency ω_0 belongs to the underlying undamped system. The rate β has units s^{-1} and controls an exponential amplitude decay, rather than a frequency measured in cycles per second. Define the dimensionless damping ratio $\zeta = \beta/\omega_0$. Holding m and k_s fixed while varying b changes ζ without changing ω_0 .

15.5.2 Energy loss: an exact balance before an approximation

Multiply Eq. (15.33) in its dimensional form by $\dot{x}$. The terms $m\ddot{x}\dot{x}$ and $k_s x\dot{x}$ are derivatives of kinetic and elastic potential energy, respectively. Thus

$$\frac{dE_{\text{mech}}}{dt} = -b\dot{x}^2 \leq 0, \quad E_{\text{mech}} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k_s x^2. \quad (15.34)$$

This balance is exact for the stated model. A change in sign of x or $\dot{x}$ cannot make the dissipation positive because the speed is squared. At a turning point the instantaneous loss rate vanishes, although the energy has decreased during the preceding motion. Between two times, the dissipated mechanical energy is $b \int \dot{x}^2 dt$, a positive quantity. Mechanical energy is not conserved; energy accounting is recovered by including the surroundings.

For weak damping one often writes an approximate cycle-averaged energy law $\overline{E_{\text{mech}}} \propto e^{-2\beta t}$. Its reasoning is that the amplitude changes little in one cycle, so the usual near-harmonic relation between energy and squared amplitude may be applied locally. It does not make the instantaneous energy exactly exponential. In particular, release from rest has $\dot{E}_{\text{mech}}(0) = 0$, whereas $E_0 e^{-2\beta t}$ has a nonzero initial slope. Figure 15.16 makes this distinction visible. Use the exact balance when damping is not weak or when time-resolved energy is required.

15.5.3 Solving the equation and imposing the initial state

A second-order equation needs two independent initial data, such as $x(0) = x_0$ and $\dot{x}(0) = v_0$. To find its functional form, try $x = Ce^{\lambda t}$. Substitution gives the characteristic equation

$$\lambda^2 + 2\beta\lambda + \omega_0^2 = 0, \quad \lambda_{\pm} = -\beta \pm \sqrt{\beta^2 - \omega_0^2}. \quad (15.35)$$

The nature of the square root divides the responses into three regimes. Figure 15.14 connects the algebra to the damping ratio.

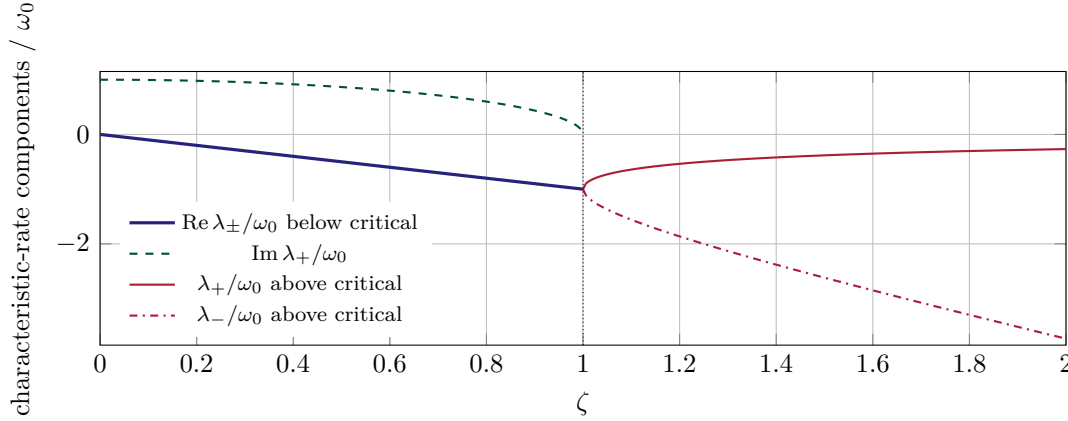


Figure 15.14: Characteristic rates as damping changes. Below critical damping, the conjugate rates have real part $-\zeta\omega_0$ and opposite imaginary parts; only the positive imaginary part is shown. At $\zeta = 1$ the rates coincide. Above critical damping both are real and negative; the less negative rate becomes small in magnitude, explaining the slow late return of an overdamped oscillator.

For $0 < \beta < \omega_0$, define the real damped angular frequency $\omega_d = \sqrt{\omega_0^2 - \beta^2}$. Combining the conjugate exponential solutions into real sines and cosines gives

$$x(t) = e^{-\beta t} [C \cos(\omega_d t) + D \sin(\omega_d t)], \quad C = x_0, \quad D = \frac{v_0 + \beta x_0}{\omega_d}. \quad (15.36)$$

The velocity condition contains βx_0 because differentiating the exponential contributes $-\beta x_0$. Omitting that term changes the initial velocity and produces an inconsistent comparison. Alternatively, $x = R e^{-\beta t} \cos(\omega_d t + \phi)$, with $R = \sqrt{C^2 + D^2}$. For $R > 0$, the phase is $\phi = \text{atan2}(-D, C)$. If $R = 0$, the motion is identically zero and its phase is undefined. The bounding curves are $\pm R e^{-\beta t}$. The coefficient R need not equal the initial displacement or a measured first maximum.

At critical damping, $\beta = \omega_0$, the two exponential solutions merge. A second independent solution contains a factor of t :

$$x(t) = [x_0 + (v_0 + \omega_0 x_0)t] e^{-\omega_0 t}. \quad (15.37)$$

This expression can be verified directly by two differentiations and substitution. For overdamping, $\beta > \omega_0$, use the two distinct real rates in Eq. (15.35):

$$x(t) = C_+ e^{\lambda_+ t} + C_- e^{\lambda_- t}, \quad (15.38)$$

$$C_+ = \frac{v_0 - \lambda_- x_0}{\lambda_+ - \lambda_-}, \quad C_- = \frac{\lambda_+ x_0 - v_0}{\lambda_+ - \lambda_-}.$$

Both rates are negative for $m, k_s, b > 0$, so every free solution tends to equilibrium. Neither critical nor overdamped motion contains a repeating oscillation. However, an initially fast body can cross equilibrium once. “Nonoscillatory” does not mean that every possible initial state approaches equilibrium monotonically. Likewise, critical damping’s familiar rapid return without overshoot refers to a specified release or step experiment, not a universal shortest-time statement for arbitrary initial conditions and tolerance criteria.

Example 15.11: Compare damping regimes at equal initial conditions

Statement. Three oscillators have identical $m, k_s, x(0) = A > 0$ and $\dot{x}(0) = 0$, but damping ratios .20, 1.00 and 1.50. Derive the three normalized trajectories used in Fig. 15.15, identify

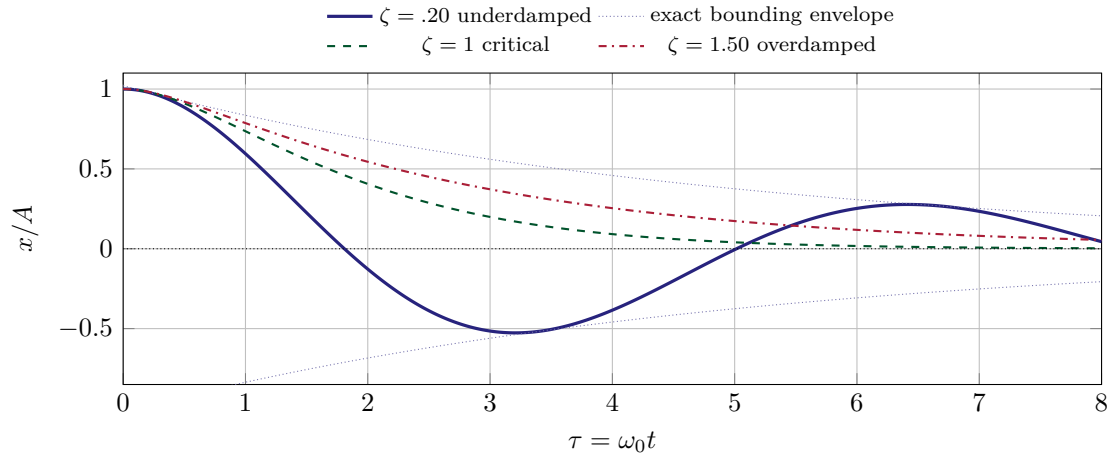


Figure 15.15: F15-05. Analytical free responses with the same initial state $x(0) = A$, $\dot{x}(0) = 0$ and the same ω_0 . The underdamped trajectory crosses equilibrium repeatedly, whereas these release-from-rest critical and overdamped trajectories remain positive. The exact underdamped bounding pair is $x/A = \pm e^{-.2\tau}/\sqrt{.96}$, not $\pm e^{-.2\tau}$; its coefficient follows from the nonzero sine term needed for zero initial velocity.

their bounding or decay rates, and compare their energies at $\tau = \omega_0 t = 2$.

Solution. Put $y = x/A$ and $u = dy/d\tau$. For $\zeta = .20$, Eq. (15.36) gives

$$y = e^{-.2\tau} \left[\cos(\sqrt{.96}\tau) + \frac{.2}{\sqrt{.96}} \sin(\sqrt{.96}\tau) \right], \quad u = -\frac{e^{-.2\tau}}{\sqrt{.96}} \sin(\sqrt{.96}\tau).$$

Thus $\omega_d = .97990\omega_0$ and the envelope coefficient is $R/A = 1/\sqrt{.96} = 1.02062$. Setting $\tau = 0$ explicitly gives $y = 1$, $u = 0$. Critical damping gives $y = (1 + \tau)e^{-\tau}$ and $u = -\tau e^{-\tau}$. For $\zeta = 1.50$, the normalized rates are $-1.5 \pm \sqrt{1.25}$, so

$$y = 1.170820e^{-.381966\tau} - .170820e^{-2.618034\tau}.$$

The coefficients add to one; their rate-weighted sum is zero. The slow $-.381966$ term dominates at late times.

The initial energy is $E_0 = \frac{1}{2}k_s A^2$. Since $\dot{x} = A\omega_0 u$ and $m\omega_0^2 = k_s$,

$$\frac{E_{\text{mech}}}{E_0} = y^2 + u^2.$$

At $\tau = 2$, the pairs (y, u) are approximately $(-.12748, -.63308)$, $(.40601, -.27067)$ and $(.54450, -.20595)$, respectively. Their normalized energies are .41705, .23810 and .33889. They started with the same energy, but damping changed their velocity histories and therefore their integrated losses. A larger b alone does not order the energy loss at every time, because the rate $b\dot{x}^2$ contains both b and the response velocity. Each individual energy history is non-increasing.

Example 15.12: Recover coefficients from a displacement and velocity

Statement. A 1.00 kg oscillator has $k_s = 25.0 \text{ N m}^{-1}$, $b = 6.00 \text{ kg s}^{-1}$ and initial state $x_0 = .0400 \text{ m}$, $v_0 = .0800 \text{ m s}^{-1}$. Find its displacement, amplitude coefficient and phase. Then find the critical and overdamped trajectories for the same m , k_s and initial state when b is changed to 10.0 and 12.0 kg s⁻¹.

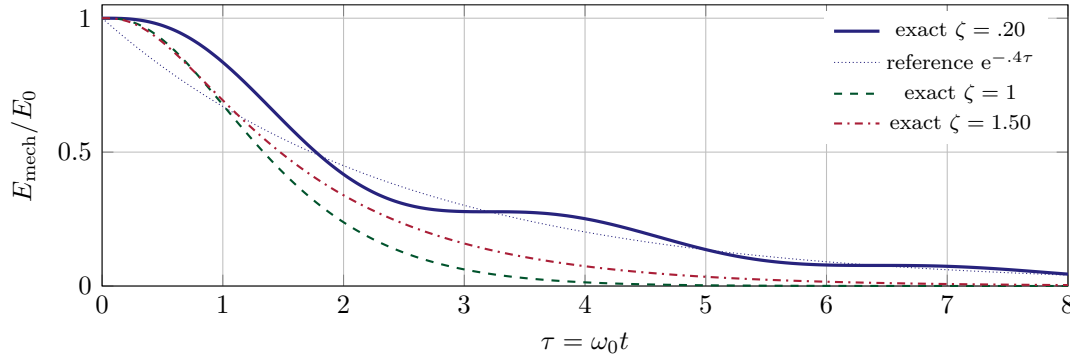


Figure 15.16: Exact energy histories for the common initial state in F15-05. All obey $d(E/E_0)/d\tau = -4\zeta u^2 \leq 0$. The dotted exponential is a reference decay scale, not the exact instantaneous underdamped energy. The smooth low-slope intervals of the blue curve correspond to small velocity near turning points.

Solution. The undamped frequency is $\omega_0 = 5.00 \text{ rad s}^{-1}$. With $b = 6.00 \text{ kg s}^{-1}$, $\beta = 3.00 \text{ s}^{-1}$, $\zeta = .600$ and $\omega_d = 4.00 \text{ rad s}^{-1}$. Hence

$$C = .0400 \text{ m}, \quad D = \frac{.0800 + 3.00(.0400)}{4.00} \text{ m} = .0500 \text{ m},$$

$$x(t) = e^{-3.00t} [.0400 \cos(4.00t) + .0500 \sin(4.00t)] \text{ m}.$$

Here t is expressed in seconds. The amplitude coefficient is $R = .06403 \text{ m}$ and $\phi = -.89606 \text{ rad}$, so an equivalent form is $x = .06403e^{-3.00t} \cos(4.00t - .89606) \text{ m}$. The envelope starts above the actual initial displacement, which is physically consistent with a nonzero initial phase. Differentiation gives $\dot{x}(0) = -3(.0400) + 4(.0500) = .0800 \text{ m s}^{-1}$.

For critical damping, $\beta = 5.00 \text{ s}^{-1}$, so

$$x(t) = (.0400 + .2800t)e^{-5.00t} \text{ m}.$$

For $b = 12.0 \text{ kg s}^{-1}$, $\beta = 6.00 \text{ s}^{-1}$ and $\lambda_{\pm} = -6.00 \pm \sqrt{11} \text{ s}^{-1}$. Solving the two initial-state equations gives

$$x(t) = [.0682418e^{-2.683375t} - .0282418e^{-9.316625t}] \text{ m}.$$

The coefficient sum is $.0400 \text{ m}$ and its rate-weighted sum is $.0800 \text{ m s}^{-1}$. All three bodies initially move farther from equilibrium because $v_0 > 0$; a nonoscillatory response need not decrease immediately. The initial energy is $\frac{1}{2}(1.00)(.0800)^2 + \frac{1}{2}(25.0)(.0400)^2 = .0232 \text{ J}$ in every case.

15.5.4 Phase portraits and the meaning of decay

The (x, v) portrait records state rather than time. For a conservative oscillator the trajectory is an ellipse of constant energy. Damping moves the state inward across lower-energy ellipses, since Eq. (15.34) forbids outward energy growth. An underdamped portrait spirals inward; critical and overdamped portraits approach the origin without repeated encirclement. Figure 15.17 uses $y = x/A$ and $u = v/(A\omega_0)$ so that circles $y^2 + u^2 = \text{constant}$ represent constant energy. A crossing of $y = 0$ is passage through equilibrium, not arrival at rest, because u is generally nonzero there.

15.5.5 Peak ratios and a measurable damping rate

A free-decay record offers an experimentally useful route to β . For an underdamped motion, corresponding same-sign maxima are separated by $T_d = 2\pi/\omega_d$. The sinusoidal phase at these

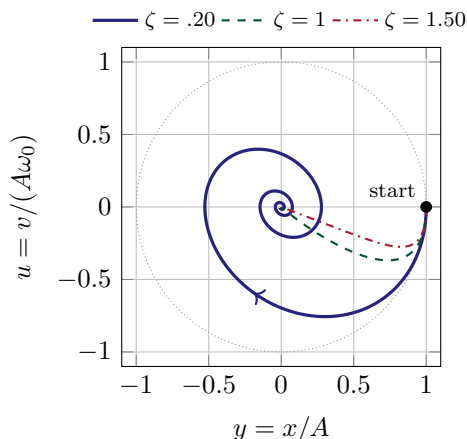


Figure 15.17: Phase portraits for the same release used in F15-05. Time starts at $(1, 0)$ and proceeds inward; the blue arrow indicates its direction on the underdamped trajectory. The dotted unit circle has the initial energy. No damped trajectory can cross outward through a constant-energy circle.

maxima is the same, so their displacement magnitudes differ by the exponential factor:

$$\ell = \ln \frac{p_n}{p_{n+1}} = \beta T_d, \quad \zeta = \frac{\ell}{\sqrt{4\pi^2 + \ell^2}}. \quad (15.39)$$

The symbol ℓ denotes the logarithmic decrement here; δ will denote the steady forced-response phase lag. Positive maxima separated by one negative minimum are one full cycle apart. Adjacent absolute extrema are only half a cycle apart, so inserting their ratio into the full-period formula would give a factor-of-two error. Figure 15.18 distinguishes these choices. Averaging the decrement over several cycles reduces sensitivity to peak-location noise, but a consistent exponential fit still needs a justified linear-damping regime.

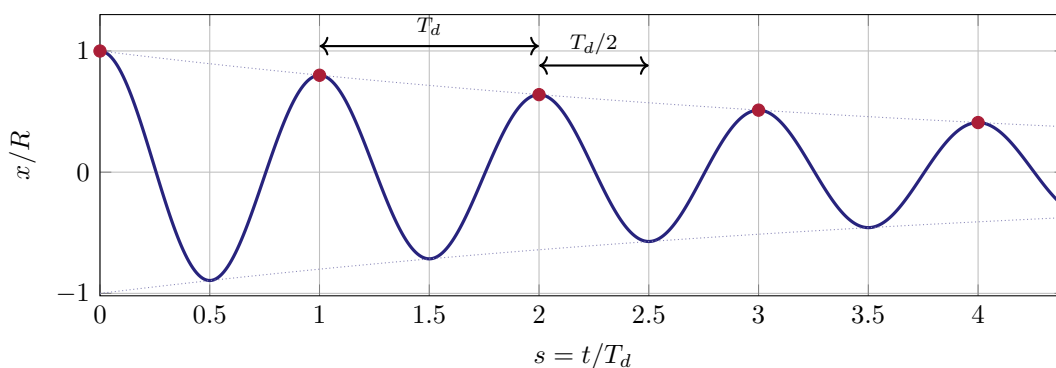


Figure 15.18: Synthetic free decay with $\ell = \ln(1.25)$ and the initial phase chosen so $s = 0$ is a positive maximum. Red points are same-sign maxima one full damped period apart, each .8 times its predecessor. The negative extremum halfway between them belongs to a different phase. The dotted curves bound the signal; their coefficient is slightly larger than the first measured maximum.

Example 15.13: Infer parameters from a synthetic ring-down record

Statement. A payload of mass .750 kg has successive positive maxima 12.0 and 9.60 mm separated by .800 s. Assume a single linear viscously damped mode. Determine β , ω_d , ω_0 , b , k_s and ζ , and predict the fifth later positive maximum. Explain the information needed if the listed maxima were not successive same-sign peaks.

Solution. Because the peaks have the same sign and are successive, $T_d = .800$ s and $\ell = \ln(12.0/9.60) = .223144$. Therefore

$$\beta = .278929 \text{ s}^{-1}, \quad \omega_d = \frac{2\pi}{.800} = 7.85398 \text{ rad s}^{-1},$$

$$\omega_0 = \sqrt{\omega_d^2 + \beta^2} = 7.85893 \text{ rad s}^{-1}.$$

Recover the physical coefficients from the definitions:

$$b = 2m\beta = .418394 \text{ kg s}^{-1}, \quad k_s = m\omega_0^2 = 46.3221 \text{ N m}^{-1}, \quad \zeta = .0354920.$$

The fifth later same-sign peak is $p_5 = 12.0(.800)^5 = 3.93216$ mm, at 4.00 s after the first listed maximum. Light damping is supported by $\zeta \ll 1$, but we have used the exact relation between ω_d and ω_0 . If several full cycles separated the two peaks, their number would be required to recover T_d ; the amplitude ratio alone does not determine a frequency. If opposite-sign extrema were supplied, the time interval would be a half period and the corresponding decrement would be $\beta T_d/2$. These numbers describe a synthetic instructional record, not a measured payload.

