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Chapter 17

Sound Waves

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Physics I

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Mechanics, Fluids, Waves, and Thermodynamics

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Preface

Sound is a mechanical wave, but it asks us to look at waves in a different way from the visible string of Chapter 16. A sound wave in a fluid is predominantly longitudinal: the material oscillates back and forth along the direction in which the pattern propagates. The same disturbance can be described through particle displacement, particle velocity, density change, or acoustic pressure. These descriptions are related, but they are not interchangeable at one instant. A pressure maximum, for example, need not occur where the displacement is maximum.

This chapter builds a quantitative acoustic model from that distinction. We connect compressibility and density to wave speed; derive the relation between displacement and pressure; define acoustic intensity and logarithmic sound level; develop interference, beats, and air-column modes; and derive a signed Doppler formula in the rest frame of the transmitting medium. A short extension then uses wavefront geometry to introduce Mach number and the Mach angle without attempting shock thermodynamics.

The optional advanced section turns a sound record into a reproducible numerical object. It introduces sampling, finite-record frequency spacing, aliasing, windowing, simple tube-end corrections, and the difference between a calibrated pressure spectrum and an arbitrary normalized spectrum. These tools are included because modern acoustic measurement is inseparable from digital signal processing, but they do not become hidden prerequisites for the chapter's core pathway.

The Guideline

Treat an acoustics problem as a problem about a *medium*, a *wave variable*, and a *measurement model*. State which quantity oscillates, distinguish propagation speed from particle velocity, identify peak versus RMS quantities, state the intensity reference before using decibels, and define all source and observer velocities relative to the transmitting medium before applying a Doppler formula.

Pedagogical boundary

This chapter does not provide hearing-safety limits, occupational exposure rules, medical advice, room-acoustic design standards, or shock-wave thermodynamics. Numerical medium properties are either stipulated in the problem or explicitly identified as ideal-model inputs. The microscopic thermodynamic derivation of the sound speed in a gas is reserved for Chapter 19, *Kinetic Theory of Gases*.

Conventions and notation

The equilibrium coordinate of a fluid element is denoted by x , and its small longitudinal displacement from equilibrium is $\xi(x, t)$. The local particle velocity is

$$u(x, t) = \frac{\partial \xi}{\partial t},$$

whereas c_w is the speed at which the acoustic pattern propagates through the medium. These are physically different velocities.

The acoustic pressure perturbation is $p'(x, t) = p(x, t) - p_{\text{eq}}$, where p_{eq} is the equilibrium pressure. A positive p' represents compression relative to equilibrium and a negative p' represents rarefaction. For sinusoidal quantities, a subscript “amp” denotes a peak amplitude and “rms” an RMS value. For a sinusoid, $p_{\text{rms}} = p_{\text{amp}}/\sqrt{2}$.

Symbol	Meaning	SI unit
ρ	equilibrium mass density of the medium	kg m^{-3}
B	relevant bulk modulus in the stated model	Pa
c_w	sound-wave propagation speed	m s^{-1}
$\xi(x, t)$	longitudinal particle displacement	m
$u = \partial \xi / \partial t$	longitudinal particle velocity	m s^{-1}
p'	acoustic pressure perturbation	Pa
$p_{\text{amp}}, p_{\text{rms}}$	peak and RMS acoustic pressure	Pa
I_{ac}	time-averaged acoustic intensity	W m^{-2}
L_I	intensity level relative to the declared I_0	dB
L_p	pressure level relative to stipulated p_{ref}	dB
$Z = \rho c_w$	plane-wave characteristic acoustic impedance	Pa s m^{-1}
$f, \omega = 2\pi f$	cyclic and angular frequency	Hz, rad s^{-1}
$\lambda, k_w = 2\pi/\lambda$	wavelength and wave number	m, m^{-1}
v_s, v_o	signed source and observer velocities in the medium frame	m s^{-1}
$\text{Ma} = v_s/c_w$	Mach number for a supersonic source speed magnitude	1
f_{samp}	digital sampling frequency	Hz
T_{rec}	record duration	s
Δf_{bin}	DFT bin spacing, $1/T_{\text{rec}}$ for a contiguous record	Hz

For a right-going sinusoidal displacement we use

$$\xi(x, t) = \xi_{\text{amp}} \cos(k_w x - \omega t + \phi).$$

The amplitude is non-negative. Directions are carried by the phase convention and by signed velocities. Unless stated otherwise, the core idealisation is a small-amplitude, one-dimensional plane progressive wave in a homogeneous, lossless medium. When spherical spreading is used, that geometrical assumption is stated explicitly.

Note

The core requires algebra, trigonometry, logarithms, derivatives, and the wave concepts developed in Chapter 16. The logarithm in a decibel formula always acts on a dimensionless ratio. The advanced section introduces the discrete-frequency spacing of a finite sampled record and the simplest consequences of windowing and aliasing.

Chapter 17

Sound Waves

Two idealised media are described only by their equilibrium densities and bulk moduli. Medium A has $\rho_A = 1.20 \text{ kg m}^{-3}$ and $B_A = 1.42 \times 10^5 \text{ Pa}$; medium B has $\rho_B = 1000 \text{ kg m}^{-3}$ and $B_B = 2.25 \times 10^9 \text{ Pa}$. Which carries a small longitudinal disturbance faster? Density alone does not answer the question. Compressibility matters at the same time. The calculation will give about 344 m s^{-1} for A but 1500 m s^{-1} for B. This opening comparison is a reminder that acoustics is built from constitutive response as well as inertia.

Learning goals

After studying the core of this chapter, you should be able to:

- explain compression, rarefaction, and longitudinal particle motion without confusing them with bulk transport of the medium;
- calculate a sound speed from a stipulated bulk modulus and density and interpret the assumptions of that model;
- relate longitudinal displacement, particle velocity, acoustic pressure amplitude, RMS pressure, and time-averaged intensity for a plane progressive sinusoid;
- use an explicitly stated intensity reference to calculate logarithmic sound levels and distinguish incoherent intensity addition from coherent phase-dependent addition;
- calculate path-difference interference and beat frequency, identifying the carrier and envelope time scales;
- derive ideal open–open and open–closed air-column mode spectra from pressure/displacement boundary conditions;
- apply a signed one-dimensional Doppler relation after defining the medium frame and the propagation direction;
- construct the wavefront envelope of a supersonic source and relate its Mach angle to the Mach number.

The optional advanced section adds sampling, spectral resolution, aliasing, windowing, end correction, and a reproducible two-tone spectrum.

Key message

A sound wave transports energy through oscillatory motion and compression of a material medium. The propagation speed c_w , the local particle velocity u , and the source/observer velocities used in Doppler problems are three different quantities and must not be merged into one symbol or one intuition.

17.1 Longitudinal sound and wave speed

17.1.1 Compression and rarefaction are moving patterns

Imagine a piston pushing slightly into a long fluid-filled tube. The first layer of fluid compresses; its increased pressure pushes the next layer, which compresses the layer beyond it. A sequence of local responses carries the disturbance down the tube. A marked fluid element need not accompany that disturbance. In a small-amplitude periodic wave it moves back and forth near its equilibrium position, whereas a surface of constant phase progresses through many such elements.

We describe the medium by an equilibrium pressure p_{eq} and density ρ . The acoustic pressure $p' = p - p_{\text{eq}}$ is a *perturbation*: it is positive in a compression and negative in a rarefaction. Negative acoustic pressure means pressure below equilibrium; the absolute pressure can remain positive throughout. The continuum displacement $\xi(x, t)$ labels the motion of a small fluid element whose equilibrium coordinate is x . Its instantaneous position is $x + \xi(x, t)$. At leading order we use the equilibrium coordinate and the ordinary spatial coordinate interchangeably; corrections caused by their difference are of higher order in the small disturbance.

Figure 17.1 makes the essential distinction visible. Displacement is a signed change in position, whereas compression is a change in *separation*. Two neighbouring elements can both have positive displacement and still move farther apart. The quantity that decides compression is the displacement gradient, not displacement itself.

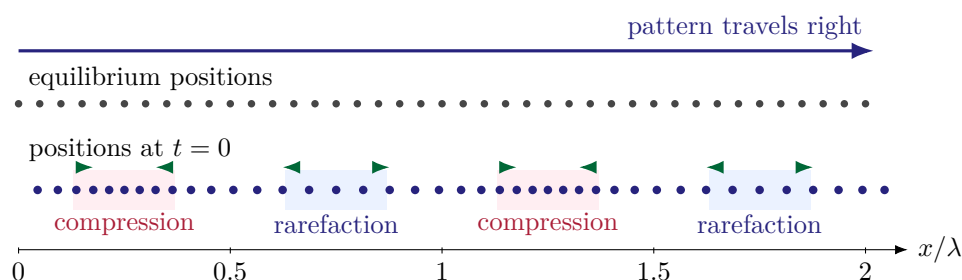


Figure 17.1: Material motion and pattern motion. Equally spaced equilibrium markers are mapped to $x + \xi$ using one sinusoidal displacement. The excursion is exaggerated; the short horizontal arrows show selected signed displacements. Compression occurs where the displacement decreases with position. The long propagation arrow describes the pattern, not a fluid trajectory. Construction adapted from Neutelings' *Waves in a gas* (TikZ.net).

The model describes the organized motion of fluid elements rather than individual thermal molecular trajectories. A gas molecule has random thermal motion in addition to the small organized acoustic motion. Sound can propagate through liquids and gases, which resist compression. A vacuum lacks the material medium needed for this mechanism. Solids can support additional elastic modes; the particular fluid equations below do not represent all waves in a solid.

17.1.2 A material slab connects displacement to compression

Choose a thin material slab of equilibrium length Δx and cross-sectional area A . Its left boundary moves from x to $x + \xi(x, t)$, and its right boundary moves from $x + \Delta x$ to $x + \Delta x + \xi(x + \Delta x, t)$. Subtraction of these two actual positions gives its new length:

$$\ell = \Delta x + \xi(x + \Delta x, t) - \xi(x, t).$$

Thus $\Delta V = A[\xi(x + \Delta x, t) - \xi(x, t)]$, with the *right* displacement minus the *left* displacement. Dividing by the original volume $V = A\Delta x$ and taking the small-slab limit gives $\Delta V/V = \partial\xi/\partial x$ to first order. A spatial derivative compares neighbouring elements at one instant; it does not describe a single element's time history.

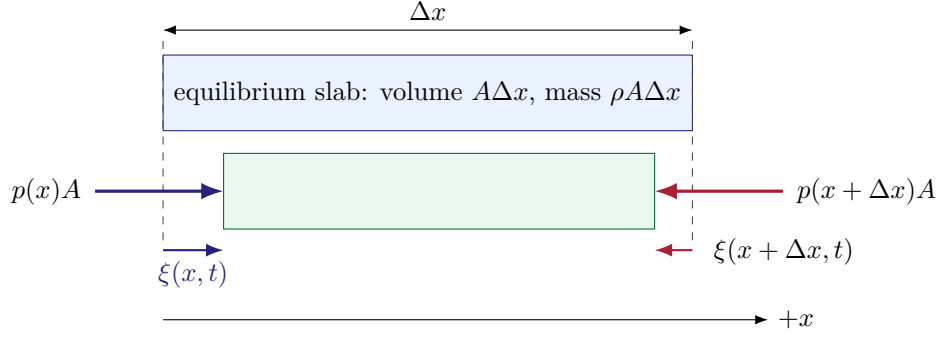


Figure 17.2: A material slab before and during a compression. Its length change is the right boundary displacement minus the left boundary displacement. Pressure acts inward at each face. The drawn displacements and force lengths are schematic; the algebra determines the net force and fractional volume change.

Mass conservation provides a second description. Write the disturbed density as $\rho + \rho'$. The mass of the slab is unchanged, so

$$(\rho + \rho')(V + \Delta V) = \rho V.$$

Neglecting the product $\rho' \Delta V$ gives $\rho'/\rho = -\Delta V/V = -\partial_x \xi$. Here and below a subscript on a derivative is shorthand for its independent variable. The linear approximation requires $|\partial_x \xi| \ll 1$ and $|\rho'| \ll \rho$; it is the strain, rather than the displacement measured in metres alone, that must be small.

17.1.3 Restoring response and Newton's law produce a wave equation

The bulk modulus describes the fluid's resistance to compression:

$$B = -V \frac{dp}{dV}, \quad (17.1)$$

$$p' = -B \frac{\Delta V}{V} = -B \frac{\partial \xi}{\partial x}. \quad (17.2)$$

The minus sign ensures $B > 0$ for a stable fluid: a decrease in volume raises pressure. The modulus has units of pascals. It must describe the physical response on the disturbance's time scale. For an ordinary acoustic oscillation in a gas, sufficiently rapid compression is approximately adiabatic, meaning little heat crosses the element during a cycle. Chapter 19 develops the ideal-gas thermodynamic and microscopic explanation. Using an isothermal static compressibility indiscriminately would change the speed prediction. We assume a specified constant B here and do not need a microscopic derivation to establish the wave mechanics.

