

Chapter 16

Mechanical Waves and Superposition

First Edition

Hamzeh Khanpour

hamzeh.khanpour@agh.edu.pl

Faculty of Physics and Applied Computer Science

AGH University of Krakow

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Preface

A disturbance can travel across a room, along a cable, through the ground, or through air even though the material that carries it does not travel with the disturbance. That separation between *propagation* and *local motion* is the central idea of this chapter. We begin with pulses and sinusoidal waves on an ideal string because the geometry is visible and Newton's laws can be applied directly. From that model we develop wave speed, the one-dimensional wave equation, energy transport, superposition, reflection, and normal modes.

The chapter then asks a deeper question: how can a complicated shape or a short packet be represented by simpler oscillatory components? Fourier synthesis provides one answer. Finally, the advanced section introduces dispersion and a benchmarked finite-difference calculation so that phase speed, group speed, and numerical phase error can be distinguished rather than blended into one vague idea of “wave speed.”

The Guideline

Treat a wave problem as a problem about a *field of local displacements* and a *propagating pattern*. Identify the independent variables, state what is held fixed, distinguish the speed of the pattern from the velocity of a material element, and check the boundary conditions before interpreting a picture.

Pedagogical boundary

This chapter develops the general mechanical-wave framework. Acoustic pressure, acoustic intensity, air-column modes, beats as a sound phenomenon, and the Doppler effect belong to Chapter 17, *Sound Waves*. Electromagnetic waves and elastic-solid tensor waves are outside Volume I's present scope.

Conventions and notation

The transverse displacement is $y(x, t)$, where x labels position along the undisturbed string and t is time. A partial derivative means that the other independent variable is held fixed. Thus

$$v_y(x, t) = \frac{\partial y}{\partial t}$$

is the transverse velocity of a particular string element at the position x ; it is not the propagation speed c_w of the wave pattern.

Symbol	Meaning	SI unit
A	displacement amplitude	m
λ	wavelength	m
$k_w = 2\pi/\lambda$	wave number	m^{-1}
$T_{\text{osc}}, f = 1/T_{\text{osc}}$	oscillation period and cyclic frequency	s, Hz
$\omega = 2\pi f$	angular frequency	rad s^{-1}
c_w	propagation speed of a wave pattern	m s^{-1}
F_T	string-tension magnitude	N
μ_ℓ	linear mass density	kg m^{-1}
u_ℓ	mechanical energy per unit string length	J m^{-1}
$\mathcal{P}$	instantaneous power crossing a fixed position	W
L	string length between boundaries	m
n	positive integer mode number	1
v_p, v_g	phase speed and group speed in an advanced dispersive model	m s^{-1}

We use the travelling-wave phase $k_w x - \omega t + \phi$ for propagation toward increasing x when $k_w > 0$ and $\omega > 0$. The amplitude A is non-negative; phase and signs carry direction. The tension magnitude is written F_T rather than T to avoid confusion with period or temperature.

Note

The core requires algebra, trigonometry, derivatives, and elementary integrals. Partial derivatives are introduced locally as rates with one variable held fixed. The Fourier section uses orthogonality integrals. The advanced section introduces $d\omega/dk$ and a finite-difference recurrence after their physical meanings are established.

Chapter 16

Mechanical Waves and Superposition

A high-speed camera records a sinusoidal disturbance on a long string. One image shows the displacement as a function of position; a sensor fixed at one point records displacement as a function of time. The two graphs look similar, but their horizontal axes are different. From the first we can obtain a wavelength; from the second, a period. Together they determine a propagation speed. Yet no marked element of the string travels one wavelength during one period: it mainly moves up and down. That is the puzzle with which we begin.

Learning goals

After studying the core of this chapter, you should be able to:

- distinguish propagation speed from medium-element velocity and acceleration;
- interpret a travelling-wave function and infer direction, wavelength, period, and phase;
- derive $c_w = \sqrt{F_T/\mu_\ell}$ and the linear one-dimensional string wave equation from a small-slope force balance;
- derive and use the mean power carried by a sinusoidal travelling wave;
- apply linear superposition and calculate phase-dependent interference;
- impose ideal fixed and free boundary conditions on reflected pulses;
- derive the fixed-end normal-mode spectrum from boundary conditions rather than memorised shapes;
- reconstruct a specified initial string shape from a finite sine series and quantify truncation error.

The optional advanced section adds dispersion, phase/group speeds, and a finite-difference propagation benchmark.

Key message

A mechanical wave transmits a disturbance and can transport energy through a medium. The material itself generally oscillates locally rather than being carried along with the pattern.

16.1 Disturbances and travelling-wave functions

16.1.1 Pulse translation and periodic waves

A mechanical wave requires a medium with inertia and an interaction that couples neighbouring parts of the medium. On a stretched string, tension provides that coupling. If one end is displaced once and released, a pulse travels away. If the end is driven periodically, a periodic wave can be produced.

A particularly useful mathematical description of an undistorted pulse moving toward increasing x is

$$y(x, t) = f(x - c_w t), \quad (16.1)$$

where f is the pulse shape. The argument $x - c_w t$ is constant at a point of fixed phase. Setting $x - c_w t = \text{constant}$ and differentiating gives $dx/dt = c_w$, so the shape translates toward $+x$. A function $f(x + c_w t)$ instead travels toward $-x$.

For a sinusoidal component we write

$$y(x, t) = A \sin(k_w x - \omega t + \phi), \quad (16.2)$$

with

$$k_w = \frac{2\pi}{\lambda}, \quad \omega = 2\pi f = \frac{2\pi}{T_{\text{osc}}}, \quad c_w = \frac{\omega}{k_w} = \lambda f. \quad (16.3)$$

The last equality is kinematic: a point of fixed phase advances one wavelength in one period. In a nondispersive model, all sinusoidal components relevant to the signal share the same c_w .

16.1.2 Snapshots versus time traces

At fixed time $t = t_0$, Eq. (16.2) is a function of x and a spatial graph reveals λ . At fixed position $x = x_0$, it is a function of t and a time graph reveals T_{osc} . The slopes have different dimensions:

$$\frac{\partial y}{\partial x} \text{ is dimensionless,} \quad \frac{\partial y}{\partial t} \text{ has units of velocity.}$$

For the sinusoidal wave,

$$v_y(x, t) = \frac{\partial y}{\partial t} = -\omega A \cos(k_w x - \omega t + \phi), \quad (16.4)$$

and

$$a_y(x, t) = \frac{\partial^2 y}{\partial t^2} = -\omega^2 y(x, t). \quad (16.5)$$

A marked string element therefore executes simple harmonic motion in the ideal sinusoidal model while the phase travels along the string.

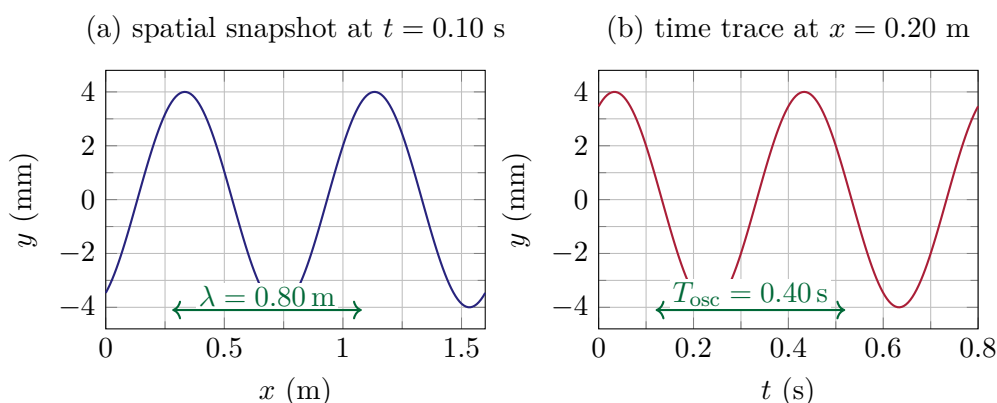


Figure 16.1: The same calculated wave, $A = 4.0$ mm, $\lambda = 0.80$ m, $T_{\text{osc}} = 0.40$ s, and $\phi = \pi/6$, viewed in two ways. Panel (a) holds time fixed and varies position; panel (b) holds position fixed and varies time. The propagation speed is $\lambda/T_{\text{osc}} = 2.0$ m s $^{-1}$; the horizontal axes are not interchangeable.

Example 16.1: Extract wavelength and period from two observations

A spatial snapshot of one sinusoidal wave gives a crest-to-crest spacing of 0.80 m. At a fixed detector position, successive upward zero crossings are separated by 0.40 s. The amplitude is

4.0 mm.

Model and representation. Treat the disturbance as one nondispersive sinusoidal component. A snapshot determines a spatial repetition length; a time trace determines a temporal repetition time.

Relate and solve.

$$\lambda = 0.80 \text{ m}, \quad T_{\text{osc}} = 0.40 \text{ s}, \quad f = 2.50 \text{ Hz}, \quad c_w = \frac{\lambda}{T_{\text{osc}}} = 2.00 \text{ m s}^{-1}.$$

Then $k_w = 2\pi/\lambda = 7.85 \text{ m}^{-1}$ and $\omega = 2\pi/T_{\text{osc}} = 15.7 \text{ rad s}^{-1}$.

Test and interpret. The units of ω/k_w are m s^{-1} . The maximum transverse material speed is $\omega A = 0.0628 \text{ m s}^{-1}$, much smaller than the propagation speed and directed transversely. The two speeds describe different motions.

Guided checks

Check 1. In a snapshot, a crest is at $x = 0.30 \text{ m}$ and 0.20 s later the same crest is at $x = 0.70 \text{ m}$. What is the propagation direction and speed? *Reasoning:* the same phase advances 0.40 m in 0.20 s , so $\Delta x/\Delta t = +2.0 \text{ m s}^{-1}$.

Check 2. If $y = A \sin(k_w x + \omega t)$ with positive k_w, ω , which way does the pattern travel? *Reasoning:* holding $k_w x + \omega t$ fixed gives $dx/dt = -\omega/k_w < 0$.

Check 3. Can $\partial y/\partial t$ be constant and equal to c_w for a sinusoidal transverse wave? *Reasoning:* differentiating at fixed x gives $-\omega A \cos(k_w x - \omega t)$, which oscillates between $\pm \omega A$ and has a different direction and meaning.

16.1.3 Following a crest and following a marked element

An observation must specify what is being followed. A crest is a location of constant phase, so its horizontal coordinate changes according to $x_{\text{crest}}(t) = x_{\text{crest}}(0) + c_w t$. A small mark painted on the string instead identifies a material element. In the linear transverse model its horizontal coordinate is approximately fixed and its displacement changes with time. Figure 16.2 makes both observations on one wave rather than treating them as unrelated pictures.

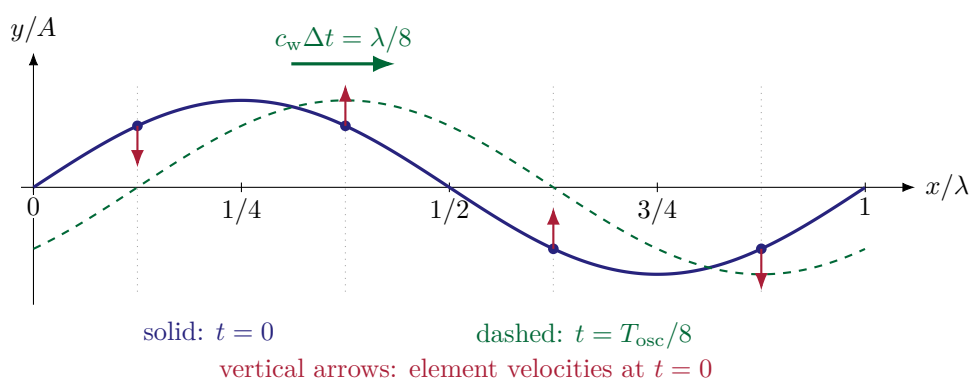


Figure 16.2: Original calculated travelling-wave observation diagram for $y = A \sin(2\pi x/\lambda - 2\pi t/T_{\text{osc}})$. The dashed later snapshot is the solid snapshot translated right by $\lambda/8$. The four dots mark material positions; their red arrows indicate the sign of $v_y = -\omega A \cos(2\pi x/\lambda)$ at $t = 0$, with a common schematic arrow length. The arrows describe transverse motion, whereas the upper arrow describes translation of the pattern. The displacement scale is normalized and exaggerated relative to the horizontal physical scale.

At the first and fourth marked positions the element velocity is downward; at the middle two it is upward. This follows directly from the derivative, not from whether the element lies above or below equilibrium. For a right-going pulse $y = f(x - c_w t)$ the same chain rule gives $v_y = -c_w \partial_x y$:

a rising portion of the spatial profile moves downward at a fixed detector as the pattern passes. A crest has $\partial_x y = 0$, so the element at the crest is instantaneously at rest transversely even though the crest location advances at c_w .

The model makes a useful quantitative prediction. For the wave of Figure 16.1, $A = 4.0$ mm, $\lambda = 0.80$ m, and $T_{\text{osc}} = 0.40$ s, the largest transverse element speed is $\omega A = 0.0628$ m s⁻¹, whereas the pattern speed is 2.0 m s⁻¹. Their ratio is $\omega A/c_w = 2\pi A/\lambda = 0.0314$. It is the same small parameter that controls the maximum string slope. This relation helps explain why the elements may move gently while a disturbance propagates rapidly.

16.1.4 Reading phase without confusing distance and time

A phase is a dimensionless argument that identifies the state of an oscillation. The coefficients of x and t therefore carry inverse-length and inverse-time units. Radians are retained in angular-frequency units to signal the trigonometric convention, although a radian is dimensionless in dimensional analysis. When differentiating a sine with respect to a physical coordinate, the argument must be in radians. A plotting program that evaluates trigonometric functions in degrees needs an explicit conversion; that software convention does not alter the physics.

For a sine wave, crests occur when

$$k_w x - \omega t + \phi = \frac{\pi}{2} + 2\pi j, \quad j \in \mathbb{Z}.$$

Adding π instead would alternate crests and troughs. Solving the crest equation gives $x_j(t) = c_w t + (\pi/2 - \phi + 2\pi j)/k_w$. All crests advance at the same speed; their separation is $2\pi/k_w$. The position of a particular material element is not $x_j(t)$: its equilibrium coordinate x_0 remains fixed in this transverse approximation, and its displacement is $y(x_0, t)$.

At a crest the instantaneous transverse velocity is zero, but its acceleration is not zero. Since $a_y = -\omega^2 y$, a positive crest is accelerating downward. Half a period later the same element is at a trough and accelerates upward. This is the local oscillator reasoning from Chapter 15 applied separately to each equilibrium coordinate, without treating each element as an independent spring for energy calculations.

Example 16.2: Reconstruct a phase from a measured event

Statement. A right-going sine wave has $A = 3.00$ mm, $\lambda = 0.600$ m, and $f = 8.00$ Hz. At $x = 0, t = 0$ its displacement is 1.50 mm and it is moving downward. Find a phase in $[0, 2\pi)$, a compatible wave function, and its local acceleration at that event.

Strategy. Displacement specifies $\sin \phi$, but two phases give the same sine. The velocity sign selects the branch. Here $\sin \phi = 1/2$, so $\phi = \pi/6$ or $5\pi/6$. For a right-going wave $v_y = -\omega A \cos \phi$; downward motion requires $\cos \phi > 0$, selecting $\pi/6$.

$$y = (0.00300 \text{ m}) \sin \left[\left(\frac{10\pi}{3} \text{ m}^{-1} \right) x - (16\pi \text{ s}^{-1}) t + \frac{\pi}{6} \right].$$

The phase speed is $\lambda f = 4.80$ m s⁻¹. At the event,

$$v_y = -(16\pi)(0.00300) \frac{\sqrt{3}}{2} = -0.1306 \text{ m s}^{-1}; \quad a_y = -(16\pi)^2(0.00150) = -3.790 \text{ m s}^{-2}.$$

Check. $Ak_w = 0.0314$ is small, and the velocity is indeed downward. Displacement alone would not have selected the phase. Even a complete snapshot by itself does not establish travel direction; a second snapshot or local velocity information is needed.

16.1.5 A pulse need not be sinusoidal

A localised disturbance may have no single wavelength or period. It still has a translation speed if its shape moves rigidly. For example, the prescribed Gaussian

$$y(x, t) = A \exp \left[- \left(\frac{x - x_c - c_w t}{\sigma} \right)^2 \right] \quad (16.6)$$

has amplitude A , initial centre x_c , and width parameter $\sigma > 0$. Its displacement falls to A/e a distance σ from its centre. A Gaussian has infinite, exponentially small tails; calling it a finite pulse refers to its finite energy, not exact compact support.

The chain rule gives $\partial_t y = -c_w \partial_x y$. Therefore the material moves downward on the rising left flank of a right-going positive pulse and upward on the falling right flank. Its centre travels to the right even though the element momentarily at the peak has zero transverse velocity. Figure 16.3 keeps the spatial axes common so that translation can be inferred from the actual positions.

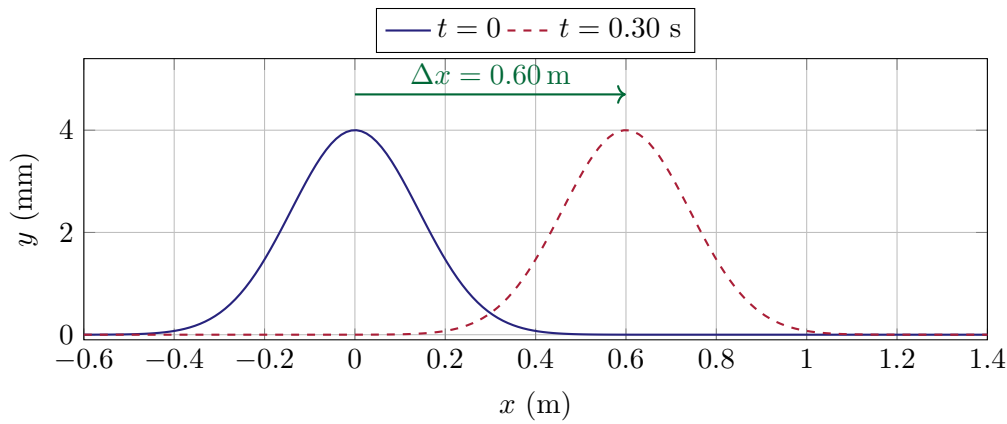


Figure 16.3: Rigid translation of the Gaussian model with $A = 4.0$ mm, $\sigma = 0.20$ m, $x_c = 0$, and $c_w = 2.0$ m s⁻¹. The vertical graph is displacement, not a trajectory of one material element. Both snapshots have the same amplitude and width.

Example 16.3: Determine a pulse speed from a detector

Statement. The pulse in Figure 16.3 reaches its maximum at a detector $x = 0.800$ m. Find the time of maximum, the two times when $y = A/e$, and the greatest magnitude of transverse velocity.

Solution. The centre reaches the detector at $t_c = x/c_w = 0.400$ s. At this position $y = A \exp[-(c_w(t - t_c)/\sigma)^2]$. Setting $y = A/e$ gives $|t - t_c| = \sigma/c_w = 0.100$ s, hence $t = 0.300$ and 0.500 s.

Write $q = (x - c_w t)/\sigma$. Then $v_y = 2Ac_w q e^{-q^2}/\sigma$. The magnitude of $q e^{-q^2}$ is largest at $|q| = 1/\sqrt{2}$, obtained by differentiating it. Thus

$$|v_y|_{\max} = \frac{\sqrt{2}Ac_w}{\sigma} e^{-1/2} = 0.0343 \text{ m s}^{-1}.$$

This is smaller than the pattern speed and is directed transversely. The maximum slope is $\sqrt{2}(A/\sigma)e^{-1/2} = 0.0172$, so this pulse can be consistent with the small-slope string model. There is no need to invent a wavelength for it.

16.2 Wave speed and the wave equation

The travelling function tells us how a pattern moves once its speed is known. It does not yet explain why a particular string has a particular speed. For that we need dynamics.

16.2.1 String-element force balance

Consider a taut uniform string with equilibrium direction along x . Let its tension magnitude be approximately constant and equal to F_T , and let its linear mass density be μ_ℓ . We isolate a short element from x to $x + \Delta x$. If the slopes are small, $|\partial y/\partial x| \ll 1$, the horizontal components of the two tension forces nearly cancel, while the net vertical force is the difference of their vertical components.

The force on the left end points toward decreasing x , while the force on the right points toward increasing x . Their signed transverse components are

$$F_{L,y} \simeq -F_T \left. \frac{\partial y}{\partial x} \right|_x, \quad F_{R,y} \simeq +F_T \left. \frac{\partial y}{\partial x} \right|_{x+\Delta x}.$$

Therefore

$$F_{y,\text{net}} \simeq F_T \left[\left. \frac{\partial y}{\partial x} \right|_{x+\Delta x} - \left. \frac{\partial y}{\partial x} \right|_x \right]. \quad (16.7)$$

The element mass is $\Delta m \simeq \mu_\ell \Delta x$. Newton's second law gives

$$F_T \left[\left. \frac{\partial y}{\partial x} \right|_{x+\Delta x} - \left. \frac{\partial y}{\partial x} \right|_x \right] = \mu_\ell \Delta x \frac{\partial^2 y}{\partial t^2}.$$

Divide by Δx and take the continuum limit:

$$\mu_\ell \frac{\partial^2 y}{\partial t^2} = F_T \frac{\partial^2 y}{\partial x^2}. \quad (16.8)$$

This is the linear wave equation for the ideal string model.

16.2.2 Linear wave equation and speed

Equation (16.8) can be written

$$\frac{\partial^2 y}{\partial t^2} = c_w^2 \frac{\partial^2 y}{\partial x^2}, \quad c_w = \sqrt{\frac{F_T}{\mu_\ell}}. \quad (16.9)$$

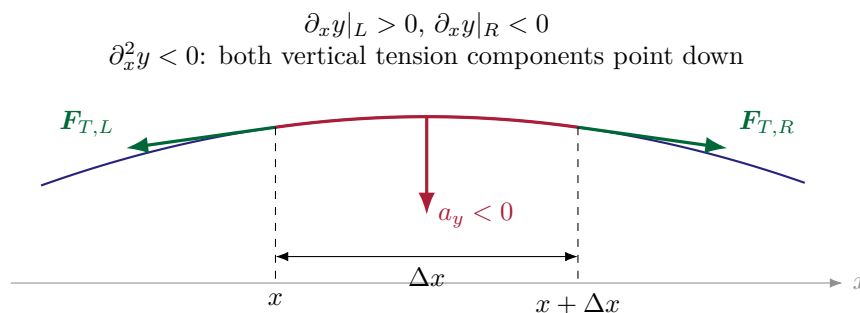


Figure 16.4: An isolated string element with concave-down displacement. The external tension vectors point away from the element along the endpoint tangents, and their vertical sum is downward. The highlighted segment is curved, rather than a chord. Geometry and arrow lengths are schematic; a local physical element requires small slopes and Δx short compared with the variation scale.

The dimensions are

$$\left[\frac{F_T}{\mu_\ell} \right] = \frac{\text{kg m s}^{-2}}{\text{kg m}^{-1}} = \text{m}^2 \text{s}^{-2},$$

so the square root has the dimensions of speed. Dimensional analysis supports the combination but does not determine its numerical coefficient; that coefficient arose from the force balance.

Any twice differentiable $f(x - c_w t)$ satisfies the wave equation:

$$\frac{\partial^2 y}{\partial t^2} = c_w^2 f''(x - c_w t), \quad \frac{\partial^2 y}{\partial x^2} = f''(x - c_w t).$$

The same is true for $g(x + c_w t)$. By linearity, their sum is also a solution.

Caution

The ideal string equation is not exact for arbitrary deformation. It assumes a uniform string, small transverse slope, approximately constant tension, negligible bending stiffness, and negligible dissipation. A mathematically valid sinusoidal function need not satisfy those physical assumptions. For $y = A \sin(k_w x - \omega t)$, the maximum slope is Ak_w ; checking $Ak_w \ll 1$ is a direct model test.

Example 16.4: Check an assumed string waveform

A uniform string has $F_T = 120 \text{ N}$ and $\mu_\ell = 0.030 \text{ kg m}^{-1}$. A proposed travelling wave has $A = 8.0 \text{ mm}$ and $\lambda = 1.20 \text{ m}$.

Wave speed.

$$c_w = \sqrt{\frac{120}{0.030}} = 63.25 \text{ m s}^{-1}.$$

The corresponding frequency for this wavelength is

$$f = \frac{c_w}{\lambda} = 52.70 \text{ Hz}, \quad \omega = 331.2 \text{ rad s}^{-1}.$$

Small-slope check. Here $k_w = 2\pi/1.20 = 5.236 \text{ m}^{-1}$, so

$$Ak_w = (8.0 \times 10^{-3})(5.236) = 0.0419.$$

The maximum slope is about 4.2%, which is consistent with a small-slope teaching model. If the amplitude were 0.24 m at the same wavelength, $Ak_w \simeq 1.26$ and the linear string interpretation would no longer be justified even though the sine function itself would remain mathematically well defined.

Method

Model check for an ideal string wave.

1. Identify F_T , μ_ℓ , A , and λ in SI units.
2. Compute $c_w = \sqrt{F_T/\mu_\ell}$ and $k_w = 2\pi/\lambda$.
3. Test $Ak_w \ll 1$ if the linear small-slope derivation is being used.
4. Check whether the model also requires uniform tension, negligible stiffness, or negligible damping for the question at hand.
5. Only then use $\omega = c_w k_w$ for the ideal nondispersive string.

16.2.3 Curvature gives the sign of acceleration

The derivative $\partial_x y$ is the slope of the snapshot, while $\partial_x^2 y$ tells how that slope changes along the string. A concave-down segment has a decreasing slope. The upward component of the right-hand tension is then smaller than the contribution required to cancel the left-hand tension, so the net force points down. This is the sign visible in Figure 16.4. At a positive sinusoidal crest $\partial_x^2 y = -k_w^2 y < 0$, and the dynamical equation predicts downward acceleration, agreeing with the oscillator result $a_y = -\omega^2 y$.

The two tension arrows in a body-only diagram are the forces *on* the isolated element from the string outside it. They point away from the element. The equal and opposite forces that the element exerts on its neighbours are not added to the same free-body diagram. Likewise, slope components should be used to calculate the resultant; they are not extra physical forces in addition to the full tension vectors.

16.2.4 What controls the speed, and what controls the frequency?

In the ideal uniform string model, tension and line density determine c_w . A drive imposes a temporal frequency at its location. If the medium remains unchanged, changing that frequency changes $\lambda = c_w/f$, rather than the speed. The source amplitude controls the displacement amplitude in the linear response, subject to boundary reflection and resonance. These three roles should be kept separate.