Caution

The decay time $1/\beta$, the damped period $2\pi/\omega_d$ and the undamped period $2\pi/\omega_0$ describe different features. Overdamped motion has no real ω_d and no oscillation period. A settling time also needs a stated tolerance and initial state; it is not automatically equal to one decay time.

15.6 Forced response and resonance

15.6.1 Specifying what the actuator controls

A force-driven experiment imposes a known sinusoidal force $F(t) = F_0 \cos(\Omega t)$ on the mass while the spring and damper supports remain fixed. Let $F_0 > 0$ be held constant during a frequency sweep. The equation is

$$m\ddot{x} + b\dot{x} + k_s x = F_0 \cos(\Omega t). \quad (15.40)$$

The drive angular frequency Ω is an input; it need not equal either ω_0 or ω_d . A prescribed support displacement is a different experiment, derived later. Specifying the input is essential because a plot against frequency also depends on what is being held fixed.

The complete solution is a particular forced response plus a homogeneous solution. The latter is determined by the initial displacement and velocity. For positive damping it decays, leaving a periodic steady state at the drive frequency. During start-up both contributions matter. A simple linear superposition can look complicated because it combines two frequencies, yet it is not chaotic. For weak damping, several or many drive cycles may be needed before a measured amplitude represents the steady-state response.

15.6.2 Deriving amplitude and phase with real trigonometry

Write the steady response in the form $x_{ss} = P \cos(\Omega t) + Q \sin(\Omega t)$. The second derivative contributes $-\Omega^2 x_{ss}$. Equating coefficients of cosine and sine in Eq. (15.40) gives

$$(k_s - m\Omega^2)P + b\Omega Q = F_0, \quad (k_s - m\Omega^2)Q - b\Omega P = 0. \quad (15.41)$$

The symbols P, Q here are displacement coefficients, not power or a quality factor. Solving these two linear equations, with $\mathcal{D} = (k_s - m\Omega^2)^2 + (b\Omega)^2$, yields $P = F_0(k_s - m\Omega^2)/\mathcal{D}$ and $Q = F_0 b\Omega/\mathcal{D}$.

Rewriting the same signal as

$$x_{ss}(t) = A(\Omega) \cos(\Omega t - \delta) \quad (15.42)$$

identifies $P = A \cos \delta$ and $Q = A \sin \delta$. Therefore

$$A = \frac{F_0}{\sqrt{(k_s - m\Omega^2)^2 + (b\Omega)^2}} = \frac{F_0/m}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\beta\Omega)^2}}, \quad (15.43)$$

$$\delta = \text{atan2}(b\Omega, k_s - m\Omega^2), \quad 0 \leq \delta \leq \pi. \quad (15.44)$$

Using a one-argument inverse tangent without choosing the quadrant gives the wrong phase above ω_0 . The displacement is then nearly opposite in phase to the applied force, rather than nearly in phase with it.

Figure 15.19 visualizes the same algebra without requiring complex-number notation. A phasor is an arrow representing the amplitude and phase of a sinusoid at a common frequency. In the equation's left side, the spring contribution is aligned with displacement, the acceleration contribution is opposite, and the velocity contribution is a quarter cycle ahead of displacement. Their vector sum must reproduce the force phasor. These are equation-term phasors; the actual spring and damping forces on the mass carry minus signs when placed on the right side of Newton's law.

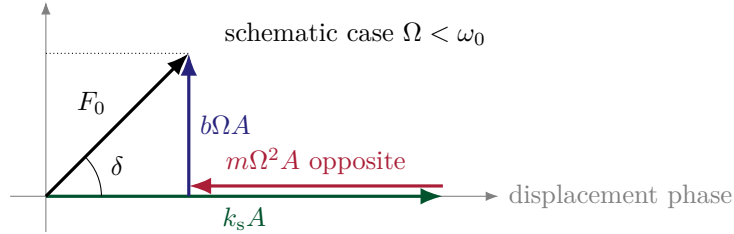


Figure 15.19: Phase construction for the steady force-driven response. The horizontal resultant is $(k_s - m\Omega^2)A$ and the perpendicular component is $b\Omega A$, giving the amplitude denominator and phase lag in Eqs. (15.43)–(15.44). At $\Omega = \omega_0$ the horizontal terms cancel; above it their resultant points left and the phase lag lies between $\pi/2$ and π . Arrow lengths are schematic, not a dimensional scale.

15.6.3 Displacement, velocity and absorbed power peak differently

Introduce the frequency ratio $r = \Omega/\omega_0$ and static displacement scale $X_{st} = F_0/k_s$. The displacement magnification is

$$M_x = \frac{A}{X_{st}} = \frac{1}{\sqrt{(1 - r^2)^2 + 4\zeta^2 r^2}}. \quad (15.45)$$

It approaches one as $r \rightarrow 0$: a slowly changing force produces a nearly static spring extension. At high frequency inertia dominates and $A \sim F_0/(m\Omega^2)$. Thus a finite static amplitude at low frequency and a decreasing displacement at high frequency are important checks of the transfer function.

To find the displacement peak, minimize its squared denominator $D(r) = (1 - r^2)^2 + 4\zeta^2 r^2$. Its derivative is $D'(r) = 4r(r^2 - 1 + 2\zeta^2)$. Apart from the endpoint $r = 0$, a stationary point occurs only if

$$\Omega_{x,\max} = \omega_0 \sqrt{1 - 2\zeta^2} = \sqrt{\omega_0^2 - 2\beta^2}, \quad \zeta < \frac{1}{\sqrt{2}}. \quad (15.46)$$

For $\zeta \geq 1/\sqrt{2}$, D increases for all $r > 0$, so the displacement amplitude decreases from its static value with no nonzero frequency peak. In the weak-damping limit the shift below ω_0 is small, which explains why identifying these frequencies approximately can be useful; it is not an exact identity.

The velocity amplitude is $V = \Omega A$. Its normalized value is $M_v = r/\sqrt{D(r)}$. Differentiating $M_v^2 = r^2/D$ gives a numerator proportional to $1 - r^4$, so for any $b > 0$

$$\Omega_{V,\max} = \omega_0, \quad V_{\max} = \frac{F_0}{b}. \quad (15.47)$$

Velocity vanishes in both low- and high-frequency limits. Acceleration, if chosen instead, weights displacement by another factor of Ω and has its own behavior. The response observable must accompany a quantitative statement about resonance. Figure 15.20 compares displacement and velocity under exactly the same fixed-force input.

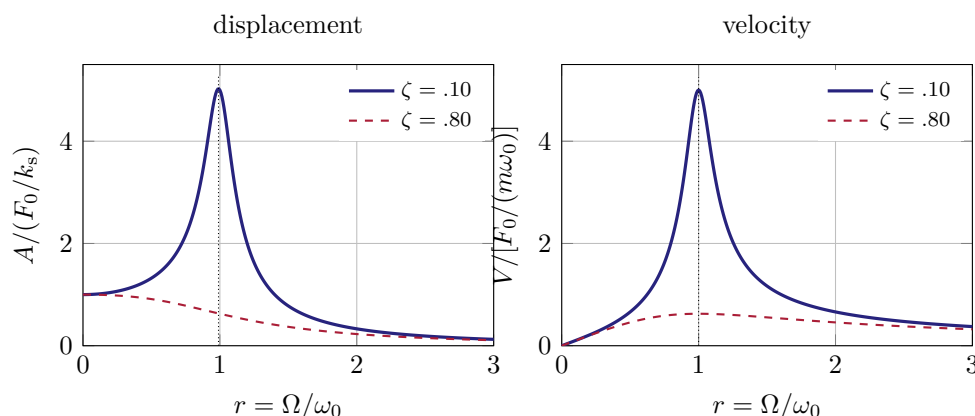


Figure 15.20: F15-06. Analytical steady-state responses to a *fixed sinusoidal force amplitude* F_0 . For $\zeta = .10$, displacement peaks at $r = \sqrt{.98}$, while velocity peaks at $r = 1$. For $\zeta = .80 > 1/\sqrt{2}$, no nonzero displacement peak exists, but velocity still peaks at $r = 1$. Dotted vertical lines mark the weakly damped peak frequencies; they are not separate response curves.

Example 15.14: Frequency sweep of a force-driven oscillator

Statement. A system has $m = 2.00 \text{ kg}$, $k_s = 200 \text{ N m}^{-1}$ and $b = 4.00 \text{ kg s}^{-1}$. The sinusoidal force amplitude is fixed at $F_0 = 10.0 \text{ N}$. Find the static displacement, the displacement-peak frequency and amplitude, and the velocity-peak frequency and amplitude. Repeat the peak classification when $b = 30.0 \text{ kg s}^{-1}$ and explain the difference.

Solution. The parameters are

$$\omega_0 = 10.0 \text{ rad s}^{-1}, \quad \beta = 1.00 \text{ s}^{-1}, \quad \zeta = .100, \quad X_{\text{st}} = \frac{10.0}{200} = .0500 \text{ m}.$$

Equation (15.46) gives $\Omega_{x,\max} = \sqrt{98} = 9.89949 \text{ rad s}^{-1}$. At that frequency the dynamic stiffness magnitude is $\sqrt{(200 - 2 \times 98)^2 + (4\sqrt{98})^2} = 39.7995 \text{ N m}^{-1}$. Thus $A_{\max} = .251259 \text{ m}$, about 5.03 times the static scale. At $\Omega = \omega_0$, $A = F_0/(b\omega_0) = .2500 \text{ m}$ and $V_{\max} = F_0/b = 2.500 \text{ m s}^{-1}$. The frequencies are close but distinct, and the damped free frequency $\omega_d = \sqrt{99} = 9.94987 \text{ rad s}^{-1}$ is different from both.

For $b = 30.0 \text{ kg s}^{-1}$, $\beta = 7.50 \text{ s}^{-1}$ and $\zeta = .750 > 1/\sqrt{2}$. The displacement amplitude has no nonzero peak; its largest value is the zero-frequency static limit $.0500 \text{ m}$. Nevertheless, velocity still peaks at 10.0 rad s^{-1} , with $V_{\max} = .3333 \text{ m s}^{-1}$ and $A(\omega_0) = .03333 \text{ m}$. The system remains underdamped in free motion because $.750 < 1$. Absence of a displacement peak and absence of free oscillations therefore have different damping thresholds. A real spring must remain sufficiently linear over the predicted motion; large magnification can reveal a limit of the small-vibration model rather than promise an unlimited response.

15.6.4 Phase as a frequency-dependent physical measurement

At low frequency, $\delta \rightarrow 0$ and the displacement follows the force. At $\Omega = \omega_0$, $\delta = \pi/2$, so displacement lags by a quarter cycle. The velocity $-\Omega A \sin(\Omega t - \delta)$ is then $\Omega A \cos(\Omega t)$, exactly in phase with the force. At high frequency, $\delta \rightarrow \pi$: force and displacement are almost opposite, consistent with acceleration rather than extension dominating Newton's law. These are time-phase relations, not directions of permanent forces. Figure 15.21 gives three damping ratios. Greater damping smooths the transition between the limiting phases.

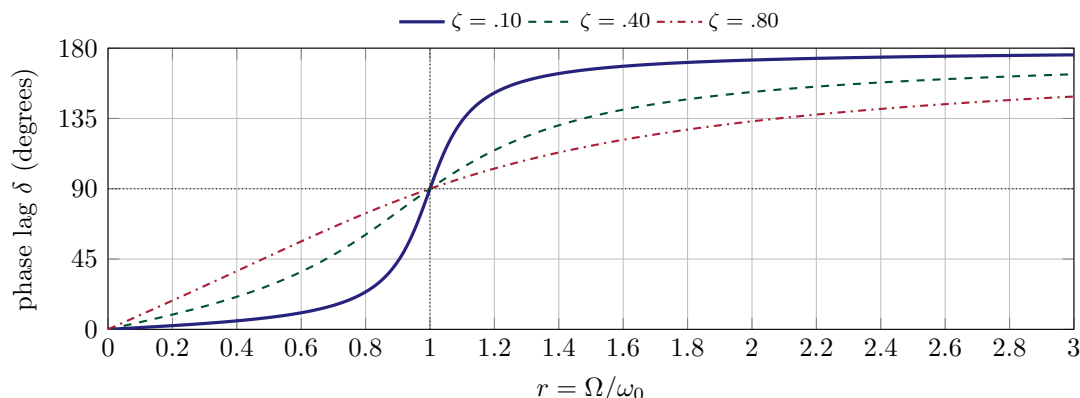


Figure 15.21: Displacement phase lag for a positive-frequency input. Every curve passes through $(r, \delta) = (1, 90^\circ)$ and tends to 180° at high frequency. The correct quadrant is retained through atan2 ; a principal inverse tangent alone would jump to a negative angle above $r = 1$.

15.6.5 Transient plus steady state after switching on a force

For an underdamped system, the complete solution is

$$x = x_{ss} + e^{-\beta t} [C_h \cos(\omega_d t) + D_h \sin(\omega_d t)], \quad (15.48)$$

where the initial conditions must be imposed on the *sum*. Since $x_{ss}(0) = A \cos \delta$ and $\dot{x}_{ss}(0) = \Omega A \sin \delta$,

$$C_h = x_0 - A \cos \delta, \quad D_h = \frac{v_0 - \Omega A \sin \delta + \beta C_h}{\omega_d}. \quad (15.49)$$

The sine sign follows from differentiating $\cos(\Omega t - \delta)$. A drive switched on at its cosine maximum can have a nonzero force at $t = 0$ while displacement and velocity remain continuous. Its acceleration changes immediately; a finite force step does not create a velocity jump. The following example and Fig. 15.22 show how a homogeneous contribution cancels the initial steady-state displacement and velocity.

Example 15.15: Match a switched sinusoid to an initially resting mass

Statement. Use the $m = 2.00 \text{ kg}$, $k_s = 200 \text{ N m}^{-1}$, $b = 4.00 \text{ kg s}^{-1}$ system of Example 15.14. At $t = 0$ a force $10.0 \cos(8.00t) \text{ N}$ is switched on while the mass is at equilibrium and at rest. Find the complete displacement for $t \geq 0$, verify both initial conditions, and determine the initial acceleration.

Solution. Here $\beta = 1.00 \text{ s}^{-1}$, $\omega_d = \sqrt{99} = 9.94987 \text{ rad s}^{-1}$ and $\mathcal{D} = (200 - 2 \times 64)^2 + (4 \times 8)^2 = 6208 \text{ N}^2 \text{ m}^{-2}$. The cosine and sine coefficients are $P = .1159794 \text{ m}$ and $Q = .0515464 \text{ m}$. The homogeneous cosine coefficient must be $C_h = -P$. Its sine coefficient is

$$D_h = \frac{-8Q + \beta C_h}{\omega_d} = -.0531012 \text{ m}.$$

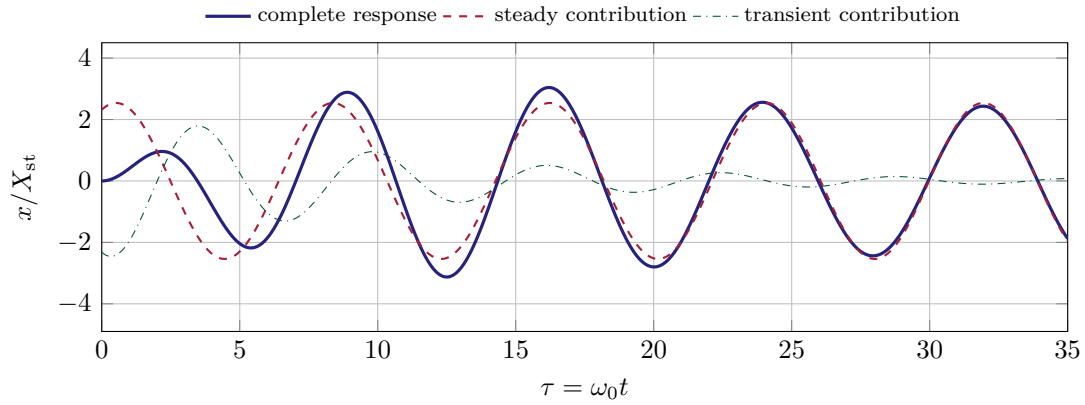


Figure 15.22: Analytical switch-on response for $\zeta = .10$, $r = .80$ and $x(0) = \dot{x}(0) = 0$, with a cosine force beginning at $t = 0$. The complete curve is the sum of the two plotted contributions, not a fit or measured signal. Initially the transient cancels the steady contribution. At late times it decays and the drive-frequency response remains.

Consequently, with t in seconds,

$$x(t) = .1159794 \cos(8t) + .0515464 \sin(8t) - e^{-t} \left[.1159794 \cos(\sqrt{99}t) + .0531012 \sin(\sqrt{99}t) \right] \quad \text{m.}$$

At $t = 0$ the cosine terms cancel. The initial velocity is $8(.0515464) + .1159794 - \sqrt{99}(.0531012) = 0$ to the displayed rounding. Newton's law then gives $\ddot{x}(0^+) = F_0/m = 5.00 \text{ m s}^{-2}$, because the spring and damping forces initially vanish. Thus the response begins quadratically in time, $x \simeq \frac{1}{2}(5.00)t^2$, even though the steady contribution alone begins at a nonzero displacement. The eventual amplitude is .126918 m and phase lag is .418224 rad; the transient changes the start-up, not these late-time values. An exponential factor of e^{-t} reduces its amplitude coefficient to about five percent in 3.00 s, although instantaneous cancellation or reinforcement still affects the observed displacement at particular times.

15.6.6 The undamped limit and finite-time resonant growth

The steady response formula assumed that a transient eventually decays. Setting $b = 0$ removes that assumption: the homogeneous oscillation persists. For an initially resting body and a cosine force switched on at $t = 0$, an off-resonant undamped solution is

$$x(t) = \frac{F_0/m}{\omega_0^2 - \Omega^2} [\cos(\Omega t) - \cos(\omega_0 t)], \quad \Omega \neq \omega_0. \quad (15.50)$$

The first cosine is a particular solution; the second cancels its initial displacement without adding an initial velocity. Both remain indefinitely. When the frequencies are close, their subtraction gives a slow modulation of a faster oscillation. The apparent start-up envelope is therefore not evidence of energy loss. It results from superposition of two persistent motions with slightly different frequencies.

At exact resonance, a constant-amplitude cosine cannot be a particular solution because its inertial and spring terms cancel. Try $x = Ct \sin(\omega_0 t)$ instead. Differentiation gives $\ddot{x} + \omega_0^2 x = 2C\omega_0 \cos(\omega_0 t)$, hence

$$x(t) = \frac{F_0}{2m\omega_0} t \sin(\omega_0 t) \quad (b = 0, \Omega = \omega_0, x_0 = v_0 = 0). \quad (15.51)$$

Its oscillation coefficient grows linearly with time. It is finite at every finite time, and it begins with the same acceleration F_0/m as the damped switch-on solution. Calling the amplitude “infinite at resonance” is a shorthand for failure of a bounded infinite-time steady-state solution, not an instantaneous physical event. Any real apparatus eventually encounters finite travel, nonlinearity, actuator saturation or a loss mechanism, but those additional mechanisms must be modeled before making quantitative claims.

For the system with $m = 2.00$ kg, $\omega_0 = 10.0$ rad s⁻¹ and $F_0 = 10.0$ N, the undamped resonant expression is $x = .2500t \sin(10.0t)$ m when t is in seconds. Its coefficient is .2500 m at $t = 1.00$ s, but its instantaneous displacement then is $-.136005$ m. A coefficient is not the position at every time. The distinction is the same one encountered when separating a damped envelope from a trajectory.