Let $u = \partial_t \xi$ denote the signed local fluid velocity. The net pressure force on the slab in Fig. 17.2 is

$$F_x = A[p(x, t) - p(x + \Delta x, t)] \simeq -A \frac{\partial p'}{\partial x} \Delta x.$$

The equilibrium pressure cancels. Dividing Newton's law by the equilibrium mass $\rho A \Delta x$ gives

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p'}{\partial x}, \quad \frac{\partial \rho'}{\partial t} = -\rho \frac{\partial u}{\partial x}. \quad (17.3)$$

The second relation is the time derivative of $\rho' = -\rho \partial_x \xi$ and is the linear continuity equation. Together with $p' = (B/\rho) \rho'$, these are the closed linear acoustic model. Products of acoustic perturbations and advective acceleration $u \partial_x u$ have been discarded. The equations apply in the medium-rest frame, with no mean background flow.

Substituting $p' = -B\partial_x\xi$ into the momentum equation produces

$$\frac{\partial^2\xi}{\partial t^2} = \frac{B}{\rho} \frac{\partial^2\xi}{\partial x^2}, \quad (17.4)$$

$$c_w = \sqrt{\frac{B}{\rho}}. \quad (17.5)$$

The same wave equation holds for p' , u , and ρ' when B and ρ are constant. A function $F(x - c_w t)$ satisfies it because two time derivatives produce $c_w^2 F''$ and two space derivatives produce F'' . This checks the propagation interpretation directly. Dimensions also agree: $[B/\rho] = \text{m}^2 \text{s}^{-2}$. Elastic response supplies the restoring force; density supplies inertia. Increasing B at fixed density raises speed, while increasing density at fixed B lowers it.

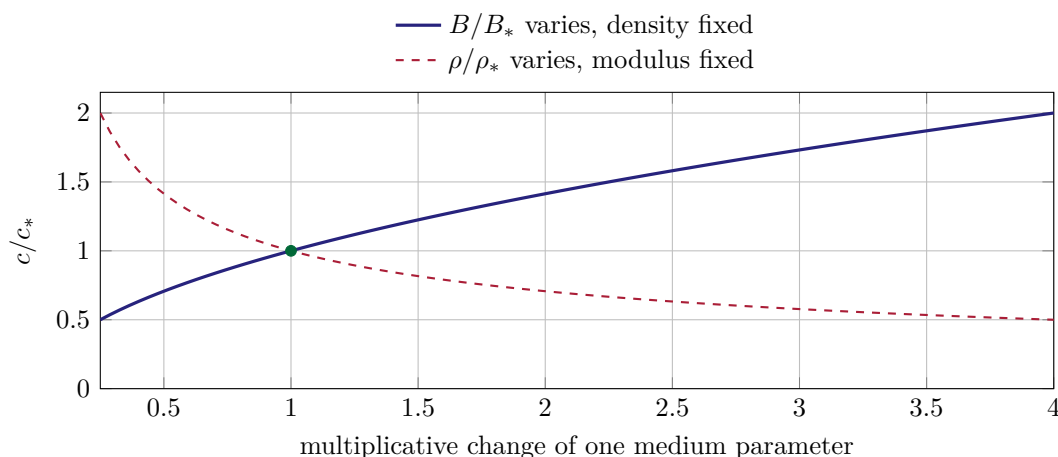


Figure 17.3: Separate sensitivities of sound speed to modulus and density. The reference medium has parameters B_* , ρ_* , c_* . Each curve changes one parameter while holding the other fixed; comparing different named materials requires both parameters.

Example 17.1: Compare two stipulated media

Statement and model. Medium A has $\rho_A = 1.20 \text{ kg m}^{-3}$ and $B_A = 1.42 \times 10^5 \text{ Pa}$; medium B has $\rho_B = 1000 \text{ kg m}^{-3}$ and $B_B = 2.25 \times 10^9 \text{ Pa}$. Predict their sound speeds and the wavelength of a 500 Hz tone in each. These are stipulated homogeneous fluid parameters, not measured properties of named substances.

Reasoning. The modulus must compete with density, so comparing densities alone is insufficient. Applying Eq. (17.5) gives

$$c_A = \sqrt{\frac{1.42 \times 10^5}{1.20}} = 343.996 \text{ m s}^{-1}, \quad c_B = \sqrt{\frac{2.25 \times 10^9}{1000}} = 1500 \text{ m s}^{-1}.$$

At the stated frequency $\lambda = c/f$, so $\lambda_A = 0.688 \text{ m}$ and $\lambda_B = 3.00 \text{ m}$. The frequency is fixed by the drive, while the wavelength adjusts to the medium. B is approximately 833 times denser, but its modulus is approximately 1.58×10^4 times larger; hence $c_B/c_A = 4.36$.

Interpretation and check. Substituting the speeds back into ρc^2 recovers both stipulated moduli. Multiplying both B and ρ by the same factor would leave c unchanged. The fluid expression is not a universal formula for a solid: longitudinal waves in a slender rod involve a different deformation constraint from bulk waves in a three-dimensional solid.

17.1.4 Wavefronts, frequency, and a speed measurement

A wavefront is a surface of constant phase. In a straight tube with predominantly axial motion it can be approximated by a plane perpendicular to the tube. Around a compact isotropic source the outgoing phase fronts are spheres; a small distant patch can look nearly plane without the entire wave having constant amplitude. For a harmonic wave,

$$c_w = f\lambda = \frac{\omega}{k_w}, \quad k_w = \frac{2\pi}{\lambda}, \quad \omega = 2\pi f. \quad (17.6)$$

Constant phase $k_w x - \omega t = \text{constant}$ gives $dx/dt = \omega/k_w > 0$. This establishes direction without guessing from a plotted crest. The frequency f counts cycles per second; angular frequency ω counts radians per second. They must not be substituted for one another in amplitude or Doppler calculations.

Example 17.2: Infer modulus from a pulse delay

Statement. Two detectors in a uniform tube are separated by $d = 1.20$ m. A weak pulse reaches the second $\Delta t = 3.50$ ms after the first. The density is stipulated as 1.20 kg m^{-3} . Infer c and B , and predict the wavelength of a 700 Hz tone. Assume no mean flow, identifiable matching pulse features, and negligible dispersion.

Strategy and solution. The source's launch time cancels because we use a difference between arrival times. Thus

$$c = \frac{d}{\Delta t} = \frac{1.20}{0.00350} = 342.857 \text{ m s}^{-1}, \quad B = \rho c^2 = 1.41 \times 10^5 \text{ Pa}.$$

The harmonic wavelength is $\lambda = c/f = 0.490$ m. If the timing uncertainty is 0.05 ms and distance and density uncertainty are negligible, first-order propagation gives $\delta c/c = \delta t/\Delta t = 0.0143$ and $\delta B/B = 2\delta t/\Delta t = 0.0286$. Accordingly $\delta c = 4.90 \text{ m s}^{-1}$ and $\delta B = 4.03 \times 10^3 \text{ Pa}$.

Check and limitation. The inferred speed returns $d/c = 3.50$ ms. Echoes or frequency-dependent pulse distortion could bias which feature is identified; a small quoted timing uncertainty cannot remove that systematic effect. This inference determines an effective acoustic modulus under the stated conditions.

17.2 Pressure, displacement, and intensity

17.2.1 Spatial phase and the history of a single element

The wave equation establishes propagation. To connect a moving pattern with microphone or displacement measurements, choose the right-going sinusoid

$$\xi(x, t) = \xi_{\text{amp}} \cos \psi, \quad \psi = k_w x - \omega t. \quad (17.7)$$

Differentiating with respect to position and time separately gives

$$p' = Bk_w \xi_{\text{amp}} \sin \psi, \quad (17.8)$$

$$u = \omega \xi_{\text{amp}} \sin \psi, \quad (17.9)$$

$$p_{\text{amp}} = Bk_w \xi_{\text{amp}} = \rho c_w \omega \xi_{\text{amp}}, \quad u_{\text{amp}} = \omega \xi_{\text{amp}}. \quad (17.10)$$

The time derivative has a positive sign: differentiating $\cos(k_w x - \omega t)$ produces two minus signs. The pressure and particle velocity are in phase for this right-going wave. Both are a quarter cycle away from displacement. At a maximum compression, displacement is zero but the neighbouring elements are displaced toward that position. At a displacement maximum, the displacement gradient is zero, so the pressure perturbation is zero.

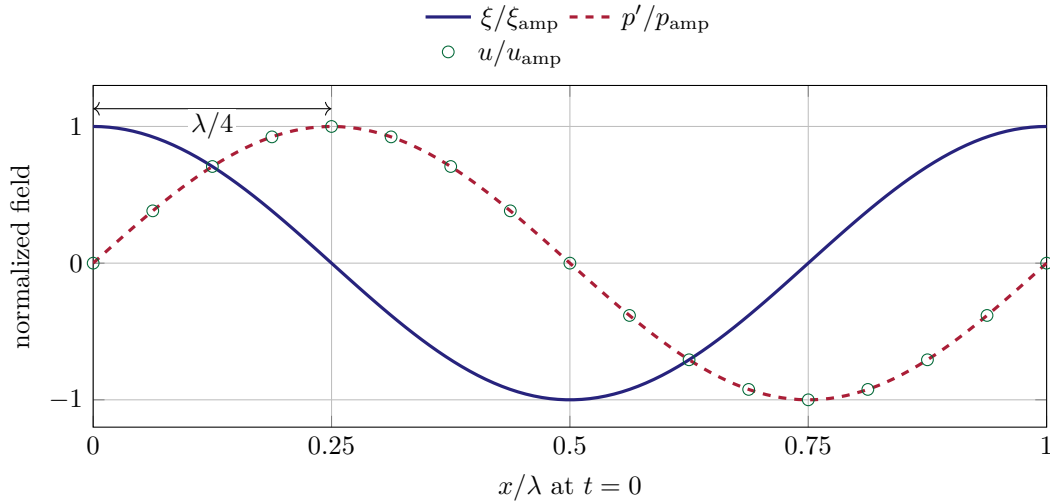


Figure 17.4: A snapshot of a right-going sinusoid. Pressure and velocity overlap, while the displacement curve reaches a maximum one quarter wavelength before the pressure maximum. The normalized vertical scale describes different physical quantities; it does not give them the same units.

The time history at a fixed equilibrium position is a different graph. At $x = 0$, $\xi = \xi_{\text{amp}} \cos \omega t$ but $p' = -p_{\text{amp}} \sin \omega t$ and $u = -u_{\text{amp}} \sin \omega t$. A rising time coordinate samples phase in the opposite direction from a rising position coordinate. Figure 17.5 shows the element initially at its greatest positive displacement, starting to move back toward equilibrium. Statements such as “pressure leads displacement” are ambiguous unless the independent variable and propagation convention are specified.

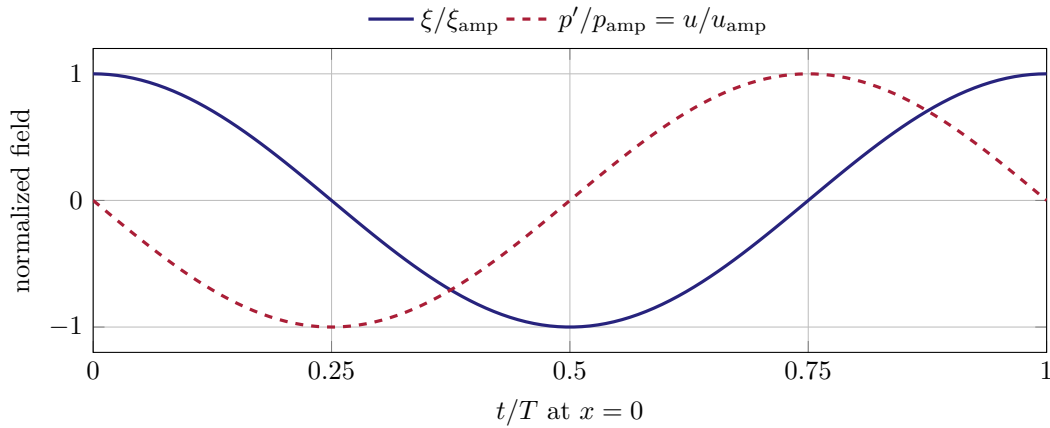


Figure 17.5: A single fluid element's time history. Its velocity equals the slope of its displacement history, and is initially negative just after $t = 0$. The graph is not a trajectory of a wavefront.

17.2.2 Energy density and energy flux

Acoustic intensity measures energy transferred through an area per unit time. A displacement amplitude measured in metres or pressure amplitude measured in pascals cannot by itself be an intensity. We derive the energy relation to keep track of what is transported.

The kinetic energy density of the organized motion is $e_{\text{kin}} = \rho u^2/2$. For the compression energy, start with the incremental work on an element, $dW = -p'd(\Delta V)$. Since $p' = -B\Delta V/V$, integration from zero perturbation gives $W = B(\Delta V)^2/(2V)$. Dividing by the equilibrium volume yields

$$e_{\text{comp}} = \frac{B}{2}(\partial_x \xi)^2 = \frac{p'^2}{2B}, \quad e = \frac{1}{2}\rho u^2 + \frac{p'^2}{2B}. \quad (17.11)$$

These are quadratic acoustic energies, measured in J m^{-3} . The equilibrium pressure contribution is excluded from this perturbation energy. For a right-going plane wave, $p' = \rho c_w u$ and $B = \rho c_w^2$, so the kinetic and compression terms are equal at each point and time. This equality is a property of a single progressive wave; it will not hold pointwise in a standing wave.