Doubling tension multiplies speed by $\sqrt{2}$, not by two. Doubling line density divides it by $\sqrt{2}$. If both are doubled together the speed is unchanged. The functional dependence is shown in Figure 16.5; the graph concerns independently varied stipulated parameters, not a stretching experiment in which length, density, and tension may change together.

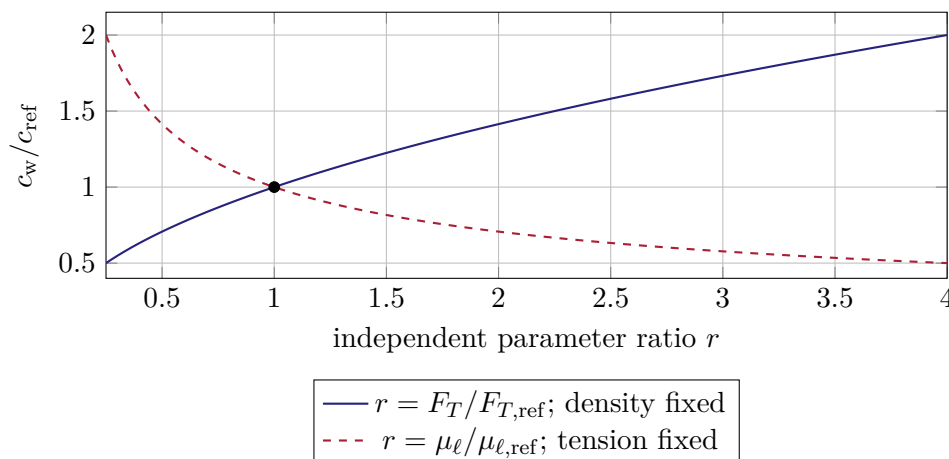


Figure 16.5: The square-root speed law normalised by $c_{\text{ref}} = \sqrt{F_{T,\text{ref}}/\mu_{\ell,\text{ref}}}$. Each curve varies one parameter while holding the other fixed. Both pass through the reference state $(r, c_w/c_{\text{ref}}) = (1, 1)$.

Example 16.5: Recover line density from propagation

Statement. A pulse takes 0.0250 s to travel between sensors separated by 1.50 m on a string with stipulated tension 72.0 N. Under the ideal model, infer its line density and the mass of a 4.00 m segment. Explain one limitation of the inference.

Solution. A transit measurement compares the same recognisable pulse feature at the two sensors. It gives $c_w = d/\Delta t = 60.0 \text{ m s}^{-1}$. Solving $c_w^2 = F_T/\mu_\ell$ yields

$$\mu_\ell = \frac{72.0}{60.0^2} = 0.0200 \text{ kg m}^{-1}, \quad m = \mu_\ell \ell = 0.0800 \text{ kg}.$$

Substituting this density back into the speed law recovers the measured transit speed. If a broad pulse distorts because of stiffness or nonuniform tension, different features can have different apparent delays. The single transit speed would then be inadequate to infer a unique ideal density without further model checks.

16.2.5 How small is a small slope?

There is no universal numerical threshold that guarantees an approximation for every requested accuracy. The discarded geometric terms provide a scale estimate. With $s = \partial_x y$, the exact excess length factor is $\sqrt{1 + s^2} - 1$, whereas the quadratic approximation is $s^2/2$. Expansion gives

$$\sqrt{1 + s^2} - 1 = \frac{s^2}{2} - \frac{s^4}{8} + \dots$$

Thus the relative error of the excess-length approximation is approximately $s^2/4$ at small nonzero slope. For the slope 0.0419 in Example 16.4, the geometric error is about 0.044%. This estimate concerns that geometric expansion; it does not prove that tension remains constant or that bending stiffness is negligible.

A sine function with $Ak_w \gtrsim 1$ is still a perfectly defined function and can solve a mathematical linear PDE. What fails is its justification as a small-slope physical string. Model validity and algebraic validity are different checks. A drawing may exaggerate transverse scale to make a small physical displacement visible, provided that the exaggeration is declared and force directions remain correct.

16.3 Energy and power transport

A wave can carry energy even when the average material position does not change. For a string element of length dx , the transverse kinetic energy is

$$dK = \frac{1}{2}(\mu_\ell dx) \left(\frac{\partial y}{\partial t} \right)^2.$$

The increase of string length caused by a small slope is

$$ds = \sqrt{1 + (\partial_x y)^2} dx \simeq \left[1 + \frac{1}{2}(\partial_x y)^2 \right] dx,$$

so the extra length is approximately $\frac{1}{2}(\partial_x y)^2 dx$. Work against the nearly constant tension gives elastic energy

$$dU \simeq \frac{1}{2}F_T \left(\frac{\partial y}{\partial x} \right)^2 dx.$$

Thus the local energy per unit string length is

$$u_\ell(x, t) = \frac{1}{2}\mu_\ell \left(\frac{\partial y}{\partial t} \right)^2 + \frac{1}{2}F_T \left(\frac{\partial y}{\partial x} \right)^2. \quad (16.10)$$

Both terms are non-negative.

For a right-going wave, the transverse force exerted across a fixed cut is approximately $-F_T \partial_x y$ on the material to the right, while the transverse velocity at the cut is $\partial_t y$. The instantaneous power crossing the cut toward $+x$ is therefore

$$\mathcal{P}(x, t) = -F_T \frac{\partial y}{\partial x} \frac{\partial y}{\partial t}. \quad (16.11)$$

For $y = A \sin(k_w x - \omega t + \phi)$, the derivatives have opposite signs at every phase, so $\mathcal{P} \geq 0$ for a right-going wave. Using $F_T = \mu_\ell c_w^2$ and $\omega = c_w k_w$,

$$\mathcal{P} = \mu_\ell c_w \omega^2 A^2 \cos^2(k_w x - \omega t + \phi). \quad (16.12)$$

Averaging over a cycle gives

$$\langle \mathcal{P} \rangle = \frac{1}{2} \mu_\ell \omega^2 A^2 c_w. \quad (16.13)$$

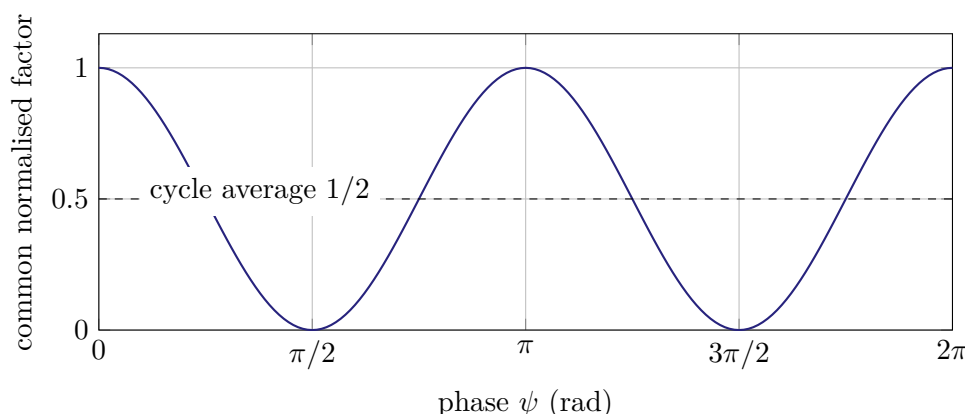


Figure 16.6: One curve represents the identical factors $u_K/(\mu_\ell \omega^2 A^2/2)$, $u_U/(\mu_\ell \omega^2 A^2/2)$, and $\mathcal{P}/(\mu_\ell c_w \omega^2 A^2)$ for a single right-going sine wave. They all equal $\cos^2 \psi$. At a displacement crest the slope and transverse velocity vanish, so both local energies vanish within this linear model; energy peaks at displacement zero crossings.

Example 16.6: Power carried by two amplitude settings

For the string in Example 16.4, let the frequency be 50.0 Hz and compare waves of amplitudes 2.00 mm and 4.00 mm. The speed is 63.25 m s^{-1} and $\omega = 314.159 \text{ rad s}^{-1}$.

$$\langle \mathcal{P} \rangle = \frac{1}{2} (0.030) (314.159)^2 A^2 (63.25).$$

For $A = 2.00 \text{ mm}$, $\langle \mathcal{P} \rangle = 0.375 \text{ W}$. For $A = 4.00 \text{ mm}$, $\langle \mathcal{P} \rangle = 1.50 \text{ W}$. The amplitude doubles while all other model parameters remain fixed, so the mean power increases by a factor of four.

Check. $[\mu_\ell \omega^2 A^2 c_w] = (\text{kg/m})(1/\text{s}^2)(\text{m}^2)(\text{m/s}) = \text{kg m}^2/\text{s}^3 = \text{W}$.

Note

The A^2 scaling is a result within the linear sinusoidal string model. It does not mean that arbitrary large-amplitude waves can be assigned arbitrarily large power while retaining the same linear assumptions.

16.3.1 A local conservation law for energy

The power expression has a sign convention: $\mathcal{P} > 0$ means energy crossing a cut toward increasing x . It is defined for any solution of the linear string equation, including a superposition; it is not restricted to right-going waves. A left-going wave $g(x + c_w t)$ has $\partial_t y = +c_w \partial_x y$, so $\mathcal{P} = -F_T c_w (\partial_x y)^2 \leq 0$. Its energy density remains nonnegative.

Differentiating the line-energy density and using $\mu_\ell y_{tt} = F_T y_{xx}$ gives

$$\begin{aligned}\partial_t u_\ell &= \mu_\ell y_t y_{tt} + F_T y_x y_{xt} \\ &= F_T (y_t y_{xx} + y_x y_{xt}) = F_T \partial_x (y_x y_t) = -\partial_x \mathcal{P}.\end{aligned}\quad (16.14)$$

Here subscripts denote partial differentiation, so $y_{xt} = \partial_x \partial_t y$ for a smooth solution. Integrate this equation from a to b :

$$\frac{d}{dt} \int_a^b u_\ell dx = \mathcal{P}(a, t) - \mathcal{P}(b, t). \quad (16.15)$$

The energy stored in a segment increases when more enters its left boundary than exits its right boundary. No energy disappearance is needed to describe a changing local displacement or a cancelling snapshot.

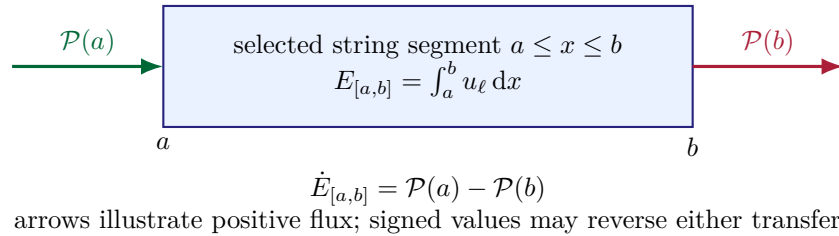


Figure 16.7: Energy bookkeeping for a fixed spatial segment. The box marks the control interval, not a rigid container. Power is signed relative to $+x$, while the stored energy is nonnegative.

For a pure right-going profile $f(x - c_w t)$, the relation $y_t = -c_w y_x$ makes the kinetic and elastic densities equal: $u_K = u_U = F_T y_x^2 / 2$. Therefore $\mathcal{P} = c_w u_\ell$. The energy distribution moves with the pulse. This equality is a special result for one direction of travel; it does not hold for a general standing wave.

16.3.2 Cycle averaging and the energy in one wavelength

For a sinusoid, a cycle average at fixed x is explicitly

$$\langle \cos^2(k_w x - \omega t + \phi) \rangle = \frac{1}{T_{\text{osc}}} \int_{t_0}^{t_0 + T_{\text{osc}}} \cos^2(k_w x - \omega t + \phi) dt = \frac{1}{2}.$$

Using $\cos^2 \psi = (1 + \cos 2\psi)/2$, the oscillatory part integrates to zero over a complete period. Averaging over one spatial wavelength also gives $1/2$, but over an arbitrary partial period or partial wavelength it generally does not. The instantaneous maximum power is twice the mean.

The total energy in any one-wavelength interval of a single travelling sinusoid is

$$E_\lambda = \frac{1}{2} \mu_\ell \omega^2 A^2 \lambda, \quad \langle \mathcal{P} \rangle = \frac{E_\lambda}{T_{\text{osc}}}. \quad (16.16)$$

The second relation is a physical cross-check: in one period a wavelength of the travelling energy pattern passes a cut. It connects an energy to a power rather than treating them as interchangeable quantities.

Example 16.7: Select an amplitude for a power limit

Statement. An ideal string has $\mu_\ell = 0.0100 \text{ kg m}^{-1}$ and $c_w = 40.0 \text{ m s}^{-1}$. At $f = 25.0 \text{ Hz}$, require mean travelling power 0.200 W . Find the amplitude, instantaneous maximum power, energy per wavelength, and small-slope parameter.

Solution. Solve the mean-power relation before substituting:

$$A = \sqrt{\frac{2\langle \mathcal{P} \rangle}{\mu_\ell \omega^2 c_w}} = \sqrt{\frac{0.400}{(0.0100)(50\pi)^2(40.0)}} = 6.37 \text{ mm}.$$

Since $\cos^2 \psi$ peaks at one, $\mathcal{P}_{\max} = 0.400 \text{ W}$. With $\lambda = c_w/f = 1.60 \text{ m}$, Eq. (16.16) gives $E_\lambda = 0.00800 \text{ J}$, independently equal to $0.200 \text{ W} \times 0.0400 \text{ s}$. Finally $Ak_w = (0.006366)(2\pi/1.60) = 0.0250$.

Interpretation. The square root is the nonnegative amplitude branch. If frequency doubles at the same power and unchanged string parameters, the amplitude must halve. The maximum material speed ωA then stays unchanged, and so does Ak_w in this nondispersive model.

16.3.3 Finite energy in a Gaussian pulse

For Eq. (16.6), let $\xi = x - x_c - c_w t$ and $q = \xi/\sigma$. Then $y_x = -2Aqe^{-q^2}/\sigma$. The line-energy density of this right-going pulse is

$$u_\ell = F_T y_x^2 = \frac{4F_T A^2}{\sigma^2} q^2 e^{-2q^2}.$$

It has two maxima at $q = \pm 1/\sqrt{2}$ and vanishes at the displacement peak. Integrating over the entire ideal infinite string gives

$$E_{\text{pulse}} = \frac{4F_T A^2}{\sigma} \int_{-\infty}^{\infty} q^2 e^{-2q^2} dq = \sqrt{\frac{\pi}{2}} \frac{F_T A^2}{\sigma}. \quad (16.17)$$

The integral follows by differentiating $\int e^{-\alpha q^2} dq = \sqrt{\pi/\alpha}$ with respect to $\alpha > 0$. Equivalently, it can be evaluated numerically and checked for independence of the pulse centre. The finite energy is constant while the pulse translates.

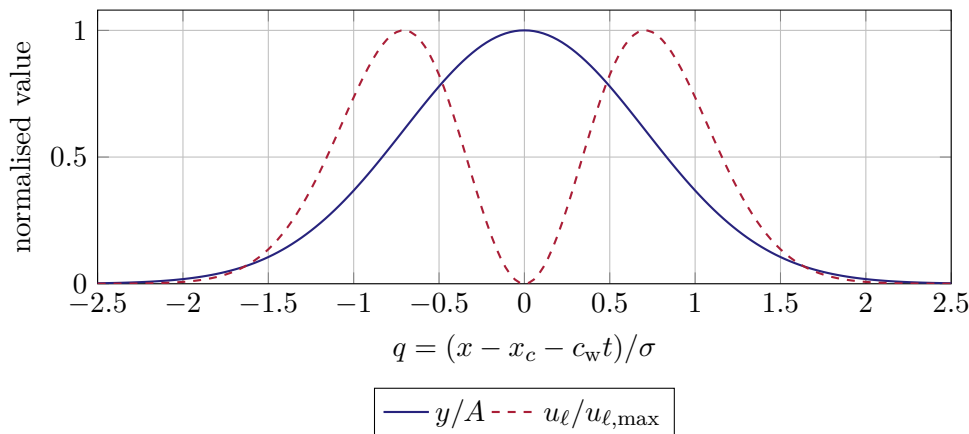


Figure 16.8: A Gaussian displacement pulse and its line-energy density. Energy is concentrated on the flanks where slope and transverse speed are large. The energy maximum is $u_{\ell,\max} = 2F_T A^2/(e\sigma^2)$; both plotted variables are dimensionless and use separate normalisations.

Example 16.8: Energy of a finite pulse

Statement. A right-going Gaussian on an ideal long string has $F_T = 20.0 \text{ N}$, $\mu_\ell = 0.0500 \text{ kg m}^{-1}$, $A = 2.00 \text{ mm}$, and $\sigma = 0.100 \text{ m}$. Find its energy and the integral of power through one fixed cut over its complete passage.

Solution. The speed is $\sqrt{20.0/0.0500} = 20.0 \text{ m s}^{-1}$, and

$$E_{\text{pulse}} = \sqrt{\pi/2} \frac{(20.0)(0.00200)^2}{0.100} = 1.003 \times 10^{-3} \text{ J}.$$

Because $\mathcal{P} = c_w u_\ell$ and $d\xi = -c_w dt$ at fixed x , substitution in the time integral gives $\int_{-\infty}^{\infty} \mathcal{P}(x, t) dt = \int_{-\infty}^{\infty} u_\ell(\xi) d\xi = E_{\text{pulse}}$. This confirms that the energy passing the cut equals the pulse energy. The maximum slope is 0.0172, and the amplitude is sufficiently small for the ideal teaching model. A finite observation window misses the small tails, so a measured time integral would need a stated window and a tail estimate.

16.4 Superposition and interference

16.4.1 Linear superposition

Equation (16.8) is linear in y . If y_1 and y_2 separately satisfy it under the same F_T and μ_ℓ , then

$$y = y_1 + y_2 \quad (16.18)$$

also satisfies it. This is the superposition principle for the present model. It is a statement about displacement, not about adding energy or power as if they were signed wave amplitudes.

For two waves with the same k_w and ω ,

$$y_1 = A_1 \sin \psi, \quad y_2 = A_2 \sin(\psi + \Delta\phi),$$

the sum can be written as a single sinusoid with amplitude A_R satisfying

$$A_R^2 = A_1^2 + A_2^2 + 2A_1 A_2 \cos \Delta\phi. \quad (16.19)$$

This is the law of cosines in phasor form. Equal amplitudes give $A_R = 2A |\cos(\Delta\phi/2)|$.

16.4.2 Energy cross terms

Energy is quadratic. If $y = y_1 + y_2$, then for example

$$\left(\frac{\partial y}{\partial t}\right)^2 = \left(\frac{\partial y_1}{\partial t}\right)^2 + \left(\frac{\partial y_2}{\partial t}\right)^2 + 2 \frac{\partial y_1}{\partial t} \frac{\partial y_2}{\partial t}.$$

The last term is an interference cross term. It can redistribute energy in space and time. Superposition therefore does not imply that the energy of a coherent sum is obtained by adding two separate intensities point by point.

Example 16.9: Phase-controlled interference on a string

Two right-going waves have the same frequency and wavelength, with amplitudes $A_1 = 3.0 \text{ mm}$ and $A_2 = 2.0 \text{ mm}$.

For $\Delta\phi = 0$, $A_R = 5.0 \text{ mm}$ (fully constructive). For $\Delta\phi = \pi/2$,

$$A_R = \sqrt{3.0^2 + 2.0^2} = 3.61 \text{ mm}.$$

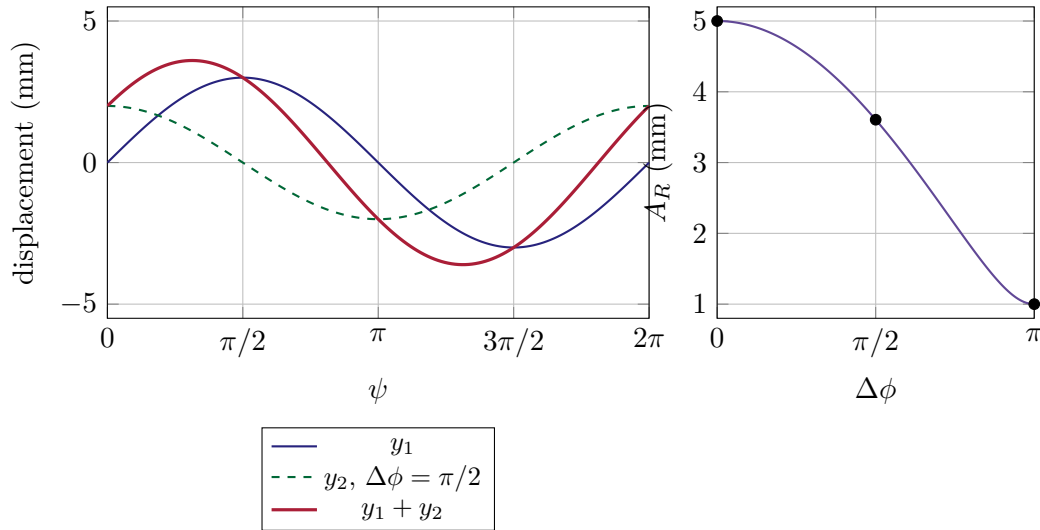


Figure 16.9: Phase-controlled interference for $A_1 = 3$ mm and $A_2 = 2$ mm. The left panel shows two coherent components and their calculated displacement sum for $\Delta\phi = \pi/2$. The right panel shows the resultant amplitude from Eq. (16.19). Because the amplitudes are unequal, a phase difference π leaves a residual amplitude of 1 mm rather than exact cancellation.

For $\Delta\phi = \pi$, $A_R = |3.0 - 2.0| = 1.0$ mm. Full destructive interference would require both $\Delta\phi = \pi$ and $A_1 = A_2$.

Caution

The word “interference” does not mean that the component waves permanently damage or consume one another in this linear model. After overlapping, each travelling component continues according to its own propagation law. What changes during overlap is the resultant displacement and the quadratic energy distribution.

16.4.3 A flat snapshot can conceal moving energy

Displacement cancellation is not disappearance of the physical state. A string configuration includes both $y(x, t)$ and $\partial_t y(x, t)$. Consider two opposite Gaussian pulses on a long uniform lossless string. Let $w > 0$ be their width parameter and put $\xi = x/w$, $s = c_w t/w$. Choose

$$\frac{y_1}{A} = \exp[-(\xi + 2 - s)^2], \quad \frac{y_2}{A} = -\exp[-(\xi - 2 + s)^2]. \quad (16.20)$$

The positive pulse moves right and the negative pulse moves left. Their initial centres are $-2w$ and $+2w$, and they coincide at $t_* = 2w/c_w$. The string displacement is the pointwise sum at every stage.

To identify the hidden motion, set $f(x) = A \exp[-(x/w)^2]$. At the collision time the two translated arguments equal x , so

$$y(x, t_*) = f(x) - f(x) = 0, \quad \partial_t y(x, t_*) = -c_w f'(x) - c_w f'(x) = -2c_w f'(x).$$

The elastic line-energy density then vanishes, but the kinetic density is $u_K = \frac{1}{2}\mu_\ell(2c_w f')^2 = 2F_T[f'(x)]^2$. Before collision the energy of each travelling constituent contains equal kinetic and elastic parts. At collision that same total energy is present in transverse motion. The field $y = 0$ alone therefore cannot decide whether a string is at rest. This conclusion uses a linear, nondissipative model; it does not assert that real pulses pass through one another unchanged at arbitrarily large amplitudes.

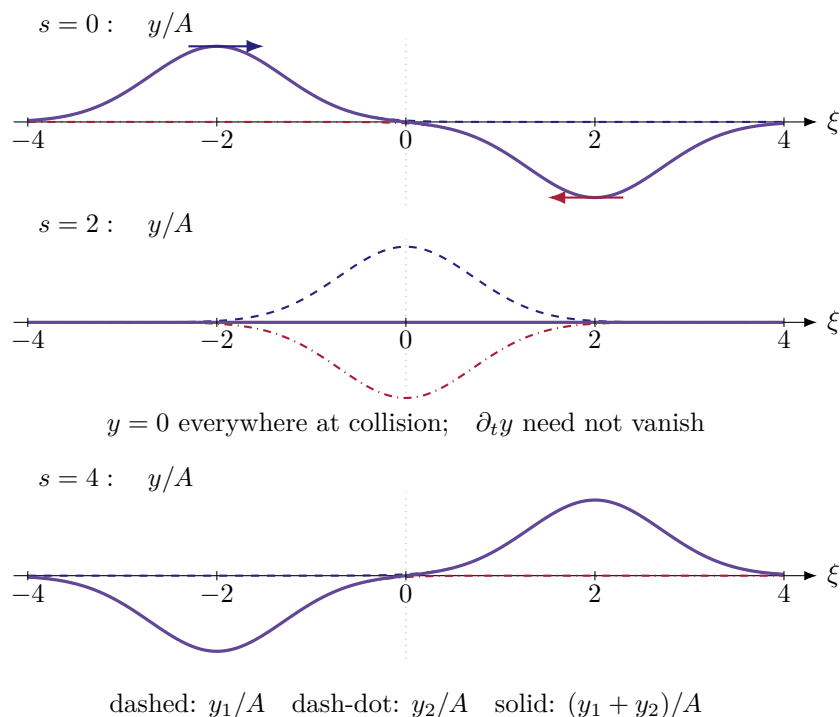


Figure 16.10: Original analytical three-stage superposition for Eq. (16.20); common normalized axes are used in all rows. At $s = 2$ the positive and negative displacement profiles cancel exactly, but the material continues to move. At $s = 4$ the two pulses have passed through each other and retain their individual shapes and speeds. The Gaussian tails extend beyond the displayed window. The isolated direction arrows in the first row indicate pulse propagation.

Example 16.10: Reconstruct the velocity hidden by pulse cancellation

Two pulses have the form in Eq. (16.20), with $A = 2.00 \text{ mm}$, $w = 0.200 \text{ m}$, $c_w = 40.0 \text{ m s}^{-1}$, and $\mu_\ell = 0.0100 \text{ kg m}^{-1}$. Find the collision time, the transverse velocity at $x = w/\sqrt{2}$ at that instant, and the local energy density. Explain how a flat string can have nonzero energy.

Solution. The initial half-separation is $2w$, so $t_* = 2w/c_w = 0.0100 \text{ s}$. Differentiating the pulse, rather than inferring the velocity from the snapshot, gives

$$v_y(x, t_*) = \frac{4c_w A x}{w^2} \exp[-(x/w)^2].$$

At the specified point,

$$v_y = \frac{4(40.0)(0.00200)}{0.200\sqrt{2}} e^{-1/2} = 0.686 \text{ m s}^{-1},$$

directed upward. Since the total displacement and slope vanish at that instant, $u_U = 0$ and

$$u_K = \frac{1}{2}(0.0100)(0.686)^2 = 2.35 \times 10^{-3} \text{ J m}^{-1}.$$

The maximum slope of one isolated Gaussian is $\sqrt{2}(A/w)e^{-1/2} = 0.00858$, consistent with the linear idealization. The nonzero kinetic energy resides in the moving elements; a flat instantaneous shape is compatible with that motion.