For positive but small damping, the eventual resonant steady amplitude is $F_0/(b\omega_0)$, and the amplitude build-up time is of order $1/\beta$. Making b small increases both quantities. Consequently, a frequency sweep that changes the drive too rapidly can record a response quite different from Eq. (15.43): the oscillator has not had time to settle. The operations “wait for steady state” and “remove damping” cannot be interchanged without attention to this diverging time scale. This is a model interpretation, not an additional experimental noise effect.

15.6.7 Work, power and the width of a resonance

With forcing, the mechanical-energy balance becomes

$$\frac{dE_{\text{mech}}}{dt} = F(t)\dot{x} - b\dot{x}^2. \quad (15.52)$$

In steady state, average over one drive period $T_{\text{drive}} = 2\pi/\Omega$. The stored energy returns to its initial value, so its average derivative is zero. Since the mean square of a sinusoidal velocity is $V^2/2$, the mean input power equals the mean dissipated power:

$$\overline{\mathcal{P}} = \frac{1}{2}F_0\Omega A \sin \delta = \frac{1}{2}b\Omega^2 A^2. \quad (15.53)$$

The first expression follows by multiplying the force and velocity sinusoids and averaging their product; their relative phase controls energy transfer. Instantaneous input power can be negative during part of a cycle, meaning the oscillator returns energy to the actuator. Net work over a steady cycle is positive, $W_{\text{cycle}} = \overline{\mathcal{P}}T_{\text{drive}}$, and exactly replaces damping loss. A large displacement does not by itself establish maximum energy absorption.

For fixed F_0 , power is proportional to V^2 and peaks exactly at $\Omega = \omega_0$, with $\overline{\mathcal{P}}_{\text{max}} = F_0^2/(2b)$. Its half-maximum frequencies are found by setting $(\omega_0^2 - \Omega^2)^2 = (2\beta\Omega)^2$. The positive roots are

$$\Omega_- = \sqrt{\omega_0^2 + \beta^2} - \beta, \quad \Omega_+ = \sqrt{\omega_0^2 + \beta^2} + \beta. \quad (15.54)$$

Their separation is exactly $\Delta\Omega = 2\beta = b/m$ in this model. Define the dimensionless quality factor

$$Q_{\text{osc}} = \frac{\omega_0}{2\beta} = \frac{m\omega_0}{b} = \frac{1}{2\zeta} = \frac{\omega_0}{\Delta\Omega}. \quad (15.55)$$

The bandwidth here is a *half-power* width. The velocity amplitude at these frequencies is $1/\sqrt{2}$ of its maximum, not half its maximum. Applying the same width to an arbitrary displacement-amplitude plot gives a different definition and need not yield the exact relation above. For light damping, Q_{osc} also approximately measures 2π times stored energy divided by energy lost per cycle; that familiar interpretation assumes only a small fractional change during one free cycle.

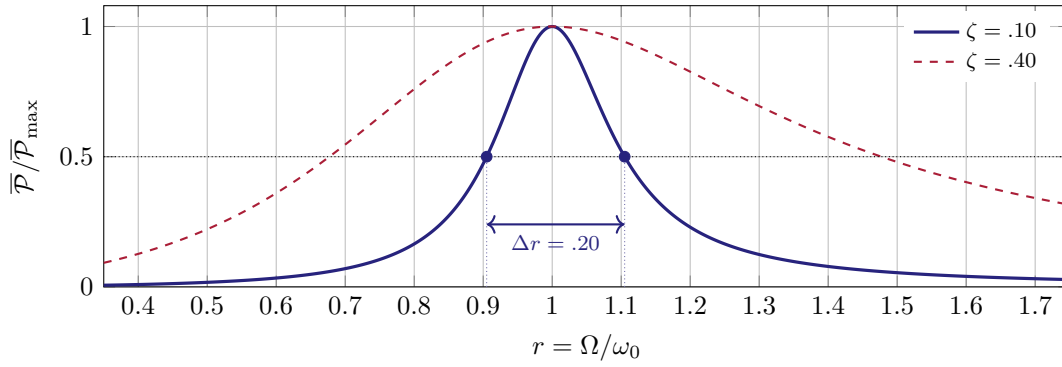


Figure 15.23: Normalized mean absorbed power for fixed force amplitude. Both damping ratios peak at $r = 1$. For $\zeta = .10$, the half-power points are $r_- = \sqrt{1.01} - .10$ and $r_+ = \sqrt{1.01} + .10$, with exact separation .20. The points are slightly asymmetric around the peak in a linear frequency coordinate, even though their product is one. Each curve is normalized by its own maximum; their dimensional peak powers differ.

Example 15.16: Use phase to check the work supplied in one cycle

Statement. For the oscillator in Example 15.14, find the phase lag, mean input power and work per cycle at $\Omega = 8.00 \text{ rad s}^{-1}$. Then find the maximum mean power, half-power frequencies and Q_{osc} . Verify power by two different expressions.

Solution. At $\Omega = 8.00 \text{ rad s}^{-1}$, $A = .126918 \text{ m}$ and

$$\delta = \text{atan2}(32, 72) = .418224 \text{ rad} = 23.9625^\circ, \quad V = 1.01535 \text{ m s}^{-1}.$$

The mean damping loss is $\frac{1}{2}(4.00)(1.01535)^2 = 2.06186 \text{ W}$. Independently, the phase formula gives $\frac{1}{2}(10.0)(8.00)(.126918) \sin(.418224) = 2.06186 \text{ W}$. With $T_{\text{drive}} = 2\pi/8.00 = .785398 \text{ s}$, $W_{\text{cycle}} = 1.61938 \text{ J}$. A positive mean power is consistent with $b > 0$ and $0 < \delta < \pi$. The power peak occurs at $\omega_0 = 10.0 \text{ rad s}^{-1}$, giving $\bar{P}_{\max} = 10.0^2/(2 \times 4.00) = 12.50 \text{ W}$. The half-power frequencies are

$$\Omega_- = \sqrt{101} - 1 = 9.04988 \text{ rad s}^{-1}, \quad \Omega_+ = \sqrt{101} + 1 = 11.0499 \text{ rad s}^{-1}.$$

Their difference is $2.00 \text{ rad s}^{-1} = b/m$, hence $Q_{\text{osc}} = 10.0/2.00 = 5.00$. At the displacement peak from Example 15.14, the mean power is 12.3737 W , slightly less than 12.50 W . This directly checks that the displacement and power peaks are different observables. Phase measurement provides a useful consistency test when amplitude calibration alone is uncertain.

15.6.8 A constant force step is not a sinusoidal steady state

A sudden constant force F_s shifts equilibrium by $X_s = F_s/k_s$. For $t > 0$, putting $z = x - X_s$ reduces the forced equation to the free damped equation. An initially resting mass at the old equilibrium has $z(0) = -X_s$ and $\dot{z}(0) = 0$. For $0 < \zeta < 1$,

$$\frac{x(t)}{X_s} = 1 - e^{-\beta t} \left[\cos(\omega_d t) + \frac{\beta}{\omega_d} \sin(\omega_d t) \right]. \quad (15.56)$$

Differentiating shows that the first maximum occurs at $t_p = \pi/\omega_d$ and exceeds the final displacement by the fraction $\exp[-\pi\zeta/\sqrt{1-\zeta^2}]$. At critical damping the response is $x/X_s = 1 - (1 + \omega_0 t)e^{-\omega_0 t}$, which rises monotonically from zero to one. These results use the same homogeneous solutions as free decay, but a different equilibrium and initial condition. Their overshoot is a start-up property, not a steady sinusoidal resonance amplitude.

For the lightly damped 2.00 kg system and a 10.0 N step, $X_s = .0500 \text{ m}$, $t_p = .315742 \text{ s}$, and the fractional overshoot is $.729248$. Hence its first maximum is $.0864624 \text{ m}$. This large overshoot is still

far below the .2500 m sinusoidal steady amplitude at $\Omega = \omega_0$: one sustained resonant input can supply energy over many cycles, while a constant input eventually becomes balanced by a static spring force. The calculation also answers an immediate model check: a constant force does not cause unbounded displacement when the spring has positive stiffness.

15.6.9 Moving the support: a different transfer function

Let a common spring and damper support have prescribed displacement $y(t) = Y \cos(\Omega t)$, and let $x(t)$ be the mass's displacement in an inertial frame. The elements act through relative extension $x - y$ and relative velocity $\dot{x} - \dot{y}$. With no separate force actuator,

$$m\ddot{x} + b(\dot{x} - \dot{y}) + k_s(x - y) = 0. \quad (15.57)$$

The transfer function for absolute displacement is most compactly written using a complex amplitude, where $i^2 = -1$ and real parts give physical sinusoids:

$$H_{xy}(\Omega) = \frac{\hat{x}}{\hat{y}} = \frac{k_s + ib\Omega}{k_s - m\Omega^2 + ib\Omega}, \quad \frac{|\hat{x}|}{Y} = \frac{\sqrt{1 + 4\zeta^2 r^2}}{\sqrt{(1 - r^2)^2 + 4\zeta^2 r^2}}. \quad (15.58)$$

This notation packages two real coefficient equations; it introduces no new physical approximation. A slowly moving base carries the mass with it, so $|\hat{x}|/Y \rightarrow 1$ as $r \rightarrow 0$. At high frequency the absolute mass response tends to zero. For nonzero b its leading ratio is $2\zeta/r$; if $b = 0$ the leading ratio instead scales as $1/r^2$. The high-frequency limit does not justify identifying base displacement with a fixed force amplitude.

The relative coordinate $z = x - y$ obeys

$$m\ddot{z} + b\dot{z} + k_s z = -m\ddot{y}, \quad \frac{|\hat{z}|}{Y} = \frac{r^2}{\sqrt{(1 - r^2)^2 + 4\zeta^2 r^2}}. \quad (15.59)$$

For fixed Y , the effective force amplitude in this relative-coordinate equation is $m\Omega^2 Y$, which changes during a sweep. At low frequency $|\hat{z}|/Y \rightarrow 0$; at high frequency it tends to one, because the base moves beneath an almost stationary mass. Figure 15.24 contrasts absolute and relative displacement with fixed-force magnification. The different numerators explain the different shapes and limits.

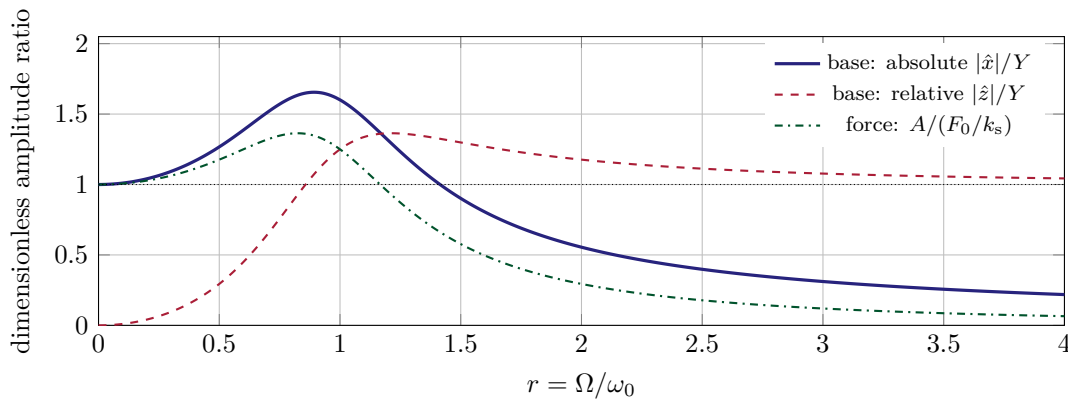


Figure 15.24: Analytical input-model comparison at $\zeta = .40$. The blue and red curves assume fixed base displacement amplitude and refer to different coordinates; the green curve assumes fixed force amplitude and a fixed support. Their normalization scales differ, so the comparison shows transfer-function shape and limits, not equal dimensional displacements. The relative base response tends to one at high frequency while the absolute mass response tends to zero.

Example 15.17: Distinguish payload motion from mount deformation

Statement. An ideal mount has $m = 2.00 \text{ kg}$, $k_s = 200 \text{ N m}^{-1}$ and $b = 16.0 \text{ kg s}^{-1}$. Both the spring and damper base are moved with amplitude $Y = 2.00 \text{ mm}$ at $\Omega = 20.0 \text{ rad s}^{-1}$. Determine the absolute payload-displacement amplitude and relative mount deformation amplitude. Compare these with the amplitude produced by a fixed-support force actuator of amplitude $F_0 = k_s Y$ at the same frequency. Explain why the two displacement amplitudes cannot be subtracted as scalars to find the third.

Solution. Here $\omega_0 = 10.0 \text{ rad s}^{-1}$, $\beta = 4.00 \text{ s}^{-1}$, $\zeta = .400$ and $r = 2.00$. The common dimensionless denominator is $\sqrt{(1-4)^2 + (2 \times .4 \times 2)^2} = \sqrt{11.56} = 3.400$. Thus

$$|\hat{x}| = Y \frac{\sqrt{1 + 1.6^2}}{3.400} = 1.10988 \text{ mm}, \quad |\hat{z}| = Y \frac{4}{3.400} = 2.35294 \text{ mm}.$$

For the separate force experiment, $F_0 = k_s Y = .400 \text{ N}$ and $X_{\text{st}} = Y$, so $A = Y/3.400 = .588235 \text{ mm}$. The base-driven absolute amplitude is larger because its numerator includes the support-velocity contribution through the damper.

Although the instantaneous coordinates satisfy $z(t) = x(t) - y(t)$, their complex amplitudes have different phases. At $r = 2$, $H_{xy} = (1 + 1.6i)/(-3 + 1.6i) = -.0380623 - .553633i$. Subtracting one gives $H_{zy} = -1.0380623 - .553633i$, whose magnitude is $4/3.4$. It is therefore invalid to replace $|\hat{x} - \hat{y}|$ by $|\hat{x}| - |\hat{y}|$. Relative deformation governs spring and damper loads. Absolute motion describes inertial-frame payload displacement; a measurement or engineering specification must identify its reference and which coordinate is limited.

Key message

For sinusoidal forcing, identify the controlled input, observed coordinate, response quantity and initial or steady-state condition. In the fixed-force model with $b > 0$, velocity and mean absorbed power peak at ω_0 ; displacement peaks below it only when $\zeta < 1/\sqrt{2}$. These results follow from the complete transfer functions and are not universal rules for a moving base or every observable.

15.7 Experimental identification and vibration

An oscillation record is useful because it can reveal properties that are difficult to measure directly. The time between peaks estimates a frequency; the reduction of their magnitudes estimates dissipation. Those inferences require a model and a calibrated coordinate. An excellent fit to an inappropriate model can produce precise but misleading parameters. We therefore separate what the instrument observes from what the equation allows us to infer.

15.7.1 From a decay record to model parameters

For an underdamped free response, write

$$x(t) = e^{-\beta t} [C \cos(\omega_d t) + D \sin(\omega_d t)].$$

Its extrema satisfy $\dot{x} = 0$. They do not generally occur where the cosine in an amplitude-phase representation equals one: differentiating the exponential contributes another term. Nevertheless, the entire state repeats with a factor $e^{-\beta T_d}$ after a full damped period $T_d = 2\pi/\omega_d$. Consequently successive same-sign extrema have precisely that ratio. If their positive magnitudes are A_n and A_{n+1} ,

$$\ell = \ln(A_n/A_{n+1}) = \beta T_d, \quad \beta = \frac{\ell}{T_d}, \quad \omega_0 = \sqrt{\omega_d^2 + \beta^2}. \quad (15.60)$$

Here ℓ denotes logarithmic decrement, so that it cannot be confused with the driven-response phase lag δ . Both are dimensionless but they describe different experiments. Alternating-sign extrema are half a period apart; using that interval in a same-sign formula doubles the inferred frequency.

The fit in Figure 15.25 starts at a positive turning point. A waveform consistent with that event is

$$x_{\text{fit}}(t) = A_p e^{-\beta t} \left[\cos(\omega_d t) + \frac{\beta}{\omega_d} \sin(\omega_d t) \right]. \quad (15.61)$$

Its derivative is $-A_p(\omega_0^2/\omega_d)e^{-\beta t} \sin(\omega_d t)$, which vanishes exactly at $t = nT_d$. Thus its positive peak sequence is $A_p e^{-\beta n T_d}$. Its rigorous sinusoidal bound is instead $\pm A_p \sqrt{1 + (\beta/\omega_d)^2} e^{-\beta t}$. The distinction is small here, but it matters when drawing a graph that claims exact initial conditions.

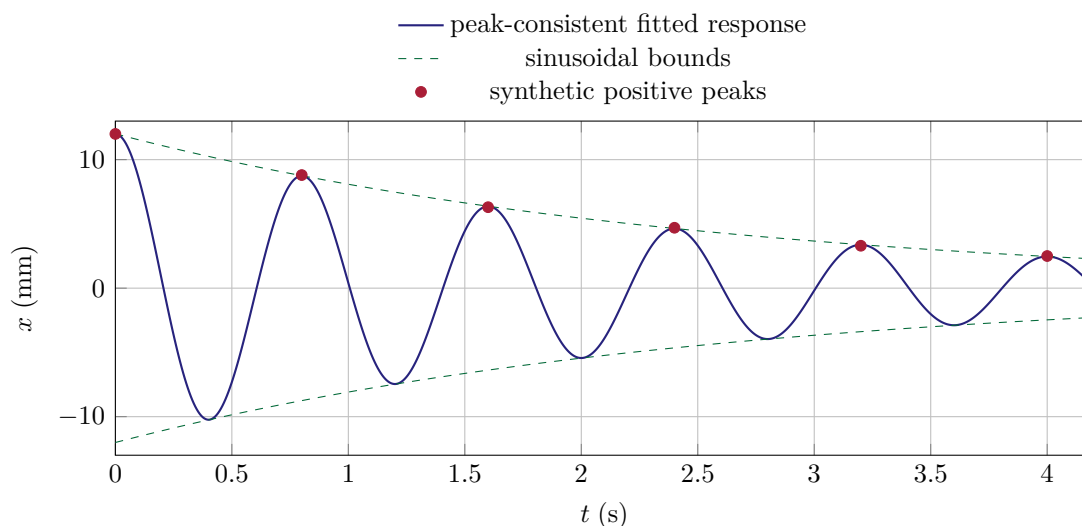


Figure 15.25: Synthetic same-sign peak readings at intervals $T_d = 0.80$ s and a fitted free response whose positive extrema occur at those times. The green dashed lines are the exact sinusoidal bounds of the fitted waveform, rather than interpolating curves through its turning points. No apparatus measurements are claimed.

Example 15.18: Infer damping from a decay

A synthetic displacement record has ideal sampling interval 0.01 s and peak values reported to 0.1 mm. Six positive peaks, separated by 0.80 s, have magnitudes 12.0, 8.8, 6.3, 4.7, 3.3, and 2.5 mm. Infer the damping rate and natural frequency; assess the fit rather than simply reporting its parameters.

Strategy. Fit $z_n = \ln[A_n/(1 \text{ mm})]$ against $t_n = 0.80n$ s. The explicit reference unit makes the logarithm dimensionless. Ordinary unweighted least squares gives

$$s = \frac{\sum_n (t_n - \bar{t})(z_n - \bar{z})}{\sum_n (t_n - \bar{t})^2}, \quad c = \bar{z} - s\bar{t}, \quad \beta = -s.$$

Using $\bar{t} = 2.0$ s gives $\beta = 0.395663 \text{ s}^{-1}$ and $A_p = \exp(c) \text{ mm} = 11.99299 \text{ mm}$. The peak separation independently yields

$$\omega_d = \frac{2\pi}{0.80} = 7.85398 \text{ rad s}^{-1}, \quad \omega_0 = \sqrt{7.85398^2 + 0.395663^2} = 7.86394 \text{ rad s}^{-1}.$$

The inferred damping ratio is $\zeta = 0.05031$, consistent with an underdamped record. This inference uses the linear viscous model, not the visual appearance of the trace alone.

t (s)	observed A (mm)	predicted A (mm)	residual (mm)
0.0	12.0	11.993	0.007
0.8	8.8	8.739	0.061
1.6	6.3	6.368	−0.068
2.4	4.7	4.640	0.060
3.2	3.3	3.381	−0.081
4.0	2.5	2.464	0.036

These are constructed teaching readings with scatter, not exact exponential values rounded to a decimal place. Several residuals exceed the half-unit rounding interval 0.05 mm; quantisation alone does not explain them. The model is broadly consistent with the data, but a physical experiment would need a declared uncertainty model before assigning a statistical goodness of fit. The synthetic nature of the readings must not be lost when interpreting the residuals.