The flux follows directly from conservation. Differentiate Eq. (17.11), then use the linear momentum equation and $\partial_t p' = -B \partial_x u$:

$$\frac{\partial e}{\partial t} = \rho u \frac{\partial u}{\partial t} + \frac{p'}{B} \frac{\partial p'}{\partial t} = -u \frac{\partial p'}{\partial x} - p' \frac{\partial u}{\partial x} = -\frac{\partial(p'u)}{\partial x}.$$

Thus

$$\frac{\partial e}{\partial t} + \frac{\partial \mathcal{I}}{\partial x} = 0, \quad \boxed{\mathcal{I} = p'u}, \quad I_x = \langle p'u \rangle. \quad (17.12)$$

The instantaneous signed flux $\mathcal{I}$ can be negative when energy flows toward decreasing x . In three dimensions the intensity is a vector, $\mathbf{I} = \langle p' \mathbf{u} \rangle$; power crossing a surface uses its normal component. For the single right-going plane sinusoid $\mathcal{I} = \rho c_w u^2 = c_w e \geq 0$, so the energy travels at the wave speed even though the local fluid speed is much smaller.

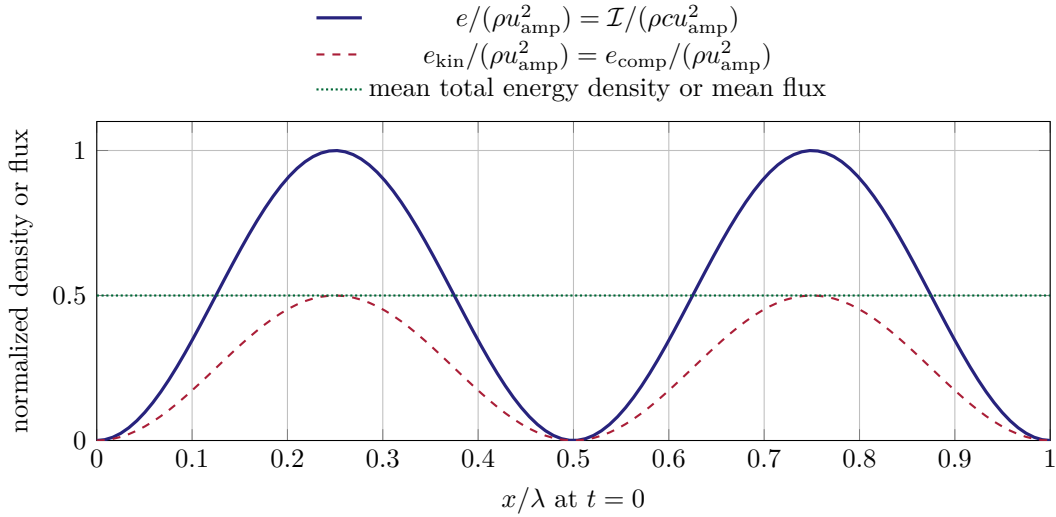


Figure 17.6: Progressive-wave energy and flux. Both energy contributions peak at the same locations and each contributes half the instantaneous total. The total-energy and flux curves coincide only after the stated normalizations; averaging $\sin^2$ over a wavelength or a period gives $1/2$.

17.2.3 Peak, RMS, and an averaging interval

For a periodic quantity q , define $q_{\text{rms}} = \sqrt{T^{-1} \int_0^T q(t)^2 dt}$. A sinusoid with zero mean has $q_{\text{rms}} = q_{\text{amp}}/\sqrt{2}$. The square, not the signed signal, is averaged: averaging p' alone would give zero without implying zero acoustic energy. Consequently a progressive sinusoid has

$$\boxed{I_{\text{ac}} = \frac{p_{\text{amp}}^2}{2\rho c_w} = \frac{p_{\text{rms}}^2}{\rho c_w}}, \quad (17.13)$$

$$I_{\text{ac}} = \frac{1}{2} \rho c_w \omega^2 \xi_{\text{amp}}^2. \quad (17.14)$$

The product $Z = \rho c_w$, measured in Pa s m^{-1} , is the characteristic plane-wave impedance. The relation $p' = Zu$ carries information about both amplitude and phase. For a wave traveling toward negative x , the corresponding signed relation is $p' = -Zu$.

For a pulse or other finite record use an explicitly stated duration T_R , with $p_{\text{rms}}^2 = T_R^{-1} \int_{t_0}^{t_0+T_R} p'^2 dt$. The peak-to-RMS ratio need not be $\sqrt{2}$: long quiet intervals reduce RMS without changing a pulse's

peak. In a uniform, lossless, one-way plane-wave field, $I_x = p_{\text{rms}}^2/Z$ still holds for that time average even when the waveform is not sinusoidal. In a reflected or mixed-direction field it generally does not. A microphone records local pressure; it cannot establish signed energy flow from pressure alone without further field information.

Example 17.3: Convert displacement to pressure and intensity

Statement. A right-going plane sinusoid has $\rho = 1.20 \text{ kg m}^{-3}$, $c = 343 \text{ m s}^{-1}$, $f = 500 \text{ Hz}$ and $\xi_{\text{amp}} = 2.0 \times 10^{-7} \text{ m}$. Find pressure amplitude, RMS pressure, intensity, particle-speed amplitude, and the size of the linear strain.

Solution. The angular frequency is $\omega = 2\pi f = 3141.59 \text{ rad s}^{-1}$ and $Z = 411.6 \text{ Pa s m}^{-1}$. Therefore

$$u_{\text{amp}} = \omega \xi_{\text{amp}} = 6.28 \times 10^{-4} \text{ m s}^{-1}, \quad p_{\text{amp}} = Z u_{\text{amp}} = 0.259 \text{ Pa}.$$

The RMS pressure is 0.183 Pa and

$$I = \frac{p_{\text{amp}}^2}{2Z} = 8.12 \times 10^{-5} \text{ W m}^{-2}.$$

Here $\lambda = 0.686 \text{ m}$, $k = 9.159 \text{ m}^{-1}$, and $k\xi_{\text{amp}} = 1.83 \times 10^{-6}$. The maximum fractional compression is about two parts per million. The maximum organized fluid speed is also only 1.83×10^{-6} of the sound speed.

Independent check. The mean total acoustic energy density is $\rho u_{\text{amp}}^2/2 = 2.37 \times 10^{-7} \text{ J m}^{-3}$; multiplying by c gives the same intensity. This checks the energy interpretation rather than merely repeating the pressure formula. The very small strain and speed ratio support the linear approximation.

Example 17.4: Compare two amplitude reports

Statement. A sinusoidal plane tone is reported as a peak pressure of 0.300 Pa by one instrument. A second report quotes 0.300 Pa RMS in the same stipulated medium, $Z = 411.6 \text{ Pa s m}^{-1}$. Do the reports describe equal intensities? What changes if the first signal is present for only one quarter of a record, with zero pressure during the remaining three quarters?

Solution. For the continuous peak-amplitude report,

$$p_{\text{rms},1} = 0.300/\sqrt{2} = 0.212 \text{ Pa}, \quad I_1 = 0.300^2/(2Z) = 1.09 \times 10^{-4} \text{ W m}^{-2}.$$

For the second report $I_2 = 0.300^2/Z = 2.19 \times 10^{-4} \text{ W m}^{-2} = 2I_1$. Equal printed pressure numbers do not mean equal mean-square pressure if one is peak and the other RMS.

For a sinusoidal burst whose active interval contains an integer number of periods, the full-record mean square is $Dp_{\text{amp}}^2/2$, where the duty fraction is $D = 1/4$. The full-record RMS is 0.106 Pa and average intensity is $2.73 \times 10^{-5} \text{ W m}^{-2}$. During the active interval the intensity remains I_1 . The burst peak is still 0.300 Pa.

Check. Integrating p'^2 over the active interval and dividing by the four-times-longer record gives precisely the factor D . Switching transients have been idealized away; a real waveform should be integrated over the actual record rather than assigned an assumed sine-wave conversion.

17.2.4 Geometrical spreading and acoustic power

An ideal source radiates a time-averaged acoustic power P . In an unbounded homogeneous lossless medium, isotropy makes the radial intensity constant on each sphere. Every sphere surrounding the

source intercepts the same outgoing power, so

$$P = \int \mathbf{I} \cdot d\mathbf{A} = 4\pi r^2 I(r), \quad \boxed{I(r) = \frac{P}{4\pi r^2}}. \quad (17.15)$$

No energy disappears: it crosses an increasing area. If far-field pressure is related to intensity by the local progressive-wave impedance, its amplitude decreases as $1/r$, while intensity decreases as $1/r^2$. A source's electrical input power is not automatically its acoustic radiated power; conversion efficiency would be additional information.

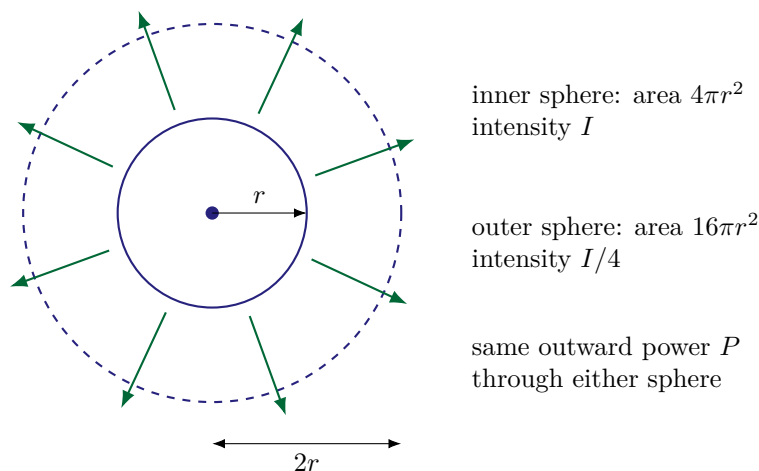


Figure 17.7: Conservation through nested spherical surfaces. Circles are cross-sections of spheres, not cylindrical wavefronts. Doubling radius quadruples the receiving area and reduces intensity to one quarter. Arrows identify radial energy flow.

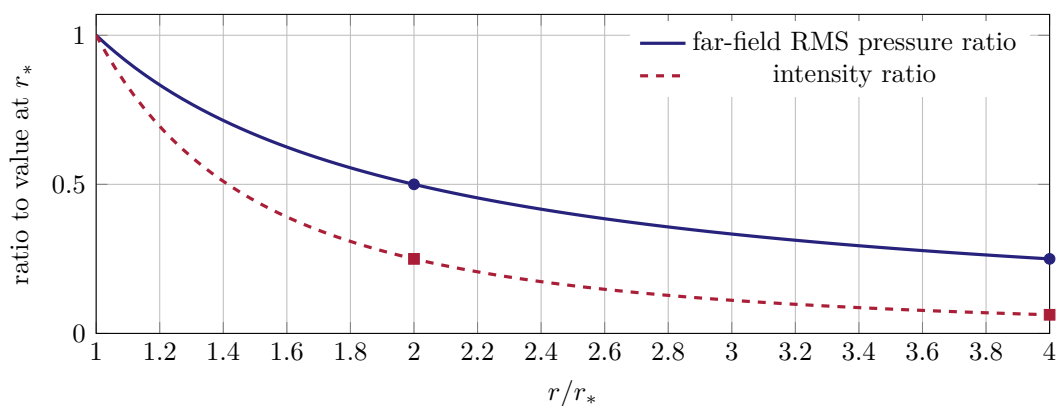


Figure 17.8: Amplitude and intensity respond differently to spherical spreading. The domain starts at a stipulated reference radius outside the source region; the point-source law is not extended to its unphysical divergence at $r = 0$. Pressure ratios assume the far-field progressive-wave relation in the same medium.

Example 17.5: Predict free-field spreading

Statement. An ideal isotropic source emits $P = 20 \text{ mW}$ into a lossless unbounded medium. Find intensities at 2.0 m and 4.0 m, and, using $Z = 411.6 \text{ Pa s m}^{-1}$, estimate RMS pressures in the far-field plane-wave approximation. What power crosses a small 0.010 m^2 receiving patch normal to the radial direction at 2.0 m?

Solution. The conservation model gives

$$I(2) = \frac{0.020}{4\pi(2.0)^2} = 3.98 \times 10^{-4} \text{ W m}^{-2}, \quad I(4) = 9.95 \times 10^{-5} \text{ W m}^{-2}.$$

The corresponding RMS pressures are $\sqrt{ZI(2)} = 0.405 \text{ Pa}$ and $\sqrt{ZI(4)} = 0.202 \text{ Pa}$. For a patch small enough to have approximately uniform intensity, $P_{\text{patch}} = I(2)A = 3.98 \mu\text{W}$.