16.4.4 Deriving the resultant phase as well as amplitude

Expand the shifted sine:

$$y = (A_1 + A_2 \cos \Delta\phi) \sin \psi + (A_2 \sin \Delta\phi) \cos \psi.$$

Writing $y = A_R \sin(\psi + \delta)$ gives $A_R \cos \delta = A_1 + A_2 \cos \Delta\phi$ and $A_R \sin \delta = A_2 \sin \Delta\phi$. Squaring and adding yields Eq. (16.19). The phase is chosen by the quadrant-aware function

$$\delta = \text{atan2}(A_2 \sin \Delta\phi, A_1 + A_2 \cos \Delta\phi).$$

A single ordinary arctangent can choose the wrong quadrant. When $A_R = 0$, the resultant phase is undefined: there is no oscillation whose phase could be measured.

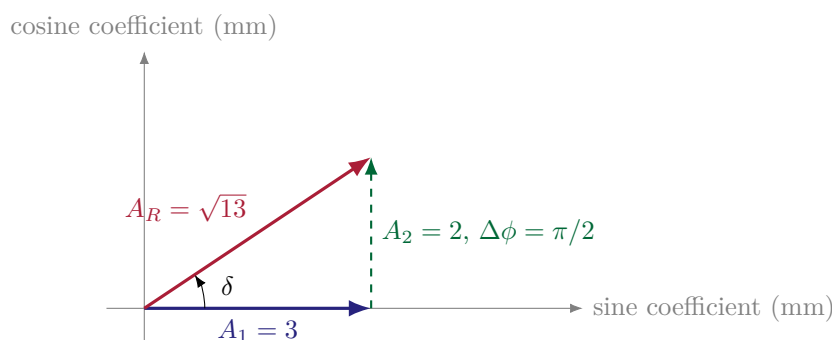


Figure 16.11: Coefficient-space construction for the same waves as Figure 16.9. This is an amplitude phasor diagram, not physical string geometry. Its component coordinates determine both the resultant amplitude and phase, $\delta = \tan^{-1}(2/3) = 0.5880$ rad in the first quadrant.

For coherent same-direction waves in the same ideal medium, the mean resultant power is proportional to A_R^2 . A phase-dependent power sum includes $2A_1A_2 \cos \Delta\phi$. For an ensemble of monochromatic components with phase fixed within each realisation but independently random between realisations, $\langle \cos \Delta\phi \rangle = 0$. The cross term then averages away and the mean powers add. Slowly varying random phases can permit a similar long-time approximation if variation is slow compared with the carrier period. Random-phase averaging is an additional statistical assumption, not a property of all overlapping waves.

The apparent question of where energy goes during destructive interference is resolved only by specifying the complete physical preparation and boundaries. Two imposed coherent drives can exchange different amounts of work when their phase changes; their individual outputs cannot automatically be treated as fixed independent power supplies. For two *opposite*-going pulses the local displacement can cancel while material velocity remains nonzero, and energy conservation follows directly from Eq. (16.14).

16.5 Reflection and boundary conditions

16.5.1 Fixed and free boundaries

A boundary condition is a physical constraint on the allowed displacement. At an ideal fixed endpoint $x = L$,

$$y(L, t) = 0. \quad (16.21)$$

At an ideal free endpoint, the endpoint has no transverse external force. In the small-slope string model this gives

$$\left. \frac{\partial y}{\partial x} \right|_{x=L} = 0. \quad (16.22)$$

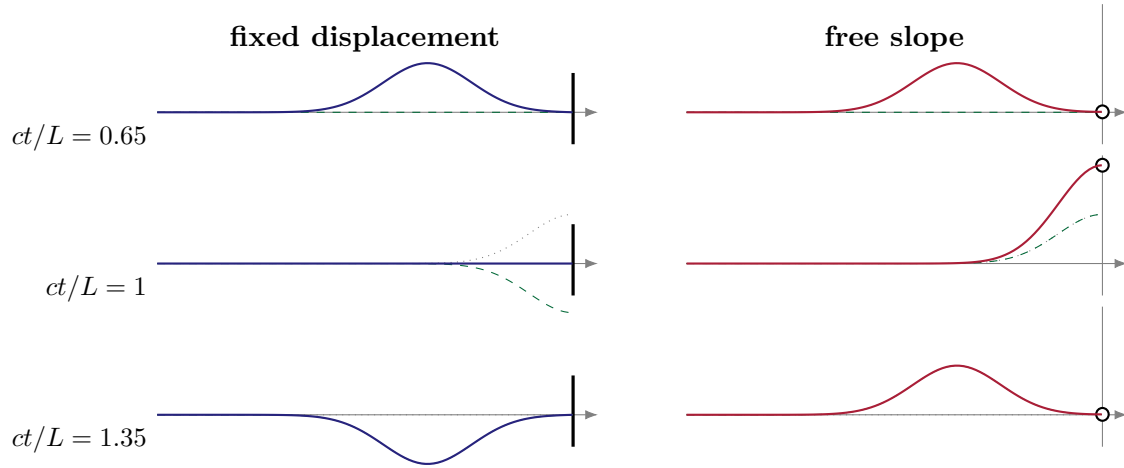
Let an incident right-going pulse be $y_i = f(x - c_w t)$. A reflected left-going pulse can be written $y_r = \sigma f(2L - x - c_w t)$, where σ is chosen to satisfy the endpoint condition. At $x = L$,

$$y_i + y_r = (1 + \sigma)f(L - c_w t).$$

Thus $\sigma = -1$ for a fixed end: the displacement pulse is inverted. For the derivative,

$$\partial_x(y_i + y_r) = f'(x - c_w t) - \sigma f'(2L - x - c_w t),$$

so $\sigma = +1$ satisfies the free-end slope condition: no displacement inversion.



dotted: incident; dashed: reflected; solid: sum
horizontal coordinate x/L ; displacement scaled by incident amplitude A

Figure 16.12: Gaussian reflection with centre initially at $x = 0$, width $\sigma = 0.15L$, and centre advance ct . A fixed endpoint enforces a zero displacement sum. An ideal massless ring sliding on a frictionless vertical guide enforces zero slope; at full overlap its displacement is $2A$. The ring/guide is schematic. The incident and reflected mathematical components are shown even while their tails overlap.

Example 16.11: Reflect a finite pulse at two ideal terminations

Take $f(\xi) = A \exp[-(\xi/\sigma)^2]$ on a lossless string ending at $x = L$. The total fixed-end solution is

$$y_{\text{fixed}} = f(x - c_w t) - f(2L - x - c_w t),$$

so $y_{\text{fixed}}(L, t) = 0$ exactly. The free-end solution is

$$y_{\text{free}} = f(x - c_w t) + f(2L - x - c_w t),$$

and differentiating shows $\partial_x y_{\text{free}}(L, t) = 0$ exactly. These algebraic residuals are stronger checks than drawing an inverted or non-inverted pulse by eye.

16.5.2 Reflection and transmission at a string junction

We now develop the interface statement rather than leaving it as a named impedance. Put the junction at $x = 0$. Assume two uniform strings under the same tension F_T , a massless junction with no external transverse force, and a single angular frequency ω . Their speeds and impedances are

$$c_j = \sqrt{F_T/\mu_j}, \quad Z_j = \frac{F_T}{c_j} = \mu_j c_j = \sqrt{F_T \mu_j}.$$

Z_j has units kg s^{-1} . It measures the magnitude ratio of the transverse force at a cut to the transverse velocity for one travelling component; the signed ratio also depends on travel direction and which side of the cut receives the force. It is distinct from tension itself.

Let the incident, reflected, and transmitted waves be

$$y_i = A_i \cos(k_1 x - \omega t), \quad y_r = A_r \cos(k_1 x + \omega t), \quad y_t = A_t \cos(k_2 x - \omega t).$$

Here A_r is a *signed* displacement coefficient, so a negative value represents a phase inversion; its magnitude is the reflected amplitude. Junction displacement continuity gives $A_i + A_r = A_t$. For the massless junction, the transverse tensions cancel, requiring $F_T(y_{i,x} + y_{r,x}) = F_T y_{t,x}$ at the junction. The sine-time coefficients then give $k_1(A_i - A_r) = k_2 A_t$, or $Z_1(A_i - A_r) = Z_2 A_t$. Solving these two simultaneous relations yields

$$r = \frac{A_r}{A_i} = \frac{Z_1 - Z_2}{Z_1 + Z_2}, \quad t = \frac{A_t}{A_i} = \frac{2Z_1}{Z_1 + Z_2} = 1 + r. \quad (16.23)$$

The symbol t in this equation denotes a transmission coefficient only; elsewhere t is time. Context and the ratio definition distinguish them.

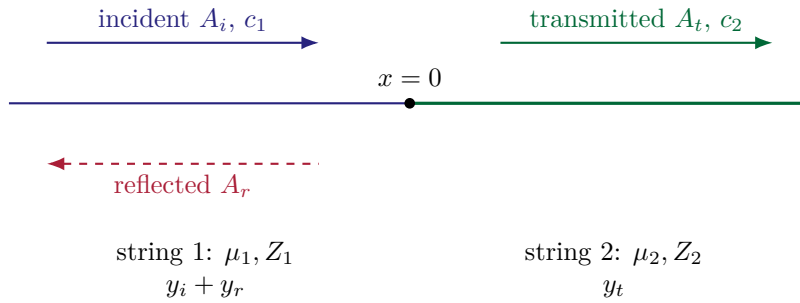


Figure 16.13: A massless junction under common tension. The arrows show propagation directions, rather than material velocity. String thickness is a schematic indicator of different line densities. Continuity of displacement and transverse force determines the signed amplitude coefficients.

A coefficient $t > 1$ is possible when the second string has smaller impedance. This does not mean more energy is transmitted than arrives. For a sinusoidal travelling component, $\langle \mathcal{P} \rangle = Z\omega^2 A^2/2$; therefore the reflected and transmitted *fractions of incident power* are

$$R = r^2, \quad \mathcal{T} = \frac{Z_2}{Z_1} t^2 = \frac{4Z_1 Z_2}{(Z_1 + Z_2)^2}, \quad R + \mathcal{T} = 1. \quad (16.24)$$

The reflected flux has a negative sign in the $+x$ convention, while R is its nonnegative magnitude fraction. The transmitted medium has the same temporal frequency, because junction motion cannot have two incompatible periods; its wavelength changes to $\lambda_2 = c_2/f$.

Example 16.12: A heavier string and a power check

Statement. Two strings have common tension 100 N and line densities $\mu_1 = 0.0100$ and $\mu_2 = 0.0400 \text{ kg m}^{-1}$. A wave of amplitude $A_i = 2.00 \text{ mm}$ and frequency 50.0 Hz arrives from string 1. Find both speeds, signed coefficients, transmitted wavelength, and power partition.

Solution. $c_1 = 100 \text{ m s}^{-1}$, $c_2 = 50.0 \text{ m s}^{-1}$, $Z_1 = 1.00$ and $Z_2 = 2.00 \text{ kg s}^{-1}$. Therefore

$$r = -\frac{1}{3}, \quad t = \frac{2}{3}, \quad A_r = -0.667 \text{ mm}, \quad A_t = 1.333 \text{ mm}.$$

The reflected displacement is inverted. Frequency is unchanged, so $\lambda_1 = 2.00 \text{ m}$ and $\lambda_2 = 1.00 \text{ m}$. Mean incident power is $Z_1(2\pi f)^2 A_i^2/2 = 0.1974 \text{ W}$. Its reflected magnitude is 0.02193 W and transmitted power 0.1755 W, corresponding to $R = 1/9$ and $\mathcal{T} = 8/9$.

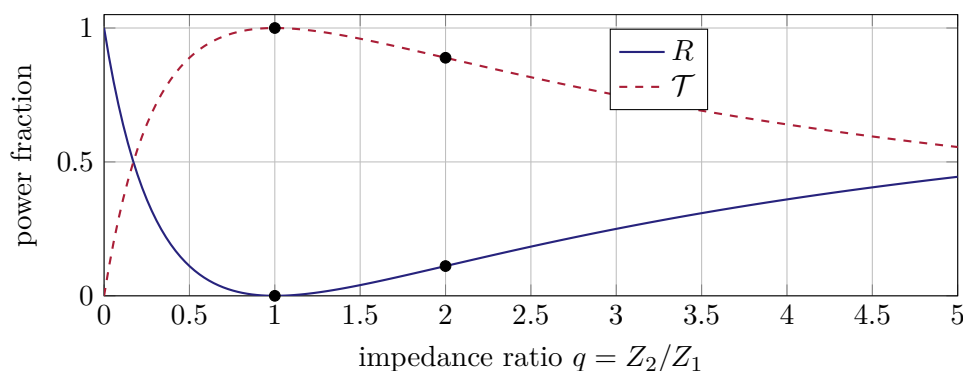


Figure 16.14: Energy partition at an ideal common-tension junction. Matched impedances ($q = 1$) produce no reflection. The $q \rightarrow 0$ and $q \rightarrow \infty$ limiting cases recover free and fixed reflection, respectively; finite physical line density corresponds to $q > 0$. The two curves sum to one at every q .

Checks. $A_i + A_r = A_t$, and $Z_1(A_i - A_r) = Z_2 A_t$. The two outgoing powers add to the incident power. The incident slope amplitude is 0.00628, and the transmitted one is 0.00838, both small. A junction with appreciable mass or an external support would require a different force balance.

16.6 Standing waves and normal modes

16.6.1 Standing waves from counter-propagation

Two equal-amplitude waves travelling in opposite directions,

$$y_+ = A \sin(k_w x - \omega t), \quad y_- = A \sin(k_w x + \omega t),$$

superpose to give

$$y(x, t) = 2A \sin(k_w x) \cos(\omega t). \quad (16.25)$$

This is a standing wave: the spatial factor and temporal factor separate. A node is a position where the displacement vanishes for *all* times, so $\sin(k_w x) = 0$. A point that happens to cross $y = 0$ in one snapshot is not necessarily a node.

16.6.2 Allowed modes from boundaries

For a string of length L fixed at both ends, $y(0, t) = 0$ is already satisfied by Eq. (16.25). The second boundary requires

$$\sin(k_w L) = 0 \implies k_w L = n\pi, \quad n = 1, 2, 3, \dots$$

Thus

$$k_{w,n} = \frac{n\pi}{L}, \quad \lambda_n = \frac{2L}{n}, \quad \omega_n = \frac{n\pi c_w}{L}, \quad f_n = \frac{nc_w}{2L}. \quad (16.26)$$

A convenient mode form is

$$y_n(x, t) = A_n \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t + \phi_n). \quad (16.27)$$

The spectrum came from the boundary conditions; the familiar pattern of loops is a consequence, not the starting rule.

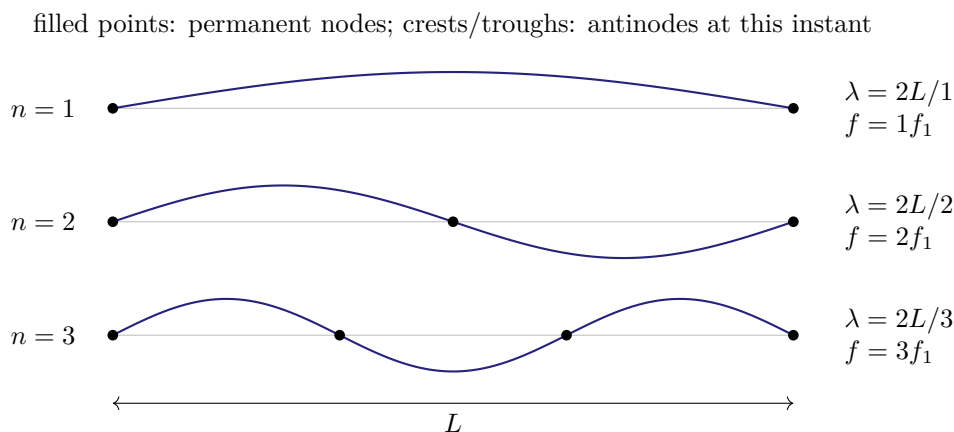


Figure 16.15: The first three fixed–fixed eigenfunctions at maximum displacement. The n th pattern has n half-wavelengths, $n + 1$ nodes including the endpoints, and n displacement antinodes. The same schematic vertical amplitude permits shape comparison; it does not prescribe equal modal energy.

Example 16.13: Choose tension for a prescribed mode frequency

A string of length $L = 0.800$ m and linear density $\mu_\ell = 0.0120$ kg m^{−1} is to have its third fixed-end mode at $f_3 = 150$ Hz. What ideal tension is required?

From Eq. (16.26),

$$c_w = \frac{2Lf_3}{3} = \frac{2(0.800)(150)}{3} = 80.0 \text{ m s}^{-1}.$$

Then

$$F_T = \mu_\ell c_w^2 = (0.0120)(80.0)^2 = 76.8 \text{ N}.$$

If the third-mode antinode amplitude is 2.0 mm, the maximum slope scale is

$$Ak_{w,3} = A \frac{3\pi}{L} = 0.0236,$$

so the small-slope condition is consistent with the selected amplitude.

Boundary check. $\sin(3\pi x/L)$ vanishes at both $x = 0$ and $x = L$. The mode contains three half-wavelengths in the length L .

Note

A pure standing mode has no net time-averaged travelling power through every cross-section in the same sense as a single right-going travelling wave. Energy is exchanged locally between kinetic and elastic forms and between neighbouring regions. Applying Eq. (16.13) blindly to the standing-wave amplitude would ignore the opposite-going component and its interference cross terms.

16.6.3 Standing-wave snapshots and the definition of a node

The counter-propagating wave identity chooses a time origin at a displacement maximum. Write the resulting mode as $y = B \sin(n\pi x/L) \cos(\omega_n t)$, where B is the antinode amplitude of the standing wave. For the equal travelling constituents used earlier, $B = 2A$. Keeping these two amplitudes distinct avoids a factor-of-four error when an energy formula contains the square of amplitude.

A node must satisfy $y(x_N, t) = 0$ for *every* t , which gives $x_N = jL/n$ for $j = 0, 1, \dots, n$. Antinodes occur where the spatial amplitude has magnitude B : $x_{AN} = (2j + 1)L/(2n)$ for $j = 0, \dots, n - 1$. Adjacent loops oscillate with opposite signs. They are π out of phase, but the pattern does not

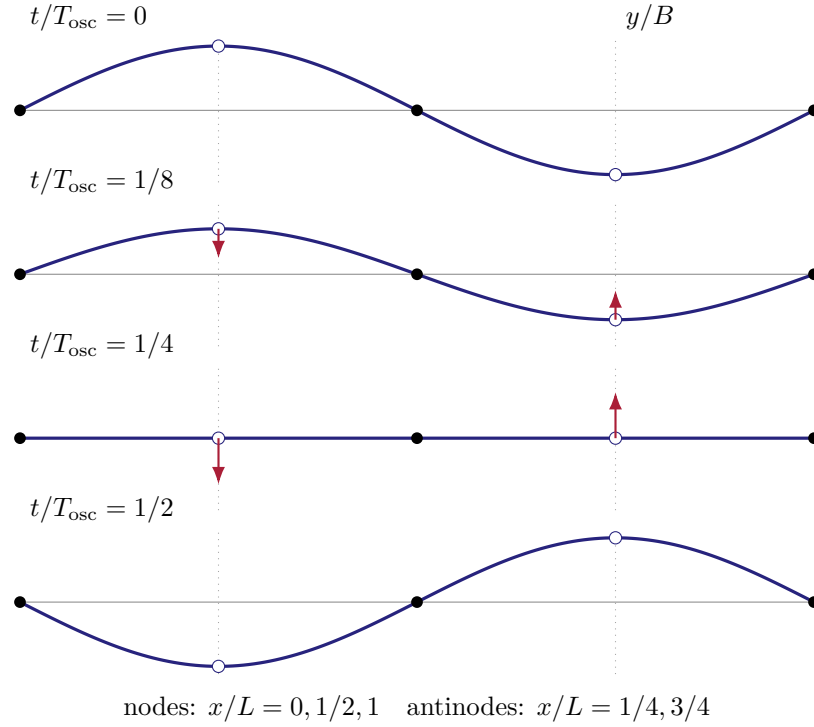


Figure 16.16: Original calculated snapshots of the second fixed–fixed mode, $y/B = \sin(2\pi x/L) \cos(2\pi t/T_{\text{osc}})$. Filled dots identify nodes; open dots mark the same two antinode elements in every row. Red arrows show transverse velocity directions at $T_{\text{osc}}/8$ and $T_{\text{osc}}/4$; they are schematic, with a larger common length at the equilibrium crossing. At $T_{\text{osc}}/4$ every point has zero displacement, but only the filled node positions remain at zero for all times. The rows share the same normalized spatial and displacement scales.

translate from one loop to the next. The antinode elements execute the largest local motion, while the mode frequency is common to all moving elements.

The velocity $v_y = -\omega_n B \sin(n\pi x/L) \sin(\omega_n t)$ provides an independent check on a snapshot sequence. At $t = 0$ all elements are instantaneously at rest. At $t = T_{\text{osc}}/4$ the string is straight and the antinode speed is maximal. At $t = T_{\text{osc}}/2$ the shape is the negative of its initial shape and the elements are again instantaneously at rest. A standing-wave animation that has maximal particle speed at a displacement extreme cannot represent this phase convention.

16.6.4 Where standing-wave energy is stored

The energy formulas for an ideal string apply to standing waves too, but the travelling-wave equality $u_K = u_U$ point by point does not. Put $k = n\pi/L$, $\omega = c_w k$, and $y = B \sin(kx) \cos(\omega t)$. Differentiating gives

$$u_K(x, t) = \frac{1}{2} \mu_\ell \omega^2 B^2 \sin^2(kx) \sin^2(\omega t), \quad (16.28)$$

$$u_U(x, t) = \frac{1}{2} F_T k^2 B^2 \cos^2(kx) \cos^2(\omega t) = \frac{1}{2} \mu_\ell \omega^2 B^2 \cos^2(kx) \cos^2(\omega t). \quad (16.29)$$

Thus antinodes can store kinetic energy, whereas displacement nodes can have nonzero elastic energy associated with the neighbouring slope. Calling a node an energy-free point confuses displacement with strain. At maximum deflection all energy is elastic; at the straight-string instant all energy is kinetic. At intermediate times both forms are present, with different spatial distributions.

Because $kL = n\pi$, both squared spatial factors have integral $L/2$. Integrating Eqs. (16.28) and (16.29) over the entire string therefore yields

$$E_K(t) = E \sin^2(\omega t), \quad E_U(t) = E \cos^2(\omega t), \quad E = \frac{\mu_\ell \omega^2 B^2 L}{4}. \quad (16.30)$$

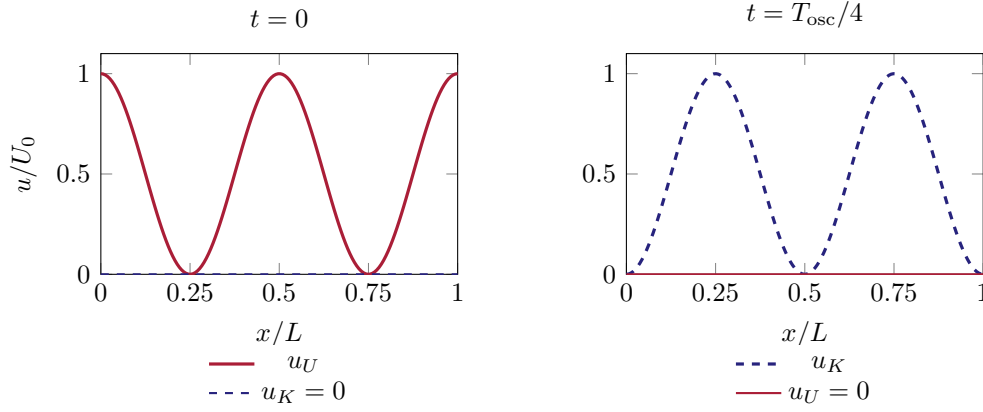


Figure 16.17: Original calculated line-energy densities of the second fixed–fixed mode, normalized by $U_0 = \mu_\ell \omega^2 B^2/2$ in J m^{-1} . At maximum deflection the elastic energy is largest near displacement nodes because the slope is largest there. A quarter period later the straight string has kinetic energy largest at antinodes. Each nonzero curve has spatial integral $L/2$ before multiplication by U_0 ; the total mode energy is unchanged.

The result has units of joules: $\mu_\ell L$ is a mass and ωB is a speed. It describes the conserved energy in one undamped normal mode, not the energy added each period by a continuously operating source.

Example 16.14: Energy and local motion in a standing mode

A fixed–fixed string has $L = 0.800 \text{ m}$, $\mu_\ell = 0.0120 \text{ kg m}^{-1}$, and $F_T = 76.8 \text{ N}$. Its third mode has antinode amplitude $B = 2.00 \text{ mm}$. Find the frequency, total energy, kinetic and elastic energies at $t = T_{\text{osc}}/8$, and maximum transverse element speed. Compare the total energy with the energies of the two equal counter-propagating constituents.

Solution. The wave speed is $c_w = \sqrt{76.8/0.0120} = 80.0 \text{ m s}^{-1}$, so $f_3 = 3c_w/(2L) = 150 \text{ Hz}$ and $\omega_3 = 300\pi \text{ s}^{-1}$. Hence

$$E = \frac{(0.0120)(300\pi)^2(0.00200)^2(0.800)}{4} = 8.53 \times 10^{-3} \text{ J}.$$

At $T_{\text{osc}}/8$, $\omega_3 t = \pi/4$ and both squared temporal factors equal $1/2$, so $E_K = E_U = 4.26 \text{ mJ}$. The maximum element speed is $\omega_3 B = 1.88 \text{ m s}^{-1}$. Each travelling constituent has amplitude $A = B/2 = 1.00 \text{ mm}$, so the energy associated with one constituent over a length L containing an integer number of half-wavelengths is $E_\pm = \mu_\ell \omega_3^2 A^2 L/2 = 4.26 \text{ mJ}$. Their sum is the standing-mode energy. Using B as the amplitude of each constituent would make this comparison four times too large. Finally, $Bk = 0.00200(3\pi/0.800) = 0.0236$, so the selected displacement is consistent with the small-slope approximation.