Figure 15.26 shows why residuals deserve their own representation. A single good-looking decay curve can conceal a constant calibration offset or a change in the friction mechanism as amplitude falls. Here the residuals alternate irregularly about zero and do not establish a specific physical mechanism; there are only six synthetic points.

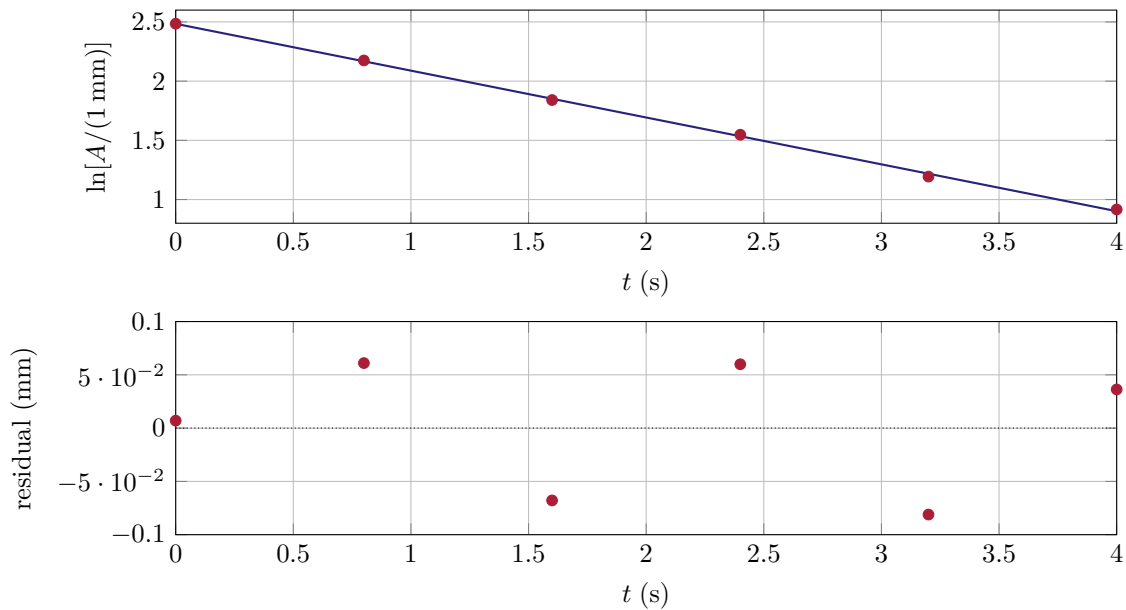


Figure 15.26: Logarithmic peak fit and displacement residuals for Example 15.18. The straight-line slope identifies $-\beta$; the lower panel tests what the fitted line leaves unexplained. All data are synthetic, with no statistical uncertainty assigned.

15.7.2 What can be identified, and what remains unknown

Dividing $m\ddot{x} + b\dot{x} + k_s x = 0$ by m shows that a free-decay record identifies k_s/m and b/m . It cannot determine all three dimensional parameters separately. Multiplying m , b , and k_s by the same positive factor leaves every displacement history unchanged for the same initial state. An independent mass measurement, a static force calibration, or another known perturbation supplies the missing dimensional scale.

Example 15.19: Recover the missing dimensional scale

Use the decay fit of Example 15.18. What are k_s and b if the proof mass is independently known to be 0.60 kg? Could the record alone distinguish that system from one with twice its mass?

The frequency and damping rate determine $k_s/m = \omega_0^2 = 61.84158 \text{ s}^{-2}$ and $b/m = 2\beta = 0.791325 \text{ s}^{-1}$. Hence

$$k_s = (0.60)(61.84158) = 37.105 \text{ N m}^{-1}, \quad b = 2(0.60)(0.395663) = 0.474795 \text{ kg s}^{-1}.$$

A system with $m = 1.20 \text{ kg}$, $k_s = 74.210 \text{ N m}^{-1}$, and $b = 0.949590 \text{ kg s}^{-1}$ has the same normalized differential equation. The same displacement record therefore cannot distinguish them. The different masses would respond differently to a prescribed force, which breaks this scale invariance. Units provide a check: squared frequency times mass has stiffness units, while damping rate times mass has viscous-coefficient units.

15.7.3 A proposed measurement and its uncertainty

A useful introductory experiment defines and calibrates the measured displacement, records many cycles, identifies same-sign peaks, and compares fits at several initial amplitudes. Timing ten cycles reduces the contribution of a fixed start–stop timing error per cycle. It does not remove drift, incorrect counting, nonlinear amplitude dependence, or a systematic length error. When the pendulum is used to estimate g , the relevant length is the pivot-to-bob-centre distance, not the exposed string alone.

Example 15.20: Timing several pendulum cycles

In a proposed small-angle pendulum investigation, let $L = 1.000 \text{ m}$ with independent standard uncertainty $u_L = 0.002 \text{ m}$. A stipulated ten-cycle time is $t_{10} = 20.06 \text{ s}$ with standard uncertainty $u_t = 0.10 \text{ s}$. Estimate g and its first-order standard uncertainty, neglecting nonlinear period corrections and air resistance.

Counting ten complete cycles gives $T = t_{10}/10 = 2.006 \text{ s}$ and $u_T = u_t/10 = 0.010 \text{ s}$. Solving $T = 2\pi\sqrt{L/g}$ gives $g = 4\pi^2 L/T^2 = 9.81065 \text{ m s}^{-2}$. For independent uncertainties, differentiating $g(L, T)$ gives

$$\left(\frac{u_g}{g}\right)^2 \simeq \left(\frac{u_L}{L}\right)^2 + 4\left(\frac{u_T}{T}\right)^2.$$

The fractional uncertainty is 0.01017, so $u_g \simeq 0.10 \text{ m s}^{-2}$. A suitable report is $g = (9.81 \pm 0.10) \text{ m s}^{-2}$, where the quoted uncertainty is a standard uncertainty under the stipulated independent-error model. This agreement with the chapter's teaching value is a consistency check, not evidence that the proposed investigation has been performed. A non-negligible initial angle lengthens the period and biases this estimator downward; repeat at smaller amplitudes to test that limitation.

For the decay fit, an additive displacement error becomes a larger fractional error at late, small peaks. An unweighted logarithmic fit implicitly gives those points as much weight as larger peaks. If the sensor has a known nearly constant absolute standard uncertainty u_A , then $u_{\ln A} \simeq u_A/A$ and weighted fitting becomes appropriate. A common multiplicative scale error cancels from a ratio A_n/A_{n+1} ; a common additive zero offset does not. These differences matter more than displaying extra decimal places in β .

Guided checks

Identification check. Opposite-sign extrema at 0.40 s and 0.80 s are only a half-period apart in the viscous underdamped model. The same-sign spacing would be 0.80 s; using 0.40 s in the full-period frequency relation gives a factor-of-two error.

Computation check. Halving a time step changes the estimated steady amplitude by 6%. The original computation has not established convergence to a stricter accuracy than that change. Further refinement and comparison with the analytic steady response are needed.

15.7.4 Numerical integration with a physical reference test

To compute a non-sinusoidal force history, replace the second-order equation by a first-order state system,

$$\dot{x} = v, \quad \dot{v} = \frac{F(t) - bv - k_s x}{m}. \quad (15.62)$$

The state contains displacement and velocity with different units. A solver tolerance must therefore be interpreted for each component, or the state should be nondimensionalized. For the unforced undamped test use $X = x/A$, $V = v/(A\omega_0)$, and $\tau = \omega_0 t$. Then $X' = V$, $V' = -X$, with exact solution $X = \cos \tau$, $V = -\sin \tau$ for release from rest. This provides a strong reference for an algorithm before it is applied to unknown motion.

For a state derivative $f(\tau, y)$, fourth-order Runge–Kutta evaluates four slopes: $K_1 = f(\tau, y)$, $K_2 = f(\tau + h/2, y + hK_1/2)$, $K_3 = f(\tau + h/2, y + hK_2/2)$, and $K_4 = f(\tau + h, y + hK_3)$. The update is $y_{n+1} = y_n + h(K_1 + 2K_2 + 2K_3 + K_4)/6$. The intermediate slopes approximate the bending of the trajectory within a step. The method's name does not guarantee accuracy at an arbitrary h .

Listing 15.1: Optional RK4 step; no software execution is needed to compile the chapter.

```
def rk4_step(t, y, h, rhs):
    def add(a, q, scale):
        return tuple(ai + scale*qi for ai, qi in zip(a, q))
    k1 = rhs(t, y)
    k2 = rhs(t+h/2, add(y, k1, h/2))
    k3 = rhs(t+h/2, add(y, k2, h/2))
    k4 = rhs(t+h, add(y, k3, h))
    return tuple(yi+h*(a+2*b+2*c+d)/6
                for yi,a,b,c,d in zip(y,k1,k2,k3,k4))
```

Example 15.21: Check an integrator before a sweep

Integrate the dimensionless undamped oscillator from $\tau = 0$ to 20, with initial state (1, 0). Compare RK4 steps $h = 0.20$, 0.10, and 0.05 against the exact state at every grid time. What evidence supports convergence, and what does that evidence leave untested?

Use the maximum absolute error over both dimensionless state components and all grid times. The resulting errors are 2.532×10^{-4} , 1.580×10^{-5} , and 9.853×10^{-7} . Halving the step reduces them by about 16, consistent with fourth-order global accuracy over this fixed interval. The maximum error in $E/E_0 = X^2 + V^2$ is respectively 8.844×10^{-5} , 2.774×10^{-6} , and 8.678×10^{-8} . The energy error is especially small here but cannot replace the state test: a method could preserve energy while accumulating a wrong phase.

The convergence is numerical evidence for this equation and interval. For a driven calculation, use the same physical inputs, resolve both drive and natural time scales, and discard a sufficiently long transient before fitting a sinusoidal response. Compare the fitted amplitude and phase with the analytic fixed-force expressions, then halve h . A physically incorrect forcing model will still converge to its own incorrect result. Strong overdamping can introduce a much faster

decaying time scale than the observed slow return, so a step adequate for the slow trajectory alone may be inadequate for an explicit integrator.

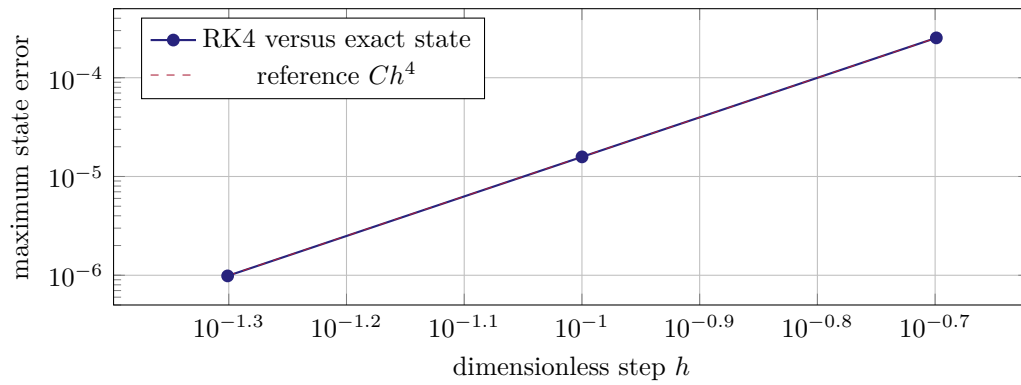


Figure 15.27: Computed RK4 convergence over $0 \leq \tau \leq 20$ for the undamped oscillator released from rest. The dashed fourth-order guide is anchored at $h = 0.20$, not fitted as an independent physical model. Grid refinement tests state accuracy, including phase.

15.8 Advanced Topics for Technical and Engineering Students

The selected enrichment is a symmetric pair of coupled linear oscillators. It preserves the single-oscillator model while introducing a new physical possibility: motion initiated at one mass can reach another through an interaction. The mathematical bridge is modest. We add and subtract two equations, then interpret the resulting independent coordinates. Readers need the linear restoring-force model, energy, and the elementary trigonometric identities used below. Core practice does not require this section; Problems 8 and 29–32 are explicitly advanced.

15.8.1 The system and the coupling force

Two equal masses m move horizontally on an ideal frictionless guide. Each is connected to its adjacent fixed wall by a spring of stiffness k_s , and a spring of stiffness k_c joins the masses. The equilibrium lengths have already absorbed any static preload. Define both displacements x_1, x_2 as positive to the right from their own equilibria. Increasing $x_2 - x_1$ stretches the coupling spring; its forces on the two masses are equal and opposite.

Figure 15.28 separates the displacement coordinates from the spring extensions. The coupling force on mass 1 is $+k_c(x_2 - x_1)$, and that on mass 2 is its negative. The wall springs contribute $-k_s x_i$. Newton's laws therefore give

$$m\ddot{x}_1 = -(k_s + k_c)x_1 + k_c x_2, \quad (15.63)$$

$$m\ddot{x}_2 = k_c x_1 - (k_s + k_c)x_2. \quad (15.64)$$

If both masses shift equally, the coupling spring's extension does not change. If the masses shift oppositely, that spring changes length by twice either displacement. These two elementary observations anticipate the normal-mode frequencies without requiring a matrix calculation.

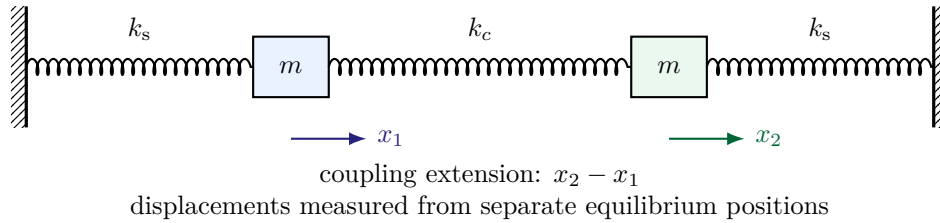


Figure 15.28: The two-mass system, drawn schematically with both displacement axes positive rightward. The central spring responds to the difference of displacements, rather than either displacement alone. Spring lengths and arrow lengths are not physical scales.

15.8.2 Normal coordinates and mode frequencies

Introduce the orthonormal combinations

$$q_+ = \frac{x_1 + x_2}{\sqrt{2}}, \quad q_- = \frac{x_1 - x_2}{\sqrt{2}}, \quad x_1 = \frac{q_+ + q_-}{\sqrt{2}}, \quad x_2 = \frac{q_+ - q_-}{\sqrt{2}}. \quad (15.65)$$

Adding the two Newton equations eliminates the coupling forces, while subtracting doubles their restoring contribution. Dividing by m produces

$$\ddot{q}_+ + \omega_+^2 q_+ = 0, \quad \ddot{q}_- + \omega_-^2 q_- = 0, \quad \omega_+ = \sqrt{\frac{k_s}{m}}, \quad \omega_- = \sqrt{\frac{k_s + 2k_c}{m}}. \quad (15.66)$$

The plus and minus subscripts refer to symmetric and antisymmetric coordinates, not to frequency ordering: for $k_c > 0$, $\omega_- > \omega_+$. Every motion is a superposition of these two modes. A pure mode has a fixed ratio x_2/x_1 , so every mass oscillates with the same modal frequency and a fixed phase relation.

The same result can be expressed as an eigenvalue problem. Substituting $\mathbf{x} = \mathbf{a} \cos \omega t$ gives

$$\begin{pmatrix} k_s + k_c & -k_c \\ -k_c & k_s + k_c \end{pmatrix} \mathbf{a} = m\omega^2 \mathbf{a}.$$

Multiplying the matrix by $(1, 1)^T$ gives $k_s(1, 1)^T$; multiplying it by $(1, -1)^T$ gives $(k_s + 2k_c)(1, -1)^T$. Thus the coordinate method and matrix method describe the same physical decomposition. The normalized eigenvectors have length one and are perpendicular, which is why the kinetic energy also separates cleanly.

Figure 15.29 shows the displacement patterns at one positive-phase instant. These arrows are modal displacement components, not force vectors. In the in-phase mode, the central spring retains its equilibrium length. In the out-of-phase mode it is compressed or stretched, adding stiffness to each mass.

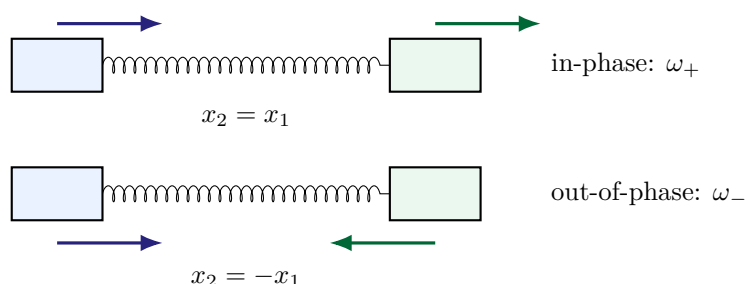


Figure 15.29: Schematic displacement patterns of the two normal modes. Equal rightward displacements leave the coupling extension unchanged; opposite displacements change it by twice the individual scale. Arrow direction distinguishes the modes in grayscale as well as in color.

Example 15.22: Select a mode by the initial state

For $m = 0.50 \text{ kg}$, $k_s = 80 \text{ N m}^{-1}$, and $k_c = 20 \text{ N m}^{-1}$, compare release from $x_1 = x_2 = a$ with release from $x_1 = a, x_2 = -a$, both from rest. Which frequency appears, and which springs initially store energy?

In the first case $q_+(0) = \sqrt{2}a$ and $q_-(0) = 0$. Only the symmetric mode is excited: $x_1 = x_2 = a \cos(12.6491t)$. The coupling spring is unchanged, so $E_0 = k_s a^2$, the sum of two outer-spring energies. In the second case $q_+(0) = 0$, $q_-(0) = \sqrt{2}a$. Then $x_1 = a \cos(15.4919t)$ and $x_2 = -a \cos(15.4919t)$. Its initial energy is $k_s a^2 + \frac{1}{2} k_c (2a)^2 = (k_s + 2k_c) a^2$. The added restoring energy explains the higher frequency. If $a = 0.020 \text{ m}$, the energies are 0.0320 J and 0.0480 J respectively. The two releases have equal individual displacement magnitudes but different stored energies; comparing them at equal displacement is not comparing them at equal total energy.

15.8.3 Release one mass: interference and slow envelopes

For $x_1(0) = a$, $x_2(0) = 0$ and zero velocities, both modes start with $q_{\pm}(0) = a/\sqrt{2}$. Transforming back gives

$$x_1(t) = \frac{a}{2} (\cos \omega_+ t + \cos \omega_- t), \quad (15.67)$$

$$x_2(t) = \frac{a}{2} (\cos \omega_+ t - \cos \omega_- t). \quad (15.68)$$

Define $\bar{\omega} = (\omega_+ + \omega_-)/2$ and $\Delta\omega = \omega_- - \omega_+ > 0$. The sum and difference identities give

$$x_1 = a \cos(\bar{\omega}t) \cos(\Delta\omega t/2), \quad x_2 = a \sin(\bar{\omega}t) \sin(\Delta\omega t/2).$$

These expressions separate a fast carrier from slowly changing amplitudes when $k_c \ll k_s$. At stronger coupling the algebra remains exact, although calling the second factor “slow” becomes less useful. Figure 15.30 uses the retained ratio $k_c/k_s = 0.25$ and shows actual displacements, rather than identifying envelope functions as trajectories.