Check and interpretation. Multiplying either intensity by its spherical area recovers 20 mW. The pressure halves when radius doubles, while intensity falls by four. Using $I_0 = 10^{-12} \text{ W m}^{-2}$, the intensity levels defined next are 86.0 dB and 80.0 dB, a 6.02 dB decrease. These are ideal model predictions. Boundaries, reflection, directivity, absorption, and near-field phase differences would need additional models.

17.3 Logarithmic sound levels

17.3.1 A logarithm compares ratios

The base-ten logarithm asks which exponent of ten gives a positive number: $\log_{10}(10^a) = a$. Its rules $\log(ab) = \log a + \log b$ and $\log(a/b) = \log a - \log b$ convert multiplication and division into addition and subtraction. Acoustic levels use this compression of ratios to express many orders of magnitude compactly.

For positive time-averaged intensity magnitude I , define the intensity level by

$$L_I = 10 \log_{10} \left(\frac{I_{\text{ac}}}{I_0} \right), \quad I_0 = 10^{-12} \text{ W m}^{-2}. \quad (17.16)$$

The ratio is dimensionless, and a decibel records a logarithmic comparison rather than an energy flux. Zero decibels means $I = I_0$, not silence. An intensity below the reference has a negative level. Exactly zero intensity has no finite logarithmic level; the limit tends toward $-\infty$. For a signed intensity component, first specify a direction and use a positive relevant magnitude; taking a logarithm of a negative component is undefined.

Subtracting two levels eliminates the reference:

$$\Delta L = 10 \log_{10} \frac{I_2}{I_1}, \quad \frac{I_2}{I_1} = 10^{\Delta L/10}. \quad (17.17)$$

Thus twice the intensity gives 3.010 dB more, four times gives 6.021 dB more, and ten times gives 10 dB more. Frequency and subjective loudness are separate matters; a decibel calculation does not predict a person's perception or establish an exposure limit.

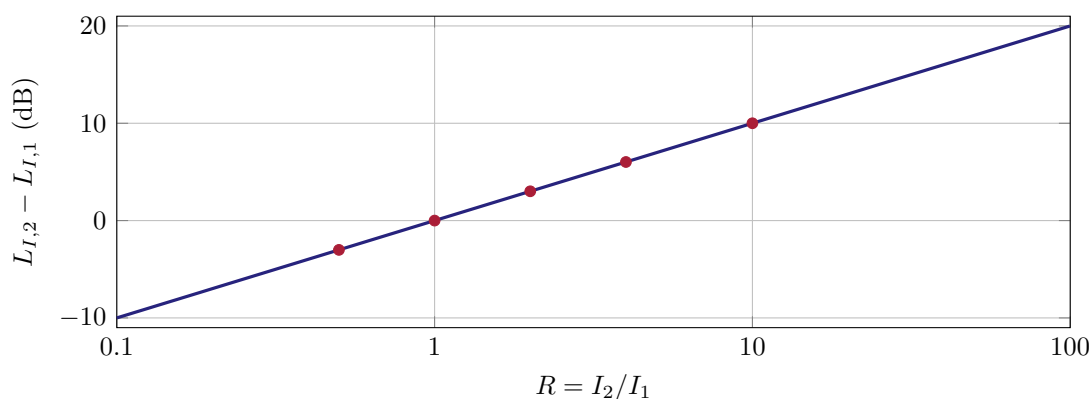


Figure 17.9: Intensity-ratio to level-difference conversion. A logarithmic horizontal axis makes the curve straight: equal multiplicative changes in intensity produce equal additive level changes. The reference cancels in this difference.

17.3.2 Intensity level and pressure level have different references

A calibrated microphone naturally provides pressure rather than power flow. The conventional air reference is $20 \mu\text{Pa}$ [4]. Define a pressure level for a stated RMS pressure and reference pressure p_{ref} by

$$L_p = 20 \log_{10} \frac{p_{\text{rms}}}{p_{\text{ref}}}, \quad p_{\text{ref}} = 20 \mu\text{Pa} \quad \text{in our air examples.} \quad (17.18)$$

The factor twenty arises because intensity in a progressive plane wave is proportional to the square of RMS pressure. It is not appropriate to use $20 \log(I/I_0)$ for intensity. The pressure reference must accompany a level, particularly when comparing different media or reports. Our air examples use the conventional $20 \mu\text{Pa}$ reference; a different stipulated reference would assign a different numerical pressure level to the same physical pressure.

Even with these reference choices L_I and L_p are not exactly identical. In a progressive plane wave,

$$L_I - L_p = 10 \log_{10} \frac{p_{\text{ref}}^2}{ZI_0}. \quad (17.19)$$

The ratio's units cancel because ZI_0 has units of pressure squared. For $Z = 411.6 \text{ Pa s m}^{-1}$, the offset is -0.124 dB . Their close numerical agreement in this medium follows from the reference choices and approximate air impedance, rather than an identity of the quantities. In a standing wave a pressure level can be finite where net mean intensity is zero; then the conversion above does not apply.

Example 17.6: Read a pressure level without changing its meaning

Statement. A calibrated measurement gives $L_p = 70.0 \text{ dB}$ relative to $20 \mu\text{Pa}$. Infer RMS pressure. If the field is a single progressive plane wave with $Z = 411.6 \text{ Pa s m}^{-1}$, infer intensity and L_I relative to $10^{-12} \text{ W m}^{-2}$. State what changes if the field is standing.

Solution. Inverting the pressure-level definition gives

$$p_{\text{rms}} = p_{\text{ref}} 10^{L_p/20} = (20 \times 10^{-6}) 10^{3.5} = 0.0632 \text{ Pa}.$$

Under the progressive-wave assumption,

$$I = \frac{p_{\text{rms}}^2}{Z} = 9.72 \times 10^{-6} \text{ W m}^{-2}, \quad L_I = 69.876 \text{ dB} \simeq 69.9 \text{ dB}.$$

This agrees with $L_I = L_p - 0.124 \text{ dB}$. For an ideal standing field the same measured L_p still determines RMS pressure, but it does not determine $\langle p'u \rangle$. One must measure or model the velocity and its phase relative to pressure. An uncalibrated audio amplitude cannot determine either absolute pressure level or absolute intensity.

17.3.3 Independent sources: add physical mean squares first

For independent random pressure signals the cross-correlation vanishes on a sufficiently representative averaging interval: $\langle p'_1 p'_2 \rangle = 0$. Hence $p_{\text{rms},R}^2 = p_{\text{rms},1}^2 + p_{\text{rms},2}^2$. In a common one-way plane-wave geometry, the resulting intensities add as

$$I_{\text{tot}} = I_1 + I_2, \quad L_{I,\text{tot}} = 10 \log_{10} (10^{L_{I,1}/10} + 10^{L_{I,2}/10}). \quad (17.20)$$

The same logarithmic sum applies to pressure levels with a common reference when pressure cross-correlation vanishes. For independent fields traveling in different directions, the *intensity vectors* add; pressure mean squares can still add while the magnitude or a chosen component of intensity behaves differently. The word independent addresses correlation, not direction.

17.3.4 Coherent sources: phase and direction both matter

Two same-frequency coherent pressure signals have a stable relative phase $\Delta\phi$. Squaring and averaging the sum gives a cross term. If p_1, p_2 denote their peak pressure amplitudes,

$$p_R^2 = p_1^2 + p_2^2 + 2p_1p_2 \cos \Delta\phi. \quad (17.21)$$

This local pressure result applies regardless of direction. Converting it into an intensity using $I = p_R^2/(2Z)$ additionally requires that the resultant field be progressive with the same impedance relation, as for two co-propagating plane waves. For equal component intensities in that case,

$$I_R = 2I(1 + \cos \Delta\phi) = 4I \cos^2(\Delta\phi/2). \quad (17.22)$$

An opposing pair instead has $u = (p'_+ - p'_-)/Z$, so $\langle p'u \rangle = (\langle p'^2_+ \rangle - \langle p'^2_- \rangle)/Z$. Equal opposing waves have zero net flux, even at a large pressure antinode. The next sections develop this standing field. Oblique waves similarly require $\mathbf{I} = \langle p'\mathbf{u} \rangle$; their local pressure amplitude is not sufficient to infer its direction or magnitude.

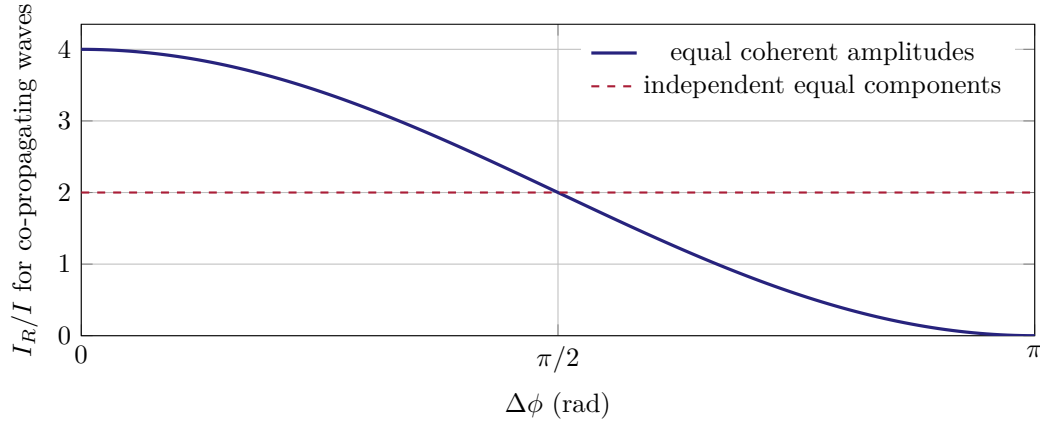


Figure 17.10: Conditional intensity addition. Two coherent equal-frequency plane waves traveling in the same direction yield phase-dependent mean intensity; independent equal components yield a fixed doubled intensity. The pressure sum alone cannot supply this intensity curve for opposing waves.

Co-propagating

$$\longrightarrow p'_+, \quad u_+ = p'_+/Z$$

$$\longrightarrow p'_2, \quad u_2 = p'_2/Z$$

$$\text{in phase, equal waves: } p' = 2p'_+, \quad u = 2p'_+/Z$$

Opposing

$$\longrightarrow p'_+, \quad u_+ = p'_+/Z$$

$$\longleftarrow p'_-, \quad u_- = -p'_-/Z$$

$$\text{in phase at the chosen point: } p' = 2p'_+, \quad u = 0$$

Figure 17.11: Direction is an additional condition beyond pressure phase. At an in-phase point, equal opposing waves can give the same doubled pressure as equal co-propagating waves, but their velocities cancel. This schematic compares local fields; arrow lengths do not encode amplitudes.

Example 17.7: Diagnose a combined sound level

Statement. Each source separately produces a progressive intensity $I = 1.0 \times 10^{-6} \text{ W m}^{-2}$ at a chosen point. Use $I_0 = 10^{-12} \text{ W m}^{-2}$. Compare independent equal sources with coherent equal-frequency co-propagating waves at phase differences 0 , $\pi/2$ and π . Also combine independent sources of levels 60.0 and 55.0 dB.

Solution. Each equal source alone has 60.0 dB. Independence gives $2I$ and $60.0 + 10 \log_{10} 2 = 63.01$ dB. For the coherent co-propagating pair, Eq. (17.22) gives respectively $4I$, $2I$ and zero. The first two levels are 66.02 dB and 63.01 dB; ideal cancellation has no finite logarithmic level. A real residual signal prevents exact zero, but a noise floor must be measured rather than invented.

For the unequal independent pair, $I_1 = 10^{-6} \text{ W m}^{-2}$ and $I_2 = 10^{-6.5} \text{ W m}^{-2}$. Their sum is $1.316 \times 10^{-6} \text{ W m}^{-2}$, giving 61.19 dB. The weaker source increases the level by 1.19 dB rather than by its printed 55 dB.

Check and interpretation. Squaring a doubled pressure produces four times the intensity only when the velocity also follows the progressive-wave relation. Equal opposing waves carry zero net mean flux. Their local pressure level can still be large; labeling it as a positive net-intensity level would answer a different physical question.

Guided checks

GC17-A1. Does a fluid element travel a wavelength while one wavelength of sound passes it? **Answer:** No. The pattern advances by λ in one period; the element completes a local oscillation and returns to its initial position in the linear periodic model.

GC17-A2. A microphone at a standing-wave pressure antinode records a large RMS pressure. Does this prove large outward net intensity? **Answer:** No. Intensity requires the pressure–velocity correlation. Equal opposing waves can have large local pressure and zero time-averaged net flux. State the field model before using p_{rms}^2/Z .

The pressure–displacement relation explains how a local pattern becomes a measured signal. Energy conservation explains intensity, while logarithms encode a stated-reference comparison. The remaining task is to determine the phases and directions generated by multiple paths, boundary reflections, and moving sources; those determine which of the addition rules is physically appropriate.

17.4 Interference and beats

17.4.1 Two paths: add arriving pressures before squaring

Interference begins with two disturbances at the *same* observation point. A source can feed two branches of a duct, or two speakers can be driven from one oscillator. Both arrangements give a controllable relative phase. The source amplitudes alone do not determine cancellation: attenuation, spreading, reflections, and the receiving geometry determine the amplitudes that actually arrive.