16.6.5 Zero mean flux does not mean zero instantaneous flux

At a fixed cross-section the signed mechanical power remains $\mathcal{P} = -F_T \partial_{xy} \partial_t y$. For the standing mode,

$$\mathcal{P}(x, t) = \frac{F_T k \omega B^2}{4} \sin(2kx) \sin(2\omega t). \quad (16.31)$$

This can be positive or negative. Positive power carries energy toward increasing x across the chosen cut, negative power toward decreasing x . Over one cycle the time average is zero at every cut because $\langle \sin(2\omega t) \rangle = 0$. Local regions nevertheless exchange energy during parts of that cycle.

One may check the flux against the local conservation equation $\partial_t(u_K + u_U) + \partial_x \mathcal{P} = 0$. With

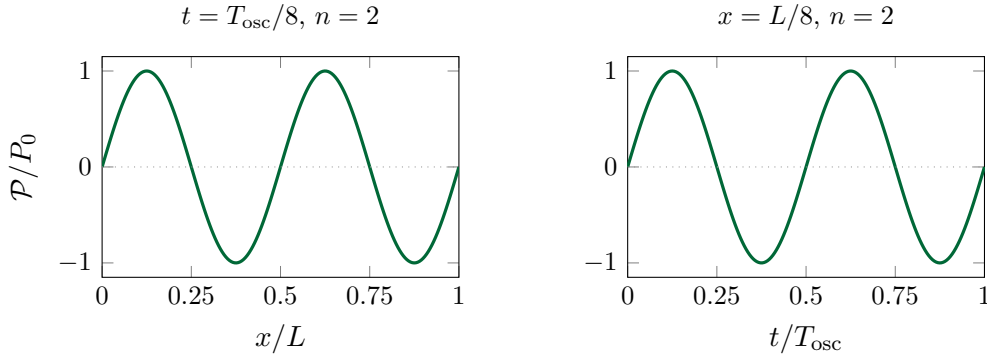


Figure 16.18: Original analytical standing-mode flux plots, normalized by $P_0 = F_T k \omega B^2 / 4$ in watts. Left: the spatial pattern of signed flux at $t = T_{\text{osc}}/8$ for $k = 2\pi/L$. Right: flux at $x = L/8$ during one displacement period; its two positive and two negative lobes have equal signed area. Flux vanishes at fixed ends and has zero time average at every position, while nonzero instantaneous values describe reversible energy exchange within the string.

$$U_0 = \mu_\ell \omega^2 B^2 / 2,$$

$$\partial_t(u_K + u_U) = -U_0 \omega \cos(2kx) \sin(2\omega t), \quad \partial_x \mathcal{P} = U_0 \omega \cos(2kx) \sin(2\omega t).$$

The terms cancel exactly because $F_T k^2 = \mu_\ell \omega^2$. An element at a node has no kinetic energy, but energy flux and the elastic energy in a nearby finite interval must be considered before drawing a conclusion about storage or transfer in that region.

16.6.6 A fixed end and an ideal free end select a different spectrum

Replace the right fixed support by a massless ring that can slide vertically without friction on a guide. In the linear model the guide balances horizontal tension but exerts no transverse force. The boundary conditions are therefore $Y(0) = 0$ and $Y'(L) = 0$ for the spatial mode shape. Starting from $Y = C \sin(kx) + D \cos(kx)$, the left condition sets $D = 0$ and the right condition gives $Ck \cos(kL) = 0$. A nontrivial oscillating mode needs $C \neq 0$ and $k > 0$, so

$$k_m L = \frac{(2m-1)\pi}{2}, \quad \lambda_m = \frac{4L}{2m-1}, \quad f_m = \frac{(2m-1)c_w}{4L}, \quad m = 1, 2, 3, \dots \quad (16.32)$$

Here m is the ordinal *mode index*. The associated harmonic numbers relative to the fundamental are 1, 3, 5, ... Calling $m = 2$ the “third mode” confuses its frequency ratio with its place in the allowed sequence.

At the free end the displacement is an antinode and the transverse tension component vanishes. A real end fitting with non-negligible mass has a different dynamical boundary condition; it cannot automatically be treated as either a perfectly fixed or a perfectly free endpoint. The ideal model is useful precisely because its limiting assumptions are stated.

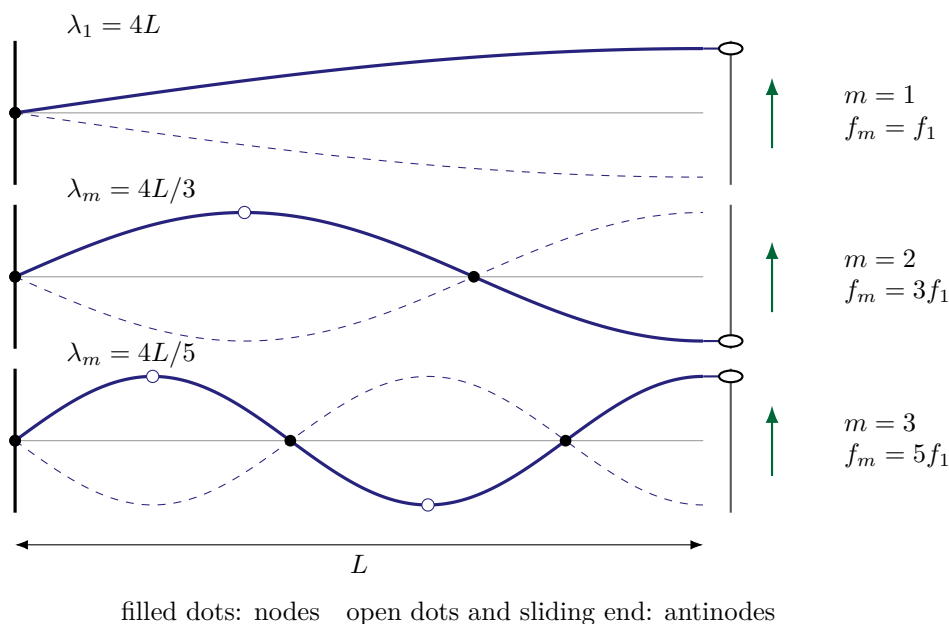


Figure 16.19: Original analytical first three fixed-free mode families: $Y_m/B = \sin[(2m-1)\pi x/(2L)]$, shown at positive and negative temporal extrema. The left endpoint is fixed, and the right endpoint is an ideal massless ring sliding on a frictionless vertical guide. Every curve has zero displacement at the fixed end and zero slope at the free end. The guide, ring and green motion arrow are schematic; vertical displacement is exaggerated. The allowed frequency ratios are $1 : 3 : 5$, although the mode indices are $m = 1, 2, 3$.

Example 16.15: Infer a mixed-boundary string from two resonances

A string with one fixed end and one ideal free end has $L = 0.600 \text{ m}$ and $\mu_\ell = 0.00800 \text{ kg m}^{-1}$. Its first two resonances are measured as 50.0 Hz and 150.0 Hz in the idealized dataset. Find the wave speed and tension, predict the third resonance, and locate the internal node and antinode of the second mode. Then predict the fundamental if the free end is clamped without changing L , F_T , or μ_ℓ .

Solution. The ratio $150.0/50.0 = 3$ is consistent with the first two mixed-boundary modes. Their fundamental gives

$$c_w = 4Lf_1 = 4(0.600)(50.0) = 120 \text{ m s}^{-1}, \quad F_T = \mu_\ell c_w^2 = (0.00800)(120)^2 = 115.2 \text{ N}.$$

The third allowed frequency is $f_3 = 5f_1 = 250 \text{ Hz}$, not $3f_1$. For the second mode, $Y_2 \propto \sin(3\pi x/(2L))$. The internal node is at $x = 2L/3 = 0.400 \text{ m}$, and its internal antinode at $x = L/3 = 0.200 \text{ m}$; the free endpoint at $x = L$ is also an antinode, with the opposite sign of displacement. Differentiation verifies $Y_2'(L) \propto \cos(3\pi/2) = 0$. Clamping the right end changes the boundary condition to $Y(L) = 0$, so the new fundamental is $c_w/(2L) = 100 \text{ Hz}$. The propagation speed stays at 120 m s^{-1} under the stated unchanged string parameters; the altered resonance comes from a different admissible mode shape. The resonance measurements establish consistency with the stipulated boundary model, rather than uniquely proving the absence of all other physical effects in a real apparatus.

16.7 Wave packets and Fourier synthesis

A single sinusoid extends across all space or across the entire idealised string. A localised pulse or a plucked-string shape is not a single sinusoid. In a linear system, however, a complicated shape can be represented as a sum of normal modes.

16.7.1 Orthogonality and modal coefficients

For fixed ends, the spatial eigenfunctions are $\sin(n\pi x/L)$. They satisfy the orthogonality relation

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0, & m \neq n, \\ L/2, & m = n. \end{cases} \quad (16.33)$$

Suppose the initial displacement is a prescribed function $g(x)$ that vanishes at both ends. We seek

$$g(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right). \quad (16.34)$$

Multiply by the m th sine function, integrate from 0 to L , and use orthogonality. Every term vanishes except $n = m$, giving

$$A_m = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{m\pi x}{L}\right) dx. \quad (16.35)$$

The initial velocity must be specified separately. If $\partial_t y(x, 0) = 0$, each mode can start at a displacement extremum and the subsequent motion is

$$y(x, t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t). \quad (16.36)$$

16.7.2 A triangular pluck

Let a string of length L be displaced to height h at $x = a$ and held piecewise straight to the two fixed ends:

$$g(x) = \begin{cases} hx/a, & 0 \leq x \leq a, \\ h(L-x)/(L-a), & a \leq x \leq L. \end{cases} \quad (16.37)$$

Evaluating Eq. (16.35) on the two intervals gives

$$A_n = \frac{2hL^2}{\pi^2 n^2 a(L-a)} \sin\left(\frac{n\pi a}{L}\right). \quad (16.38)$$

The $1/n^2$ envelope reflects the fact that $g(x)$ is continuous but has a corner at the pluck point. A finite-mode truncation is an approximation to the prescribed initial shape, not a new physical force.

Example 16.16: Reconstruct a plucked-string shape

Take $L = 1.00$ m, $a = 0.35$ m, and $h = 10.0$ mm, with zero initial velocity. Equation (16.38) gives the first coefficients

$$A_1 = 7.9365 \text{ mm}, \quad A_2 = 1.8016 \text{ mm}, \quad A_3 = -0.1548 \text{ mm}, \quad A_4 = -0.5295 \text{ mm}.$$

The signs matter: a negative coefficient means that the corresponding sine mode enters with opposite phase at $t = 0$.

A numerical quadrature over $0 \leq x \leq L$ gives the root-mean-square reconstruction errors

$$\varepsilon_{\text{RMS}} = 0.452 \text{ mm} \quad (N = 3), \quad 0.116 \text{ mm} \quad (N = 7), \quad 0.0432 \text{ mm} \quad (N = 15).$$

The sequence shows convergence for this continuous shape. It does not imply that every visible local feature decreases monotonically with each additional mode.

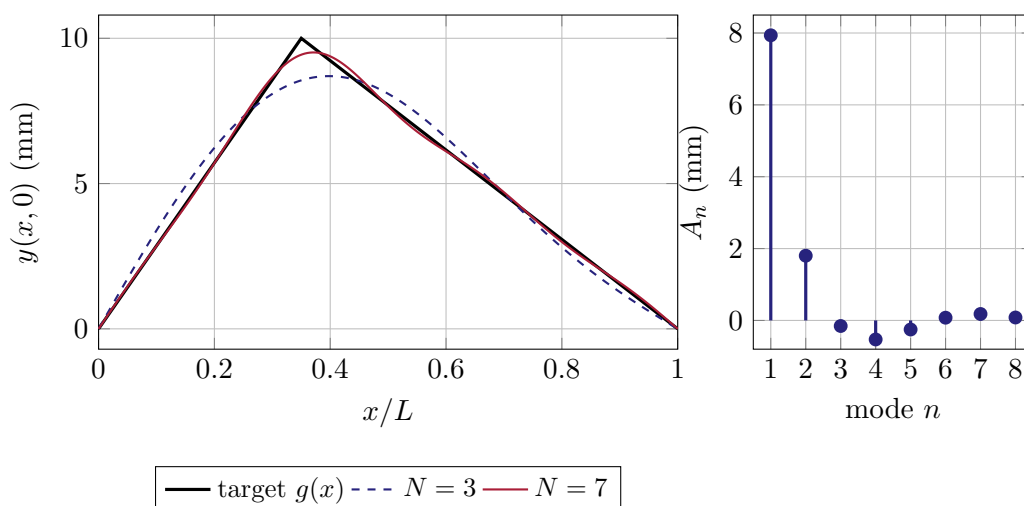


Figure 16.20: Fourier reconstruction of a synthetic triangular pluck with $L = 1.00$ m, $a = 0.35L$, and $h = 10.0$ mm. The left panel compares the exact specified initial shape with finite $N = 3$ and $N = 7$ sine sums. The right panel shows the first eight calculated coefficients. All curves satisfy the fixed-end displacement condition.

Guided checks

Check 1. Why do all finite sine sums satisfy $y(0, t) = y(L, t) = 0$ automatically? *Reasoning:* $\sin(0) = \sin(n\pi) = 0$ for every integer n , so the weighted sum also vanishes at each end at all times.

Check 2. Does specifying $g(x)$ determine the future motion completely? *Reasoning:* a second-order time equation needs both displacement and velocity at release. Different initial velocity fields give different sine-time coefficients even with the same $g(x)$.

Check 3. If the string is plucked exactly at $a = L/2$, what happens to even- n coefficients? *They vanish because $\sin(n\pi/2) = 0$ for even n .*

16.7.3 Why orthogonality isolates one mode

The product-to-sum identity gives

$$\sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) = \frac{1}{2} \left[\cos\left(\frac{(n-m)\pi x}{L}\right) - \cos\left(\frac{(n+m)\pi x}{L}\right) \right].$$

For unequal positive integers m, n , both cosine integrals vanish over $[0, L]$. For $m = n$, the first cosine is one, so its integral is L , while the second still integrates to zero. This proves Eq. (16.33) and explains the factor $2/L$ in each coefficient. Projection is a mathematical filtering operation: each integral measures how much of one spatial shape is present in the initial displacement.

For the triangular pluck, define $K = n\pi/L$. The two contributions are

$$\begin{aligned} \int_0^a \frac{hx}{a} \sin Kx \, dx &= -\frac{h \cos Ka}{K} + \frac{h \sin Ka}{aK^2}, \\ \int_a^L \frac{h(L-x)}{L-a} \sin Kx \, dx &= \frac{h \cos Ka}{K} + \frac{h \sin Ka}{(L-a)K^2}. \end{aligned}$$

The $\cos Ka/K$ terms cancel because the shape is continuous at the join. Multiplying the sum by $2/L$ gives Eq. (16.38). The slopes h/a and $-h/(L-a)$ should also be small if this prescribed shape is assigned to an ideal taut string. A sharply localised pluck near an endpoint may violate that condition even when h/L appears small.

16.7.4 Both initial fields determine the motion

Write the initial velocity as $h_v(x) = \partial_t y(x, 0)$, using a subscript to distinguish it from the pluck height h . A compatible modal expansion is

$$y(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) [a_n \cos \omega_n t + b_n \sin \omega_n t], \quad (16.39)$$

where

$$a_n = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad b_n = \frac{2}{L\omega_n} \int_0^L h_v(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

The units of both coefficients are metres. The factor $1/\omega_n$ converts a velocity projection into a displacement coefficient. Release from rest sets every $b_n = 0$; it is an independent initial condition, rather than a property inferred from a photograph of the pluck.

Example 16.17: A two-mode initial-value problem

Statement. A fixed-fixed string with $L = 1.00$ m and $c_w = 10.0$ m s⁻¹, with $\mu_\ell = 0.0200$ kg m⁻¹, starts with $g(x) = (2.00$ mm) $\sin(\pi x/L)$ and $h_v(x) = (0.100$ m s⁻¹) $\sin(2\pi x/L)$. Obtain the exact motion within the model and the total energy.

Solution. Only $a_1 = 0.00200$ m and $b_2 = 0.100/\omega_2$ are nonzero. Since $\omega_1 = 10\pi$ and $\omega_2 = 20\pi$ s⁻¹, $b_2 = 1.59155$ mm, and

$$y = 0.00200 \sin(\pi x/L) \cos(10\pi t) + \frac{0.100}{20\pi} \sin(2\pi x/L) \sin(20\pi t),$$

with displacement in metres and time in seconds. It satisfies both boundary conditions and, by evaluating it and its time derivative at zero, both initial fields.

Energy. For a specified $\mu_\ell = 0.0200$ kg m⁻¹, orthogonality gives

$$E = \frac{\mu_\ell L}{4} \sum_n \omega_n^2 (a_n^2 + b_n^2) = \frac{0.0200}{4} \left[(10\pi)^2 (0.00200)^2 + 0.100^2 \right] = 6.974 \times 10^{-5} \text{ J}.$$

At release the first contribution is elastic and the second kinetic. Their different initial roles do not imply that the modes subsequently cease to exchange their own kinetic and elastic energies.

16.7.5 Displacement convergence does not guarantee energy convergence

For the triangular pluck released from rest, the initial energy follows directly from its piecewise constant slopes:

$$E = \frac{F_T h^2}{2} \left(\frac{1}{a} + \frac{1}{L-a} \right) = \frac{F_T h^2 L}{2a(L-a)}. \quad (16.40)$$

The energy of its finite N -mode representation is $E_N = \mu_\ell L \sum_{n=1}^N \omega_n^2 A_n^2 / 4$. Since A_n has a $1/n^2$ envelope while $\omega_n \propto n$, modal energy has a $1/n^2$ envelope. The energy tail decays more slowly than the mean-square displacement tail, whose terms scale as $1/n^4$. A truncated shape can therefore look close to the target while retaining an appreciable error in slope and stored energy.

Figure 16.21 uses the same $a/L = 0.35$ as Figure 16.20. It shows the retained energy fraction, which is independent of the common multiplicative factors F_T and h^2 . A 100% reference line is a conservation target, not a fitted data line. Every finite partial sum has $E_N \leq E$ because the discarded modal energies are nonnegative.

16.8 Engineering, computation, and measurement

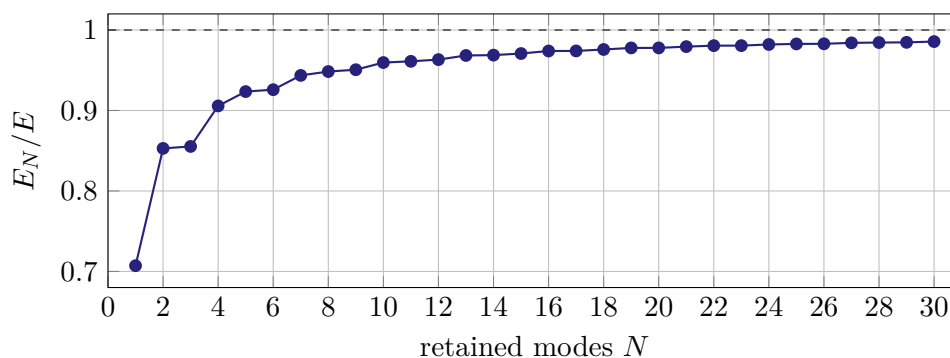


Figure 16.21: Energy retained by the finite sine representation of a triangular pluck at $a = 0.35L$. The values are calculated from $\sum_{n \leq N} \omega_n^2 A_n^2$ and divided by the direct piecewise-slope energy. The slow tail is associated with the corner of the specified initial shape; it is not numerical dissipation during propagation.

16.8.1 Tensioned-wire resonance as a parameter-sensitivity model

Consider an ideal string or light wire of fixed length L and line density μ_ℓ . The fixed-end frequencies are

$$f_n = \frac{n}{2L} \sqrt{\frac{F_T}{\mu_\ell}}.$$

Taking a logarithmic differential gives

$$\frac{df_n}{f_n} = -\frac{dL}{L} + \frac{1}{2} \frac{dF_T}{F_T} - \frac{1}{2} \frac{d\mu_\ell}{\mu_\ell}. \quad (16.41)$$

Thus a 1% increase in tension changes the ideal frequency by about 0.5%, while a 1% increase in length changes it by about -1% if the other parameters are independently held fixed.

This calculation is useful for a tensioned-wire sensor or string-resonance estimate, but it is not a complete vibration specification for a real cable. Bending stiffness, sag, nonuniform tension, end compliance, damping, and coupled supports can shift or broaden resonances.

Example 16.18: Sensitivity of an ideal resonance frequency

A wire has $L = 1.20$ m, $\mu_\ell = 5.0 \times 10^{-3} \text{ kg m}^{-1}$, and $F_T = 180$ N. Its ideal fundamental frequency is

$$f_1 = \frac{1}{2(1.20)} \sqrt{\frac{180}{5.0 \times 10^{-3}}} = 79.1 \text{ Hz}.$$

If the tension increases by 4.0% with L and μ_ℓ fixed, the linear sensitivity estimate predicts $\Delta f/f \simeq 2.0\%$. The exact square-root factor is $\sqrt{1.04} = 1.01980$, so the exact increase is 1.98%. The first-order estimate is adequate here.

16.8.2 Computational investigation: boundary residuals before animations

A useful computation is to construct a finite sum of exact fixed-end modes and verify three properties before plotting it:

1. the boundary residuals $|y(0, t)|$ and $|y(L, t)|$ are near machine round-off;
2. each component satisfies $\partial_t^2 y = c_w^2 \partial_x^2 y$ analytically;
3. the sum satisfies the same equation because the model is linear.

The output can then include spatial snapshots, time traces, and modal energy shares. A visually smooth animation is not a verification test by itself.

16.8.3 Measurement investigation: observing low-amplitude modes

A proposed educational investigation can use a light string under a controlled, modest tension and a low-amplitude periodic drive, or it can analyse a supplied recording. The measured quantities should include the vibrating length L , tension F_T , and line density $\mu_\ell = m/\ell$ independently. The observed resonance peaks are compared with $f_n = n\sqrt{F_T/\mu_\ell}/(2L)$.

Method

A defensible resonance comparison records: the measured length and its endpoint definition; how tension was inferred or measured; how line density was obtained; drive frequency resolution; end-condition uncertainty; repeatability; and whether the measured peak width is dominated by damping or instrumental resolution. A finite resonance width is not the undamped normal frequency itself.

High-tension improvised apparatus is unnecessary for this chapter. If a real setup is not available, a clearly labelled synthetic dataset with known parameters is preferable to invented measurements.

16.8.4 A developed synthetic calibration and residual test

The following dataset is synthetic teaching data, not measurements from a physical wire. A hypothetical string has $L = 1.00$ m and nominal $\mu_\ell = 0.0100$ kg m⁻¹. The ideal fundamental obeys

$$f_1^2 = \frac{F_T}{4L^2\mu_\ell}.$$

A plot of f_1^2 against tension should therefore be linear through the origin if the ideal assumptions apply. This representation tests the square-root law more directly than an arbitrary straight-line fit of f_1 against tension.

F_T (N)	36	49	64	81	100
f_1 (Hz)	30.2	34.8	40.1	44.9	50.2
f_1^2 (Hz ²)	912.04	1211.04	1608.01	2016.01	2520.04

The five frequencies were obtained by adding the stipulated deviations +0.2, -0.2, +0.1, -0.1, +0.2 Hz to the ideal values. This construction makes the source of the scatter explicit. It supplies no empirical distribution or measured uncertainty by itself.

For an illustrative equal-weight fit constrained through the origin, let $s_i = f_i^2$ and minimise $\sum_i (s_i - bF_i)^2$. Differentiation with respect to b gives

$$b = \frac{\sum_i F_i s_i}{\sum_i F_i^2} = 25.0631 \text{ Hz}^2 \text{ N}^{-1}.$$

The density inferred from that model is $\mu_\ell^{\text{fit}} = 1/(4L^2b) = 0.00997481$ kg m⁻¹. This is close to the synthetic generating value, as expected. The fit procedure has stipulated equal weights in f^2 , rather than asserting that real frequency observations have equal variance after squaring.

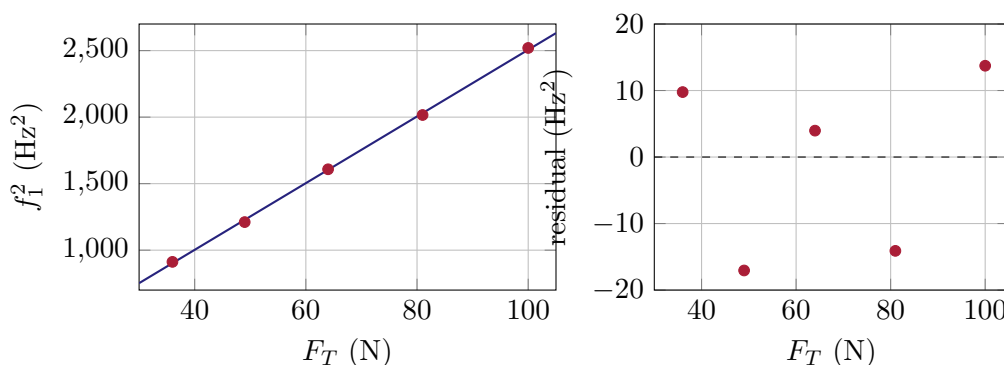


Figure 16.22: Synthetic tension–frequency calibration: observations and constrained fit (left), and $f_i^2 - bF_i$ (right). Residuals display the deviations hidden by the scale of the main graph. They provide a model diagnostic; five synthetic points do not establish a statistical distribution or prove the validity of an actual string apparatus.

Example 16.19: Interpret a resonance calibration

Statement. Use the displayed synthetic dataset to predict the fundamental at $F_T = 70.0\text{ N}$, estimate line density from the constrained fit, and explain how an endpoint error could bias the result. Give a reasoned assessment of the fit’s limitations.

Solution. The fitted square-frequency prediction is $f_1 = \sqrt{bF_T} = \sqrt{25.0631(70.0)} = 41.886\text{ Hz}$, and the inferred density is $0.00997481\text{ kg m}^{-1}$. The prediction lies within the tension range, so it is interpolation rather than extrapolation. If the effective vibrating length is larger than the recorded length while the frequencies are otherwise ideal, the inferred density satisfies $\mu_\ell^{\text{fit}} = \mu_\ell^{\text{true}}(L_{\text{effective}}/L_{\text{recorded}})^2$ and is biased high.

Interpretation and criteria. A defensible analysis identifies which endpoints actually constrain displacement, checks each mode’s integer spacing, documents tension and density measurements, and inspects residuals versus tension and mode number. Systematic curvature can suggest varying tension, sag, or stiffness; a frequency calibration bias can shift all inferred parameters coherently. The fit is useful for teaching parameter inference but does not distinguish these alternatives with only the five supplied points.