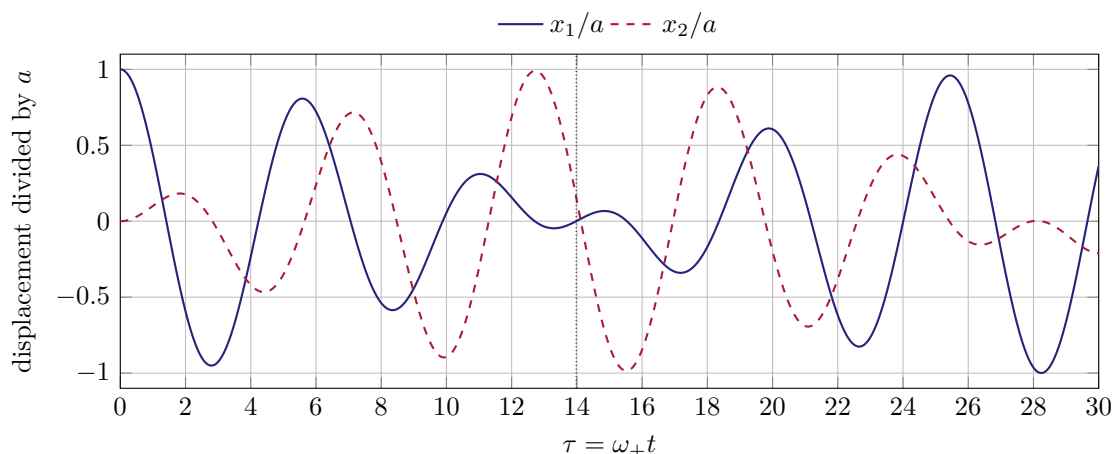


Figure 15.30: Analytic motion after releasing mass 1 alone, with $k_c/k_s = 0.25$. The dotted line marks $\Delta\omega t = \pi$, the first zero of coordinate 1’s modulation factor. It does not assert that every energy contribution of mass 1 is zero. The physical setup is drawn separately in Figure 15.28.

Example 15.23: Excite one mass in a coupled pair

Let $m = 0.50 \text{ kg}$, $k_s = 80 \text{ N m}^{-1}$, $k_c = 20 \text{ N m}^{-1}$, and $a = 20 \text{ mm}$. Release mass 1 from $x_1 = a$ while mass 2 starts at equilibrium, both from rest. Find the two motions, the first envelope-cancellation time, and the total energy.

The frequencies are $\omega_+ = \sqrt{160} = 12.6491 \text{ rad s}^{-1}$ and $\omega_- = \sqrt{240} = 15.4919 \text{ rad s}^{-1}$. Thus

$$x_1 = 0.010[\cos(12.6491t) + \cos(15.4919t)] \text{ m}, \quad x_2 = 0.010[\cos(12.6491t) - \cos(15.4919t)] \text{ m}.$$

The first zero of $\cos(\Delta\omega t/2)$ occurs at

$$t_{\text{tr}} = \frac{\pi}{\omega_- - \omega_+} = 1.10510 \text{ s}.$$

At release, the left outer spring and the coupling spring each change length by a ; the right outer spring does not. Therefore $E_0 = \frac{1}{2}(k_s + k_c)a^2 = 0.0200 \text{ J}$. Substitution at $t = 0$ recovers both initial displacements and zero velocities, checking that the superposition is the required one. In the limit $k_c \rightarrow 0$, the two frequencies become equal, x_2 vanishes identically, and the transfer time diverges.

The envelope event does give $x_1 = 0$. Differentiating x_1 shows $v_1(t_{\text{tr}}) = 0.02807 \text{ m s}^{-1}$ for these inputs; mass 1 still has kinetic energy. The event is therefore an exact displacement-envelope cancellation, with only approximate near-complete energy transfer when coupling is sufficiently weak. This qualification follows from the velocity, not from a visual judgment about the plot.

15.8.4 Conserved modal energy and physical energy transfer

For the two masses and all three springs together, the energy is

$$E = \frac{1}{2}m(v_1^2 + v_2^2) + \frac{1}{2}k_s(x_1^2 + x_2^2) + \frac{1}{2}k_c(x_2 - x_1)^2. \quad (15.69)$$

Orthogonality implies $v_1^2 + v_2^2 = \dot{q}_+^2 + \dot{q}_-^2$ and $x_1^2 + x_2^2 = q_+^2 + q_-^2$, while $x_2 - x_1 = -\sqrt{2}q_-$. Hence

$$E = \frac{1}{2}m\dot{q}_+^2 + \frac{1}{2}m\omega_+^2 q_+^2 + \frac{1}{2}m\dot{q}_-^2 + \frac{1}{2}m\omega_-^2 q_-^2. \quad (15.70)$$

Each modal energy is constant because each coordinate obeys an independent undamped oscillator equation. Physical-mass energies vary because each mass mixes both modes. To avoid an arbitrary allocation of the central spring, define $E_1 = \frac{1}{2}mv_1^2 + \frac{1}{2}k_s x_1^2$, $E_2 = \frac{1}{2}mv_2^2 + \frac{1}{2}k_s x_2^2$, and $E_c = \frac{1}{2}k_c(x_2 - x_1)^2$ separately. Then

$$\dot{E}_1 = k_c(x_2 - x_1)v_1, \quad \dot{E}_2 = -k_c(x_2 - x_1)v_2, \quad \dot{E}_c = k_c(x_2 - x_1)(v_2 - v_1),$$

whose sum vanishes. These expressions explicitly track the exchange pathway without claiming the central spring belongs entirely to either mass.

Figure 15.31 uses the same release and parameter ratio as Figure 15.30. Normalize time by ω_+ and write $X_i = x_i/a$, $V_i = \dot{x}_i/(a\omega_+)$. Then $E_1/E_0 = (X_1^2 + V_1^2)/1.25$, $E_2/E_0 = (X_2^2 + V_2^2)/1.25$, and $E_c/E_0 = 0.25(X_2 - X_1)^2/1.25$. The sum is one at every time, including the envelope event.

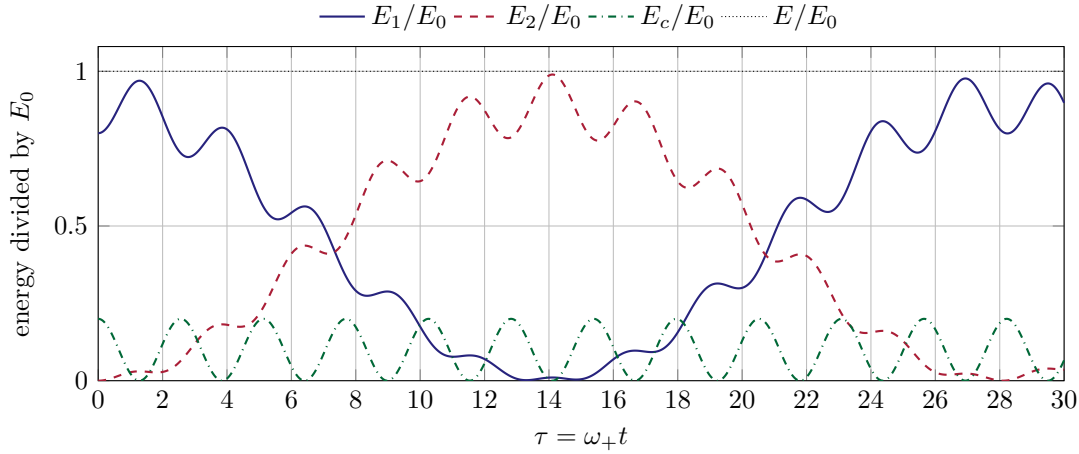


Figure 15.31: Energy bookkeeping for the same coupled release. The central spring is kept as its own energy store. The three variable curves sum exactly to the constant dotted line. Modal energies are constant even while energy moves between the physical coordinates and coupling spring.

Example 15.24: Audit the envelope event energetically

For Example 15.23, evaluate the separately defined energies at t_{tr} . Does coordinate 1 possess zero mechanical energy?

At the event $x_1 = 0$, $x_2 = 0.00316035 \text{ m}$, $v_1 = 0.0280711 \text{ m s}^{-1}$, and $v_2 = -0.277875 \text{ m s}^{-1}$. Thus $E_1 = 0.0001970 \text{ J}$, $E_2 = 0.0197031 \text{ J}$, and $E_c = 0.00009988 \text{ J}$. Their sum is 0.0200000 J to the displayed precision. Coordinate 1 still carries about 0.985% of the total in kinetic form. The central spring carries another 0.499%. These small but finite contributions explain why an envelope-zero time should not be labelled an exact complete-energy-transfer time. The modal expression checks conservation independently: $E_+ = k_s a^2/4 = 0.0080 \text{ J}$ and $E_- = (k_s + 2k_c)a^2/4 = 0.0120 \text{ J}$ remain constant.

Theoretical point 15.1: From a pair to a distributed system

A longer chain has more independent displacement coordinates and more normal modes. The continuum limit adds a spatial variable and permits propagation, reflection, and standing patterns. The finite pair establishes the essential logic: the interaction determines the possible

frequencies and shapes, while the initial conditions determine their mixture. Chapter 16 develops the wave description of that same interaction-and-inertia mechanism.

15.9 Chapter summary and working formula sheet

Summary

Oscillations begin with equilibrium and a restoring interaction. A smooth nondegenerate potential minimum has positive curvature, and its small-displacement dynamics is harmonic. Initial displacement and velocity determine a motion's amplitude and phase; displacement alone does not determine its direction. The phase circle represents the same algebra geometrically, while a phase portrait displays both state variables. Neither representation adds a new physical force.

Gravity shifts an ideal vertical spring's equilibrium. It provides the restoring torque for simple and physical pendulums, whose sinusoidal model requires small angles. Linear viscous damping removes energy at the exact rate bv^2 and separates free responses into three regimes. A driven system has both transient and steady contributions. For fixed force amplitude, displacement, velocity, acceleration and absorbed power require distinct response comparisons. Prescribed support motion changes the input model.

A decay record estimates ratios of parameters unless a dimensional scale is known independently. Synthetic data, measurement uncertainty and model discrepancy have different meanings. Numerical integration becomes evidence only through initial-condition, exact-limit and refinement tests. A coupled pair reorganizes the same oscillator physics into independent normal modes; physical coordinates can exchange energy while modal energies remain constant.

Formula summary

Ideal motion and energy, with displacement from equilibrium

$$\ddot{x} + \omega_0^2 x = 0, \quad \omega_0 = \sqrt{k_s/m}, \quad x = x_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t.$$

$$A = \sqrt{x_0^2 + (v_0/\omega_0)^2}, \quad E = \frac{1}{2}mv^2 + \frac{1}{2}k_s x^2 = \frac{1}{2}k_s A^2, \quad |v| = \omega_0 \sqrt{A^2 - x^2}.$$

Angular models, within their stated linear ranges

$$T_{\text{simple}} = 2\pi\sqrt{L/g}, \quad T_{\text{physical}} = 2\pi\sqrt{I_O/(Mgd)}, \quad T_{\text{torsion}} = 2\pi\sqrt{I_O/\kappa_\theta}.$$

Free viscous response and decay identification

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0, \quad \beta = b/(2m), \quad \zeta = \beta/\omega_0, \quad \dot{E} = -bv^2.$$

$$\omega_d = \sqrt{\omega_0^2 - \beta^2} \quad (\beta < \omega_0), \quad \ell = \ln(A_n/A_{n+1}) = \beta T_d \quad (\text{same-sign peaks}).$$

Fixed-force steady response, with $b > 0$

$$A(\Omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\beta\Omega)^2}}, \quad \delta = \text{atan2}(2\beta\Omega, \omega_0^2 - \Omega^2).$$

$$\Omega_{x,\text{max}} = \sqrt{\omega_0^2 - 2\beta^2} \quad (\zeta < 1/\sqrt{2}), \quad \Omega_{V,\text{max}} = \omega_0, \quad Q = \frac{m\omega_0}{b}.$$

Optional symmetric coupled pair

$$q_{\pm} = (x_1 \pm x_2)/\sqrt{2}, \quad \omega_+ = \sqrt{k_s/m}, \quad \omega_- = \sqrt{(k_s + 2k_c)/m}.$$

Before using a formula, identify its coordinate origin, force model, input, observable, initial conditions and approximation. Check signs, dimensions, limiting behavior and the physical meaning of the result. A correctly substituted formula can still answer the wrong model.

15.10 Exercises and problems

These tasks progress from selecting a physical model to evaluating measurements and designing a response. The final group extends graphical, inverse, experimental and computational reasoning. Problems 8, 29–32 and 40 use the optional coupled-oscillator section; Problem 8 is explicitly an advanced conceptual extension. All other tasks use the core material. Use $g = 9.81 \text{ m s}^{-2}$, ideal springs and negligible spring mass unless a statement specifies otherwise. Distinguish angular frequency from frequency, use a coordinate measured from equilibrium, and identify an approximation before using it. Complete solutions follow the statements.

A. Concepts and model selection

- 1. Coordinate consistency.** A mass hangs from an ideal vertical spring. A student writes $m\ddot{x} = mg - k_s x$ but says that $x = 0$ is the static equilibrium. Diagnose the inconsistency and write the correct equation for a coordinate measured from equilibrium. Explain which force has cancelled and why gravity has not been discarded.
- 2. Phase quadrant.** At $t = 0$, an oscillator has $x_0 > 0$ and $v_0 > 0$. In the convention $x = A \cos(\omega_0 t + \phi)$, identify the quadrant of ϕ in $0 \leq \phi < 2\pi$. Explain why $\tan^{-1}[-v_0/(\omega_0 x_0)]$ alone can be ambiguous when used as a general phase-reconstruction rule.
- 3. Auxiliary circular motion.** An auxiliary phase point moves uniformly around a circle while a spring mass moves on a line. Which quantities are projections of the auxiliary motion? Which force from circular motion must *not* be added to the spring-mass model? Explain how the projected acceleration remains consistent with Newton's law.
- 4. Pendulum mass and amplitude.** Two simple pendulums have the same length but bob masses 0.20 kg and 2.0 kg. Compare their small-angle periods. Then explain why a large-amplitude period can differ from the small-angle prediction and whether changing bob mass alone produces that difference in the ideal model.
- 5. A non-crossing decay.** A damped oscillator released from rest at positive displacement returns to equilibrium without crossing it. Is this description compatible with underdamped motion? Explain the compatible regimes and the limitation of inferring a regime from a short experimental record.
- 6. Two definitions of resonance.** For a fixed-amplitude sinusoidal force, a displacement sensor and a velocity sensor report their largest steady amplitudes at different frequencies. Is either sensor necessarily wrong? State the theoretical peak frequencies in terms of ω_0 and β , including the condition for a nonzero displacement peak.
- 7. Peak spacing.** A decay record has alternating positive and negative extrema every 0.40 s. What time interval should be used between successive *same-sign* peaks in the logarithmic-decrement formula? Explain the factor-of-two issue.
- 8. Coupling stiffness (optional advanced).** For two identical coupled oscillators, explain why the in-phase mode does not stretch the coupling spring and why the out-of-phase mode has the higher frequency when $k_c > 0$. Identify the incremental coupling-spring extension in both modes.

B. Ideal oscillators and pendulums

9. A 0.40 kg mass is attached to a 64 N m^{-1} spring. Find ω_0 , T and f . Explain why these predictions require a linear spring and a negligible spring mass.
10. An oscillator has $\omega_0 = 5.00 \text{ rad s}^{-1}$, $x_0 = -0.030 \text{ m}$ and $v_0 = +0.200 \text{ m s}^{-1}$. Find A and a phase ϕ in $0 \leq \phi < 2\pi$. Write $x(t)$ and verify both initial conditions.
11. A 0.50 kg oscillator with $k_s = 72 \text{ N m}^{-1}$ has amplitude 0.080 m. Find its total energy and its speed magnitude when $x = 0.050 \text{ m}$. Explain why the displacement alone does not specify the sign of velocity.
12. A harmonic oscillator has $A = 12 \text{ mm}$ and $f = 4.0 \text{ Hz}$. Find $v_{\max}$ and $a_{\max}$. State where each magnitude is attained and whether both maxima occur simultaneously.
13. Find the small-angle period of a 1.20 m simple pendulum. By what factor does the period change if the length is multiplied by four? State the assumptions that allow the mass-independent expression.
14. A uniform rod of length 0.90 m oscillates through small angles about a horizontal pivot at one end. Derive and evaluate its period. Identify the moment of inertia and the centre-of-mass lever arm rather than treating the rod as a point bob.
15. A torsional oscillator has $I_O = 6.0 \times 10^{-3} \text{ kg m}^2$ and $\kappa_\theta = 0.150 \text{ N m rad}^{-1}$. Find ω_0 and T . Specify the torque law being assumed.
16. A 1.2 kg mass extends a vertical spring by 0.18 m at static equilibrium. Infer k_s and the small-oscillation period. Explain why the static extension changes if g changes while the period does not when k_s and m remain fixed.

C. Damping and forced response

17. A system has $m = 1.5 \text{ kg}$, $k_s = 96 \text{ N m}^{-1}$ and $b = 6.0 \text{ kg s}^{-1}$. Find ω_0 , β , ζ and, if applicable, ω_d . Classify the free response.
18. For $m = 0.80 \text{ kg}$ and $k_s = 50 \text{ N m}^{-1}$, find the critical damping coefficient b_c . Explain why a larger coefficient is not simply a faster oscillatory decay.
19. In a lightly damped oscillator, the displacement envelope falls by a factor e^{-1} after 2.5 s. Find β . By what factor has the squared-amplitude, cycle-averaged energy scale fallen at that time? Distinguish this scale from the instantaneous mechanical energy at an arbitrary phase.
20. For an underdamped response with $\beta = 0.30 \text{ s}^{-1}$, how long does the displacement envelope take to fall from 20 mm to 5.0 mm? Explain why the units used for both amplitudes cancel in the ratio.
21. A fixed-force oscillator has $m = 1.0 \text{ kg}$, $k_s = 100 \text{ N m}^{-1}$, $b = 2.0 \text{ kg s}^{-1}$, $F_0 = 4.0 \text{ N}$ and drive angular frequency $\Omega = 8.0 \text{ rad s}^{-1}$. Find the steady displacement amplitude and phase lag relative to $F_0 \cos \Omega t$. State why the release transient need not share that amplitude or frequency.
22. For the same m, k_s, b, F_0 as Problem 21, find the displacement-response peak frequency and its amplitude. Compare that amplitude with the static deflection F_0/k_s .
23. At what drive angular frequency is the steady velocity amplitude maximal for Problem 21? Evaluate that maximum velocity amplitude. Explain the role of viscous resistance at that frequency.

24. A force-driven oscillator has $\zeta = 0.80$. Does its displacement amplitude have a nonzero-frequency maximum? Does its velocity amplitude? Explain how an underdamped free response can coexist with an absent displacement-resonance peak.

D. Measurement, computation and engineering

25. Same-sign peaks in a synthetic free decay are 9.0 mm and 6.0 mm one damped period 0.70 s apart. Estimate the logarithmic decrement ℓ , β and ω_d . Explain what assumption makes the logarithmic decrement an estimate of a constant viscous damping rate.
26. A 0.60 kg mass has an independently known $k_s = 54 \text{ N m}^{-1}$ and a fitted damping rate $\beta = 0.45 \text{ s}^{-1}$. Infer b and compare ω_d with ω_0 as a percentage difference, using ω_0 as the reference value.
27. A numerical undamped oscillator calculation gives periods 1.0120 s, 1.0030 s and 1.0008 s when the time step is successively halved. The analytical period is 1.0000 s. Calculate the absolute errors and the two error-reduction ratios. Is convergence toward the analytical limit evident? Explain what these data do and do not establish about accuracy.
28. A force-driven proof mass has $\omega_0 = 50 \text{ rad s}^{-1}$ and $\zeta = 0.05$. If only displacement is measured, explain why a sweep must include frequencies slightly below ω_0 and evaluate the predicted peak frequency. If only velocity is measured, where should the peak occur? State one reason this conclusion cannot be transferred unchanged to base excitation.