Represent two arriving, equal-frequency pressure signals by

$$p'_1 = A_1 \cos(\omega t + \theta_1), \quad p'_2 = A_2 \cos(\omega t + \theta_2).$$

Here $A_1, A_2 \geq 0$ are *peak arriving amplitudes*. In a lossless nondispersive path of length L_i , the signal emitted with phase ϕ_i arrives as $A_i \cos(\omega t - k_w L_i + \phi_i)$. Therefore

$$\Delta\theta = \theta_2 - \theta_1 = (\phi_2 - \phi_1) - k_w(L_2 - L_1), \quad k_w = \frac{2\pi}{\lambda}. \quad (17.23)$$

The minus sign belongs to this delayed-signal convention. Defining a propagation phase *lag* instead

gives $+k_w \Delta L$. Either convention gives the same cosine in the resultant amplitude when the source-phase term is handled consistently. A reflection phase, if present, must also be included.

Squaring the sum and averaging over a period uses $\langle \cos^2 \rangle = 1/2$ and $\langle \cos(\omega t + \theta_1) \cos(\omega t + \theta_2) \rangle = \frac{1}{2} \cos \Delta\theta$. Thus the resultant peak amplitude obeys

$$A_R^2 = A_1^2 + A_2^2 + 2A_1A_2 \cos \Delta\theta, \quad p_{\text{rms}} = \frac{A_R}{\sqrt{2}}. \quad (17.24)$$

This relation is equally a phasor construction: two rotating vectors of lengths A_1, A_2 have a fixed included angle, and their vector sum has length A_R . It is a pressure identity and does not itself establish an energy-flow direction.

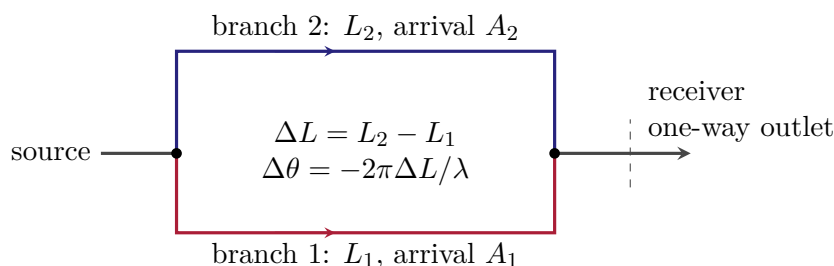


Figure 17.12: Two coherent paths recombine into a common outlet. Branch lengths are schematic, not to scale. With equal source phases, a path difference of half a wavelength reverses one arrival's pressure phase. Cancellation additionally requires $A_1 = A_2$. The common one-way outlet is the geometry in which a progressive-wave pressure-to-intensity conversion is appropriate.

For in-phase sources, maxima occur at $\Delta L = m\lambda$ and minima at $\Delta L = (m + \frac{1}{2})\lambda$, where m is an integer. At a maximum $A_R = A_1 + A_2$; at a minimum $A_R = |A_1 - A_2|$. Only equal arriving amplitudes permit exact cancellation. Figure 17.13 makes this amplitude requirement visible independently of the phase requirement. Maintaining a fixed phase difference over the averaging interval is what coherence means here. If it wanders randomly over many cycles, the cross term averages away rather than producing a stable minimum.

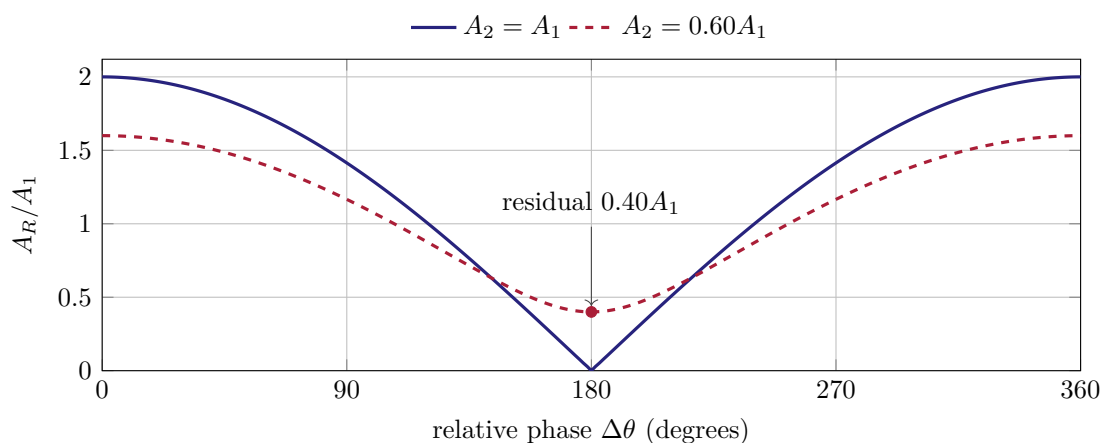


Figure 17.13: Resultant *peak pressure* versus phase for equal and unequal arriving amplitudes. Both curves reach their minimum at 180° , but the unequal pair has a nonzero residual. Taking a logarithm before adding pressures would lose this phase information.

Example 17.8: A two-path minimum that does not vanish

A common-outlet duct carries a 500 Hz tone through a medium with $c_w = 340 \text{ m s}^{-1}$ and $\rho = 1.20 \text{ kg m}^{-3}$. Its branches have $L_1 = 1.20 \text{ m}$ and $L_2 = 1.54 \text{ m}$, with equal source phases

and no additional reflection phase. Their arriving peak amplitudes are 0.120 Pa and 0.080 Pa. Find the residual pressure and the one-way outlet intensity; compare the minimum with a phase-aligned maximum at unchanged amplitudes.

The wavelength is $\lambda = 340/500 = 0.680$ m, so $\Delta L = 0.340$ m = $\lambda/2$ and $\Delta\theta = -\pi$. Equation (17.24) gives $A_R = 0.040$ Pa and $p_{\text{rms}} = 0.0283$ Pa. The acoustic impedance is $Z = \rho c_w = 408$ Pa s m⁻¹. In the stipulated progressive outlet,

$$I_{\min} = \frac{(0.040)^2}{2(408)} = 1.96 \times 10^{-6} \text{ W m}^{-2}.$$

The aligned maximum has $A_R = 0.200$ Pa and $I_{\max} = 4.90 \times 10^{-5} \text{ W m}^{-2}$. Hence $I_{\max}/I_{\min} = 25.0$, corresponding to $10 \log_{10} 25.0 = 14.0$ dB. The unequal attenuation limits the depth of the minimum even when the phase is exactly destructive. This comparison holds the arriving amplitudes fixed; altering a real branch may also alter them.

17.4.2 A microphone measures pressure, not automatically net flux

At a crossing of two free-field waves, the scalar pressures still add, but their particle velocities point along different propagation directions. In a locally plane-wave approximation with common impedance Z , let $\mathbf{n}_i$ be the unit propagation direction of component i . Then $\mathbf{u}_i = p'_i \mathbf{n}_i / Z$. Averaging the actual energy-flux vector gives

$$\langle \mathbf{I} \rangle = \frac{A_1^2 \mathbf{n}_1 + A_2^2 \mathbf{n}_2 + A_1 A_2 \cos \Delta\theta (\mathbf{n}_1 + \mathbf{n}_2)}{2Z}. \quad (17.25)$$

This follows by expanding $(p'_1 + p'_2)(\mathbf{u}_1 + \mathbf{u}_2)$; the two cross terms have different directions. If $\mathbf{n}_1 = \mathbf{n}_2$, Eq. (17.25) reduces to $A_R^2 \mathbf{n}_1 / (2Z)$. If $\mathbf{n}_2 = -\mathbf{n}_1$, the cross terms cancel and the net flux is $(A_1^2 - A_2^2) \mathbf{n}_1 / (2Z)$, regardless of the local pressure interference pattern.

Caution

For crossing or opposing waves, p_{rms}^2/Z is generally *not* the net acoustic intensity. A microphone can establish a pressure interference minimum and a pressure level. Net energy flow requires the associated particle velocity and its direction, or an explicitly justified progressive-wave model.

A geometric interference experiment should therefore distinguish the pressure measurement from an inferred power flow. Figure 17.14 shows two spherical arrivals sampled on a line of receivers. A point away from the midline has both a path difference and unequal arriving amplitudes, even when the sources themselves are identical. The plot is calculated from a declared monopole far-field model, not from measured speaker directivity.

Example 17.9: Read a pressure interference pattern from geometry

For Fig. 17.14, calculate the RMS pressure at the midline $y = 0$ and at $y = 0.50$ m. Decide whether a lower pressure at the second point demonstrates zero net energy flow there.

At $y = 0$, both distances are $\sqrt{3.00^2 + 0.50^2} = 3.0414$ m. The arrivals are in phase and each has peak amplitude 0.03288 Pa. Their sum has $A_R = 0.06576$ Pa, or $p_{\text{rms}} = 0.04650$ Pa.

At $y = 0.50$ m, $r_1 = 3.0000$ m and $r_2 = \sqrt{10} = 3.1623$ m. With $\lambda = 0.500$ m, the phase difference is $-2\pi(0.16228/0.500) = -2.0392$ rad. The arriving amplitudes are 0.03333 and 0.03162 Pa. Substitution in Eq. (17.24) gives $A_R = 0.03405$ Pa and $p_{\text{rms}} = 0.02408$ Pa. The pressure falls because the arrivals partly cancel. This is not a zero-flux conclusion: neither the pressure is

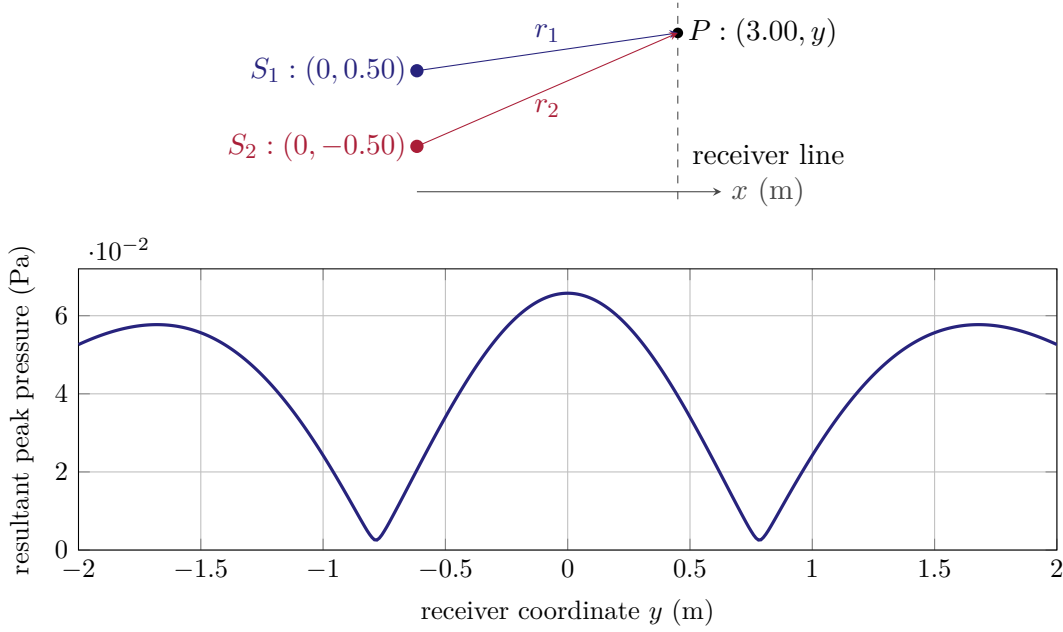


Figure 17.14: Unequal paths from identical coherent point-source models. Coordinates are in metres; each source has $A_i = (0.10 \text{ Pa m})/r_i$, $f = 680 \text{ Hz}$, and $c_w = 340 \text{ m/s}$. The pressure pattern includes both spreading and phase. Its vertical quantity is pressure amplitude; it is not a plot of net intensity. The upper geometry is schematic in vertical scale.

zero nor the particle-velocity vectors have been included. Equation (17.25) would be required for the local far-field flux estimate.

17.4.3 Two nearby frequencies: three distinct envelopes

Fixed phase produces stationary interference. A small frequency difference instead makes the relative phase drift. At one point, choose equal pressure amplitudes and initially aligned phases:

$$p'(t) = p_a \cos(2\pi f_1 t) + p_a \cos(2\pi f_2 t).$$

The sum-to-product identity yields

$$p'(t) = \underbrace{2p_a \cos[\pi(f_2 - f_1)t]}_{\text{signed envelope}} \underbrace{\cos(2\pi f_c t)}_{\text{carrier}}, \quad f_c = \frac{f_1 + f_2}{2}. \quad (17.26)$$

The identity is exact. Interpreting the second factor as a fast carrier and the first as a slow envelope additionally requires $|f_2 - f_1| \ll f_c$. Figure 17.15 resolves individual cycles rather than connecting a sparse sample into a misleading curve.