16.8.5 Uncertainty and model discrepancy have different roles

If length, tension, and density have independent standard uncertainties, first-order propagation gives

$$\left(\frac{u(f_n)}{f_n}\right)^2 \simeq \left(\frac{u(L)}{L}\right)^2 + \frac{1}{4}\left(\frac{u(F_T)}{F_T}\right)^2 + \frac{1}{4}\left(\frac{u(\mu_\ell)}{\mu_\ell}\right)^2.$$

This formula does not follow from adding maximum errors: it follows from the variance of a linearised model and the stated independence. If the density was inferred using the same measured length, correlations may enter, and the general covariance formula must be used. A common frequency timebase error also affects several peaks together; it need not show up as random scatter between them.

A proposed experiment can record low-amplitude motion with a camera or displacement sensor, fit a sinusoid at a marked coordinate, and compare frequency with the resonance spectrum. To estimate transit speed, use the lag of the same pulse feature at two locations. To estimate mean power, independently obtain μ_ℓ , f , A , and c_w and propagate their uncertainties; observing a standing mode does not by itself measure the mean power of one travelling component. An acceptable report states observable, inference, model, uncertainty, residuals, and what evidence would contradict the model. These are evaluation criteria for the proposed investigation, not a claim that it has been performed.

16.9 Advanced Topics for Technical and Engineering Students

The ideal linear string has $\omega = c_w k_w$: every harmonic component travels with the same phase speed. A spatially localised waveform therefore translates without changing shape in that model. Real wave systems can disperse because stiffness, geometry, microstructure or coupling supplies an additional scale. Here we preserve the ideal-string result and examine two different extensions: a *stipulated* dispersive comparison, and the errors introduced when the *nondispersive* equation is solved on a grid. Separating these questions is essential when interpreting a computed waveform.

The prerequisites are superposition, differentiation and the interpretation of sinusoidal phase. The additional mathematics is introduced as needed: a local Taylor expansion of frequency, the complex notation $e^{i\phi} = \cos \phi + i \sin \phi$, and a discrete Fourier trial solution. The complex notation is a calculation device; its real part represents displacement. Neither the comparison law nor the numerical discretisation below establishes a new constitutive law for a real cable.

16.9.1 Phase speed, envelope speed and a controlled comparison

For a right-moving harmonic component with $k_w > 0$, the phase $k_w x - \omega t$ is constant along a crest. Differentiating that constant-phase condition gives $k_w dx - \omega dt = 0$. Consequently,

$$v_p = \frac{\omega}{k_w}, \quad v_g = \frac{d\omega}{dk_w}. \quad (16.42)$$

The second definition requires explanation: it describes the envelope of a sufficiently narrow spectrum, not an individual sinusoidal crest. For two components centred on k_0 and separated by δk , the cosine addition identity gives a rapidly oscillating carrier multiplied by a slow modulation. The modulation phase is $(\delta k x - \delta \omega t)/2$, so its speed is $\delta \omega / \delta k$. When the separation becomes small, this quotient becomes $\omega'(k_0)$.

The envelope moves with v_g only within the assumptions of the narrowband construction. An arbitrary sharply switched signal contains a broad range of wave numbers; a group-speed calculation alone does not establish the speed of its leading disturbance. Nor should group speed automatically be identified with energy speed without an energy-transport law for the specified physical system.

Take the prescribed comparison

$$\omega(k_w) = c_0 k_w \sqrt{1 + \ell_d^2 k_w^2}, \quad k_w > 0, \quad (16.43)$$

where $c_0 > 0$ is a *reference speed* and $\ell_d \geq 0$ a length. The symbol c_0 does not denote the speed of light. This law is used as a mathematical model; no claim is made that its parameters describe the ideal taut string. Dimensional consistency requires $\ell_d k_w$ to be dimensionless. Dividing by k_w gives

$$v_p = c_0 \sqrt{1 + \ell_d^2 k_w^2}, \quad (16.44)$$

while the product and chain rules give

$$\frac{d\omega}{dk_w} = c_0 \sqrt{1 + \ell_d^2 k_w^2} + \frac{c_0 \ell_d^2 k_w^2}{\sqrt{1 + \ell_d^2 k_w^2}}.$$

Combining the numerator terms therefore yields

$$v_g = c_0 \frac{1 + 2\ell_d^2 k_w^2}{\sqrt{1 + \ell_d^2 k_w^2}}. \quad (16.45)$$

The difference $v_g - v_p = c_0 \ell_d^2 k_w^2 / \sqrt{1 + \ell_d^2 k_w^2}$ is positive when $\ell_d k_w \neq 0$. Both speeds approach c_0 as $\ell_d k_w \rightarrow 0$. At large $\ell_d k_w$, their ratio tends to two. This high-wave-number limit is a statement about the stipulated function, not evidence that a real cable follows it at arbitrarily short wavelengths.

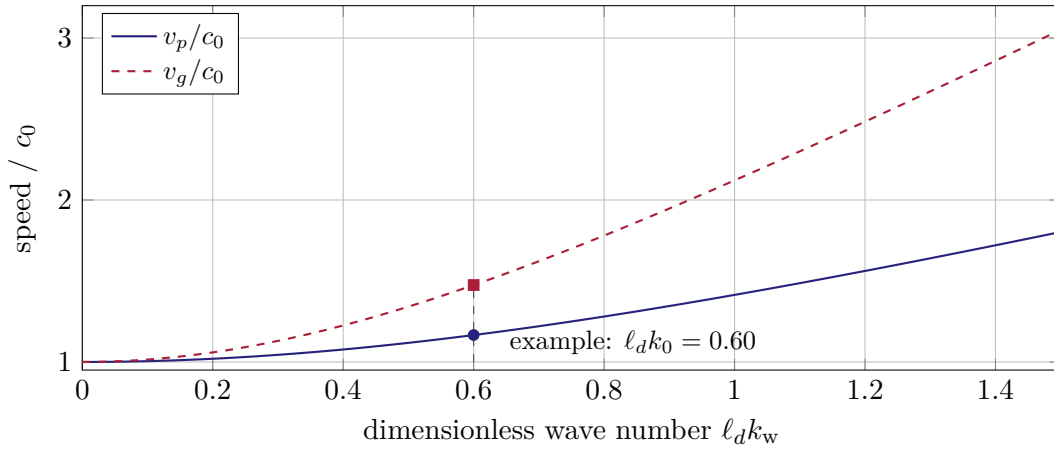


Figure 16.23: Phase and group speeds for the prescribed comparison law. The horizontal variable is dimensionless; both curves use the same reference speed. The marked carrier has $v_g > v_p$, so carrier crests drift backward relative to its moving envelope. The common limit at the origin recovers nondispersive translation.

Example 16.20: Compare carrier and envelope motion

A narrow spectrum is centred at $k_0 = 15.0 \text{ m}^{-1}$ in the stipulated model with $c_0 = 20.0 \text{ m s}^{-1}$ and $\ell_d = 0.040 \text{ m}$. Find the carrier frequency, the two speeds, and their predicted displacements in 0.050 s .

Since $(\ell_d k_0)^2 = 0.36$,

$$\omega_0 = 20(15)\sqrt{1.36} = 349.86 \text{ rad s}^{-1}, \quad f_0 = \frac{\omega_0}{2\pi} = 55.68 \text{ Hz}.$$

Equations (16.44) and (16.45) give

$$v_p = 23.3238 \text{ m s}^{-1}, \quad v_g = 29.4978 \text{ m s}^{-1}.$$

Thus a constant carrier phase advances by $v_p t = 1.1662 \text{ m}$, whereas the first-order envelope advances by $v_g t = 1.4749 \text{ m}$. In the envelope frame the carrier speed is $v_p - v_g = -6.1739 \text{ m s}^{-1}$: a crest can pass through the packet rather than remaining its centre. These are predictions for the specified narrowband comparison, with the displayed precision useful for checking a plot. They do not replace $c_w = \sqrt{F_T/\mu_\ell}$ in an ideal-string calculation.

16.9.2 A finite spectrum and the limits of rigid envelope translation

A concrete packet must specify its amplitudes, wave numbers and frequencies. Define

$$y(x, t) = \sum_{j=-24}^{24} B_j \cos[k_j x - \omega(k_j)t], \quad (16.46)$$

with

$$k_j = k_0 + j\delta k, \quad \delta k = 0.25 \text{ m}^{-1}, \quad B_j = \frac{\exp[-(k_j - k_0)^2/(2\sigma_k^2)]}{\sum_{r=-24}^{24} \exp[-(k_r - k_0)^2/(2\sigma_k^2)]}, \quad \sigma_k = 1.50 \text{ m}^{-1}.$$

Here $k_0 = 15.0 \text{ m}^{-1}$, all 49 components have positive wave number ($9 \leq k_j \leq 21 \text{ m}^{-1}$), and $\sum B_j = 1$ normalises the initial peak. The Gaussian spectral weights are cut at four standard deviations. A finite set of sinusoids is not a function of strictly finite spatial support. This particular set repeats

after $L_p = 2\pi/\delta k = 25.133$ m; it represents a localised packet in a periodic cell. The plotted window shows one packet, far from its periodic replicas. Its squared amplitude is integrable over that cell, while infinitely repeating the same wave would not give finite total energy on the whole real line.

An exact modulation envelope for this specified sum is obtained from

$$z(x, t) = \sum_j B_j e^{i[k_j x - \omega(k_j)t]}, \quad y = \Re z, \quad E(x, t) = |z(x, t)|.$$

This is the analytic-signal envelope of the positive-wave-number construction. It bounds the plotted real displacement, $|y| \leq E$, without assigning a separate envelope displacement to the material.

Writing $q = k - k_0$, a first-order Taylor expansion gives

$$\omega(k_0 + q) = \omega_0 + v_g q + O(q^2).$$

If the neglected phase remains small over the significant spectrum and time interval, then

$$z(x, t) \simeq e^{i(k_0 x - \omega_0 t)} \sum_j B_j e^{iq_j(x - v_g t)}.$$

The envelope is approximately translated while the carrier phase follows v_p . A useful local check is $\sigma_k/k_0 \ll 1$, together with $|\omega''(k_0)|t\sigma_k^2 \ll 1$. The first condition compares spectral width with the carrier wave number; the second limits accumulated quadratic phase. Narrowness at $t = 0$ does not guarantee rigid translation at arbitrarily late times.

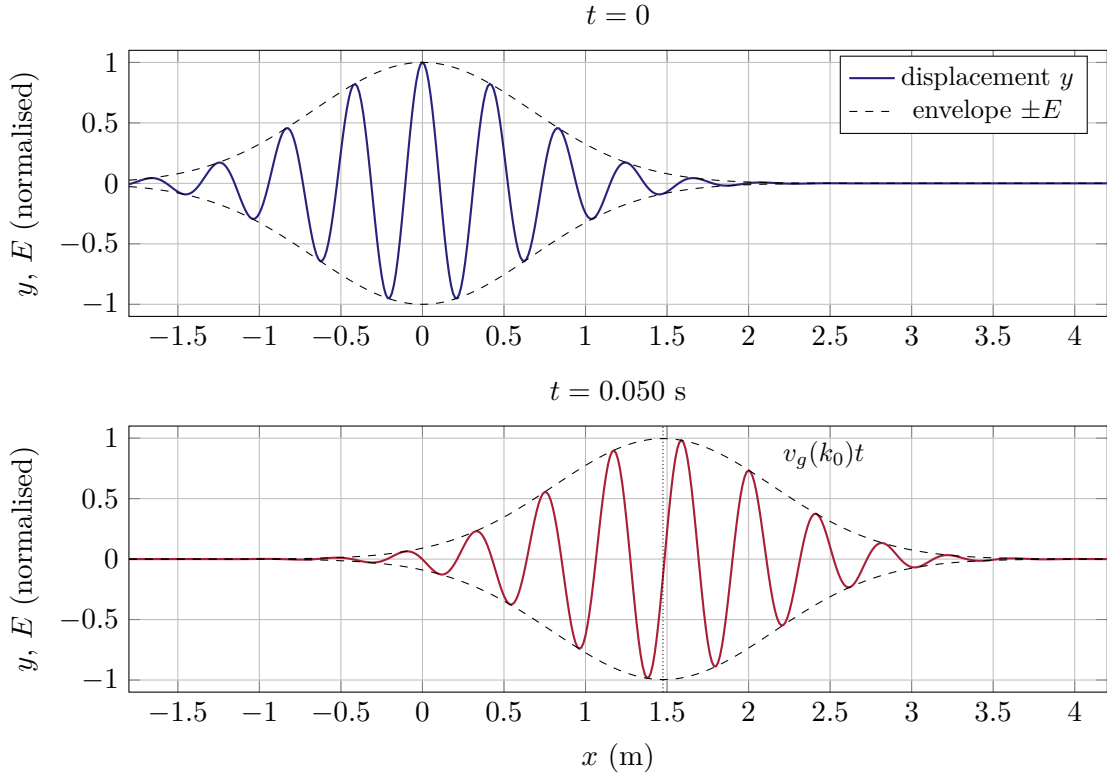


Figure 16.24: Explicit 49-component summation for Eq. (16.43), with the *same* amplitudes and wave numbers at both times. Solid curves show the real displacement and dashed curves its exact positive and negative envelope $\pm|z|$. The envelope travels right approximately 1.475 m in 0.050 s; the carrier moves through it. This is a periodic finite-spectrum calculation, not a numerical solution of the ideal-string PDE.

The finite sum supplies a useful conserved check: orthogonality of distinct periodic Fourier components gives $\int_{-L_p/2}^{L_p/2} |z|^2 dx = L_p \sum_j B_j^2$, independent of time. This squared-envelope integral is a mathematical spectral invariant. Calling it the mechanical energy would require an additional energy model, which the stipulated frequency law alone does not supply.

Example 16.21: Estimate packet spreading and verify its scale

For the same spectrum, estimate the envelope spreading by $t = 0.050$ s. Then compare the estimate with the exact finite-component sum.

Differentiate Eq. (16.45) once more:

$$\omega''(k_0) = c_0 \ell_d^2 k_0 \frac{3 + 2\ell_d^2 k_0^2}{(1 + \ell_d^2 k_0^2)^{3/2}} = 1.12584 \text{ m}^2 \text{ s}^{-1}.$$

Retaining the quadratic Taylor term and replacing the finely spaced spectral sum by a Gaussian integral gives an envelope proportional to

$$\frac{1}{\sqrt{1 + i\beta}} \exp\left[-\frac{\sigma_k^2(x - v_g t)^2}{2(1 + i\beta)}\right], \quad \beta = \omega''(k_0)t\sigma_k^2.$$

This follows by completing the square in the integral of $\exp[-q^2/(2\sigma_k^2) + iq(x - v_g t) - i\omega''tq^2/2]$. Taking its modulus gives an amplitude width

$$\sigma_x(t) = \frac{\sqrt{1 + \beta^2}}{\sigma_k}, \quad \sigma_x(0) = 0.66667 \text{ m}.$$

Here $\beta = 0.126657$ and $\sigma_x(t) = 0.67199$ m, so the predicted width increases by 0.80%. The intensity E^2 has RMS width $\sigma_x/\sqrt{2}$: its predicted final value is 0.47517 m.

For the exact finite sum, direct integration over the periodic cell gives initial and final RMS intensity widths 0.47140 m and 0.47516 m. Its intensity centroid is at 1.47614 m, close to $v_g(k_0)t = 1.47489$ m. The small difference is expected: finite spectral width samples slightly different group speeds, and the full law contains higher Taylor terms. A narrow packet is an approximation whose error can be measured, rather than a guarantee of an exactly Gaussian envelope moving forever at one speed.

16.9.3 A centred scheme for the nondispersive equation

Now return to $\frac{\partial^2 y}{\partial t^2} = c_w^2 \frac{\partial^2 y}{\partial x^2}$ with constant c_w . This is a separate model. Let $x_j = j\Delta x$, $t^n = n\Delta t$, and $y_j^n \simeq y(x_j, t^n)$. Centred second differences give

$$\frac{y_j^{n+1} - 2y_j^n + y_j^{n-1}}{\Delta t^2} = c_w^2 \frac{y_{j+1}^n - 2y_j^n + y_{j-1}^n}{\Delta x^2}.$$

Solving for the new time level gives the explicit update

$$y_j^{n+1} = 2y_j^n - y_j^{n-1} + C^2(y_{j+1}^n - 2y_j^n + y_{j-1}^n), \quad C = \frac{c_w \Delta t}{\Delta x}. \quad (16.47)$$

The recurrence uses two previous time levels. Displacement data alone cannot initialise a second-order wave equation. If $y(x, 0) = I(x)$ and $\partial_t y(x, 0) = V(x)$, the PDE and Taylor's theorem give the consistent first step

$$y_j^0 = I_j, \quad y_j^1 = I_j + \Delta t V_j + \frac{C^2}{2}(I_{j+1} - 2I_j + I_{j-1}). \quad (16.48)$$

This is second-order accurate for sufficiently smooth data. Simply setting $y_j^1 = y_j^0$ generally imposes the wrong startup acceleration, even if the intended initial velocity is zero.

Boundary conditions are part of the problem. In a periodic domain $[0, L)$, use N distinct points, $\Delta x = L/N$, and interpret indices modulo N ; do not duplicate $x = L$ as an independent point. Fixed or free ends require different endpoint rules matching their displacement or slope conditions. Stability of a periodic interior calculation does not certify an arbitrary boundary implementation.

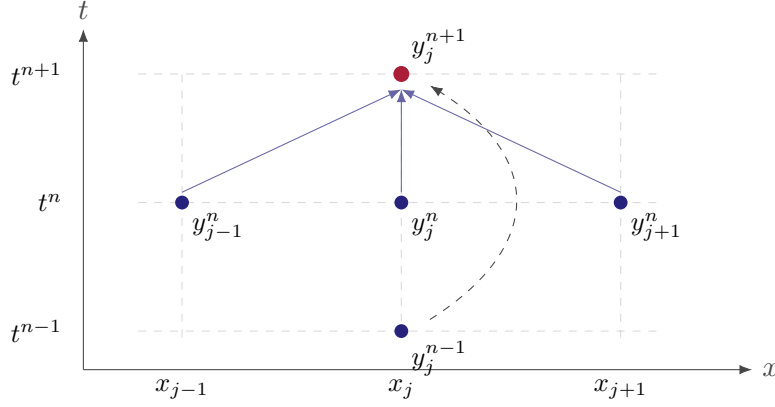


Figure 16.25: Stencil for the centred nondispersive wave update. Three values on the current time level and one on the preceding level determine the next value. The arrows indicate data dependence, not material trajectories. For the periodic benchmark, indices wrap around the domain; another physical endpoint would require its own boundary update.

16.9.4 Fourier stability and the two frequency branches

To examine each resolvable Fourier component, insert $y_j^n = q^n e^{ikj\Delta x}$ into Eq. (16.47). The spatial second difference contributes $-4 \sin^2(k\Delta x/2)$, giving

$$q^2 - 2(1 - 2S)q + 1 = 0, \quad S = C^2 \sin^2\left(\frac{k\Delta x}{2}\right).$$

When $0 < S < 1$, the roots are a conjugate pair of unit modulus: write $q = e^{\mp i\theta}$, where $\cos \theta = 1 - 2S$. They describe the two temporal propagation branches rather than two independently chosen amplitudes. When $S > 1$, the roots are real and reciprocal, and one has modulus greater than one. Its numerical amplitude grows exponentially even though the physical ideal wave has no such growth.

Requiring all resolvable modes to have $S \leq 1$ gives the usual one-dimensional restriction

$$0 < C \leq 1. \quad (16.49)$$

The highest resolvable wave number is $|k| = \pi/\Delta x$, the Nyquist mode. At $C = 1$ this mode has a repeated root $q = -1$, for which a general recurrence solution can contain $(a + bn)(-1)^n$. Thus the equality case is marginal: compatible modal initialisation removes the growing partner, but it is prudent to choose $C < 1$ and resolve wavelengths with substantially more than two points. The zero-wave-number repeated root represents the physical uniform displacement plus uniform velocity; it also deserves separate interpretation rather than a claim of oscillatory phase speed.

For oscillatory modes substitute $q = e^{-i\Omega\Delta t}$. The result is

$$\sin^2\left(\frac{\Omega\Delta t}{2}\right) = C^2 \sin^2\left(\frac{k\Delta x}{2}\right). \quad (16.50)$$

Taking a square root must retain the two signs. On the principal temporal branch, for $0 < k\Delta x < \pi$ and $0 < C < 1$,

$$\Omega_+(k) = \frac{2}{\Delta t} \arcsin\left[C \sin\left(\frac{k\Delta x}{2}\right)\right], \quad \Omega_-(k) = -\Omega_+(k).$$

Frequencies differing by $2\pi/\Delta t$ have the same sampled time sequence. Choosing the principal branch prevents such temporal aliases from being mistaken for additional physical modes. Negative wave numbers are handled with the corresponding direction and reality partners; the benchmark below uses a real standing combination.

Let $p = k\Delta x/2$. The positive-branch numerical phase-speed ratio is

$$r(C, p) = \frac{\Omega_+}{c_w k} = \frac{\arcsin(C \sin p)}{Cp} = 1 - \frac{1 - C^2}{6} p^2 + O(p^4).$$

The limiting value at $p = 0$ is one. At fixed $C < 1$, shorter resolved waves lag more; refining Δx at fixed physical k reduces the leading error quadratically. Taking a smaller time step alone does not eliminate the spatial error. For $C = 1$ and well-resolved interior modes, $\arcsin(\sin p) = p$ on this interval, so the discrete frequency agrees exactly with the continuum one. This special identity does not remove aliasing, justify arbitrary startup data, or replace a boundary test. The standard scheme and its dispersion relation are also discussed in the numerical-wave reference [4]; the derivation here makes the branches and the endpoint qualifications explicit.

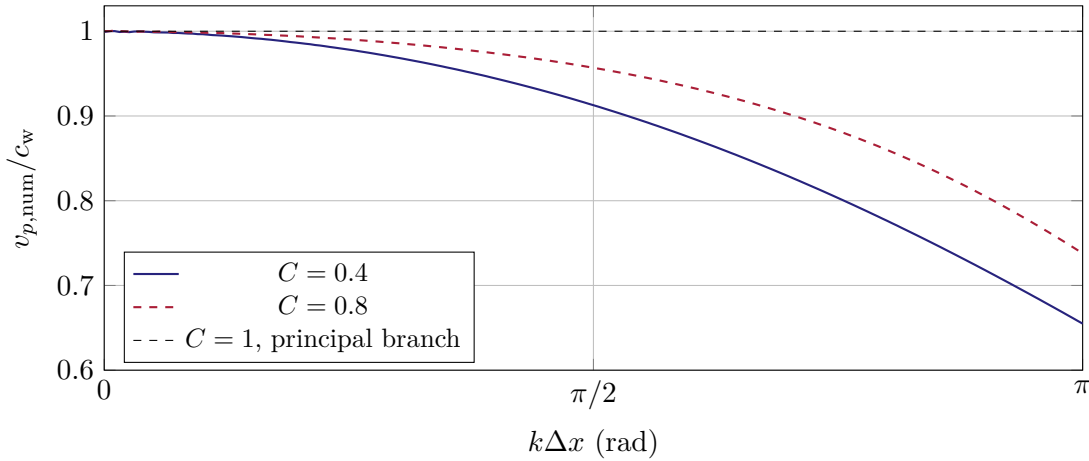


Figure 16.26: Numerical phase speed of the centred scheme for a physically nondispersive equation. Increasing $k\Delta x$ means fewer grid points per wavelength; $k\Delta x = \pi$ means only two. The curves demonstrate that stability and accuracy are distinct. The $C = 1$ line refers to the principal resolved-mode frequency and does not certify the marginal Nyquist root or endpoint treatments.

16.9.5 An actual time-stepping benchmark and mesh refinement

A benchmark compares the computed update with both the physical solution and the analytic *discrete* solution. Choose a periodic cell of length $L = 1.00$ m, $c_w = 2.00$ ms⁻¹, mode $m = 4$, $k = 2\pi m/L = 8\pi$ m⁻¹, and $C = 0.8$. Specify displacement $I(x) = A \cos(kx)$ and initial velocity $V(x) = 0$. The continuum solution is

$$y_{\text{cont}}(x, t) = A \cos(kx) \cos(c_w kt).$$

With Eq. (16.48), the first numerical step is exactly $y_j^1 = A \cos(kx_j) \cos \theta$. The recurrence therefore predicts the discrete solution

$$y_{j,\text{disc}}^n = A \cos(kx_j) \cos(n\theta), \quad \theta = 2 \arcsin[C \sin(k\Delta x/2)].$$

The time-stepping algorithm must agree with this expression up to floating-point arithmetic. Its disagreement with the continuum then measures the scheme's phase error, rather than a coding error hidden inside an attractive plot.

Example 16.22: Benchmark the numerical wave scheme

For the periodic standing-mode test just specified, use $N = 50, 100, 200, 400$. Calculate phase-speed errors and compare actual time stepping at $t = 0.240$ s with both reference solutions.

Here $\Delta x = 1/N$ m, $\Delta t = 0.4/N$ s and the final step is $n = 0.6N$, an integer for every grid. The numerical phase-speed ratios are respectively

$$0.9961525, \quad 0.9990489, \quad 0.9997629, \quad 0.9999408,$$

and the corresponding relative errors are

$$3.84755 \times 10^{-3}, \quad 9.51053 \times 10^{-4}, \quad 2.37093 \times 10^{-4}, \quad 5.92315 \times 10^{-5}.$$

For $N = 50$, $\Omega_+ = 50.07208$ rad s⁻¹, compared with $c_w k = 50.26548$ rad s⁻¹. The accumulated phase lag is $(c_w k - \Omega_+)t = 0.046416$ rad.

Define a waveform error relative to the initial amplitude norm,

$$\varepsilon_y = \frac{\left[\sum_j (y_j^n - y_{\text{cont}}(x_j, t^n))^2 \right]^{1/2}}{\left[\sum_j I_j^2 \right]^{1/2}}.$$

It equals $|\cos(n\theta) - \cos(c_w k t^n)|$ for this mode. The actual recurrence gives $\varepsilon_y = 0.023297, 0.0055848, 0.0013815$ and 0.00034446 . Agreement with the discrete expression is within $9 \times 10^{-15}A$ throughout these runs. Thus the code passes its discrete benchmark while the physical error decreases under refinement. The approximately fourfold error reduction confirms the expected second-order regime for this smooth, fixed-mode, fixed-time test. It does not certify every boundary condition, wavelength or simulation duration.