E. Advanced coupled-oscillator challenges

29. Two equal masses $m = 0.25 \text{ kg}$ have outer springs $k_s = 36 \text{ N m}^{-1}$ and coupling $k_c = 9 \text{ N m}^{-1}$. Find the in-phase and out-of-phase mode frequencies in rad s^{-1} and Hz. Explain the ordering.
30. For Problem 29, let $x_1(0) = 10 \text{ mm}$, $x_2(0) = 0$, with both initial velocities zero. Write $x_1(t)$ and $x_2(t)$ explicitly and check all four initial conditions.
31. Show algebraically that $k_c \rightarrow 0$ makes the two normal frequencies degenerate and that the solution of Problem 30 then leaves mass 2 at rest. Explain the physical meaning and the consequence for a beat-transfer time.
32. For the general initial state $x_1(0) = a$, $x_2(0) = 0$, with zero initial velocities, derive the first time when the slow displacement envelope of coordinate 1 vanishes. Verify that the initial total energy is $\frac{1}{2}(k_s + k_c)a^2$ and remains constant using modal energies. Explain why an envelope zero is not automatically an instant when all energy is assigned to mass 2 alone.

F. Graphical, inverse and engineering synthesis

33. **A local harmonic approximation.** A 0.75 kg slider has potential energy

$$U(y) = (3.0 \text{ N m}^{-1})y^2 + (200 \text{ N m}^{-3})y^4,$$

where y is measured in metres and $U(0) = 0$. Figure 15.32 compares this potential with its quadratic part. (a) Find the equilibrium, assess stability, and derive the small-oscillation stiffness and period. (b) At $|y| = 0.050 \text{ m}$, evaluate both potential energies and the fractional excess relative to the quadratic prediction. (c) Require the cubic contribution to the restoring force to be at most 5% of the linear contribution. Find the allowed displacement range. Explain why the energy-based and force-based approximation tests differ.

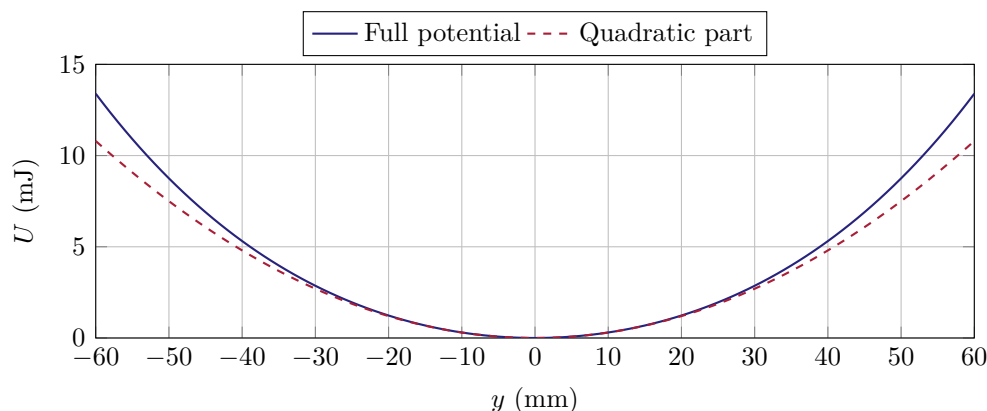


Figure 15.32: Given potential for Problem 33. Both curves use the stated coefficients and common axes; their separation tests the range of a local harmonic model.

- 34. Read a displacement record.** Figure 15.33 is a synthetic undamped record with exactly known amplitude and period. The record starts at $x(0) = +10.0$ mm while increasing; the next positive maximum occurs at $t = 1/3$ s. (a) Read A and T from the graph and reconstruct ω_0 and ϕ in the cosine convention. (b) Find $v(0)$, $a(0)$ and the first positive time when $x = 0$, specifying the crossing direction. (c) For $m = 0.25$ kg, infer k_s , total energy and the kinetic-energy fraction at $t = 0$. Show how the initial state lies on a phase-space ellipse.

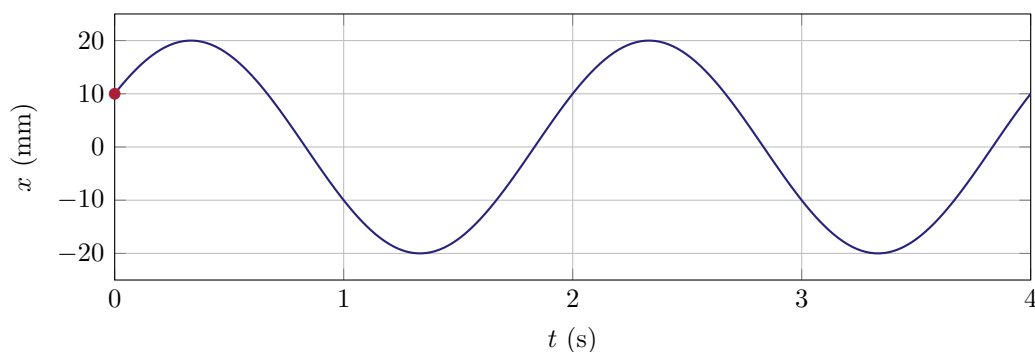


Figure 15.33: Given displacement record for Problem 34. Time, displacement and the increasing initial value are supplied; velocity and acceleration must be reconstructed.

- 35. Uncertainties in inferred quantities.** Independent measurements give $m = (0.500 \pm 0.005)$ kg, $k_s = (72.0 \pm 1.4)$ N m⁻¹ and $A = (0.0800 \pm 0.0010)$ m. The quoted values are standard uncertainties. (a) Calculate ω_0 , T and E . (b) Propagate the uncertainties to these quantities using first-order derivatives and independence. (c) Explain the relation between the fractional uncertainties in ω_0 and T , and state how a correlation between m and k_s would alter the frequency result. Do not assume the uncertainties are worst-case bounds.
- 36. Separate stiffness from sensor zero.** A vertical spring is read by a displacement sensor with an unknown constant offset z_0 . Two static records are $(m, z_{eq}) = (0.40$ kg, 105.4 mm) and (0.80 kg, 170.8 mm); see Figure 15.34. Assume a linear spring, constant g , and $z_{eq} = z_0 + mg/k_s$. (a) Determine k_s and z_0 using both records. (b) Predict the undamped oscillation periods for the two masses. (c) Explain why one static record alone does not determine both unknowns, and why a fitted z_0 need not identify the spring's natural length without geometrical calibration.
- 37. A base-isolated instrument.** A 5.0 kg instrument is mounted on a spring $k_s = 2000$ N m⁻¹ and viscous damper $b = 20$ kg s⁻¹. Its platform moves as $y(t) = Y \cos \Omega t$, where $Y = 2.0$ mm. The absolute mass displacement is x ; the spring and damper sense $x - y$. (a) Derive the

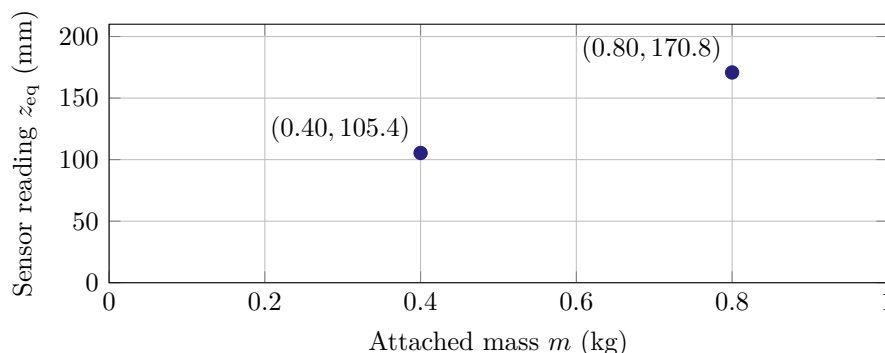


Figure 15.34: Given static calibration points for Problem 36. No fitted slope or intercept is drawn; these are quantities to infer from the spring model.

forced equation for x and obtain the absolute transmissibility $|X/Y|$. (b) Evaluate the absolute amplitude and the relative amplitude $|X - Y|$ at $\Omega = 10$ and 40 rad s^{-1} . Here X, Y are complex amplitudes; subtract them as phasors, not magnitudes. (c) Use Figure 15.35 and the formula to determine the frequency-ratio threshold for $|X/Y| < 1$. Explain what the model says about absolute isolation and mount travel.

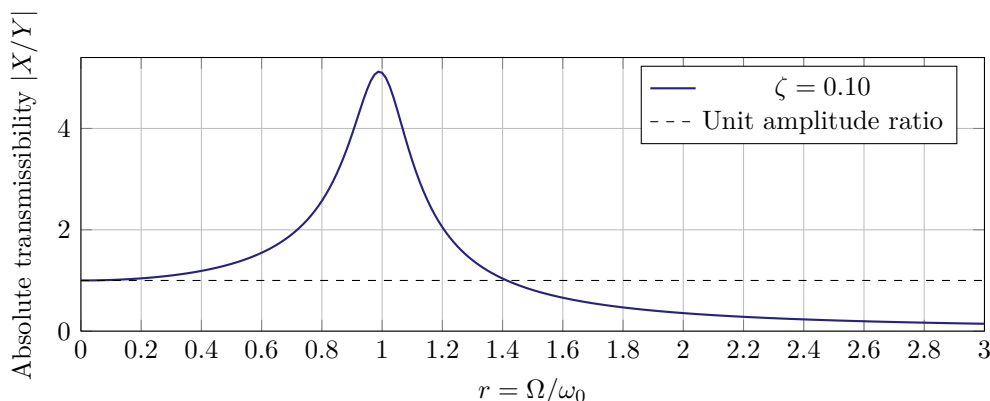


Figure 15.35: Base-displacement response used in Problem 37. The numerator represents the combined spring and damper input; this is a different transfer function from a prescribed external force.

- 38. Test a damping model against a residual.** A synthetic record lists same-sign positive peaks at $t = (0, 0.50, 1.00, 1.50) \text{ s}$ with $x = (12.000, 9.600, 7.680, 6.500) \text{ mm}$, as plotted in Figure 15.36. (a) Use only the first three peaks to estimate β , ω_d , ω_0 and ζ for a viscously damped free oscillator. (b) Predict the fourth peak and compute the observed-minus-predicted residual and its percentage of the prediction. (c) Assess whether these four observations alone prove nonviscous damping. Specify additional information needed for a justified conclusion.
- 39. Numerical energy growth.** An undamped oscillator has $m = 1.0 \text{ kg}$, $k_s = 100 \text{ N m}^{-1}$, $x_0 = 0.020 \text{ m}$ and $v_0 = 0$. A forward-Euler algorithm updates both variables from the old state:

$$x_{n+1} = x_n + hv_n, \quad v_{n+1} = v_n - h\omega_0^2 x_n.$$

- (a) For $h = 0.010 \text{ s}$, calculate the first updated state and compare its energy with the exact initial energy. (b) Derive a general recurrence for $E_n = \frac{1}{2}m(v_n^2 + \omega_0^2 x_n^2)$ and predict E after 1.00 s for $h = 0.010$ and 0.0050 s . (c) Interpret Figure 15.38; explain why an apparently plausible oscillation trace and step-halving alone do not establish long-time energy conservation. Propose a concrete validation test for an improved algorithm.

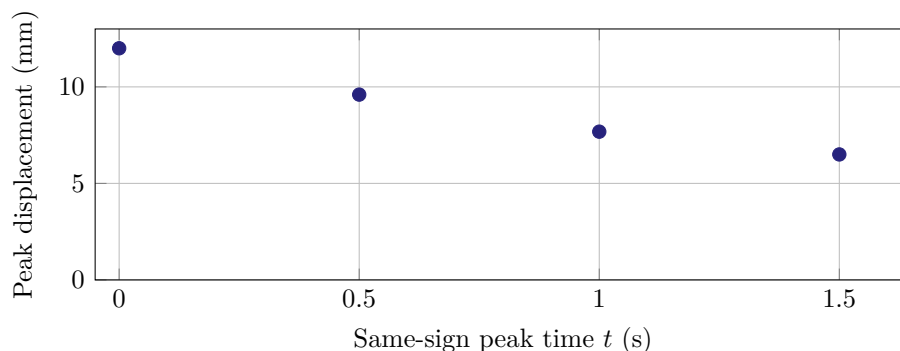


Figure 15.36: Given synthetic peak data for Problem 38. The first three points provide a reference fit; the last point tests that fit. No statistical uncertainty has been supplied.

- 40. Infer a coupling from modal measurements.** In the ideal symmetric two-oscillator model, each mass is $m = 0.20$ kg. Measurements identify an in-phase angular frequency $\omega_+ = 10.0 \text{ rad s}^{-1}$ and an out-of-phase angular frequency with $\omega_-^2 = 150 \text{ s}^{-2}$. (a) Infer the outer stiffness k_s and coupling stiffness k_c . (b) For a release $x_1(0) = a = 15$ mm, $x_2(0) = 0$, zero initial velocities, calculate the first coordinate-1 envelope-zero time and the initial total energy. (c) Explain which assumptions make this inverse problem solvable and give a diagnostic that could reveal unequal masses or unequal outer springs.

15.11 Complete reasoned solutions

The solutions retain the same physical coordinate and input convention as their problems. A numerical answer is accompanied by its model and a check. For the sinusoidal force convention, the steady response is $x = A \cos(\Omega t - \delta)$ with a nonnegative amplitude and lag $0 \leq \delta \leq \pi$. Peak-decay formulas use same-sign extrema. Values displayed below are rounded after evaluating the unrounded expressions.

Solutions 1–8: interpreting the model

1. Coordinate consistency. Take a downward extension y from the spring's natural length. Newton's law is $m\ddot{y} = mg - k_s y$; at equilibrium $y_{\text{eq}} = mg/k_s$. Thus $y = 0$ is not the static equilibrium unless $g = 0$ or a further force has been added. With the displacement from equilibrium $x = y - y_{\text{eq}}$, the constant shift has zero acceleration, so

$$m\ddot{x} = mg - k_s(x + y_{\text{eq}}) = -k_s x.$$

Gravity and the static part of the spring force cancel; gravity is retained in the equilibrium location. The check is $x = 0$, $\dot{x} = 0$: the net force and acceleration both vanish. Reversing the positive coordinate direction changes intermediate signs but still gives a restoring force proportional to $-x$.

2. Phase quadrant. At release $x_0 = A \cos \phi$ and $v_0 = -A\omega_0 \sin \phi$. Positive displacement requires $\cos \phi > 0$; positive velocity requires $\sin \phi < 0$. Hence $3\pi/2 < \phi < 2\pi$, the fourth quadrant. A principal one-argument arctangent usually returns a value only between $-\pi/2$ and $\pi/2$. It therefore requires an explicit quadrant correction even for this state, where its negative value must have 2π added. More generally, a ratio loses the separate signs that distinguish opposite quadrants. The reliable reconstruction is

$$\phi = \text{atan2}(-v_0/\omega_0, x_0) \pmod{2\pi}.$$

The two arguments have the same dimension, length. Checking the separate sine and cosine signs tests a phase choice independently of the tangent ratio.

3. Auxiliary circular motion. A mathematical point with $\mathbf{r} = A(\cos \theta, \sin \theta)$, $\theta = \omega_0 t + \phi$, has x projection $A \cos \theta$. Projecting its velocity and acceleration yields $-A\omega_0 \sin \theta$ and $-A\omega_0^2 \cos \theta = -\omega_0^2 x$. These are precisely the oscillator's x, v, a . The construction does not put the physical mass on a circular track. No additional centripetal force, circular-track normal force or radial tension belongs in its free-body diagram. The real spring alone supplies $F_x = -k_s x$, and $ma_x = -m\omega_0^2 x = -k_s x$ because $m\omega_0^2 = k_s$. The coincidence of projected kinematics is a representation of the same solution, not a second force mechanism.

4. Pendulum mass and amplitude. For a point bob, $I = mL^2$ and the gravitational torque is $-mgL \sin \theta$. Dividing $mL^2 \ddot{\theta} = -mgL \sin \theta$ by the inertia cancels m . With $|\theta| \ll 1$ in radians, $\ddot{\theta} + (g/L)\theta = 0$ and $T = 2\pi\sqrt{L/g}$. Thus the two stated masses have equal ideal small-angle periods; the tenfold mass ratio produces no tenfold or square-root change in period. At large angle the replacement $\sin \theta \simeq \theta$ overestimates the restoring torque magnitude, and the actual period increases with amplitude. That amplitude dependence occurs even with identical masses. Mass-dependent effects would require additional physics, such as air drag, finite-size bobs or different suspension mechanics; none is specified here.

5. A non-crossing decay. For an ideal underdamped oscillator, any nonzero solution is an exponentially decaying sinusoid and ultimately crosses equilibrium repeatedly. Starting from $x(0) = x_0 > 0$, $v(0) = 0$, it can be written

$$x = x_0 e^{-\beta t} \left(\cos \omega_d t + \frac{\beta}{\omega_d} \sin \omega_d t \right).$$

The trigonometric factor has zeros, so a complete return that never crosses excludes this regime. At critical damping, the same release gives $x = x_0(1 + \omega_0 t)e^{-\omega_0 t} > 0$; its derivative is negative for $t > 0$. An overdamped release also returns monotonically for these initial conditions. Thus critical and overdamped motion are compatible. A short measured segment can end before the first underdamped zero or below the sensor resolution. Such a record alone cannot establish that no crossing ever occurs; the inference needs adequate duration and displacement sensitivity.

6. Two definitions of resonance. For $b > 0$, fixed F_0 , and linear viscous damping,

$$A = \frac{F_0/m}{\sqrt{(\omega_0^2 - \Omega^2)^2 + 4\beta^2\Omega^2}}, \quad V = \Omega A.$$

Minimizing the displacement denominator gives a positive-frequency stationary minimum at $\Omega_x = \sqrt{\omega_0^2 - 2\beta^2}$ only if $\beta < \omega_0/\sqrt{2}$. Otherwise the largest displacement is the zero-frequency limit. Maximizing V gives $\Omega_v = \omega_0$ regardless of the positive damping magnitude. Hence different sensor peaks are expected, and neither sensor is necessarily wrong. The distinction follows from weighting the displacement response by Ω to obtain velocity. The undamped limit is singular at resonance: no finite steady amplitude exists there, so a bounded peak formula requires nonzero damping.

7. Peak spacing. A positive extremum and the next negative extremum are separated by half a damped period. Two successive positive extrema, or two successive negative extrema, are separated by the full period

$$T_d = 2(0.40 \text{ s}) = 0.80 \text{ s}.$$

The same-sign decrement is $\ell = \ln(|x_j|/|x_{j+1}|) = \beta T_d$. Using 0.40 s with a same-sign amplitude ratio would double the inferred damping rate, and using it as the full period would double $\omega_d = 2\pi/T_d$. Alternating extrema could also be used with their magnitudes, but then the elapsed time in the logarithmic decay relation is explicitly a half-period, not the same-sign convention.

8. Coupling stiffness. Incremental coupling extension is $x_2 - x_1$. In the in-phase mode $x_1 = x_2$, it vanishes, leaving each mass restored only by its outer spring: $m\ddot{x}_1 = -k_s x_1$. In the out-of-phase mode $x_2 = -x_1$, the extension is $-2x_1$, so the coupling contributes $-2k_c x_1$ to the force on mass 1. The effective stiffness is $k_s + 2k_c$, giving $\omega_-^2 = (k_s + 2k_c)/m > k_s/m = \omega_+^2$ for $k_c > 0$. The sign check is useful: positive x_1 and negative x_2 must make the coupling pull mass 1 toward equilibrium, not away from it.