The *signed* envelope repeats after $2/|f_2 - f_1|$ and changes sign halfway through that interval. Changing its sign reverses the carrier phase by π ; it does not create a negative amplitude magnitude. The physical pressure-amplitude envelope is $2p_a |\cos[\pi(f_2 - f_1)t]|$, whose maxima recur after $1/|f_2 - f_1|$. Consequently

$$\boxed{f_b = |f_2 - f_1|}, \quad T_b = 1/f_b. \quad (17.27)$$

The ordinary beat frequency counts successive amplitude or intensity maxima. It is not the frequency of the signed cosine factor.

For two waves sharing a one-way progressive propagation direction and approximately the same impedance Z , average over a time window much longer than $1/f_c$ but much shorter than $1/f_b$. The

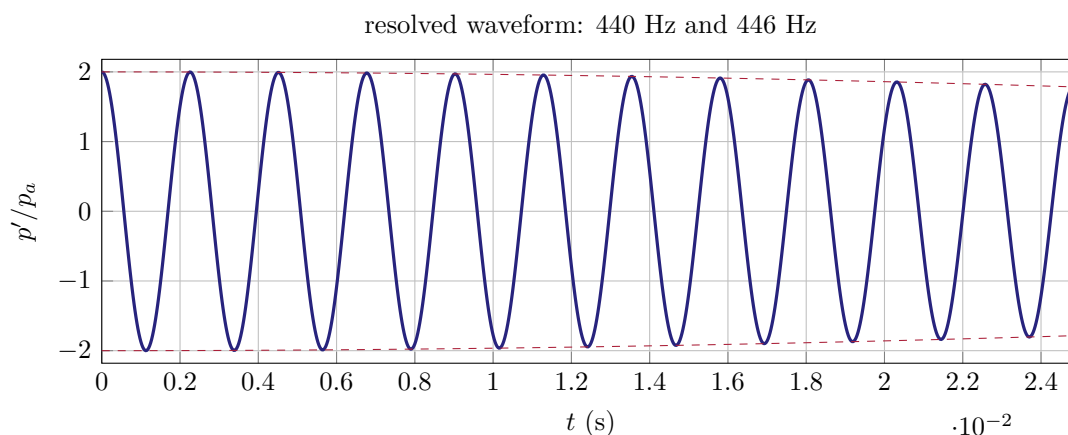


Figure 17.15: Calculated short-time two-tone pressure. The dashed curves bound the rapid oscillations; they are the positive and negative magnitude envelope. This 25-ms window contains about eleven carrier cycles and only a small part of a 1/6-s beat period.

carrier average is then approximately $1/2$, while the slowly varying envelope is nearly constant during that window. The resulting *carrier-averaged* intensity is

$$I_{\text{carr}}(t) \simeq \frac{2p_a^2}{Z} \cos^2[\pi(f_2 - f_1)t]. \quad (17.28)$$

A long average over many beat periods instead gives $\langle I \rangle = p_a^2/Z$, the sum of the two separate intensities. These are different averaging operations. A real level meter or listener may smooth the modulation further, so the mathematical recurrence should not be mistaken for a universal perceptual threshold.

Example 17.10: Two slightly detuned tones and the factor of two

Two equal-amplitude tones at one point have frequencies 440 Hz and 446 Hz. Find their carrier and beat frequencies, the interval between beat maxima, and the recurrence time of the signed envelope.

Equation (17.26) gives $f_c = (440 + 446)/2 = 443$ Hz and $f_b = 6.00$ Hz. Therefore successive magnitude or intensity maxima are $T_b = 1/6 = 0.1667$ s apart. The signed factor is $2p_a \cos(6\pi t)$, a 3-Hz cosine whose own period is $1/3$ s. Its negative extreme at $t = 1/6$ s has the same magnitude as its positive extreme at $t = 0$. A constant relative source phase shifts this pattern in time without changing f_b . A drift in either frequency changes the beat rate.

17.4.4 Unequal amplitudes leave shallow beat minima

For unequal amplitudes and a constant initial relative phase δ , the instantaneous slowly varying phase difference is $2\pi(f_2 - f_1)t + \delta$. The magnitude envelope becomes

$$A_{\text{env}}(t) = \sqrt{A_1^2 + A_2^2 + 2A_1A_2 \cos[2\pi(f_2 - f_1)t + \delta]}. \quad (17.29)$$

Its extrema are $A_1 + A_2$ and $|A_1 - A_2|$, so equal amplitudes are again necessary for zero minima. Under the same co-propagating approximation, $I_{\text{carr}} = A_{\text{env}}^2/(2Z)$. A useful measure of intensity contrast is

$$\mathcal{V} = \frac{I_{\text{max}} - I_{\text{min}}}{I_{\text{max}} + I_{\text{min}}} = \frac{2A_1A_2}{A_1^2 + A_2^2}.$$

It ranges from zero when only one tone remains to unity for equal amplitudes. This contrast is a mathematical modulation measure, not a hearing test.

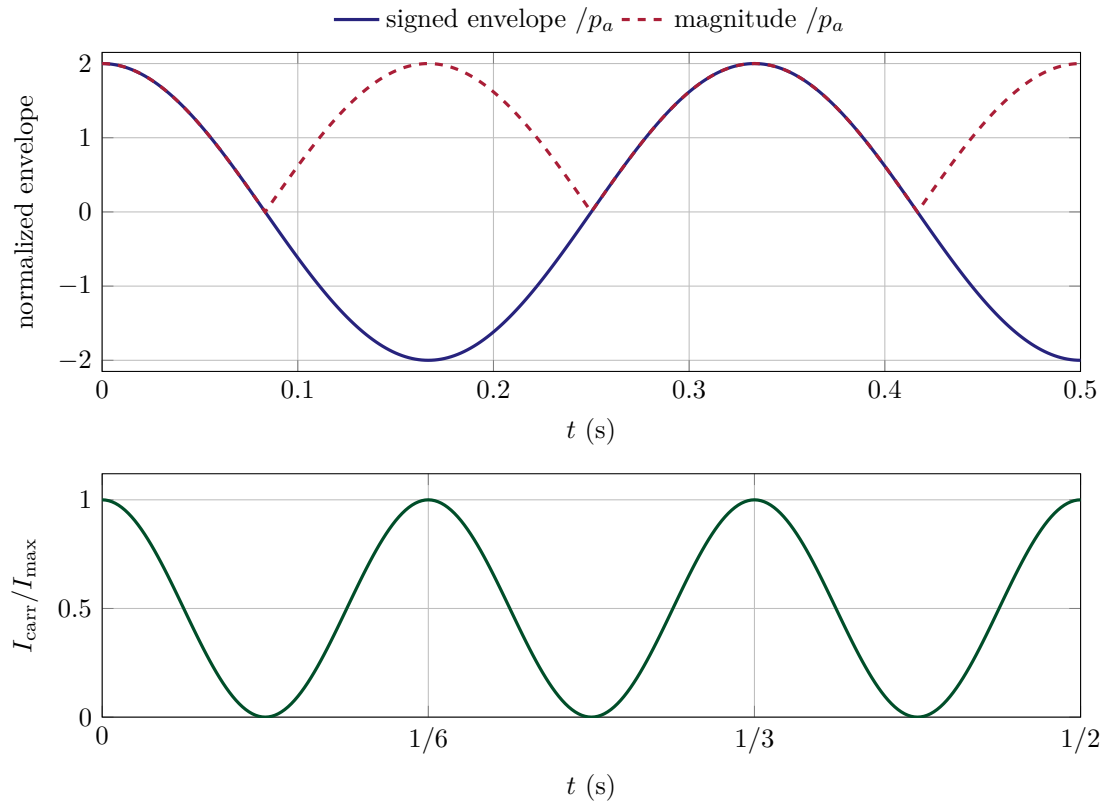


Figure 17.16: The 6-Hz beat pattern has a signed envelope, a nonnegative magnitude, and a quadratic intensity modulation. The signed envelope repeats after $1/3$ s; magnitude and intensity maxima repeat after $1/6$ s. The lower panel assumes a co-propagating progressive-wave model and separates carrier averaging from a long-time mean.

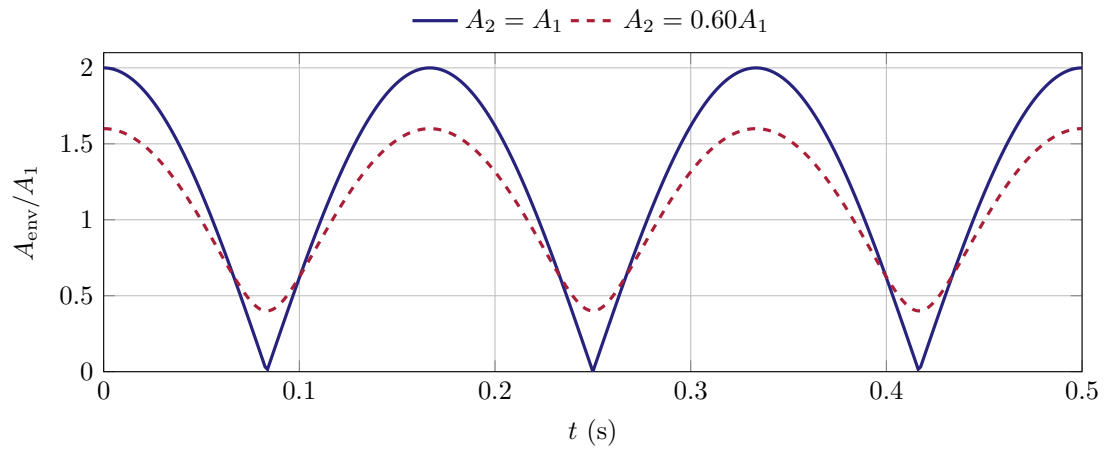


Figure 17.17: Equal and unequal two-tone amplitude envelopes with the same 6-Hz beat rate. Unequal amplitudes change the depth, not the recurrence rate, of the modulation. Both curves use $\delta = 0$.

Example 17.11: Quantify an unequal two-tone modulation

In the progressive common-outlet model, let $A_1 = 0.100$ Pa, $A_2 = 0.060$ Pa, $f_1 = 440$ Hz, $f_2 = 446$ Hz, and $Z = 408$ Pa s m⁻¹. Find the amplitude extrema, carrier-averaged intensity extrema, long-time mean, and contrast.

The pressure envelope reaches 0.160 Pa and 0.040 Pa. Hence $I_{\max} = 0.160^2/[2(408)] = 3.14 \times 10^{-5}$ W m⁻² and $I_{\min} = 0.040^2/[2(408)] = 1.96 \times 10^{-6}$ W m⁻². Averaging the cross term over many beats gives

$$\langle I \rangle = \frac{0.100^2 + 0.060^2}{2(408)} = 1.67 \times 10^{-5} \text{ W m}^{-2}, \quad \nu = \frac{2(0.100)(0.060)}{0.100^2 + 0.060^2} = 0.882.$$

The mean also equals $(I_{\max} + I_{\min})/2$, an independent algebraic check. Modulation maxima still recur every 1/6 s. The nonzero minimum follows from unequal amplitudes, not from a change in beat frequency.

Guided checks

Equal-frequency coherent tones with fixed phase do not beat: their relative phase is constant. Two close tones have a drifting relative phase and therefore a changing resultant amplitude. A signed envelope may become negative, but a pressure-amplitude magnitude and an intensity cannot. In a crossing-wave experiment, a pressure beat trace alone does not establish the net flux; its velocity field must also be specified.

17.5 Air columns and acoustic modes

17.5.1 Reflection depends on the acoustic variable

A tube supports waves traveling in both directions. The same reflected wave has different signs when represented by pressure and by longitudinal particle velocity. To see why, write right- and left-going pressure components as $p'_+(x, t)$ and $p'_-(x, t)$. In the one-dimensional linear model,

$$p' = p'_+ + p'_-, \quad u = \frac{p'_+ - p'_-}{Z}, \quad Z = \rho c_w.$$

Positive u always means motion in the positive x direction. The minus sign for the left-going component is physical: its energy propagates toward negative x .

At an ideal rigid cap, $u = 0$. Thus incident and reflected pressures are equal at the cap, with pressure reflection coefficient +1; their velocities cancel, with velocity reflection coefficient -1 . Since harmonic displacement is the time integral of velocity, its reflection coefficient is also -1 . At an ideal pressure-release opening, $p' = 0$; the reflected pressure is reversed, coefficient -1 , while velocity and displacement reflection coefficients are +1. These are lossless limiting boundary models. A real opening radiates energy and is represented more accurately by a frequency-dependent radiation impedance.