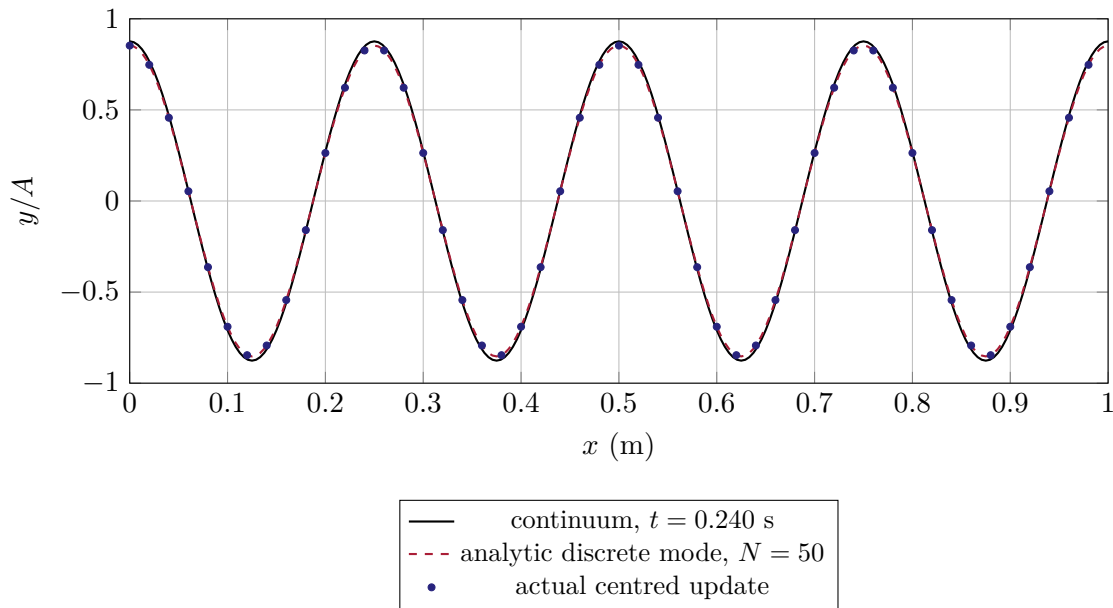


Figure 16.27: A computed standing-mode snapshot for the independently specified nondispersive PDE, with periodic boundaries, $m = 4$, $C = 0.8$ and $N = 50$. Markers are the values after 30 actual recurrence updates; they coincide with the analytic discrete solution. The solid continuum curve differs because the numerical oscillation is slightly slower. This compares complete displacement fields, not speed inferred by watching an animation.

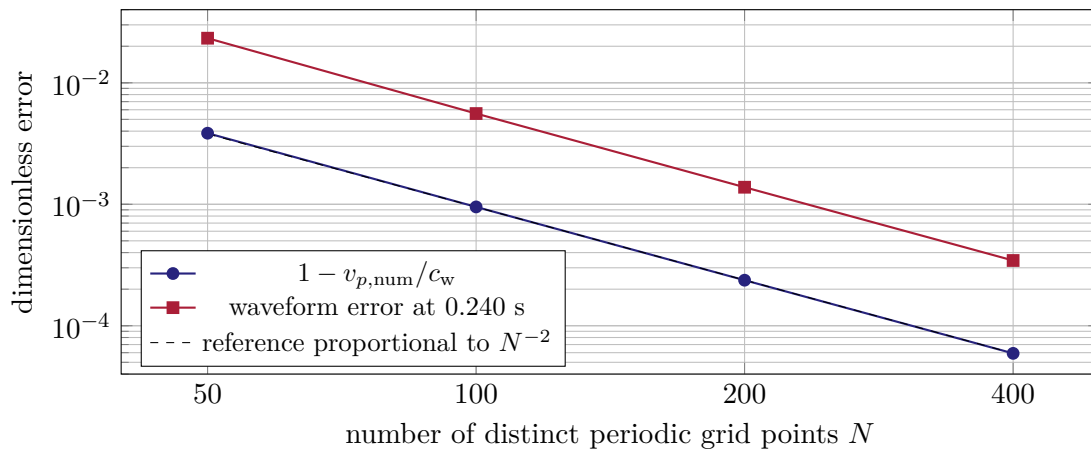


Figure 16.28: Phase-speed and actual waveform errors for the periodic nondispersive benchmark. Each refinement holds L , c_w , m , C and physical final time fixed; both Δx and Δt decrease. The approximate fourfold reduction on doubling N is consistent with second-order convergence. Instantaneous waveform error also depends on the oscillation phase, so a small error at one fortuitous time would not alone verify the phase speed.

Example 16.23: Diagnose a Courant-number failure

A proposed simulation has $\Delta x = 0.020$ m, $c_w = 2.00$ m s $^{-1}$ and $\Delta t = 0.0102$ s. Does its smooth appearance at early times establish stability? Find the highest-wave-number amplification and give a concrete repair.

The Courant number is $C = 2(0.0102)/0.020 = 1.02 > 1$. For the alternating Nyquist pattern, $k\Delta x = \pi$, so $S = C^2 = 1.0404$. The characteristic roots are

$$q_{\pm} = 1 - 2S \pm 2\sqrt{S(S-1)}.$$

The unstable root is $q_- = -1.490835$. A contaminating component in that root grows in magnitude by $(1.490835)^{20} \simeq 2.94 \times 10^3$ over 20 steps. A smooth initial waveform may contain little Nyquist content, so a short run can conceal the failure; round-off or a boundary can still seed it.

Retaining the grid and choosing $C = 0.8$ gives $\Delta t = C\Delta x/c_w = 0.0080$ s. Initialise the first step with both I and V , test the stated boundary conditions, and compare a known mode under refinement. The smaller step repairs this particular stability violation; it does not by itself guarantee that the physical wavelength is resolved accurately.

16.9.6 A reproducible implementation of the modal test

The following small program states the periodic indexing and startup explicitly. It uses only Python's standard mathematics library and implements the recurrence rather than substituting the discrete reference as its "computed" field. The displacement amplitude $A = 10^{-3}$ m is chosen only to give the field physical units; all relative errors in the linear test are independent of A .

```
from math import pi, sin, cos, asin, sqrt
N, L, c, C, m, A = 100, 1.0, 2.0, 0.8, 4, 1.0e-3
dx = L / N
dt = C * dx / c
k = 2 * pi * m / L
steps = round(0.240 / dt)
t = steps * dt
initial = [A * cos(k * j * dx) for j in range(N)]
velocity = [0.0] * N
```

```

def lap(v, j):
    return v[(j + 1) % N] - 2*v[j] + v[(j - 1) % N]

previous = initial[:]
current = [initial[j] + dt*velocity[j]
           + 0.5*C*C*lap(initial, j) for j in range(N)]
for n in range(1, steps):
    following = [2*current[j] - previous[j]
                 + C*C*lap(current, j) for j in range(N)]
    previous, current = current, following

theta = 2 * asin(C * sin(k * dx / 2))
discrete = [v * cos(steps * theta) for v in initial]
continuum = [v * cos(c * k * t) for v in initial]
reference_error = max(abs(a-b) for a,b in zip(current, discrete))
waveform_error = sqrt(sum((a-b)**2 for a,b
                           in zip(current, continuum))
                     / sum(v*v for v in initial))
print(t, reference_error / A, waveform_error)

```

For $N = 100$, the last output is approximately 0.00558483; the discrete-reference residual divided by A is of order 10^{-14} , with its precise value depending on arithmetic rounding. The time is 0.240 s. Discrete-reference disagreement tests implementation; continuum disagreement measures discretisation for the selected mode.

Each new array is formed from the complete old arrays before advancing the time level. Updating `current[j]` in place while still reading neighbours would change the specified scheme and could invalidate both the dispersion calculation and the Courant restriction. The modulo operation implements the declared periodic boundary; replacing it by zero-valued neighbours would instead create a different endpoint problem. The first-step formula also permits nonzero velocity with a compatible analytic test.

The four values of N supply Fig. 16.28. Keep m fixed to test the same physical wave. Record the actual final time `steps` Δt and compare with the continuum at that time, even when rounding places it near a requested time.

16.9.7 Interpretation: physical and numerical dispersion

The packet and grid calculations answer different questions. In the stipulated dispersive comparison, $\omega(k)$ is part of the model and is held fixed while component phases evolve. In the numerical benchmark, the continuum relation remains $\omega = c_w k$; the discretisation introduces a different $\Omega(k, \Delta x, \Delta t)$. A packet can therefore broaden either because the selected model is dispersive or because an insufficiently refined numerical scheme transports its components at different speeds.

Establish the continuous model and assumptions, compute a mode with known continuous and discrete solutions, then refine the grid at fixed physical parameters. Check boundary residuals and initial conditions as well as dispersion and the actual update. Discrepancy disappearing under refinement indicates numerical error; a converged continuous model can retain its own dispersion. Agreement with a real apparatus still requires independent justification of material assumptions.

Take-home message. Derive each propagation speed from its declared model, resolve the spectrum, and check a computed wave against both an analytic limit and mesh refinement.

16.10 Chapter summary and a working formula sheet

Summary

A travelling wave is a propagating pattern, while the medium elements execute local motion. For a sinusoid, k_w describes spatial repetition and ω temporal repetition. On an ideal taut uniform string, a small-slope Newtonian force balance gives the linear wave equation and $c_w = \sqrt{F_T/\mu_\ell}$. Travelling waves transport mechanical energy; their mean power scales as $A^2\omega^2$. Linear superposition adds displacement fields, while energy contains cross terms. Reflection follows from endpoint conditions. Fixed-end normal modes are selected by $y(0, t) = y(L, t) = 0$, and an arbitrary compatible initial shape can be expanded in those modes.

Formula summary

Travelling phase and kinematics

$$y = A \sin(k_w x - \omega t + \phi), \quad k_w = \frac{2\pi}{\lambda}, \quad \omega = \frac{2\pi}{T_{\text{osc}}}, \quad c_w = \frac{\omega}{k_w} = \lambda f.$$

Ideal taut uniform string

$$\frac{\partial^2 y}{\partial t^2} = c_w^2 \frac{\partial^2 y}{\partial x^2}, \quad c_w = \sqrt{\frac{F_T}{\mu_\ell}}, \quad |\partial_x y| \ll 1.$$

Energy and power

$$u_\ell = \frac{1}{2}\mu_\ell(\partial_t y)^2 + \frac{1}{2}F_T(\partial_x y)^2, \quad \mathcal{P} = -F_T(\partial_x y)(\partial_t y),$$

$$\langle \mathcal{P} \rangle = \frac{1}{2}\mu_\ell\omega^2 A^2 c_w \quad \text{for one sinusoidal travelling wave.}$$

Coherent interference

$$y = y_1 + y_2, \quad A_R^2 = A_1^2 + A_2^2 + 2A_1 A_2 \cos \Delta\phi.$$

Ideal endpoints

$$y(L, t) = 0 \quad \text{fixed}, \quad \partial_x y(L, t) = 0 \quad \text{free.}$$

Fixed–fixed modes

$$\lambda_n = \frac{2L}{n}, \quad f_n = \frac{nc_w}{2L}, \quad y_n = A_n \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t + \phi_n).$$

Sine-series coefficient for a specified initial shape

$$A_n = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Caution

Before accepting a string-wave result, check which speed is being discussed, whether the slope is small enough for the linear string model, whether the boundary condition matches the physical endpoint, and whether a travelling-wave power formula is being applied to a standing pattern or coherent superposition where cross terms matter.

16.11 Exercises and problems

Use SI units unless stated otherwise. For ideal-string calculations assume a uniform string, small transverse slope, approximately constant tension, and negligible damping and bending stiffness. Explain the propagation direction and boundary condition before calculating. Problems 30–31, 34–39, 46, and 48 use the advanced section. All other problems use the core or applied sections. Figures supply stated data where a task explicitly calls for them; otherwise the problem is self-contained.

A. Concepts, representations, and diagnostic reasoning

1. A pulse travels to the right on a string. At one instant the string rises with increasing x through a marked point. Is that marked string element moving upward, downward, or can the direction not be determined? Explain using a translated pulse rather than a memorised rule.
2. **X16-01.** A student reads a spatial snapshot and says, “The crest spacing is 0.60 s, so the period is 0.60 s.” Diagnose the error and state what additional graph or observation is needed to obtain the period.
3. A wave has propagation speed 18 m s^{-1} , while a marked element has instantaneous transverse velocity -0.40 m s^{-1} . Explain why there is no contradiction and identify the directions of the two velocities.
4. **X16-02.** A proposed ideal string wave has $A = 0.10 \text{ m}$ and $\lambda = 0.40 \text{ m}$. A student uses the linear string power formula without further checks. Evaluate Ak_w and diagnose the modelling problem.
5. Two solutions of the same linear string equation overlap. Does the energy density of the sum equal the sum of the two separate energy densities at every point? Explain by expanding one quadratic term.
6. **X16-05.** A reflected pulse at a fixed endpoint is drawn without inversion, but the drawing shows the endpoint remaining displaced while the pulse overlaps it. State the violated boundary condition and repair the reflected sign.
7. In one snapshot of a standing wave, five points happen to lie at $y = 0$. Are all five necessarily nodes? State the defining test for a node.
8. A resonance peak in a driven real string has finite width. Why is the width not, by itself, a measurement of an undamped normal-mode frequency? Name two effects that can contribute to the width.

B. Travelling waves and the ideal-string model

9. A wave is $y = (6.0 \text{ mm}) \sin(4\pi x - 20\pi t + \pi/3)$, with x in metres and t in seconds. Find A , λ , f , c_w , the direction of travel, and the maximum transverse element speed.
10. For the wave in Problem 9, calculate y and v_y at $x = 0.10 \text{ m}$ and $t = 0.030 \text{ s}$. Check that $|v_y| \leq \omega A$.
11. A sinusoidal wave has $A = 5.0 \text{ mm}$ and $\lambda = 0.250 \text{ m}$. Evaluate Ak_w . Is the small-slope condition plausible? Repeat for $A = 50 \text{ mm}$ with the same wavelength.
12. A string has $F_T = 64 \text{ N}$ and $\mu_\ell = 0.016 \text{ kg m}^{-1}$. Find the ideal wave speed. If the driving frequency doubles while the tension and line density remain fixed, what happens to the speed and wavelength?

13. A string has $F_T = 90 \text{ N}$, $\mu_\ell = 0.018 \text{ kg m}^{-1}$, and is driven at 75 Hz . Find c_w and λ . If $A = 4.0 \text{ mm}$, test the small-slope parameter Ak_w .
14. Show directly that $y = f(x - c_w t) + g(x + c_w t)$ satisfies the linear wave equation whenever f and g are twice differentiable. Which step uses linearity?

C. Energy, power, and interference

15. A sinusoidal travelling wave has $\mu_\ell = 0.020 \text{ kg m}^{-1}$, $c_w = 50 \text{ m s}^{-1}$, $f = 100 \text{ Hz}$, and $A = 1.50 \text{ mm}$. Calculate the mean power. What factor changes the answer if only A is tripled?
16. **X16-03.** A student writes $\langle \mathcal{P} \rangle = \mu_\ell \omega A c_w / 2$. Use dimensions and amplitude scaling to diagnose the formula.
17. Two coherent waves have amplitudes 4.0 mm and 4.0 mm and phase difference 120° . Find the resultant amplitude. Classify the interference as fully constructive, fully destructive, or intermediate.
18. Two coherent waves have amplitudes 5.0 mm and 3.0 mm . Find the largest and smallest possible resultant amplitudes. Find a phase difference in $[0, \pi]$ that gives $A_R = 4.0 \text{ mm}$.
19. **X16-04.** A student claims that a phase difference of π always gives zero resultant displacement. Give the precise amplitude condition that is also required, and construct a counterexample with unequal amplitudes.
20. Two equal waves of amplitude A travel in opposite directions. Starting from $\sin(k_w x - \omega t) + \sin(k_w x + \omega t)$, derive the standing-wave form and identify node positions.

D. Reflection, boundaries, and normal modes

21. For an incident pulse $f(x - c_w t)$ approaching a boundary at $x = L$, write an incident-plus-reflected expression that satisfies (a) a fixed end and (b) a free end. Verify the boundary algebraically.
22. A fixed-fixed string has $L = 0.750 \text{ m}$ and $c_w = 60.0 \text{ m s}^{-1}$. Find f_1 , f_2 , and f_5 .
23. A string of length 0.900 m and line density $8.00 \times 10^{-3} \text{ kg m}^{-1}$ must have its second fixed-end mode at 120 Hz . Find the ideal wave speed and tension.
24. Derive the allowed frequencies for a string fixed at $x = 0$ and ideally free at $x = L$. Use $y(0, t) = 0$ and $\partial_x y(L, t) = 0$. Express the spectrum in terms of $n = 1, 2, 3, \dots$
25. A fixed-fixed string vibrates in mode $n = 4$. List all node positions in the interval $0 \leq x \leq L$ and all antinode positions. How many half-wavelengths fit into L ?
26. **X16-06.** A student draws the third mode of a fixed-fixed string with only two half-wavelengths between the supports. Diagnose the drawing using the boundary-derived wavelength formula.

E. Fourier synthesis, computation, and measurement

27. For the triangular pluck in Eq. (16.37), verify Eq. (16.38) by integrating the two linear pieces. Check the units and the $1/n^2$ scaling.
28. A string is plucked at its midpoint, $a = L/2$. Use Eq. (16.38) to prove that all even- n coefficients vanish. What symmetry of the initial shape is being represented?

29. **X16-07.** A reconstruction matches the initial displacement well but the simulation gives a nonzero initial velocity even though the string was released from rest. What independent modal condition was omitted?
30. A finite-difference calculation uses $c_w = 40 \text{ m s}^{-1}$ and $\Delta x = 5.0 \text{ mm}$. What is the largest time step allowed by the standard condition $C \leq 1$? What time step corresponds to $C = 0.80$?
31. A code uses the recurrence in Eq. (16.47) with periodic boundaries. Describe three tests that should be passed before trusting an animation of a sinusoidal wave. At least one must be a refinement test.
32. In an ideal resonance measurement, $f_n \propto L^{-1} F_T^{1/2} \mu_\ell^{-1/2}$. If independent standard relative uncertainties are 0.5% in L , 2.0% in F_T , and 1.0% in μ_ℓ , estimate the standard relative uncertainty in f_n using first-order propagation.
33. A synthetic measurement gives resonance peaks near 82.0, 164.5, and 246.1 Hz. Test whether they are consistent with the first three harmonics of one ideal fixed–fixed string to within 0.5% relative deviation from integer multiples of the first peak. State what this test cannot establish.

F. Advanced challenges

34. For the stipulated dispersion relation in Eq. (16.43), derive Eqs. (16.44) and (16.45) and prove that $v_g - v_p > 0$ for $\ell_d > 0$ and $k > 0$.
35. Evaluate v_p and v_g for $c_0 = 30 \text{ m s}^{-1}$, $\ell_d = 0.025 \text{ m}$, and $k_0 = 20 \text{ m}^{-1}$. Compare their fractional difference.
36. **X16-08.** A finite-difference animation with $C = 1.08$ looks smooth for the first few frames. Explain why visual smoothness is not a stability test and state the relevant condition for the declared scheme.
37. For the benchmark of Example 16.22, compute the ratio of the phase-speed error at $N = 100$ to that at $N = 200$. What convergence order does this suggest?
38. Two nearby components have $k_1 = k_0 - \Delta k/2$ and $k_2 = k_0 + \Delta k/2$. Use a trigonometric sum to show that the envelope of their equal-amplitude superposition moves at the finite-difference speed $[\omega(k_2) - \omega(k_1)]/(k_2 - k_1)$, and connect the narrow-spacing limit to v_g .
39. A simulation of the physically nondispersive wave equation shows wavelength-dependent phase speed that changes noticeably when the grid is refined. Is the observed dispersion physical or numerical? Give a test that separates the two possibilities and explain what evidence would be needed to claim physical dispersion.

G. Visual interpretation, inverse problems, and synthesis

The following figures and datasets are calculated or synthetic teaching materials. Problems 40–48 extend the core; Problems 46 and 48 also use the advanced section. Plot coordinates, rather than decorative curve shapes, supply the evidence in the visual questions.

40. Figure 16.29(a) shows two snapshots of one low-amplitude sinusoid. The later trace is displaced by less than half a wavelength; follow the crest initially at $x = 0.60 \text{ m}$. (a) Read the amplitude, wavelength, crest displacement, and propagation direction. (b) Calculate the speed and frequency. (c) The initial curve rises or falls at $x = 0.075 \text{ m}$: decide whether the marked element initially moves upward or downward and explain your sign without treating the pattern speed as the element speed.

41. Figure 16.29(b) is the displacement recorded at $x = 0$ for the same wave as Problem 40. (a) Determine the period independently and reconcile it with the snapshots. (b) Write a cosine wave function consistent with both panels, with x in metres and t in seconds. (c) Calculate displacement, transverse velocity, and transverse acceleration at $x = 0.075$ m, $t = 0$, including signs. State the largest transverse speed and compare it with the propagation speed.

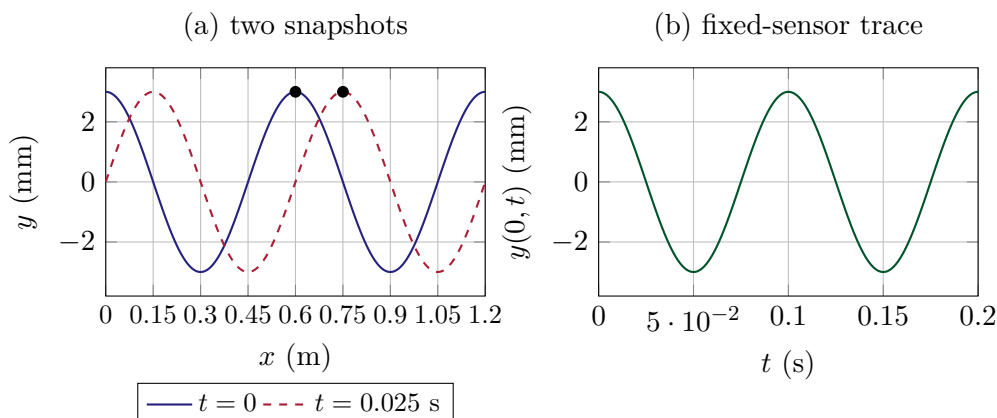


Figure 16.29: Calculated observations of one travelling sinusoid for Problems 40 and 41. Markers identify the tracked crest in the two snapshots; the sensor remains at $x = 0$. The common vertical axes give displacement in millimetres.

42. Figure 16.30 displays synthetic fixed-fixed resonance data. The four frequencies are 40.1, 79.8, 120.3, and 159.9 Hz for $n = 1, 2, 3, 4$; each has independent standard uncertainty 0.4 Hz. The length is $L = (1.200 \pm 0.003)$ m and independently measured line density is $\mu_\ell = (0.00600 \pm 0.00006)$ kg m $^{-1}$. (a) Use the model $f_n = sn$ and derive the equal-weight least-squares estimate of s constrained through the origin. (b) Calculate s , all four residuals, and the ideal tension. (c) Derive the standard uncertainty in s and propagate independent uncertainties to the tension. (d) Explain what the frequency graph supports and what it cannot prove about real end conditions or stiffness.
43. The junction in Figure 16.31 joins strings under a common tension 72.0 N. Their line densities are $\mu_{\ell,1} = 0.00800$ kg m $^{-1}$ and $\mu_{\ell,2} = 0.0320$ kg m $^{-1}$. An incident sinusoid from string 1 has amplitude 1.80 mm and frequency 25.0 Hz. (a) Determine both speeds and wavelengths. (b) Impose displacement and transverse-force continuity to derive and evaluate the displacement reflection and transmission coefficients. (c) Find both outgoing amplitudes, including the reflected sign. (d) Calculate reflected and transmitted energy fractions and powers, verify energy conservation, and test the small-slope assumption for every component.

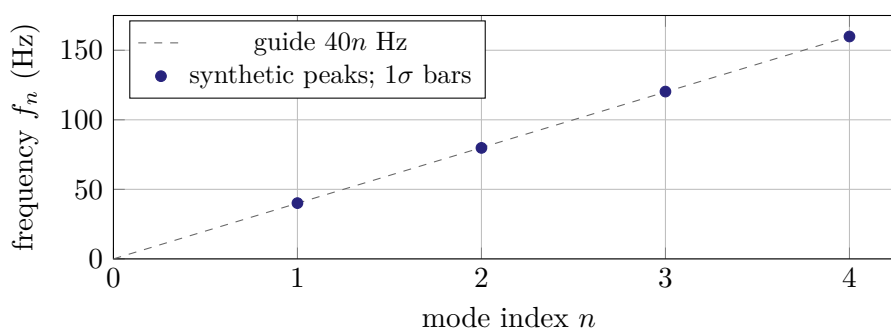


Figure 16.30: Synthetic resonance frequencies for Problems 42 and 45. Error bars are much smaller than the full frequency scale; the numerical data in the statement are needed for the fit. The guide is a comparison line, not an additional measurement.

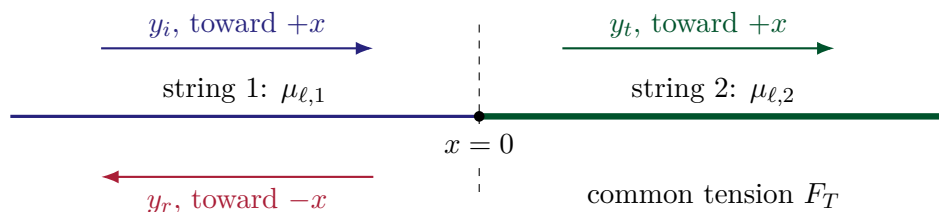


Figure 16.31: Junction geometry for Problem 43. Arrows label pattern directions; their vertical placement separates labels and does not depict transverse displacement. The junction is ideal, massless, and free of external transverse force.

44. Figure 16.32 shows the initial elastic line-energy distribution of a fixed-end mode released from a displacement extremum:

$$y = A \sin(2\pi x/L) \cos \omega t.$$

Here $L = 0.800$ m, $A = 2.00$ mm, $F_T = 80.0$ N, and $\mu_\ell = 0.0200$ kg m⁻¹; $u_0 = F_T A^2 (2\pi/L)^2 / 2$.