Solutions 9–16: ideal oscillations

9. Spring frequency, period and frequency in hertz. For a displacement measured from equilibrium, $m\ddot{x} + k_s x = 0$. Its sinusoidal solution requires $\omega_0^2 = k_s/m$, so

$$\omega_0 = \sqrt{64/0.40} = 12.649 \text{ rad s}^{-1}, \quad T = \frac{2\pi}{\omega_0} = 0.49673 \text{ s}, \quad f = \frac{\omega_0}{2\pi} = 2.0132 \text{ Hz}.$$

The stiffness-to-mass ratio has dimension s^{-2} ; angular frequency and frequency differ by 2π , not merely by a unit label. The check $fT = 1$ is exact. A linear force makes the frequency independent of amplitude. A massive spring would contribute distributed kinetic energy and change the effective inertia, so treating all inertia as 0.40 kg requires its mass to be negligible or already included appropriately.

10. Reconstructing the initial state. The initial conditions give $A \cos \phi = x_0$ and $-A\omega_0 \sin \phi = v_0$. Adding their normalized squares eliminates the phase:

$$A = \sqrt{x_0^2 + (v_0/\omega_0)^2} = \sqrt{0.030^2 + 0.040^2} = 0.0500 \text{ m}.$$

Therefore $\cos \phi = -0.600$ and $\sin \phi = -0.800$. Both are negative, selecting quadrant III:

$$\phi = 4.06889 \text{ rad} = 233.130^\circ, \quad x(t) = 0.0500 \cos(5.00t + 4.06889) \text{ m},$$

with t in seconds. Direct substitution gives $x(0) = 0.0500(-0.600) = -0.0300 \text{ m}$ and $v(0) = -0.0500(5.00)(-0.800) = +0.200 \text{ m s}^{-1}$. This check rules out the other quadrant with the same tangent. The scaled velocity $v_0/\omega_0 = 0.0400 \text{ m}$ is larger than $|x_0|$, explaining why the amplitude exceeds the initial displacement magnitude.

11. Energy and speed at a specified displacement. The undamped model conserves $E = \frac{1}{2}mv^2 + \frac{1}{2}k_s x^2$. At a turning point $v = 0$ and $|x| = A$, so

$$E = \frac{1}{2}(72)(0.080)^2 = 0.2304 \text{ J}.$$

At $x = 0.050 \text{ m}$, $U = \frac{1}{2}(72)(0.050)^2 = 0.0900 \text{ J}$ and $K = 0.1404 \text{ J}$. Consequently

$$|v| = \sqrt{\frac{2K}{m}} = \sqrt{\frac{72}{0.50}(0.080^2 - 0.050^2)} = 0.7494 \text{ m s}^{-1}.$$

The allowed displacement satisfies $|x| \leq A$, so the radicand is positive. The same displacement is visited while moving toward and away from the positive turning point; those visits have opposite velocity signs but the same kinetic energy. At $|x| = A$ this expression returns zero, and at $x = 0$ it returns the maximum speed, providing two independent limit checks.

12. Kinematic maxima. Convert the amplitude before substitution: $A = 0.012 \text{ m}$ and $\omega_0 = 2\pi f = 8\pi \text{ rad s}^{-1}$. Differentiating the displacement sinusoid gives $v = -A\omega_0 \sin(\omega_0 t + \phi)$ and $a = -A\omega_0^2 \cos(\omega_0 t + \phi)$. Hence

$$v_{\max} = A\omega_0 = 0.3016 \text{ m s}^{-1}, \quad a_{\max} = A\omega_0^2 = 7.580 \text{ m s}^{-2}.$$

Speed is largest when $x = 0$, whereas acceleration magnitude is largest at $|x| = A$, where the mass instantaneously stops. Their maxima are separated by a quarter-period and do not coincide. This explains why using $a = v^2/x$ is inappropriate here: that centripetal expression describes a different geometric motion and would be undefined at the oscillator's maximum-speed position.

13. Simple-pendulum scaling. For a point bob on a light inextensible suspension, a fixed frictionless pivot, uniform g and small angles, the angular equation is $\ddot{\theta} + (g/L)\theta = 0$. Therefore

$$T = 2\pi\sqrt{\frac{1.20}{9.81}} = 2.19754 \text{ s.}$$

Length multiplied by four gives $T' = 2\pi\sqrt{4L/g} = 2T = 4.39507 \text{ s}$. Thus the requested factor is two, derived before numerical calculation. Dimensions also support the square root: L/g has dimension time squared. The cancellation of bob mass assumes gravitational and inertial mass enter the ideal equation in the same way; damping, finite suspension elasticity or a large angle would require a modified model.

14. A uniform-rod physical pendulum. Let the rod mass be M , which is not required numerically. About its end $I_O = ML^2/3$, while its centre of mass is at $d = L/2$. The restoring torque is $-Mgd\sin\theta \simeq -Mgd\theta$, so

$$I_O\ddot{\theta} + Mgd\theta = 0, \quad T = 2\pi\sqrt{\frac{I_O}{Mgd}} = 2\pi\sqrt{\frac{2L}{3g}}.$$

With $L = 0.90 \text{ m}$,

$$T = 2\pi\sqrt{\frac{1.80}{29.43}} = 1.55389 \text{ s.}$$

The rod is equivalent in period to a simple pendulum of length $I_O/(Md) = 2L/3 = 0.60 \text{ m}$, not L or $L/2$. The mass cancels because both its rotational inertia and gravitational torque scale with M . The derivation assumes a rigid uniform rod and a horizontal pivot axis; a different axis or a nonuniform mass distribution changes I_O and possibly d .

15. Torsional stiffness. A linear restoring torque obeys $\tau = -\kappa_\theta\theta$, with angle in radians and a fixed rotational inertia about the suspension axis. Rotational dynamics gives $I_O\ddot{\theta} + \kappa_\theta\theta = 0$, directly analogous to the translational spring equation. Thus

$$\omega_0 = \sqrt{\frac{0.150}{6.0 \times 10^{-3}}} = 5.00 \text{ rad s}^{-1}, \quad T = \frac{2\pi}{5.00} = 1.25664 \text{ s.}$$

The ratio of torque stiffness to rotational inertia has dimension s^{-2} , with radians dimensionless in dimensional analysis. A longer period would result from a greater inertia or smaller torque stiffness. The calculation assumes small enough twists for the linear torque law and neglects damping; a torsion fibre driven beyond its elastic range would violate the stated law.

16. Static extension and dynamic stiffness. With downward natural-length extension y , equilibrium requires $k_s y_{\text{eq}} = mg$. The measured extension gives

$$k_s = \frac{(1.2)(9.81)}{0.18} = 65.4 \text{ N m}^{-1}.$$

Around that equilibrium the perturbation equation is $m\ddot{x} + k_s x = 0$, so

$$T = 2\pi\sqrt{\frac{1.2}{65.4}} = 2\pi\sqrt{\frac{0.18}{9.81}} = 0.85110 \text{ s.}$$

The second expression uses the particular measured Earth-gravity extension to determine the stiffness; it does not make gravity a dynamic restoring term. If g changes with the spring and mass unchanged, $y_{\text{eq}} = mg/k_s$ changes but $T = 2\pi\sqrt{m/k_s}$ remains fixed. Substituting the *new* extension into $2\pi\sqrt{y_{\text{eq}}/g}$ gives that same period, resolving the apparent contradiction.

Solutions 17–24: damping and driven response

17. Classifying a damped oscillator. The viscous model is $m\ddot{x} + b\dot{x} + k_s x = 0$, or $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$. Its parameters are

$$\omega_0 = \sqrt{96/1.5} = 8.00 \text{ rad s}^{-1}, \quad \beta = \frac{6.0}{2(1.5)} = 2.00 \text{ s}^{-1}, \quad \zeta = \beta/\omega_0 = 0.250.$$

Since $\zeta < 1$, the roots are complex and the motion is underdamped. The free oscillation angular frequency is

$$\omega_d = \sqrt{\omega_0^2 - \beta^2} = \sqrt{60} = 7.74597 \text{ rad s}^{-1}.$$

It is lower than ω_0 , as damping should make it in this model. The coefficient b has dimension mass/time, while β is a rate; confusing the two would produce an incorrect regime. At $b \rightarrow 0$, the formula returns $\omega_d \rightarrow \omega_0$.

18. Critical damping coefficient. Critical damping is the repeated-root condition $\beta = \omega_0$, so

$$b_c = 2m\omega_0 = 2\sqrt{mk_s} = 2\sqrt{(0.80)(50)} = 12.6491 \text{ kg s}^{-1}.$$

The product mk_s has dimension mass squared/time squared, yielding the correct viscous-coefficient unit after the square root. A coefficient larger than b_c makes the roots real and distinct; it removes oscillations rather than merely shortening their envelope. In the strongly overdamped limit the slow root is approximately $-k_s/b$, whose magnitude decreases as b increases. Thus increasing damping beyond critical can slow the final return. This statement compares systems with the same m, k_s and appropriate initial conditions, not arbitrary driven transients.

19. Amplitude and energy scales. An underdamped displacement envelope is $A(t) = A_0 e^{-\beta t}$. The stated e^{-1} reduction at 2.5 s gives $\beta(2.5) = 1$, hence

$$\beta = 0.400 \text{ s}^{-1}, \quad \frac{A^2(2.5)}{A_0^2} = e^{-2} = 0.135335.$$

For light damping, the cycle-averaged energy scale is proportional to A^2 and therefore falls to about 13.53% of its initial value. Instantaneous energy is $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}k_s x^2$, and differentiating it yields $\dot{E} = -b\dot{x}^2 \leq 0$. Its rate varies within a cycle, so an arbitrary instantaneous ratio need not equal the envelope-squared factor exactly. The qualification prevents substituting an amplitude decay law for a complete phase-resolved energy calculation.

20. Time to a prescribed envelope. Use the amplitude ratio, not the instantaneous displacement, because an instantaneous zero occurs much sooner without eliminating the oscillation. The model gives

$$\frac{5.0}{20} = e^{-0.30t}, \quad t = \frac{\ln(20/5.0)}{0.30} = 4.62098 \text{ s}.$$

Millimetres cancel inside the logarithm; its argument must be dimensionless. The result is positive because the final envelope is smaller than the initial one. It represents two successive halvings: each takes $\ln 2/\beta$, so their sum gives $\ln 4/\beta$. A model with amplitude-dependent damping could not use one constant β over this range without independent evidence.

21. Steady amplitude and force-relative phase. For a prescribed force $F_0 \cos \Omega t$, the complex dynamic stiffness is $k_s - m\Omega^2 + ib\Omega$. Thus

$$A = \frac{F_0}{\sqrt{(k_s - m\Omega^2)^2 + (b\Omega)^2}}, \quad \delta = \text{atan2}(b\Omega, k_s - m\Omega^2).$$

The two stiffness components are $100 - 1(8)^2 = 36 \text{ N m}^{-1}$ and $2(8) = 16 \text{ N m}^{-1}$, giving

$$A = \frac{4}{\sqrt{36^2 + 16^2}} = 0.101535 \text{ m}, \quad \delta = \text{atan2}(16, 36) = 0.418224 \text{ rad} = 23.9625^\circ.$$

Since $\Omega < \omega_0 = 10 \text{ rad s}^{-1}$, the lag is less than $\pi/2$, agreeing with the signs of the dynamic stiffness. These values describe the particular steady solution. A release generally adds a homogeneous, damped free response fixed by the initial conditions; it decays at rate $\beta = 1 \text{ s}^{-1}$ and oscillates at ω_d , not necessarily at the drive frequency.

22. Displacement-response maximum. For the stated system $\omega_0 = 10 \text{ rad s}^{-1}$ and $\beta = b/(2m) = 1 \text{ s}^{-1}$. The amplitude denominator squared is

$$D(\Omega) = (k_s - m\Omega^2)^2 + b^2\Omega^2.$$

For $\Omega > 0$, $D' = 0$ requires $\Omega_x^2 = k_s/m - b^2/(2m^2) = \omega_0^2 - 2\beta^2 = 98 \text{ s}^{-2}$. The condition for this minimum holds because $\beta/\omega_0 = 0.1 < 1/\sqrt{2}$. Therefore

$$\Omega_x = 9.89949 \text{ rad s}^{-1}, \quad A_{\max} = \frac{4}{\sqrt{(100 - 98)^2 + 4(98)}} = 0.201008 \text{ m}.$$

The static deflection is $F_0/k_s = 0.0400 \text{ m}$; the dynamic-to-static ratio is 5.02519. The peak lies slightly below the undamped natural frequency. Its large amplitude is a prediction of the ideal linear model; applying it to hardware would require checking spring linearity and available travel at approximately 0.20 m.

23. Velocity-response maximum. The velocity amplitude is $V = \Omega A$. Dividing its denominator by Ω^2 gives, for positive Ω ,

$$V = \frac{F_0}{\sqrt{(k_s/\Omega - m\Omega)^2 + b^2}}.$$

The nonnegative square is minimized to zero when $\Omega^2 = k_s/m$. Hence

$$\Omega_v = 10.0 \text{ rad s}^{-1}, \quad V_{\max} = F_0/b = 2.00 \text{ m s}^{-1}.$$

At this frequency spring and inertial terms cancel in the sinusoidal balance; the applied force is in phase with velocity and balances the viscous term. The displacement amplitude there is $V/\Omega_v = 0.200 \text{ m}$, slightly below Problem 22's displacement peak. The finite result requires $b > 0$. Letting b vanish makes it diverge, consistent with the absence of a bounded undamped resonant steady state.

24. Underdamped does not imply a displacement peak. A positive-frequency displacement maximum requires $\zeta < 1/\sqrt{2} \simeq 0.7071$. Since 0.80 exceeds this threshold, the displacement amplitude decreases from its maximum static value as drive frequency increases; there is no nonzero-frequency displacement peak. Velocity still peaks at $\Omega = \omega_0$ for fixed force and positive viscous damping. Meanwhile free motion is underdamped because its condition is only $\zeta < 1$: $\omega_d = \omega_0\sqrt{1 - 0.80^2} = 0.60\omega_0$. These are different tests of different responses. One classifies the roots of the free equation; the other locates a maximum of a steady frequency-response observable. Applying the free-motion threshold to displacement resonance would conflate them.

Solutions 25–28: measurements and computation

25. Same-sign logarithmic decrement. The synthetic data assume constant linear viscous damping, so successive same-sign peaks share a phase and their ratio is the exponential envelope ratio. Therefore

$$\ell = \ln \frac{9.0}{6.0} = 0.405465, \quad \beta = \frac{\ell}{0.70} = 0.579236 \text{ s}^{-1}, \quad \omega_d = \frac{2\pi}{0.70} = 8.97598 \text{ rad s}^{-1}.$$

The logarithm is dimensionless and the damping rate has inverse-time units. The peak spacing directly determines the damped frequency, while the amplitude ratio determines the damping rate. Treating the given spacing as a half-period would contradict the same-sign condition. For a real experiment, several peak pairs should be inspected: nearly constant decrements support this model, whereas systematic dependence on amplitude suggests additional physics or measurement bias. Synthetic consistency alone does not validate a real apparatus.

26. Inferring the viscous coefficient. From $\beta = b/(2m)$,

$$b = 2(0.60)(0.45) = 0.540 \text{ kg s}^{-1}.$$

The independently known stiffness fixes $\omega_0 = \sqrt{54/0.60} = 9.486833 \text{ rad s}^{-1}$. Since $\beta < \omega_0$, the free motion is underdamped and

$$\omega_d = \sqrt{90 - 0.45^2} = 9.476154 \text{ rad s}^{-1}.$$

Using the requested undamped reference,

$$100 \frac{\omega_0 - \omega_d}{\omega_0} = 0.112563\%.$$

The damped frequency is slightly smaller. An independent expansion gives $1 - \sqrt{1 - (\beta/\omega_0)^2} \simeq \frac{1}{2}(\beta/\omega_0)^2$, which predicts 0.1125% here and checks the small numerical difference. Comparing rounded frequencies prematurely can lose significant digits in this difference.

27. Error reduction under refinement. Subtract the exact period from each computed value and take magnitudes:

$$(e_1, e_2, e_3) = (0.0120, 0.0030, 0.0008) \text{ s}.$$

The two reductions are $e_1/e_2 = 4.00$ and $e_2/e_3 = 3.75$. Both successive values approach 1.0000 s, so convergence in this observable is evident. If the leading error behaves as Ch^p , halving h reduces it by 2^p ; the observed ratios correspond to $p = 2.00$ and approximately 1.907. Thus these data are consistent with roughly second-order period error over the tested range, allowing for the reported rounding. They do not prove the method's general order, conservation of energy, or accuracy of amplitude and phase over long times. Those properties need separate diagnostics against the analytic solution and a specified integration method.

28. A force-driven proof mass. For a prescribed force of fixed amplitude, the displacement peak occurs at

$$\Omega_x = \omega_0 \sqrt{1 - 2\zeta^2} = 50 \sqrt{1 - 2(0.05)^2} = 49.8748 \text{ rad s}^{-1}.$$

A sweep that begins at 50 rad s^{-1} misses the predicted peak on its low-frequency side. The velocity peak is at exactly $\Omega_v = \omega_0 = 50.0 \text{ rad s}^{-1}$ within this model. Practical frequency sampling should be fine enough to resolve the response width; a mathematical peak location does not ensure that coarse data will identify it. With base excitation, the input arises through spring and damping forces proportional to support displacement and support velocity. Its transfer-function numerator therefore depends on frequency and on the chosen absolute or relative output. The fixed-force peak formula cannot be reused without deriving that new response.

Solutions 29–32: normal modes and envelopes

29. Normal-mode frequencies. Identical masses and outer springs admit eigenvectors $(1, 1)$ and $(1, -1)$. In phase the coupling is unstretched, while out of phase it adds twice its stiffness to each restoring force. Thus

$$\omega_+ = \sqrt{\frac{36}{0.25}} = 12.000 \text{ rad s}^{-1}, \quad \omega_- = \sqrt{\frac{36 + 18}{0.25}} = 14.69694 \text{ rad s}^{-1}.$$

Dividing each angular frequency by 2π gives $f_+ = 1.90986$ Hz and $f_- = 2.33909$ Hz. The squared difference is $\omega_-^2 - \omega_+^2 = 2k_c/m = 72\text{ s}^{-2} > 0$, checking the ordering without rounded subtraction. Both stiffness eigenvalues are positive, so both modes are stable. A negative inferred coupling would require questioning either the physical model or the mode identification.

30. Releasing only one mass. Use orthonormal coordinates $q_+ = (x_1 + x_2)/\sqrt{2}$ and $q_- = (x_1 - x_2)/\sqrt{2}$. At release both modal displacements are $a/\sqrt{2}$ with $a = 0.010$ m, and both modal velocities vanish. Therefore $q_{\pm} = (a/\sqrt{2}) \cos \omega_{\pm} t$. Transforming back gives

$$\begin{aligned} x_1(t) &= 0.005[\cos(12.000t) + \cos(14.69694t)] \text{ m}, \\ x_2(t) &= 0.005[\cos(12.000t) - \cos(14.69694t)] \text{ m}. \end{aligned}$$

Time is in seconds, so each trigonometric argument is dimensionless. At $t = 0$ the cosine sum is two and the difference zero; hence $x_1 = 0.010$ m and $x_2 = 0$. Differentiation produces sine factors, all zero at release, checking both velocities. This initial pattern is neither eigenvector alone. Equal modal coordinate amplitudes therefore occur even though their modal energies differ because the two frequencies differ.

31. Vanishing coupling. With $k_c \rightarrow 0$, $\omega_-^2 = (k_s + 2k_c)/m \rightarrow k_s/m = \omega_+^2$. Since frequencies are positive, both tend to $\sqrt{k_s/m}$. Problem 30 then gives

$$x_1 = a \cos(\sqrt{k_s/m} t), \quad x_2 = 0$$

for every time, not merely at release. The second mass has no force from the first, so its zero displacement and velocity remain an exact solution. The mode eigenvalues are degenerate; their basis is then nonunique, but the physical initial-value solution remains unique. The transfer time $\pi/(\omega_- - \omega_+)$ tends to infinity, expressing the absence of a finite coupling-mediated transfer pathway rather than a failure of the initial conditions.