17.5.2 Derive the boundary conditions from momentum

The linear momentum and compression equations are

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p'}{\partial x}, \quad p' = -B \frac{\partial \xi}{\partial x}, \quad u = \frac{\partial \xi}{\partial t}. \quad (17.30)$$

A rigid boundary imposes $u = 0$ at all times. Its time derivative is therefore zero, so momentum gives $\partial p'/\partial x = 0$ at the wall. For a harmonic standing mode this pressure slope condition corresponds to

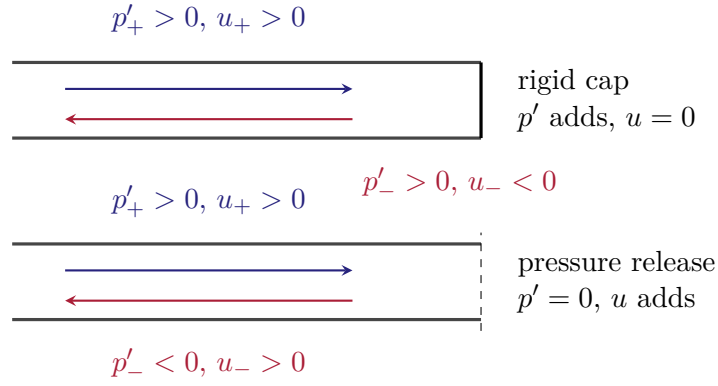


Figure 17.18: Signs at the boundary for an incident compression. Arrows indicate wave propagation, whereas the signs of $u_{\pm}$ indicate signed particle velocity along the tube. At a rigid cap compression reflects as compression; at pressure release it reflects as rarefaction. The geometry is schematic.

a pressure antinode. The displacement itself vanishes at the wall; a static offset is excluded when describing oscillations about equilibrium.

An ideal opening imposes $p' = 0$, hence $\partial\xi/\partial x = 0$ from the compression equation. For a single sinusoidal mode this is a displacement antinode. It is the *pressure perturbation*, not the absolute pressure, that vanishes: the mean pressure remains approximately the external equilibrium pressure. These ideal conditions select allowed shapes. They are not claims that an actual open end has zero radiation or that arbitrary sums of modes have one universal amplitude there.

17.5.3 Spatial complementarity and temporal quadrature

For a tube closed at $x = 0$, one convenient standing-wave convention is

$$p'(x, t) = P \cos(k_w x) \cos(\omega t), \quad (17.31)$$

$$u(x, t) = \frac{P}{Z} \sin(k_w x) \sin(\omega t), \quad (17.32)$$

$$\xi(x, t) = -\frac{P}{Z\omega} \sin(k_w x) \cos(\omega t). \quad (17.33)$$

Substitution into both equations in Eq. (17.30) checks the signs using $\omega = c_w k_w$ and $B = \rho c_w^2$. Pressure and displacement have complementary *spatial* nodes: where $\cos k_w x = 0$, $|\sin k_w x| = 1$, and conversely. Their signed spatial functions include the minus sign in Eq. (17.33). At a fixed interior point, pressure and *particle velocity* are in temporal quadrature: one is a cosine, the other a sine. Pressure and displacement in this convention share a cosine time factor, with their relative sign determined by position. A bare statement that “pressure and displacement are 90° out of phase” must specify whether it concerns a progressive wave or these spatial standing-mode patterns.

The instantaneous flux is

$$p'u = \frac{P^2}{Z} \cos(k_w x) \sin(k_w x) \cos(\omega t) \sin(\omega t). \quad (17.34)$$

Its period average vanishes. Energy moves back and forth locally, while no net energy is transported by the ideal pure standing wave. A driven real tube can have a predominantly standing field and a small traveling contribution that carries power to radiation and damping; zero mean flux applies to the ideal equal-amplitude counterpropagating limit.

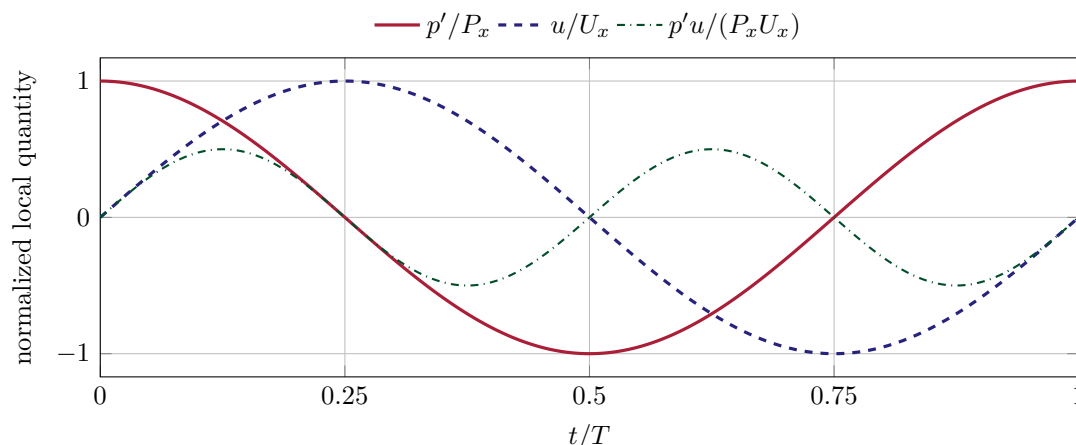


Figure 17.19: Time dependence at an interior point of the standing-wave model, where both local peak amplitudes P_x and U_x are nonzero and positive. Pressure and velocity are a quarter cycle apart. Positive and negative flux areas cancel over a period even though pressure RMS is nonzero. This is a temporal plot, distinct from the spatial mode-shape diagrams.

Example 17.12: A large pressure RMS with zero net flux

An ideal closed–open tube has standing pressure peak $P = 0.200$ Pa at its rigid end, $L = 0.850$ m, $c_w = 340$ m s^{−1}, and $Z = 408$ Pa s m^{−1}. Find pressure RMS and velocity peak at the midpoint in the fundamental. Determine the mean flux and diagnose the progressive-wave pressure shortcut.

The fundamental has $k_w L = \pi/2$, so at $x = L/2$, $k_w x = \pi/4$. Thus $P_x = P/\sqrt{2} = 0.1414$ Pa, $p_{\text{rms}} = P_x/\sqrt{2} = 0.100$ Pa, and $U_x = P/(\sqrt{2}Z) = 3.47 \times 10^{-4}$ m s^{−1}. Equations (17.31)–(17.34) give $\langle p'u \rangle = 0$: the temporal sine–cosine product averages to zero. Substituting pressure alone into p_{rms}^2/Z would give 2.45×10^{-5} W m^{−2}, a positive number that is *not* the net flux in this field. Its failure is a model error, not an RMS arithmetic error.

17.5.4 Open–open modes: all positive integer harmonics

For an open–open tube, take the separated pressure form $p'_n = P_n F_n(x) \cos(\omega_n t + \phi_n)$. The wave equation gives $F''_n + k_n^2 F_n = 0$. Its general solution is a sine plus a cosine. The condition $F_n(0) = 0$ removes the cosine term; $F_n(L) = 0$ then requires $\sin(k_n L) = 0$. A nonzero oscillation therefore has

$$k_n = \frac{n\pi}{L}, \quad \lambda_n = \frac{2L}{n}, \quad \boxed{f_n = \frac{nc_w}{2L}}, \quad n = 1, 2, 3, \dots \quad (17.35)$$

Here $n = 0$ gives no pressure oscillation and is not the acoustic fundamental. Compatible displacement and velocity are

$$\xi_n = X_n \cos(k_n x) \cos(\omega_n t + \phi_n), \quad p'_n = Bk_n X_n \sin(k_n x) \cos(\omega_n t + \phi_n),$$

$$u_n = -\omega_n X_n \cos(k_n x) \sin(\omega_n t + \phi_n).$$

The pressure nodes at both ends accompany displacement antinodes. An extra half-wavelength fits into the tube for every increase of n . The plots in Fig. 17.20 display *signed spatial coefficients* multiplying the declared time factors. A plotted blue ordinate is longitudinal displacement represented on a graph, not the vertical bending of the tube.

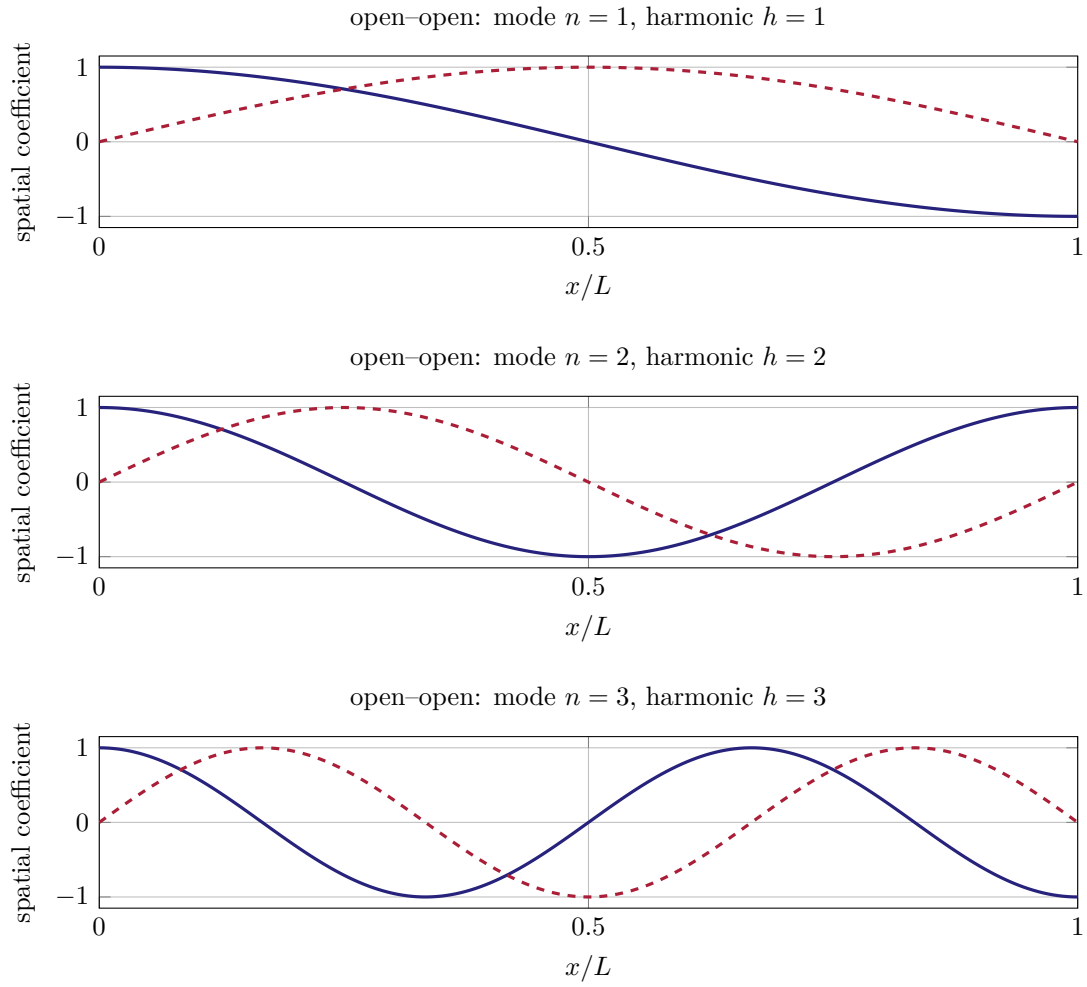


Figure 17.20: First three open-open mode families. Blue shows $\xi_{\text{shape}}/X_n = \cos(k_n x)$; red dashed shows $p'_{\text{shape}}/P_n = \sin(k_n x)$, with $P_n = Bk_n X_n$. These are signed modal spatial coefficients, independently normalized, not simultaneous oscillations of three modes or a physical transverse deflection. Each pair multiplies the same cosine time factor. Both ends are pressure nodes.

17.5.5 Closed–open modes: mode number is not harmonic number

With a rigid cap at $x = 0$, the condition $F'_n(0) = 0$ selects a cosine pressure shape. Pressure release at $x = L$ requires $\cos(k_n L) = 0$, so

$$k_n = \frac{(2n-1)\pi}{2L}, \quad \lambda_n = \frac{4L}{2n-1}, \quad \boxed{f_n = \frac{(2n-1)c_w}{4L}}, \quad n = 1, 2, 3, \dots \quad (17.36)$$

The fundamental fits one quarter-wavelength between a pressure antinode and the adjacent pressure node. Each later mode adds a half-wavelength, leaving that quarter-wave endpoint mismatch intact.

The index n counts the *allowed modes*. The harmonic number $h = f_n/f_1 = 2n-1$ counts multiples of the fundamental. Thus the second allowed mode is the third harmonic, and the third allowed mode is the fifth harmonic. Some references index this spectrum directly by $h = 1, 3, 5, \dots$ and write $f_h = hc_w/(4L)$. The two notations express the same spectrum; confusing their indices incorrectly inserts even harmonics.