- (a) Identify nodes and antinodes and explain why initial elastic energy peaks at displacement nodes. (b) Integrate to find the total initial energy and frequency. (c) Find the kinetic and elastic energy densities at a quarter period and show that the total energy is unchanged. (d) Derive the instantaneous signed power at a cut, prove that its cycle average is zero, and calculate it at $x = L/8$, $t = T_{\text{osc}}/8$.
45. Reinterpret the frequency data in Figure 16.30 as an inverse problem in which the line density is initially unknown. Use $L = (1.200 \pm 0.003)$ m as before. (a) Explain why the resonances alone cannot determine tension and line density separately. (b) A separate calibrated measurement gives $F_T = (55.0 \pm 0.5)$ N. Infer μ_ℓ and its independent standard uncertainty using the fitted slope from Problem 42. (c) With the same L , F_T , and μ_ℓ , replace the right support by an ideal free end; predict the new fundamental and the next two mode frequencies. Explain why these are not the old consecutive integer harmonics.
46. Figure 16.33 was calculated from the centred scheme's numerical dispersion relation for a periodic domain $L = 1.00$ m, $c_w = 2.00$ m s⁻¹, physical mode number $m = 3$, and $C = 0.75$. (a) Estimate the convergence order from successive plotted error ratios. (b) Derive the phase-speed ratio in terms of N and recompute the $N = 160$ point. (c) Find its time step and the phase lag in radians after one exact continuum period. (d) Explain why holding the mode number fixed during refinement matters, and why this figure measures numerical rather than physical dispersion.
47. Use the fixed-end mode shapes in Figure 16.15 to compare two separate pure modes on the same string: mode 1 with antinode amplitude 2.00 mm and mode 2 with antinode amplitude 1.00 mm. (a) Read the number of half-wavelengths and relate each wave number to L . (b) Derive the total energy of a pure standing mode and compare the two energies. (c) If both modes are present simultaneously, prove by orthogonality that the integrated total energy is the sum of their modal energies, even though their local energy densities contain cross terms. State the assumptions needed for this result.
48. Figure 16.34 gives phase and group speeds for the stipulated model in Eq. (16.43), with $z = \ell_d k$. At $k_0 = 15.0$ m⁻¹, a hypothetical measurement of the ratio gives $q = v_g/v_p = 1.26470588$; take $c_0 = 20.0$ m s⁻¹. (a) Use the graph to locate z approximately and derive an algebraic inversion for ℓ_d . (b) Calculate both speeds and their respective advances in 0.0800 s. (c) For two nearby components $k_1 = 14.0$ m⁻¹ and $k_2 = 16.0$ m⁻¹, calculate the envelope speed from the exact frequency difference and compare it with the derivative at k_0 . (d) State why this inference concerns the stipulated model and does not demonstrate that the ideal taut string is physically dispersive.

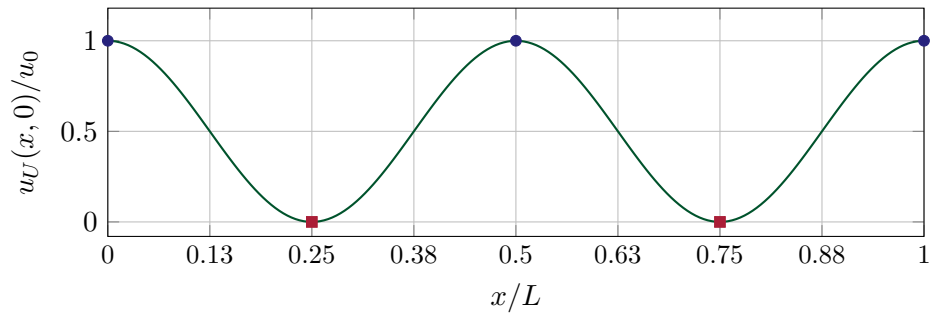


Figure 16.32: Calculated initial elastic line energy for Problem 44, normalised by u_0 . Blue circles mark displacement nodes and red squares mark displacement antinodes. Initially every element is at rest, so its kinetic energy is zero. Vertical energy values are dimensionless; u_0 carries units J m^{-1} .

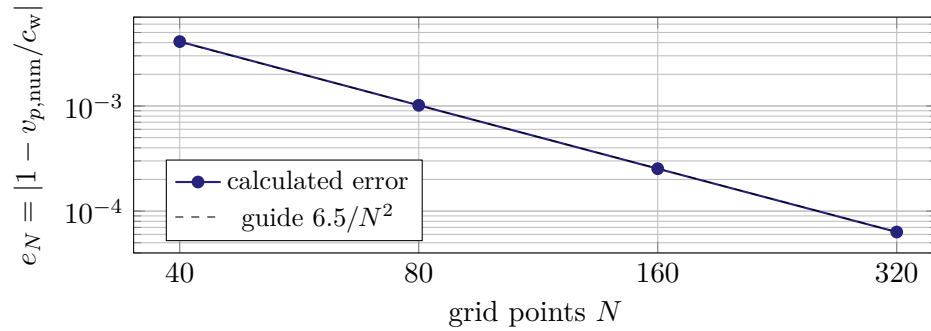


Figure 16.33: Numerical phase-speed errors for Problem 46 at a fixed physical wavelength and stable Courant number. The dashed line supplies a second-order comparison slope; it is not a fit to a measured physical string.

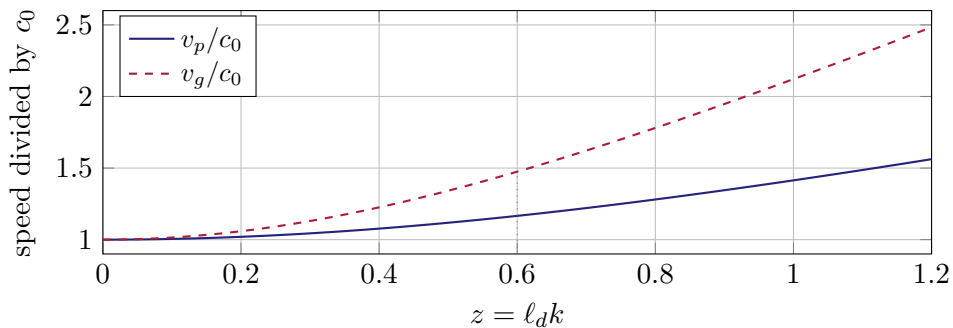


Figure 16.34: Calculated phase and group speeds for the stipulated dispersive relation, used in Problem 48. The horizontal variable is dimensionless. These curves represent a declared model rather than measured string data.

16.12 Complete reasoned solutions

Every problem and subpart is addressed below. The stated ideal-string assumptions apply unless a question explicitly concerns a different model. Numerical calculations retain extra digits internally; the displayed final precision reflects the supplied inputs. In a conceptual problem, the explanation and its limiting conditions are part of the answer.

Problem 1: motion of a marked element

For an undistorted right-going pulse, write $y(x, t) = f(x - c_w t)$ with $c_w > 0$. Differentiation while holding the other independent variable fixed gives $\partial_x y = f'(x - c_w t)$ and $\partial_t y = -c_w f'(x - c_w t)$. Thus

$$v_y = -c_w \partial_x y.$$

The stated rising snapshot has $\partial_x y > 0$, so $v_y < 0$: the marked element moves downward. An intuitive check is to translate the profile slightly right; the height at the fixed material label is then supplied by a point slightly to its left, which was lower. This conclusion uses a single right-going translated pulse. A sum of unrelated left- and right-going waves cannot be diagnosed from the instantaneous slope alone.

Problem 2: a snapshot is not a time record

A spatial snapshot holds t fixed and plots y against x . Its crest spacing is a length, so reading 0.60 from an axis in metres gives $\lambda = 0.60$ m; assigning seconds to that separation changes the meaning of the graph. A time record at one fixed sensor instead determines the interval between successive crests or equivalent zero crossings, which is T_{osc} . One may also track a crest in multiple images to measure c_w , and then use $T_{\text{osc}} = \lambda/c_w$. The snapshot alone cannot determine period because waves with different ω can share the same displacement profile at one instant. A labelled horizontal axis is therefore an essential physical datum.

Problem 3: two velocities with different meanings

The propagation speed 18 m s^{-1} describes a phase or pulse moving along the equilibrium direction of the string. The transverse velocity -0.40 m s^{-1} describes one material element moving downward, assuming positive y is upward. Their different magnitudes cause no contradiction because they are rates of different positions. In the chapter's right-going convention the pattern velocity is $+18 \text{ m s}^{-1} \hat{x}$ while the local element velocity is $-0.40 \text{ m s}^{-1} \hat{y}$. Strictly, a speed alone does not specify the sign along x ; a left-going wave would have pattern velocity $-18 \hat{x}$ with the same stated speed. The local downward motion does not select that pattern direction.

Problem 4: check the linear model before using its power law

For the proposed sinusoid,

$$Ak_w = A \frac{2\pi}{\lambda} = (0.10) \frac{2\pi}{0.40} = \frac{\pi}{2} = 1.57.$$

This is the maximum magnitude of $\partial_x y$. It is dimensionless and is not small compared with one. The linear derivation used $\sqrt{1 + (\partial_x y)^2} \simeq 1 + (\partial_x y)^2/2$ and nearly equal horizontal tension components; those approximations are not self-consistent at this slope. The sinusoidal expression is a well-defined mathematical function, but the ideal linear string power law is not justified for the claimed physical wave. A reduced amplitude or a model retaining finite deformation and appropriate tension changes is needed before interpreting a power value.

Problem 5: energy does not generally superpose

Let $y = y_1 + y_2$. Substituting into the quadratic line-energy density gives

$$u_\ell[y] = u_\ell[y_1] + u_\ell[y_2] + \mu_\ell(\partial_t y_1)(\partial_t y_2) + F_T(\partial_x y_1)(\partial_x y_2).$$

The last two terms can be positive, negative, or zero at a given event. Therefore pointwise energy additivity does not follow from displacement superposition. For example, identical in-phase fields give $y = 2y_1$ and hence $u_\ell[y] = 4u_\ell[y_1]$, whereas adding the two individual densities would give only $2u_\ell[y_1]$. This comparison describes different initial fields, not creation of energy during an isolated evolution. Orthogonal modes can have vanishing *integrated* cross terms, and suitable incoherent averages can suppress cross terms; neither exception establishes general pointwise additivity.

Problem 6: repair a fixed-end reflection

The violated condition is $y(L, t) = 0$ at every instant, including the interval of pulse overlap. If the incident contribution at the endpoint is $f(L - c_w t)$, a non-inverted equal reflected contribution doubles it instead of cancelling it. The corrected expression is

$$y(x, t) = f(x - c_w t) - f(2L - x - c_w t).$$

At $x = L$ the two arguments coincide and $y(L, t) = f(L - c_w t) - f(L - c_w t) = 0$ identically. Thus the reflected displacement changes sign. An inverted drawing made only after the pulse leaves the boundary is insufficient as verification; the cancellation must hold throughout contact. A free end would obey a different condition, namely vanishing slope, and would require the plus sign instead.

Problem 7: zero displacement at one time does not define a node

No: a node is a position x_N with $y(x_N, t) = 0$ for *all* times in the standing pattern. For $y = B \sin(k_w x) \cos \omega t$, permanent nodes satisfy $\sin(k_w x_N) = 0$, independent of the temporal factor. At times with $\cos \omega t = 0$, the whole string passes through its equilibrium position, including every antinode. Counting zero crossings in that snapshot would classify all positions as nodes. To test a candidate point, observe its displacement through a cycle or determine whether its spatial amplitude factor is zero. The five observed zeros may include genuine nodes, but the single snapshot does not establish that all five are nodes.

Problem 8: finite resonance width

The width describes a frequency range over which a driven response is appreciable; an undamped normal-mode frequency is an eigenvalue of the unforced, lossless boundary-value problem. Damping can broaden the response, and finite observation duration produces frequency-resolution broadening. Instrumental frequency resolution or finite forcing bandwidth are further possible contributions. In a simple damped oscillator the location of the displacement-response maximum can also differ from its undamped natural frequency. Thus neither the peak width nor an unmodelled peak maximum independently establishes the lossless eigenfrequency. One should identify the response observable, fit or constrain the damping and measurement response, and compare the inferred frequency with the declared normal-mode model.

Problem 9: read a travelling-wave function

Compare the supplied phase with $k_w x - \omega t + \phi$: $A = 6.0 \text{ mm}$, $k_w = 4\pi \text{ m}^{-1}$, $\omega = 20\pi \text{ rad s}^{-1}$, and $\phi = \pi/3$. The derived quantities are

$$\lambda = \frac{2\pi}{4\pi} = 0.500 \text{ m}, \quad f = \frac{20\pi}{2\pi} = 10.0 \text{ Hz}, \quad c_w = \frac{\omega}{k_w} = 5.00 \text{ m s}^{-1}.$$

Holding phase constant gives $4\pi dx - 20\pi dt = 0$, so the pattern travels toward $+x$. Meanwhile $v_y = -\omega A \cos \Phi$ oscillates locally and its largest magnitude is $\omega A = (20\pi)(0.0060) = 0.377 \text{ m s}^{-1}$. It is transverse and need not equal the pattern speed. The slope scale $Ak_w = 0.0754$ is also small enough for the stated linear idealisation.

Problem 10: displacement and element velocity at one event

Insert the specified position and time into the complete phase, including the initial phase:

$$\Phi = 4\pi(0.10) - 20\pi(0.030) + \frac{\pi}{3} = \frac{2\pi}{15} = 0.418879 \text{ rad.}$$

The displacement and velocity are

$$y = (6.0 \text{ mm}) \sin \Phi = 2.44 \text{ mm},$$

$$v_y = -\omega A \cos \Phi = -(20\pi)(0.0060) \cos \Phi = -0.344 \text{ m s}^{-1}.$$

The time derivative is taken at fixed x ; differentiating a phase trajectory instead would answer a different question. As an independent bound, $\omega A = 0.377 \text{ m s}^{-1}$ and $|\cos \Phi| \leq 1$, so the calculated magnitude is allowed. Positive displacement with negative velocity means the element lies above equilibrium and is moving downward; the pattern still travels right.

Problem 11: amplitude and small slope

At fixed wavelength $\lambda = 0.250 \text{ m}$, the wave number is $k_w = 2\pi/\lambda = 25.1327 \text{ m}^{-1}$. Thus

$$Ak_w = 0.0050(25.1327) = 0.126 \quad \text{or} \quad Ak_w = 0.050(25.1327) = 1.26.$$

The first maximum slope is plausibly small for an introductory linear model, although the required accuracy determines how small it must be. For example, a geometric correction of order $(Ak_w)^2$ is about 0.016 in the first case. The second slope is of order one, so the linear approximation is not justified. Making an amplitude ten times larger also makes the maximum slope ten times larger; keeping the same wavelength does not preserve model validity.

Problem 12: frequency changes on the same ideal string

The dynamical wave speed is

$$c_w = \sqrt{\frac{64}{0.016}} = \sqrt{4000} = 63.2 \text{ m s}^{-1}.$$

The unit ratio under the root is $\text{m}^2 \text{ s}^{-2}$. In the declared nondispersive model this speed depends on tension and line density, both of which remain fixed, so doubling the drive frequency leaves c_w unchanged. Since $\lambda = c_w/f$, it halves the wavelength. This conclusion concerns a travelling wave at an imposed frequency, rather than whether a finite resonator happens to have a resonance there. It also presumes the changed frequency does not introduce appreciable stiffness, damping, or other physics excluded from the ideal model.

Problem 13: speed, wavelength, and slope

First apply the dynamical relation, then the kinematic one:

$$c_w = \sqrt{\frac{90}{0.018}} = 70.7 \text{ m s}^{-1}, \quad \lambda = \frac{70.7107}{75} = 0.943 \text{ m}.$$

The wave number is $k_w = 2\pi/\lambda = 6.66432 \text{ m}^{-1}$, so for $A = 0.0040 \text{ m}$,

$$Ak_w = 0.0267 \ll 1.$$

This is a small dimensionless maximum slope, consistent with the linear geometry. As a cross-check, $\omega = 2\pi(75)$ and $\omega/k_w = 70.7107 \text{ m s}^{-1}$, reproducing the dynamical speed. The slope test alone does not verify negligible damping, negligible bending stiffness, or actual uniformity of the tension.

Problem 14: direct substitution and linearity

Set $\xi = x - c_w t$ and $\eta = x + c_w t$. With primes denoting ordinary derivatives of each function's argument,

$$\begin{aligned}\partial_x^2 y &= f''(\xi) + g''(\eta), \\ \partial_t y &= -c_w f'(\xi) + c_w g'(\eta), \\ \partial_t^2 y &= c_w^2 f''(\xi) + c_w^2 g''(\eta) = c_w^2 \partial_x^2 y.\end{aligned}$$

The signs of the first time derivatives distinguish the two directions, but their second derivatives have the same positive c_w^2 factor. Linearity is used when differentiating the sum term by term and adding the two zero residuals: the operator $\partial_t^2 - c_w^2 \partial_x^2$ obeys $\mathcal{L}(f + g) = \mathcal{L}f + \mathcal{L}g$. A nonlinear operator containing, for example, $(\partial_x y)^2$ would introduce cross terms and would not have this property in general.

Problem 15: mean power and amplitude scaling

For one sinusoidal travelling wave, $\omega = 2\pi f = 200\pi \text{ rad s}^{-1}$ and the amplitude in SI units is 0.00150 m. Therefore

$$\langle \mathcal{P} \rangle = \frac{1}{2}(0.020)(200\pi)^2(0.00150)^2(50) = 0.444 \text{ W}.$$

The unit product is $(\text{kg/m})(\text{s}^{-2})(\text{m}^2)(\text{m/s}) = \text{W}$. If only the amplitude is tripled, the squared factor gives $\langle \mathcal{P} \rangle' = 9\langle \mathcal{P} \rangle = 4.00 \text{ W}$. The wavelength is 0.500 m, so the original and tripled slope scales are 0.0188 and 0.0565 respectively; both remain small within this idealisation. The factor of nine is a power ratio, not a displacement ratio.

Problem 16: reject an incorrect power formula

The proposed product has dimensions

$$[\mu_\ell \omega A c_w] = \frac{\text{kg}}{\text{m}} \frac{1}{\text{s}} \text{m} \frac{\text{m}}{\text{s}} = \text{kg m s}^{-2},$$

which are force dimensions rather than power dimensions $\text{kg m}^2 \text{s}^{-3}$. Moreover, it is linear in A , whereas both kinetic and small-slope elastic energies are quadratic in the wave displacement scale. Rescaling y by a factor b must rescale energy and flux by b^2 in this model. The correct expression $\langle \mathcal{P} \rangle = \mu_\ell \omega^2 A^2 c_w / 2$ passes both tests. Dimensions alone cannot prove a candidate formula, but a dimension mismatch is enough to disprove it.

Problem 17: equal-amplitude interference

For two coherent components with the same frequency and wave number, their phasor angle is the phase difference. Using millimetres consistently,

$$A_R^2 = 4.0^2 + 4.0^2 + 2(4.0)(4.0) \cos 120^\circ = 32 - 16 = 16 \text{ mm}^2.$$

Thus $A_R = 4.0 \text{ mm}$. Equivalently, $A_R = 2A|\cos(\Delta\phi/2)| = 8.0|\cos 60^\circ| \text{ mm}$. The result lies between the fully destructive value zero and the fully constructive value 8.0 mm, so the interference is intermediate. Although the resultant amplitude equals either individual amplitude here, that equality does not mean the two coherent energy densities simply add.

Problem 18: unequal amplitudes and inverse phase

The general amplitude relation is $A_R^2 = A_1^2 + A_2^2 + 2A_1A_2 \cos \Delta\phi$. Its largest and smallest values correspond to $\cos \Delta\phi = +1$ and -1 :

$$A_{R,\max} = 5.0 + 3.0 = 8.0 \text{ mm}, \quad A_{R,\min} = |5.0 - 3.0| = 2.0 \text{ mm}.$$

For the specified 4.0 mm resultant,

$$16 = 25 + 9 + 30 \cos \Delta\phi, \quad \cos \Delta\phi = -0.600.$$

On the requested interval $[0, \pi]$, cosine is one-to-one, giving $\Delta\phi = 2.21430 \text{ rad} = 126.9^\circ$. The target lies between 2 and 8 mm, so a phase solution exists. Substitution recovers $A_R^2 = 34 - 18 = 16 \text{ mm}^2$ and provides an independent algebraic check.

Problem 19: the additional cancellation condition

At $\Delta\phi = \pi$, $\sin(\psi + \pi) = -\sin \psi$, so

$$y = (A_1 - A_2) \sin \psi, \quad A_R = |A_1 - A_2|.$$

Identically zero resultant displacement therefore additionally requires equal amplitudes, with matching frequency, wave number, and coherent phase convention. A concrete counterexample is $A_1 = 5 \text{ mm}$ and $A_2 = 3 \text{ mm}$: the resulting sinusoid has amplitude 2 mm despite the phase difference π . Unequal waves may cancel at isolated points or instants, but that is not cancellation of the entire field. The student's claim confuses opposite phase with equality of the two displacement magnitudes everywhere.

Problem 20: standing-wave form and nodes

Apply $\sin \alpha + \sin \beta = 2 \sin[(\alpha + \beta)/2] \cos[(\alpha - \beta)/2]$ with $\alpha = k_w x - \omega t$ and $\beta = k_w x + \omega t$. Including each component amplitude gives

$$y = 2A \sin(k_w x) \cos \omega t.$$

The spatial amplitude is $2A|\sin k_w x|$. Permanent nodes satisfy $k_w x_N = m\pi$, hence $x_N = m\pi/k_w = m\lambda/2$, for integer m in the domain. The factor $\cos \omega t$ can make every position momentarily zero, but does not create additional nodes. The $2A$ standing-wave antinode amplitude is twice the amplitude of either travelling component; this distinction matters in energy calculations.

Problem 21: derive reflected signs from boundary conditions

Take $y_i = f(x - c_w t)$ and $y_r = \sigma f(2L - x - c_w t)$. The reflected pulse moves left since constant $2L - x - c_w t$ gives $dx/dt = -c_w$.

(a) Fixed end. At $x = L$, $y_i + y_r = (1 + \sigma)f(L - c_w t)$. Vanishing for an arbitrary incident pulse requires $\sigma = -1$, yielding $y_{\text{fixed}} = f(x - c_w t) - f(2L - x - c_w t)$. Direct endpoint substitution gives exactly zero.

(b) Free end. Differentiation produces $\partial_x(y_i + y_r) = f'(x - c_w t) - \sigma f'(2L - x - c_w t)$. At L this is $(1 - \sigma)f'(L - c_w t)$; a zero slope for arbitrary f' requires $\sigma = +1$. Thus $y_{\text{free}} = f(x - c_w t) + f(2L - x - c_w t)$, whose boundary derivative is exactly zero. The two conditions constrain displacement and slope respectively, so their differing reflected signs are consistent.

Problem 22: fixed-end mode frequencies

Both endpoints are nodes, so n half-wavelengths fit in L : $\lambda_n = 2L/n$. Hence

$$f_n = \frac{nc_w}{2L} = \frac{n(60.0)}{2(0.750)} = 40.0n \text{ Hz}.$$

The requested values are $f_1 = 40.0 \text{ Hz}$, $f_2 = 80.0 \text{ Hz}$, and $f_5 = 200 \text{ Hz}$. Their corresponding wavelengths 1.50, 0.750, and 0.300 m each satisfy $f_n \lambda_n = 60.0 \text{ m s}^{-1}$. The integer spectrum follows from the fixed–fixed endpoint conditions together with the nondispersive relation; it is not a universal spectrum for every termination or stiff wire.

Problem 23: tension for a prescribed second mode

Invert $f_n = nc_w/(2L)$ before inserting numbers. For $n = 2$,

$$c_w = \frac{2Lf_2}{2} = (0.900)(120) = 108 \text{ m s}^{-1}.$$

The ideal tension then follows from the dynamical speed law:

$$F_T = \mu_\ell c_w^2 = (0.00800)(108)^2 = 93.3 \text{ N}.$$

The dimensions are $(\text{kg/m})(\text{m}^2/\text{s}^2) = \text{N}$. An independent spectral check gives $f_1 = c_w/(2L) = 60.0 \text{ Hz}$, so $f_2 = 2f_1 = 120 \text{ Hz}$. This is a design value within the uniform ideal-string model; an actual apparatus would also require checking end compliance and nonuniform tension.

Problem 24: fixed–free normal modes

The spatial sinusoidal solution has the form $Y(x) = B \sin(kx) + D \cos(kx)$. The fixed condition $Y(0) = 0$ forces $D = 0$. The free condition $Y'(L) = Bk \cos(kL) = 0$ then gives, for a nontrivial mode,

$$k_n L = \frac{(2n-1)\pi}{2}, \quad n = 1, 2, 3, \dots$$

Consequently $\lambda_n = 4L/(2n-1)$ and

$$f_n = \frac{(2n-1)c_w}{4L}.$$

The left end is a displacement node and the right end a displacement antinode. A zero-wave-number solution would be linear in x ; fixed displacement at zero and zero slope at L make it trivial, so there is no additional $n = 0$ vibrational mode. The allowed frequencies are odd multiples of the fundamental rather than all integer multiples.

Problem 25: fourth-mode positions

The spatial factor is $\sin(4\pi x/L)$. Setting it to zero gives $x = mL/4$ with $m = 0, 1, 2, 3, 4$ inside the closed interval. Thus the nodes are $0, L/4, L/2, 3L/4, L$. Antinodes occur when the sine has magnitude one:

$$x = \frac{(2j+1)L}{8}, \quad j = 0, 1, 2, 3,$$

namely $L/8, 3L/8, 5L/8, 7L/8$. The wavelength is $\lambda_4 = 2L/4 = L/2$, so exactly four half-wavelengths fit between the supports. Adjacent antinodes have opposite displacement signs at a nonzero snapshot, but their oscillation amplitudes have the same positive magnitude in a pure mode.

Problem 26: diagnose a mode drawing

Fixed–fixed boundaries impose $k_n L = n\pi$, so the third mode has $\lambda_3 = 2L/3$. Its half-wavelength is $L/3$, requiring three loops between the supports, with nodes at $0, L/3, 2L/3, L$. A drawing with only two half-wavelengths has $\lambda = L$ and is the second mode, not the third. Both drawings can satisfy the endpoint displacement constraints, so checking the endpoints alone is insufficient to identify the mode number. One must also count the internal nodes or compare the wavelength with the boundary-derived expression. Renaming two loops as a third mode does not change its physical frequency.

Problem 27: integrate both sides of the triangular pluck

Let $k_n = n\pi/L$. Split the coefficient integral at the corner, keeping each side's different slope:

$$A_n = \frac{2h}{L} \left[\frac{1}{a} \int_0^a x \sin(k_n x) dx + \frac{1}{L-a} \int_a^L (L-x) \sin(k_n x) dx \right].$$

Integration by parts gives $\int x \sin(k_n x) dx = -x \cos(k_n x)/k_n + \sin(k_n x)/k_n^2$. Therefore the first integral is

$$I_1 = -\frac{a \cos(k_n a)}{k_n} + \frac{\sin(k_n a)}{k_n^2}.$$

For the second, an antiderivative is $-(L-x) \cos(k_n x)/k_n - \sin(k_n x)/k_n^2$. Since $\sin(k_n L) = \sin(n\pi) = 0$, evaluation gives

$$I_2 = \frac{(L-a) \cos(k_n a)}{k_n} + \frac{\sin(k_n a)}{k_n^2}.$$

The weighted cosine terms cancel: $I_1/a + I_2/(L-a) = [1/a + 1/(L-a)] \sin(k_n a)/k_n^2$. Hence

$$A_n = \frac{2h}{a(L-a)k_n^2} \sin(k_n a) = \frac{2hL^2}{\pi^2 n^2 a(L-a)} \sin \frac{n\pi a}{L},$$

which verifies the stated coefficient. The length-squared ratio is dimensionless, leaving units of h . The coefficient envelope decays as n^{-2} , although the sine factor causes oscillating signs and sometimes exact zeros. This decay is consistent with a continuous piecewise-linear shape whose first derivative jumps at the pluck point; it does not mean every coefficient's magnitude decreases monotonically.