32. A displacement envelope and the total energy. The cosine sum identity yields

$$x_1 = a \cos(\bar{\omega}t) \cos(\Delta\omega t/2), \quad \bar{\omega} = (\omega_+ + \omega_-)/2, \quad \Delta\omega = \omega_- - \omega_+ > 0.$$

The first zero of the slow envelope is $\Delta\omega t/2 = \pi/2$, giving $t_{\text{tr}} = \pi/\Delta\omega$. This is distinct from earlier carrier zeros. Initially both velocities vanish, the first outer spring stores $\frac{1}{2}k_s a^2$, the second stores zero, and the coupling stores $\frac{1}{2}k_c a^2$. Hence

$$E(0) = \frac{1}{2}(k_s + k_c)a^2.$$

Each orthonormal mode has constant energy $E_{\pm} = \frac{1}{2}m(\dot{q}_{\pm}^2 + \omega_{\pm}^2 q_{\pm}^2) = \frac{1}{4}m\omega_{\pm}^2 a^2$. Their sum is

$$E = \frac{1}{4}m(\omega_+^2 + \omega_-^2)a^2 = \frac{1}{4}[k_s + (k_s + 2k_c)]a^2 = \frac{1}{2}(k_s + k_c)a^2,$$

proving conservation independently of the physical-coordinate beat appearance. At t_{tr} the coordinate x_1 vanishes, but differentiation of its product gives

$$\dot{x}_1(t_{\text{tr}}) = -\frac{a\Delta\omega}{2} \cos(\bar{\omega}t_{\text{tr}}),$$

which is generally nonzero. Coupling energy can also remain. Thus the envelope statement concerns displacement modulation; it is not an exact assertion that every term of the total energy belongs to mass 2 at that instant. In weak coupling the residual first-mass kinetic contribution is small, which explains the useful approximate energy-transfer language.

Solutions 33–40: graphical and applied synthesis

33. Curvature, energy and force tests. (a) The force is $F = -U' = -6y - 800y^3$ in SI units. Equilibrium requires $y(6 + 800y^2) = 0$, whose only real solution is $y = 0$. Its curvature $U''(0) = 6.0 \text{ N m}^{-1} > 0$ makes it stable. Keeping the leading restoring term gives $k_s = 6.0 \text{ N m}^{-1}$, $\omega_0 = \sqrt{6/0.75} = \sqrt{8} \text{ rad s}^{-1}$ and

$$T = \frac{2\pi}{\sqrt{8}} = 2.22144 \text{ s}.$$

This is the limiting small-amplitude period, not an exact period at any amplitude. Figure 15.32 shows increasing departure from its quadratic approximation away from the minimum.

(b) At $|y| = 0.050 \text{ m}$, the quadratic energy is $3(0.050)^2 = 0.00750 \text{ J}$ and the quartic contribution is $200(0.050)^4 = 0.00125 \text{ J}$. The total is 0.00875 J , exceeding the quadratic value by $0.00125/0.00750 = 0.166667$, or 16.67%. Both signs of displacement give the same answer because U is even.

(c) For nonzero y , the ratio of cubic to linear force magnitudes is $800|y|^3/(6|y|) = 800y^2/6$. Requiring this to be at most 0.05 gives

$$|y| \leq \sqrt{\frac{0.05(6)}{800}} = 0.0193649 \text{ m} = 19.365 \text{ mm}.$$

At zero the ratio is understood by its continuous limit. Differentiation multiplies the quartic and quadratic terms by four and two respectively; therefore the force correction is twice the relative energy correction at the same displacement. A visually close energy curve does not alone guarantee equally accurate acceleration or period.

34. Reconstructing motion from a graph. (a) The extrema are $\pm 20 \text{ mm}$ and successive positive maxima are two seconds apart: $A = 0.0200 \text{ m}$, $T = 2.00 \text{ s}$, $\omega_0 = \pi \text{ rad s}^{-1}$. The initial value gives $\cos \phi = 1/2$ and the increasing trace requires $\sin \phi < 0$. Thus $\phi = 5\pi/3$ modulo 2π , equivalent to $-\pi/3$. The reconstructed motion is $x = 0.0200 \cos(\pi t - \pi/3) \text{ m}$. Its first positive maximum at $t = 1/3 \text{ s}$ agrees with the supplied graph information.

(b) Differentiating gives

$$v(0) = 0.0200\pi \frac{\sqrt{3}}{2} = 0.0544140 \text{ m s}^{-1}, \quad a(0) = -\pi^2(0.0100) = -0.0986960 \text{ m s}^{-2}.$$

The first zero after the initial maximum has $\pi t - \pi/3 = \pi/2$, hence $t = 5/6 \text{ s} = 0.833333 \text{ s}$. There $v = -A\omega_0 < 0$, so the crossing is toward negative displacement. A later zero with positive velocity does not answer the first-crossing question.

(c) The mass fixes $k_s = m\omega_0^2 = 0.25\pi^2 = 2.46740 \text{ N m}^{-1}$. The conserved energy is

$$E = \frac{1}{2}k_s A^2 = 4.93480 \times 10^{-4} \text{ J}.$$

At release $U/E = (x_0/A)^2 = 1/4$, so $K/E = 3/4$ and $K = 3.70110 \times 10^{-4} \text{ J}$. The phase-space identity is $(x/A)^2 + [v/(A\omega_0)]^2 = 1$; the initial normalized state is $(1/2, \sqrt{3}/2)$. Figure 15.37 exposes the required velocity sign and the energy fractions through this geometry.

35. Propagating standard uncertainties. (a) The nominal values give $\omega_0 = \sqrt{72/0.500} = 12.000 \text{ rad s}^{-1}$, $T = 2\pi/12 = 0.523599 \text{ s}$ and $E = \frac{1}{2}(72)(0.0800)^2 = 0.230400 \text{ J}$. These use a linear, conservative oscillator; the uncertainty calculation does not remove possible model error.

(b) For independent input standard uncertainties, $u_q^2 = \sum_i (\partial q / \partial a_i)^2 u_{a_i}^2$. Since $\omega_0 \propto k_s^{1/2} m^{-1/2}$,

$$\left(\frac{u_\omega}{\omega_0}\right)^2 = \frac{1}{4} \left[\left(\frac{1.4}{72}\right)^2 + \left(\frac{0.005}{0.500}\right)^2 \right].$$

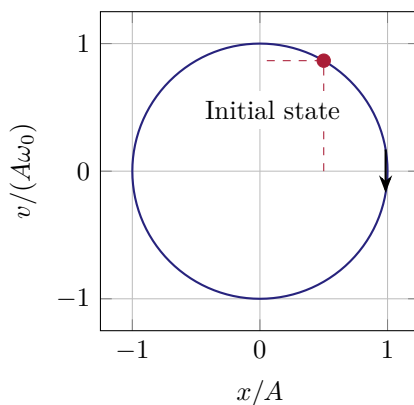


Figure 15.37: Solution representation for Problem 34. The normalized phase ellipse is a circle; the initial point fixes both signs. Time advances clockwise because $\dot{x} = v$ and $\dot{v} = -\omega_0^2 x$.

The fractional standard uncertainty is 0.0109326. Its dimensional consequences are

$$u_\omega = \omega_0(0.0109326) = 0.131191 \text{ rad s}^{-1},$$

$$u_T = T(0.0109326) = 0.005724 \text{ s}.$$

For $E \propto k_s A^2$,

$$\left(\frac{u_E}{E}\right)^2 = \left(\frac{1.4}{72}\right)^2 + 4 \left(\frac{0.0010}{0.0800}\right)^2,$$

so $u_E = 0.007297 \text{ J}$. Reasonable rounded reports are $\omega_0 = (12.00 \pm 0.13) \text{ rad s}^{-1}$, $T = (0.5236 \pm 0.0057) \text{ s}$ and $E = (0.2304 \pm 0.0073) \text{ J}$, where the uncertainties are standard uncertainties rather than guaranteed intervals.

(c) Because $T = 2\pi/\omega_0$, first-order differentiation gives equal fractional uncertainty magnitudes in period and angular frequency; their inferred fluctuations have opposite signs. If m and k_s are correlated, the frequency variance includes a covariance term:

$$\left(\frac{u_\omega}{\omega_0}\right)^2 = \frac{1}{4} \left[\frac{u_k^2}{k_s^2} + \frac{u_m^2}{m^2} - \frac{2 \text{Cov}(k_s, m)}{k_s m} \right].$$

Positive correlation can reduce the uncertainty of the ratio, while negative correlation can increase it. Adding absolute fractional errors would instead implement a different, conservative bounds calculation, not the stated independent-standard-uncertainty model.

36. An inverse calibration with an offset. (a) Subtract the two static equations to eliminate the unknown offset:

$$\Delta z = \frac{g \Delta m}{k_s}, \quad k_s = \frac{9.81(0.80 - 0.40)}{0.1708 - 0.1054} = 60.0 \text{ N m}^{-1}.$$

The offset from either record is $z_0 = 0.1054 - 0.40(9.81)/60 = 0.0400 \text{ m} = 40.0 \text{ mm}$. Using the second record gives $0.1708 - 0.80(9.81)/60 = 0.0400 \text{ m}$ again. The slope has dimension length/mass, and its reciprocal multiplied by g gives stiffness, checking the units.

(b) Dynamic displacements are measured from each equilibrium, so the constant sensor offset cancels. The predicted periods are

$$T(0.40) = 2\pi\sqrt{0.40/60} = 0.513020 \text{ s}, \quad T(0.80) = 2\pi\sqrt{0.80/60} = 0.725520 \text{ s}.$$

Their ratio is $\sqrt{2}$, because the mass doubles at fixed stiffness. A dynamic measurement could test the static stiffness inference and the negligible spring-mass assumption.

(c) One record supplies one equation for two unknowns; choosing any admissible k_s gives a corresponding $z_0 = z_{\text{eq}} - mg/k_s$. Two different masses remove that degeneracy within the linear model. However, z_0 combines instrument zero and the chosen coordinate reference. It does not identify a physical natural length without knowing where the sensor measures, its sign convention and the spring geometry. More loads would allow residuals to test linearity; two points alone always define a line and cannot establish that the spring is linear over a broad range.

37. Absolute isolation and relative travel. (a) The spring and damper exert $-k_s(x - y)$ and $-b(\dot{x} - \dot{y})$ on the instrument. Hence

$$m\ddot{x} + b\dot{x} + k_s x = b\dot{y} + k_s y.$$

With complex amplitudes in the $e^{i\Omega t}$ convention,

$$\frac{X}{Y} = \frac{k_s + ib\Omega}{k_s - m\Omega^2 + ib\Omega}, \quad \left| \frac{X}{Y} \right| = \frac{\sqrt{k_s^2 + (b\Omega)^2}}{\sqrt{(k_s - m\Omega^2)^2 + (b\Omega)^2}}.$$

The numerator differs from a fixed-force transfer function because platform velocity also drives the damper. Here $\omega_0 = \sqrt{2000/5} = 20 \text{ rad s}^{-1}$ and $\zeta = b/(2m\omega_0) = 0.10$.

(b) The relative phasor is

$$\frac{X - Y}{Y} = \frac{m\Omega^2}{k_s - m\Omega^2 + ib\Omega}.$$

At $\Omega = 10 \text{ rad s}^{-1}$, the denominator magnitude is $\sqrt{1500^2 + 200^2} = 1513.27 \text{ N m}^{-1}$. This gives $|X/Y| = 1.32823$, absolute amplitude $|X| = 2.65646 \text{ mm}$ and relative amplitude $|X - Y| = 0.66082 \text{ mm}$. At $\Omega = 40 \text{ rad s}^{-1}$, the denominator magnitude is $\sqrt{(-6000)^2 + 800^2} = 6053.10 \text{ N m}^{-1}$, giving $|X/Y| = 0.35586$, $|X| = 0.71172 \text{ mm}$ and $|X - Y| = 2.64327 \text{ mm}$. In particular, $|X - Y|$ is not $||X| - |Y||$: displacement phases matter.

(c) Put $r = \Omega/\omega_0$. Squaring the positive amplitude ratio shows

$$|X/Y|^2 < 1 \iff 1 + 4\zeta^2 r^2 < (1 - r^2)^2 + 4\zeta^2 r^2.$$

For positive r , this reduces to $r^2 > 2$, so the threshold is $r > \sqrt{2}$ or $\Omega > 28.2843 \text{ rad s}^{-1}$. The two damping terms cancel in this threshold test. Figure 15.35 shows amplification near the natural frequency and isolation farther above it. At high frequency the mass moves little in the laboratory frame, while the mount still follows the platform relative to the mass. Thus reduced absolute vibration can coexist with substantial spring travel; both amplitudes must inform mechanical design.

38. A fitted decay and a withheld point. (a) The first two same-sign ratios are $12/9.6 = 9.6/7.68 = 1.25$. Their equal spacings give

$$\ell = \ln 1.25 = 0.223144, \quad \beta = \ell/0.50 = 0.446287 \text{ s}^{-1}, \quad \omega_d = 2\pi/0.50 = 12.5664 \text{ rad s}^{-1}.$$

Using the viscous model relation, $\omega_0 = \sqrt{\omega_d^2 + \beta^2} = 12.5743 \text{ rad s}^{-1}$ and $\zeta = \beta/\omega_0 = 0.035492$. The frequency from peak timing is damped; calling it ω_0 without this correction would mix measured and inferred quantities.

(b) The fitted envelope is $A(t) = 12.000e^{-\beta t} \text{ mm}$. It predicts at 1.50 s

$$A_{\text{pred}} = 12(0.8)^3 = 6.144 \text{ mm}.$$

The withheld point has residual $6.500 - 6.144 = +0.356 \text{ mm}$, or $100(0.356/6.144) = 5.79427\%$ of the prediction. The positive residual means this point is larger than the exponential extrapolation, not that damping has necessarily become negative.

(c) No measurement uncertainties, timing uncertainties, background drift or sensor calibration have been specified. Therefore four points alone do not prove a nonviscous mechanism. A defensible test would collect more peaks over a range of amplitudes, estimate standard uncertainties, fit all relevant points, and inspect normalized residuals and decrement trends. Possible causes include amplitude-dependent friction, external forcing, sensor offset or a single erroneous point. The assessment criterion is a supported comparison of models and uncertainties, rather than a binary judgment based on one residual.

39. Forward Euler and artificial energy. (a) Here $\omega_0 = 10 \text{ rad s}^{-1}$ and the exact initial energy is $E_0 = \frac{1}{2}(100)(0.020)^2 = 0.0200 \text{ J}$. The first update gives

$$x_1 = 0.0200 \text{ m}, \quad v_1 = -0.010(100)(0.0200) = -0.0200 \text{ m s}^{-1}.$$

Thus $E_1 = \frac{1}{2}(1)(0.0200)^2 + \frac{1}{2}(100)(0.0200)^2 = 0.0202 \text{ J}$, a 1.00% increase. The physical oscillator has no energy source; this increase is a numerical artifact.

(b) Substitute both simultaneous updates into the energy:

$$\begin{aligned} 2E_{n+1}/m &= (v_n - h\omega_0^2 x_n)^2 + \omega_0^2 (x_n + hv_n)^2 \\ &= (1 + h^2\omega_0^2)(v_n^2 + \omega_0^2 x_n^2). \end{aligned}$$

The cross terms cancel exactly, so $E_n = E_0(1 + h^2\omega_0^2)^n$. After 1.00 s, the first step size gives $n = 100$ and

$$E = 0.0200(1.01)^{100} = 0.0540963 \text{ J}.$$

The halved step gives $n = 200$ and $E = 0.0200(1.0025)^{200} = 0.0329539 \text{ J}$. Both exceed the exact 0.0200 J; reducing the step improves the finite-time result but does not eliminate the amplification factor above one for any positive h .

(c) Figure 15.38 compares the energy diagnostic directly with the exact conserved value. A visually oscillatory trace can conceal a slowly growing amplitude, and period convergence alone does not test energy. One validation test is to integrate for many periods with successively halved steps, compare both $x(t)$ and $v(t)$ against the analytic initial-value solution, and track maximum relative energy error over the full interval. A velocity-Verlet or suitable symplectic method can be compared using the same conditions; its smaller or bounded error must be demonstrated, not assumed from its name. An acceptable computational report states the update ordering, step, duration, measured errors and analytic reference.

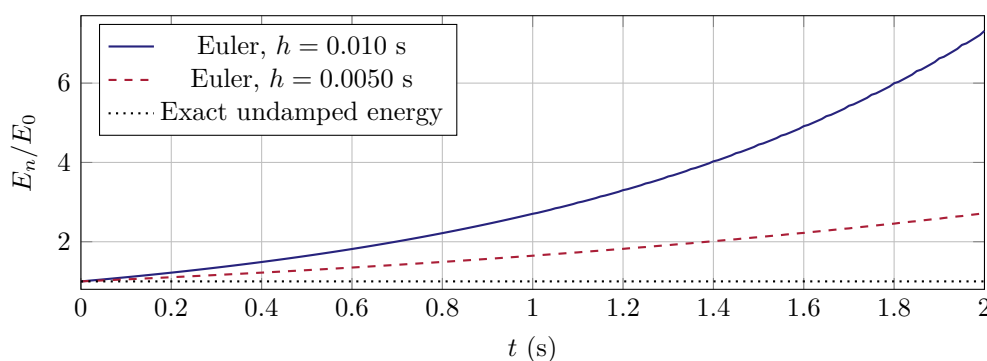


Figure 15.38: Solution diagnostic for Problem 39. Curves interpolate the exact discrete energy recurrence at the respective algorithm's sample times; they are not physical energy-growth laws. Step reduction lessens finite-time error while both Euler recurrences still amplify energy.

40. Inverting modal frequencies. (a) The symmetric model gives $\omega_+^2 = k_s/m$ and $\omega_-^2 = (k_s + 2k_c)/m$. Therefore

$$k_s = m\omega_+^2 = 0.20(100) = 20.0 \text{ N m}^{-1}, \quad k_c = \frac{m}{2}(\omega_-^2 - \omega_+^2) = \frac{0.20}{2}(150 - 100) = 5.0 \text{ N m}^{-1}.$$

Both inferred stiffnesses are positive and the out-of-phase frequency is higher, satisfying the model's stability and mode-ordering checks.

(b) The upper angular frequency is $\omega_- = \sqrt{150} = 12.2474 \text{ rad s}^{-1}$. Hence the first coordinate-1 envelope zero is

$$t_{\text{tr}} = \frac{\pi}{\sqrt{150} - 10} = 1.39785 \text{ s}.$$

The initial energy is $\frac{1}{2}(20 + 5)(0.015)^2 = 0.0028125 \text{ J}$. As in Problem 32, this time refers to a displacement envelope, not necessarily an exact exclusive allocation of all mechanical energy to the second mass.

(c) Known equal masses, equal linear outer springs, a single linear coupling and negligible damping make the two measured eigenvalues sufficient for the two stiffness unknowns. If masses or outer springs are unknown and unequal, more parameters enter than the two frequencies can determine. A useful diagnostic measures the modal displacement ratios: the symmetric modes must have $x_2/x_1 = +1$ and -1 , respectively. Systematic deviations, after sensor calibration and separation of nearby modes, challenge the symmetry assumptions. Frequency agreement alone is weaker than agreement of both frequencies and mode shapes.

15.12 Bridge to Chapter 16

An isolated oscillator stores energy locally. A coupling interaction allows neighboring coordinates to exchange it. A distributed mechanical system extends the pair to many coordinates: phase can vary with position as well as time, and disturbances can propagate. Chapter 16 retains the sinusoidal language while introducing wavelength, propagation speed, interference and boundary-dependent standing patterns. The present chapter supplies the local dynamics, initial-condition reasoning and mode interpretation needed for that transition.

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