Problem 28: midpoint symmetry and vanished even modes

With $a = L/2$ the coefficient becomes $A_n = 8h \sin(n\pi/2)/(\pi^2 n^2)$. For every even $n = 2m$, $\sin(n\pi/2) = \sin(m\pi) = 0$, so its coefficient vanishes. Odd coefficients alternate in sign. The initial triangular shape has reflection symmetry $g(L-x) = g(x)$. Meanwhile

$$\sin \frac{n\pi(L-x)}{L} = (-1)^{n+1} \sin \frac{n\pi x}{L}.$$

Even- n basis functions are antisymmetric about the midpoint, so their overlap integrals with this symmetric shape cancel between the left and right halves. Odd- n basis functions are symmetric and are allowed. The missing even modes express a displacement symmetry; with an asymmetric initial velocity, even modes could subsequently be excited despite the symmetric initial displacement.

Problem 29: specify velocity as well as displacement

The general fixed-end modal evolution is

$$y(x, t) = \sum_n \sin(k_n x) [a_n \cos \omega_n t + b_n \sin \omega_n t].$$

At $t = 0$ the displacement depends only on a_n . The initial velocity is $\partial_t y(x, 0) = \sum_n \omega_n b_n \sin(k_n x)$, an independent field. Release from rest requires this field to vanish, which orthogonality gives as $b_n = 0$ for every retained mode. Thus matching the displacement alone does not impose the missing velocity condition. In a time-stepping code, the first step must also incorporate the declared zero initial velocity consistently; arbitrarily copying a translated profile into the second time level would generally introduce motion. Both initial fields are needed for the second-order-in-time wave equation.

Problem 30: Courant time steps

For the declared explicit centred scheme, the standard stable range has $C = c_w \Delta t / \Delta x \leq 1$ with positive steps. Convert 5.0 mm to 0.0050 m before dividing:

$$\Delta t_{\max} = \frac{0.0050}{40} = 1.25 \times 10^{-4} \text{ s} = 0.125 \text{ ms}.$$

At $C = 0.80$, $\Delta t = 0.80 \Delta t_{\max} = 1.00 \times 10^{-4} \text{ s} = 0.100 \text{ ms}$. Substitution into $40 \Delta t / 0.0050$ reproduces 1.00 and 0.80 respectively. This restriction concerns stability of this particular recurrence, not a universal time-step rule. A stable step still requires sufficient spatial and temporal resolution to obtain acceptable phase accuracy.

Problem 31: verify a periodic simulation

Three substantive tests are as follows. First, initialise a resolved periodic sinusoid with its appropriate right-going initial velocity and compare the evolving solution against $A \sin[k(x - c_w t) + \phi]$. Check the direction, period, and periodic wrap-around, rather than just the first few images. Second, project the output onto that sinusoidal mode and monitor amplitude and phase over many periods; bounded amplitude and the predicted discrete frequency test the recurrence and initialisation independently of visual smoothness. Confirm $0 < C \leq 1$ for every run. Third, hold the physical wavelength, domain, observation time, and stable Courant number fixed while doubling the grid resolution. The phase-speed discrepancy should approach zero at the expected second-order rate for this smooth test. These tests do not certify unrelated nonperiodic boundary algorithms, but they provide repeatable evidence for the declared benchmark.

Problem 32: independent uncertainty propagation

Take logarithms of $f_n \propto L^{-1} F_T^{1/2} \mu_\ell^{-1/2}$. The first-order relative differential is

$$\frac{df_n}{f_n} = -\frac{dL}{L} + \frac{1}{2} \frac{dF_T}{F_T} - \frac{1}{2} \frac{d\mu_\ell}{\mu_\ell}.$$

For independent standard uncertainties the variance is the sum of the squared sensitivity-weighted terms:

$$\left(\frac{\sigma_f}{f}\right)^2 = (0.005)^2 + (0.020/2)^2 + (0.010/2)^2 = 1.50 \times 10^{-4}.$$

The standard relative uncertainty is therefore $0.0122474 = 1.22\%$. Adding absolute percentage contributions would give a worst-direction bound, not this standard uncertainty. Correlated inputs would require covariance terms. The quoted value concerns uncertain parameters within the ideal model; unknown end compliance or stiffness is a separate model discrepancy.

Problem 33: check a synthetic harmonic pattern

Using the first peak as the reference fundamental predicts $2f_1 = 164.0 \text{ Hz}$ and $3f_1 = 246.0 \text{ Hz}$. The relative deviations from those predictions are

$$\frac{|164.5 - 164.0|}{164.0} = 0.00304878 = 0.305\%, \quad \frac{|246.1 - 246.0|}{246.0} = 0.000406504 = 0.0407\%.$$

Both are below the stated 0.5% threshold, so the three synthetic peaks are consistent with the first three ideal harmonics by this criterion. This is a tolerance check, not a statistical significance test: individual uncertainties and covariances were not supplied. It cannot establish the true boundary conditions, exclude a common frequency calibration error, identify tension and line density separately, or prove the absence of small physical corrections.

Problem 34: phase and group speeds

For the stipulated $\omega(k) = c_0 k \sqrt{1 + \ell_d^2 k^2}$, division by positive k gives $v_p = c_0 \sqrt{1 + \ell_d^2 k^2}$. The product and chain rules yield

$$\begin{aligned} v_g = \frac{d\omega}{dk} &= c_0 \sqrt{1 + \ell_d^2 k^2} + \frac{c_0 \ell_d^2 k^2}{\sqrt{1 + \ell_d^2 k^2}} \\ &= c_0 \frac{1 + 2\ell_d^2 k^2}{\sqrt{1 + \ell_d^2 k^2}}. \end{aligned}$$

Subtracting with a common denominator gives $v_g - v_p = c_0 \ell_d^2 k^2 / \sqrt{1 + \ell_d^2 k^2} > 0$ for the stated positive parameters. Both expressions have speed units, and both tend to c_0 as $\ell_d \rightarrow 0$. The inequality is specific to this dispersion relation; general dispersive systems need not have group speed greater than phase speed.

Problem 35: evaluate and compare the speeds

Here $\ell_d k_0 = (0.025)(20) = 0.500$ is dimensionless and its square is 0.250. Thus

$$v_p = 30\sqrt{1.25} = 33.54 \text{ m s}^{-1}, \quad v_g = 30 \frac{1.50}{\sqrt{1.25}} = 40.25 \text{ m s}^{-1}.$$

The difference is 6.71 m s^{-1} . Relative to phase speed,

$$\frac{v_g - v_p}{v_p} = \frac{0.25}{1.25} = 0.200 = 20.0\%.$$

If group speed is chosen as the denominator instead, the relative difference is 16.7%; stating the denominator avoids ambiguity. A narrowband packet's envelope therefore advances faster than its carrier phase in this stipulated model, with the qualifications discussed in the advanced section.

Problem 36: early visual smoothness does not establish stability

The value $C = 1.08$ exceeds the declared recurrence's stable bound $C \leq 1$. To see the mechanism, a discrete spatial Fourier mode has amplification factor G satisfying

$$G^2 - [2 - 4C^2 \sin^2(k\Delta x/2)]G + 1 = 0.$$

For the highest grid wave number $k\Delta x = \pi$, $C > 1$ makes the roots real with one magnitude greater than one. With $C = 1.08$, the magnitudes are approximately 2.214 and 0.452; tiny high-frequency contamination can therefore grow exponentially by time steps. A clean initial sinusoid may postpone visible contamination, but rounding and other errors can seed it. Smooth early frames test neither the amplification spectrum nor long-time boundedness. Stability must be checked against the scheme and boundary treatment, then supplemented with benchmark evolution.

Problem 37: refinement ratio and order

The benchmark values give

$$\frac{e_{100}}{e_{200}} = \frac{9.51053 \times 10^{-4}}{2.37093 \times 10^{-4}} = 4.01130.$$

For asymptotic errors $e_N \simeq aN^{-p}$, a doubling of resolution gives $e_N/e_{2N} \simeq 2^p$. Hence $p = \log_2(4.01130) = 2.004$, consistent with second-order convergence. Errors are absolute relative phase-speed discrepancies, so their ratio is dimensionless. One pair alone does not prove the order for every problem; a sequence of refinements and smooth, fixed physical initial data provide stronger evidence. Nonsmooth pulses or different boundary algorithms can have different observed convergence behaviour.

Problem 38: derive the carrier and envelope identity

Choose the cosine representation $y = A \cos(k_1 x - \omega_1 t) + A \cos(k_2 x - \omega_2 t)$; the same argument works with the corresponding sine identity. Set $\bar{k} = (k_1 + k_2)/2 = k_0$, $\bar{\omega} = (\omega_1 + \omega_2)/2$, $\Delta k = k_2 - k_1$, and $\Delta\omega = \omega_2 - \omega_1$. Then

$$y = 2A \cos\left(\frac{\Delta k x - \Delta\omega t}{2}\right) \cos(\bar{k}x - \bar{\omega}t).$$

The second factor is the rapidly varying carrier. The first is the slowly varying modulation when $|\Delta k| \ll k_0$; the amplitude envelope is its absolute value times $2A$. Following a fixed modulation phase gives

$$\Delta k dx - \Delta\omega dt = 0, \quad v_{\text{env}} = \frac{\Delta\omega}{\Delta k}.$$

For differentiable $\omega(k)$, the symmetric difference tends to $\omega'(k_0) = v_g$. If a third derivative exists, the leading correction is $\omega'''(k_0)(\Delta k)^2/24$. Two components produce a repeating modulation, rather than a spatially isolated packet; additional nearby components localise a packet. The finite-difference envelope speed is exact for the two-component modulation, while a derivative-based packet description is a narrow-spacing approximation.

Problem 39: physical versus numerical dispersion

The continuum PDE in the question is explicitly nondispersive. Wavelength-dependent phase speed that changes with grid resolution is therefore evidence of discretisation error. Hold each physical wavelength, the continuum speed, domain, and stable Courant number fixed; compute the phase speed on successively finer grids and extrapolate toward $\Delta x = 0$. Numerical dispersion should shrink toward the common continuum value c_w , as predicted by the scheme's numerical dispersion relation. Holding only points per wavelength fixed would not refine the physical calculation. To claim physical dispersion, one would need an independently derived or measured nonlinear $\omega(k)$ appropriate to the system, followed by numerical convergence showing that the wavelength dependence survives removal of discretisation error. An attractive packet animation establishes neither requirement.

Problem 40: follow a crest and read the axes

(a) Figure 16.29(a) has amplitude 3.00 mm and crest spacing 0.600 m. The marked crest moves from 0.600 to 0.750 m in 0.0250 s, hence its displacement is +0.150 m and its direction is rightward. The less-than-half-wavelength tracking condition excludes choosing a different periodic crest.

(b) The resulting speed is $c_w = 0.150/0.0250 = 6.00 \text{ m s}^{-1}$, and $f = c_w/\lambda = 10.0 \text{ Hz}$. Thus a full spatial repetition takes 0.100 s to pass a sensor.

(c) At $x = 0.075 \text{ m}$ the initial cosine profile is falling, so $\partial_x y < 0$. A right-going wave has $\partial_t y = -c_w \partial_x y > 0$; the element moves upward. The figure's crest translates along x , whereas this element's velocity is along y . Those two directions and two rates must be kept distinct.

Problem 41: reconstruct the wave from independent records

(a) Successive maxima of the fixed-sensor trace occur at 0, 0.100, and 0.200 s, giving $T_{\text{osc}} = 0.100 \text{ s}$ and $f = 10.0 \text{ Hz}$. Together with the snapshot's $\lambda = 0.600 \text{ m}$ this yields $c_w = 6.00 \text{ m s}^{-1}$, agreeing with tracked-crest motion.

(b) A function reproducing the initial crest at zero and the rightward shift is

$$y(x, t) = (0.00300 \text{ m}) \cos\left(\frac{10\pi}{3} x - 20\pi t\right),$$

where the numerical wave-number and angular-frequency coefficients have units m^{-1} and s^{-1} respectively. Inserting $x = 0$ gives the sensor trace.

(c) At the requested event the phase is $\pi/4$. Differentiating first and then evaluating gives

$$y = 2.12 \text{ mm}, \quad v_y = \omega A \sin(\pi/4) = +0.133 \text{ m s}^{-1}, \quad a_y = -\omega^2 y = -8.37 \text{ m s}^{-2}.$$

The element is above equilibrium and moving upward, but accelerating downward. The largest transverse speed is $\omega A = 0.188 \text{ m s}^{-1}$, about 3.14% of the propagation speed. These magnitudes and signs follow from a field derivative at fixed x , rather than differentiation along the moving crest.

Problem 42: fit synthetic resonances and infer tension

(a) Equal independent uncertainties make least squares minimise $S(s) = \sum_{n=1}^4 (f_n - sn)^2$. Setting $dS/ds = -2 \sum n(f_n - sn) = 0$ gives $\hat{s} = \sum n f_n / \sum n^2$. The fit is constrained through zero because the assumed spectrum is $f_n = sn$; the $n = 0$ origin is a model constraint rather than a measured peak.

(b) Here $\sum n^2 = 30$ and $\sum n f_n = 1200.2 \text{ Hz}$, so $\hat{s} = 40.0067 \text{ Hz}$. The residuals $f_n - n\hat{s}$ are $+0.0933, -0.2133, +0.2800, -0.1267 \text{ Hz}$, each smaller than its quoted 0.4 Hz standard uncertainty. The inferred speed is $c_w = 2L\hat{s} = 96.0160 \text{ m s}^{-1}$ and

$$F_T = \mu_\ell (2L\hat{s})^2 = 55.3 \text{ N}.$$

(c) Since $\hat{s}$ is a linear combination of the independent peak frequencies, $\sigma_s^2 = \sum [n/30]^2 \sigma_f^2 = \sigma_f^2/30$. Thus $\sigma_s = 0.0730 \text{ Hz}$. The tension's logarithmic differential is $dF_T/F_T = d\mu_\ell/\mu_\ell + 2dL/L + 2ds/s$, giving

$$\frac{\sigma_T}{F_T} = \sqrt{(0.00006/0.00600)^2 + (2 \times 0.003/1.200)^2 + (2 \times 0.07303/40.0067)^2} = 0.01176.$$

Therefore $F_T = (55.3 \pm 0.7) \text{ N}$ as a standard-uncertainty result within this model.

(d) The nearly straight harmonic pattern and small residuals are consistent with the assumed spectrum over these four modes. They do not independently prove perfectly fixed ends or negligible stiffness, nor exclude systematic calibration errors. The graph has too coarse a scale to judge sub-hertz residuals by eye, which is why numerical fitting and residual comparison matter.

Problem 43: displacement matching and energy transmission

(a) The speeds are $c_1 = \sqrt{72/0.00800} = 94.9 \text{ m s}^{-1}$ and $c_2 = 47.4 \text{ m s}^{-1}$. Frequency is common across a stationary junction; hence $\lambda_1 = c_1/f = 3.79 \text{ m}$ and $\lambda_2 = 1.90 \text{ m}$.

(b) Use complex displacement amplitudes at $x = 0$, with incident and transmitted factors $e^{i(kx - \omega t)}$ and reflected factor $e^{i(-kx - \omega t)}$. If $r = A_r/A_i$ and $t = A_t/A_i$, displacement continuity gives $1 + r = t$. Equal tension and no external transverse force imply continuity of $F_T \partial_x y$, hence $k_1(1 - r) = k_2 t$. Since $k_j = \omega/c_j$, the matching condition is $Z_1(1 - r) = Z_2 t$, where $Z_j = F_T/c_j = \sqrt{F_T \mu_{\ell,j}}$. Solving yields

$$r = \frac{Z_1 - Z_2}{Z_1 + Z_2}, \quad t = \frac{2Z_1}{Z_1 + Z_2}.$$

The impedance ratio is $Z_2/Z_1 = 2$, so $r = -1/3$ and $t = 2/3$.

(c) Thus $A_r = -0.600 \text{ mm}$ as a signed coefficient, and $A_t = 1.20 \text{ mm}$. The reflected wave has positive magnitude 0.600 mm and a phase inversion. The negative coefficient does not describe negative energy.

(d) For each travelling component, the positive directed-power magnitude is $Z\omega^2|A|^2/2$. Therefore

$$R = r^2 = \frac{1}{9}, \quad \mathcal{T} = \frac{Z_2}{Z_1} t^2 = \frac{8}{9}, \quad R + \mathcal{T} = 1.$$

The incident mean power is 0.03034 W, with reflected magnitude 0.00337 W and transmitted power 0.02697 W. Signed flux toward $+x$ is $\langle \mathcal{P} \rangle_i - \langle \mathcal{P} \rangle_r = \langle \mathcal{P} \rangle_t$, confirming conservation. The slope magnitudes are $|A_i|k_1 = 0.00298$, $|A_r|k_1 = 0.000993$, and $|A_t|k_2 = 0.00397$, all small. The maximum combined left-side slope is bounded by their sum, also small. These results presume a massless, lossless junction and the common tension in the diagram.

Problem 44: standing-mode energy and signed local flux

(a) Put $k = 2\pi/L$. Nodes are $0, L/2, L$ and antinodes are $L/4, 3L/4$. At release, $\partial_t y = 0$ and $\partial_x y = Ak \cos kx$, so $u_U = u_0 \cos^2 kx$. Elastic energy depends on slope, which is largest at displacement nodes; it is not determined by the height of the string. This explains the circles and squares in Figure 16.32.

(b) Over this integer mode, $\int_0^L \cos^2 kx dx = L/2$. Thus

$$E = u_0 \frac{L}{2} = \frac{F_T A^2 k^2 L}{4} = 3.95 \times 10^{-3} \text{ J.}$$

The speed is $\sqrt{80.0/0.0200} = 63.2456 \text{ m s}^{-1}$ and $f = c_w/L = 79.1 \text{ Hz}$, with $\omega = 496.729 \text{ rad s}^{-1}$.

(c) At $t = T_{\text{osc}}/4$, $\cos \omega t = 0$ and $\sin \omega t = 1$, so $u_U = 0$ and $u_K = \mu_\ell A^2 \omega^2 \sin^2 kx/2 = u_0 \sin^2 kx$. Its integral is again $u_0 L/2 = E$. More generally $K = E \sin^2 \omega t$ and $U = E \cos^2 \omega t$, proving $K + U = E$ at every time. The energy distribution changes while the integrated energy remains constant.

(d) The signed power is

$$\mathcal{P} = -F_T (\partial_x y)(\partial_t y) = \frac{F_T A^2 k \omega}{4} \sin(2kx) \sin(2\omega t).$$

Its average over a cycle is zero because the temporal sine averages to zero at every cut. At $x = L/8$ and $t = T_{\text{osc}}/8$, both displayed sines equal one, giving $\mathcal{P} = +0.312 \text{ W}$ toward $+x$. A zero mean therefore does not imply zero instantaneous exchange between neighbouring parts of a standing wave.

Problem 45: inverse information and changed boundaries

(a) The harmonic slope determines $c_w = 2L\hat{s}$ and therefore only the ratio $F_T/\mu_\ell = c_w^2$. Multiplying both tension and line density by the same positive factor leaves the entire ideal spectrum unchanged. Consequently no number of ideal fixed-end harmonics alone separates those two parameters.

(b) The independent tension supplies the missing information:

$$\mu_\ell = \frac{F_T}{(2L\hat{s})^2} = \frac{55.0}{(96.0160)^2} = 0.00596589 \text{ kg m}^{-1}.$$

From $d\mu_\ell/\mu_\ell = dF_T/F_T - 2dL/L - 2ds/s$, independent standard uncertainties give

$$\frac{\sigma_\mu}{\mu_\ell} = \sqrt{(0.5/55.0)^2 + (2 \times 0.003/1.200)^2 + (2 \times 0.07303/40.0067)^2} = 0.01100.$$

Thus $\mu_\ell = (0.005966 \pm 0.000066) \text{ kg m}^{-1}$. The input tension here is a separate measured value, not the tension inferred using the previously supplied density in Problem 42.

(c) The same speed on a fixed-free length L gives $f'_n = (2n-1)c_w/(4L) = (2n-1)\hat{s}/2$. The first three frequencies are 20.0, 60.0, and 100.0 Hz. The free endpoint imposes a slope zero instead of a displacement zero, admitting odd quarter-wavelength patterns and only odd multiples of this new fundamental.

Problem 46: read and reproduce numerical phase error

(a) The plotted errors decrease by ratios about 4.03, 4.01, and 4.00 on successive doublings, giving $p = \log_2(e_N/e_{2N})$ close to two. The graph is consistent with second-order convergence for this smooth mode.

(b) The physical wave number is $k = 2\pi m/L = 6\pi \text{ m}^{-1}$, $\Delta x = L/N$, and $\Delta t = C\Delta x/c_w$. The resolved branch of the numerical dispersion relation yields

$$\frac{v_{p,\text{num}}}{c_w} = \frac{2 \arcsin[C \sin(\pi m/N)]}{C(2\pi m/N)}.$$

At $N = 160$ and $C = 0.75$, the ratio is 0.9997468166 and $e_{160} = 2.53183 \times 10^{-4}$, reproducing the plotted point. The numerical wave is slightly slower than the continuum wave.

(c) The time step is $\Delta t = 0.75/(2 \times 160) = 0.00234375 \text{ s}$. The exact period is $T_{\text{osc}} = 2\pi/(c_w k) = 1/6 \text{ s}$. Since $\Omega/\omega = v_{p,\text{num}}/c_w$, the lag after that physical duration is $(\omega - \Omega)T_{\text{osc}} = 2\pi e_{160} = 0.001591 \text{ rad}$. This continuous-frequency comparison can be evaluated by phase fitting if the period is not an integer number of time steps.

(d) Fixing m and L keeps the physical wavelength L/m unchanged as the mesh is refined. Increasing m with N would keep the mode equally poorly resolved instead. The underlying PDE has $\omega = c_w k$ and no physical dispersion, so the grid-dependent discrepancy here is numerical.

Problem 47: modal energy versus local cross terms

(a) The first and second shapes contain one and two half-wavelengths respectively, hence $k_1 = \pi/L$ and $k_2 = 2\pi/L$. The quoted amplitudes in the question are physical antinode amplitudes; the schematic vertical scaling of Figure 16.15 is not a measurement of them.

(b) Write the pure mode as $y_n = A_n \sin(k_n x) \cos(\omega_n t + \phi_n)$. The two spatial integrals are

$$\int_0^L \sin^2(k_n x) dx = \int_0^L \cos^2(k_n x) dx = \frac{L}{2}.$$

Integrating kinetic and elastic line energy therefore gives

$$E_n = \frac{F_T L k_n^2 A_n^2}{4} = \frac{F_T \pi^2 n^2 A_n^2}{4L}.$$

The time dependence cancels because $\mu_\ell \omega_n^2 = F_T k_n^2$. Consequently $E_2/E_1 = 2^2(1.00/2.00)^2 = 1$: the smaller second-mode amplitude exactly compensates its doubled wave number.

(c) The kinetic cross term is proportional to $\int_0^L \sin(\pi x/L) \sin(2\pi x/L) dx = 0$. The elastic cross term is proportional to $\int_0^L \cos(\pi x/L) \cos(2\pi x/L) dx = 0$. Both vanish by orthogonality, even though their pointwise integrands usually do not. Thus the simultaneous field has integrated energy $E_1 + E_2$, irrespective of their temporal phases. This uses a uniform lossless linear string, compatible fixed endpoints, the same tension and density for both modes, and integration over the whole length. Arbitrary coherent travelling waves do not necessarily have this orthogonality property.

Problem 48: invert a stipulated dispersive model

(a) The graph locates the ratio near $z = 0.60$. Algebraically, $q = v_g/v_p = (1 + 2z^2)/(1 + z^2)$, so

$$z^2 = \frac{q-1}{2-q}, \quad \ell_d = \frac{1}{k_0} \sqrt{\frac{q-1}{2-q}}.$$

With $q = 1.26470588$, this gives $z^2 \simeq 0.360000$ and $\ell_d = 0.0400 \text{ m}$. The inversion on positive k and ℓ_d requires $1 \leq q < 2$; $q = 1$ is the nondispersive limit.

(b) The phase and group speeds are $v_p = 20\sqrt{1.36} = 23.3238 \text{ m s}^{-1}$ and $v_g = 20(1.72)/\sqrt{1.36} = 29.4978 \text{ m s}^{-1}$. In 0.0800 s, their phase and envelope advances are 1.866 m and 2.360 m respectively. These are pattern advances, not material displacements.

(c) Evaluate the dispersion relation at both actual wave numbers, rather than replacing the frequency difference by a derivative:

$$v_{\text{env}} = \frac{20[16\sqrt{1 + (0.04 \times 16)^2} - 14\sqrt{1 + (0.04 \times 14)^2}]}{16 - 14} \simeq 29.51 \text{ m s}^{-1}.$$

This is about 0.0252% above the derivative-based group speed at k_0 , consistent with a small finite bandwidth correction. The trigonometric identity in Problem 38 supplies the exact interpretation of this secant speed.

(d) The length scale was inferred conditionally within a declared nonlinear dispersion relation. The core taut-string model instead has $\omega = c_w k$. Applying a stipulated dispersive inversion to hypothetical data neither derives additional string physics nor measures such physics in an actual apparatus; independent evidence and a suitable physical model would be required.

16.13 Bridge to Chapter 17

The string has made propagation visible: inertia and a restoring interaction transmit a disturbance while material elements move locally. Phase, wavelength, frequency, superposition, reflection and boundary conditions carry into sound. The oscillating fields and the medium geometry change. Chapter 17 develops longitudinal sound using displacement, pressure and density perturbations; it also treats acoustic intensity and decibels, air-column modes, beats and the Doppler effect. A displacement node in one system cannot be transferred unchanged into a pressure boundary rule in another.

Return to the opening camera-and-detector question. A spatial snapshot gives the repetition length, a fixed-position time trace gives the repetition time, and identifying the same phase across observations determines direction. Together they permit $c_w = \lambda f$, while $\partial_t y$ describes the motion of the particular marked element. Newton's law determines which wave speeds an ideal string can support. Energy conservation then explains transport and interference without confusing a flat instantaneous shape with a static state.

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