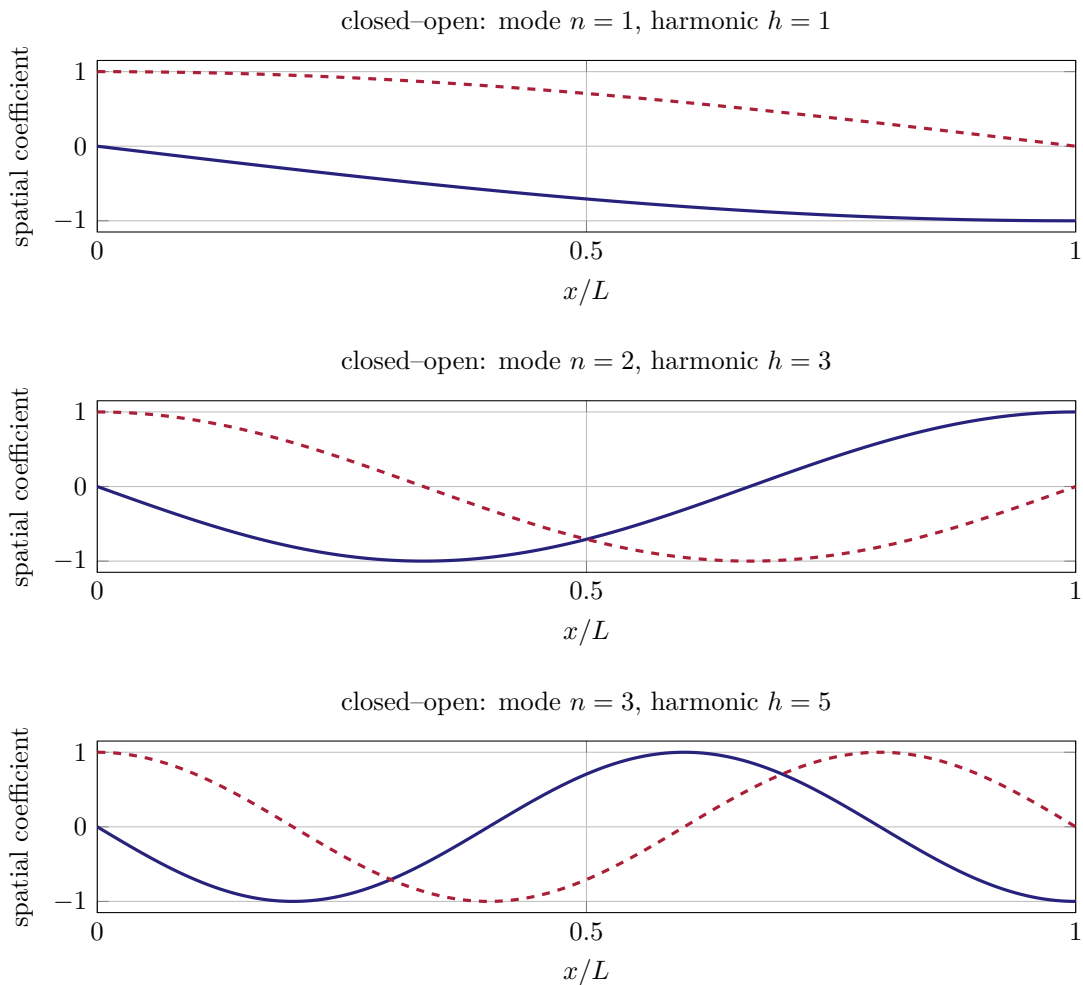


Figure 17.21: First three closed–open mode families. Blue shows $\xi_{\text{shape}}/X_n = -\sin(k_n x)$; red dashed shows $p'_{\text{shape}}/P_n = \cos(k_n x)$, with $X_n = P_n/(Z\omega_n)$. Each pair multiplies $\cos(\omega_n t + \phi_n)$, so the minus sign is required by $p' = -B\partial\xi/\partial x$. The closed end is a displacement node and pressure antinode; the open end has the complementary conditions. Curves are modal coefficients, not snapshots of simultaneous modes.

For completeness, two rigid caps impose pressure-slope zeros at both ends and allow $F_n = \cos(n\pi x/L)$, $f_n = nc_w/(2L)$ for $n \geq 1$. This closed–closed spectrum has the same ideal frequency spacing as the open–open spectrum but exchanges their pressure/displacement endpoint patterns. A frequency list alone consequently cannot distinguish all endpoint models. A uniform static pressure change is not a propagating acoustic normal mode in this discussion.

Example 17.13: Identify an ideal tube from its resonance sequence

An ideal tube has effective length $L = 0.850$ m and $c_w = 340$ m s⁻¹. Its first three observed resonances are close to 100, 300, and 500 Hz. Which endpoint model is consistent with the sequence, and what are its mode and harmonic labels?

The open–open fundamental would be $340/[2(0.850)] = 200$ Hz, giving 200, 400, 600, ... Hz. A closed–open tube instead has $f_1 = 340/[4(0.850)] = 100$ Hz and the allowed sequence 100, 300, 500, ... Hz. Thus the closed–open model agrees with both the absolute frequency scale and the spacing. These peaks are modes $n = 1, 2, 3$, corresponding to harmonics $h = 1, 3, 5$. Its pressure has an antinode at the rigid end and a node at the ideal opening. Absence of one even-frequency peak alone would be weaker evidence: excitation and detection can hide allowed modes.

17.5.6 Allowed modes, excitation, and what a sensor can miss

A mode is a boundary-compatible spatial pattern with a natural frequency. A resonance measurement also requires that the driver couple to it and that the receiver detect it. In the ideal open–open family, pressure at $x = L/2$ is proportional to $\sin(n\pi/2)$. A pressure sensor exactly at the midpoint therefore sees zero signal from every even mode, even though those modes are allowed and may carry stored acoustic energy. It is inappropriate to infer a closed end solely from that sensor’s missing even peaks.

Figure 17.22 contrasts the spectrum of allowed frequencies with this geometric pressure sensitivity. Its vertical values are mode-shape factors, not predicted spectral powers: predicting a peak height would additionally require driver position, damping, and drive amplitude. A practical inference should use several resonances, move the sensor where possible, and verify endpoint pressure/velocity behaviour.

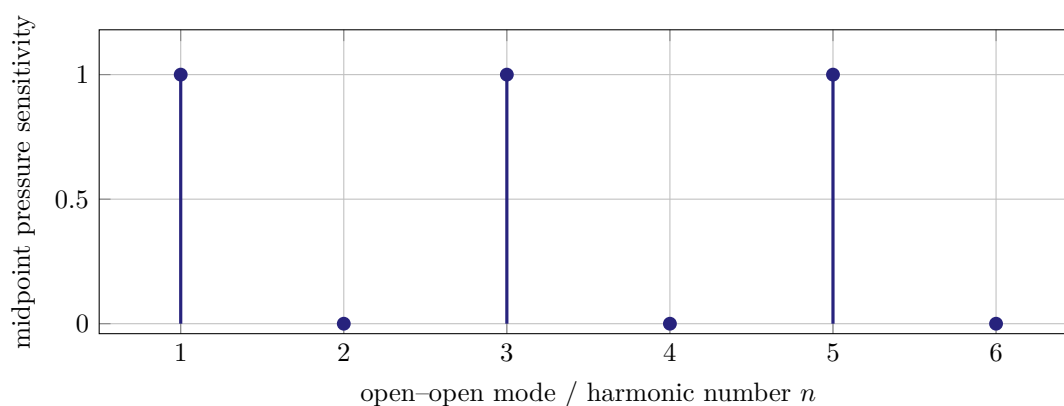


Figure 17.22: Geometric sensitivity $|F_n(L/2)| = |\sin(n\pi/2)|$ of an ideal point pressure sensor in an open–open tube. Zeros mean a sensor node, not a forbidden mode. The plot does not assign resonance linewidths or spectral peak powers.

Example 17.14: Infer wavelength from two adjacent resonance lengths

A 680 Hz tone drives a tube with one effectively rigid end and one opening. Consecutive resonance lengths are $L_a = 0.122$ m and $L_b = 0.374$ m. Assume a length-independent end correction δ at this frequency. Infer wavelength and sound speed, then estimate δ if these are the first two resonances.

The effective lengths satisfy $L_a + \delta = \lambda/4$ and $L_b + \delta = 3\lambda/4$. Subtraction eliminates the

unknown correction:

$$L_b - L_a = \lambda/2, \quad \lambda = 2(0.374 - 0.122) = 0.504 \text{ m}.$$

Thus $c_w = f\lambda = 680(0.504) = 342.72 \text{ m s}^{-1} \simeq 343 \text{ m s}^{-1}$. If the lower length is the fundamental, $\delta = \lambda/4 - L_a = 0.126 - 0.122 = 0.004 \text{ m}$. Without identifying the starting mode, the spacing still estimates wavelength under the stated model, but it does not uniquely determine δ . This differencing method also fails if the correction changes substantially between the two configurations. The inferred correction is a model estimate, not a universal value for every opening.

17.5.7 Effective length and limits of the one-dimensional model

The mode formulas use an *effective acoustic length*. A real opening's pressure minimum lies approximately outside the physical end. Replacing L by $L_{\text{eff}} = L + \delta$ for one opening, or by $L + \delta_1 + \delta_2$ for two, is a useful low-frequency approximation with explicitly specified geometry. A coefficient multiplying tube radius is not universal: flange shape, cross section, frequency, and neighboring structures matter. The advanced section quantifies a stipulated model without treating it as an exact radiation calculation.

The one-dimensional treatment also assumes transverse variations are negligible, wall motion is small, and acoustic amplitudes remain in the linear regime. Damping makes resonances finite in width and can shift their peaks. A tube with a reed, a moving termination, or strong mean flow need not match either ideal endpoint model. Within their stated limits, however, the boundary equations explain the complementary pressure/displacement shapes, the discrete frequency sequences, and the difference between an allowed mode and an observable peak.

Take-home message

At an ideal opening, pressure has a node; at a rigid cap, displacement and velocity have nodes. Standing pressure and velocity are in temporal quadrature, so the ideal pure standing wave has zero mean flux. Open–open modes include all integer harmonics; closed–open modes include odd harmonics. Sensor and driver geometry determine which allowed modes are easy to observe.

17.6 Doppler effect

17.6.1 A wavelength is set at emission; a frequency is counted at reception

An observer counts oscillations per unit time at the observer's position. A source sets the interval between successive emissions. When either moves, these two intervals need not agree. For sound the transmitting medium supplies the natural mechanical frame: a sound front moves at c_w relative to that medium, even when the source or the receiver moves. The separate moving-source and moving-observer arguments in the reference texts make this distinction especially useful [1, 2, 3].

Fix a one-dimensional x axis in the medium rest frame. First let sound travel from source to observer in $+x$. The source emits at frequency f_s and period $T_s = 1/f_s$; its signed velocity is v_s . The observer's signed velocity is v_o . Positive velocity means motion in $+x$ for either body. We assume constant velocities, a homogeneous medium, and $|v_s|, |v_o| < c_w$ throughout this derivation. Classical time is the same for source, medium, and observer.

During one emission period the earlier front advances $c_w T_s$, while the source advances $v_s T_s$. The later front starts at that displaced source position, so the separation of the forward fronts at one

instant is

$$\lambda_+ = (c_w - v_s)T_s = \frac{c_w - v_s}{f_s}. \quad (17.37)$$

The observer encounters these fronts at relative speed $c_w - v_o$. Dividing this encounter speed by their spacing gives

$$f_o = f_s \frac{c_w - v_o}{c_w - v_s} \quad \text{for sound propagating in } +x. \quad (17.38)$$

A positive source velocity reduces the forward spacing. A negative observer velocity increases the rate of encounter. Both raise f_o in this geometry. Neither statement requires an ambiguous rule about which sign means “toward.”

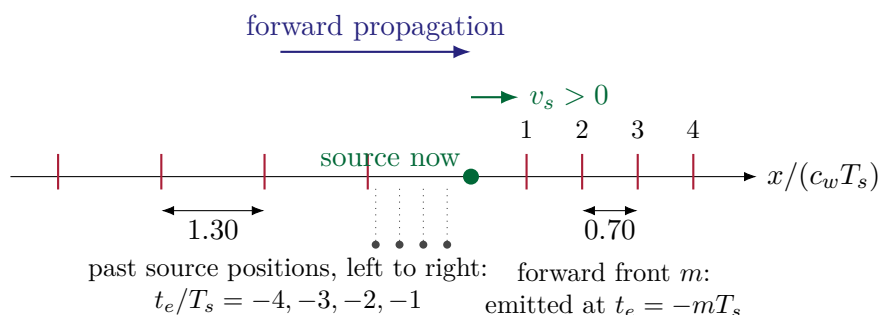


Figure 17.23: Successive emission times and front positions at one observation time. The source is at $x/(c_w T_s) = 1.20$ now, with $v_s/c_w = 0.30$. A front emitted m periods earlier started at $1.20 - 0.30m$ and has travelled distance m in these units. Its forward and backward positions are consequently $1.20 + 0.70m$ and $1.20 - 1.30m$. The forward spacing is 0.70, the backward spacing 1.30. Vertical red strokes are intersections of fronts with the axis.

Example 17.15: Moving source and observer on one line

Task. Sound travels in $+x$ through a medium with $c_w = 340 \text{ m s}^{-1}$. A source emits 800 Hz while moving at $v_s = +20 \text{ m s}^{-1}$. An observer ahead moves toward it at $v_o = -10 \text{ m s}^{-1}$. Find the received frequency, then reverse both velocities and interpret the change.

Model and strategy. Use the medium-frame front spacing, followed by the observer’s encounter rate. Both speeds are subsonic; a fixed propagation direction therefore has ordered fronts and a positive denominator.

$$\lambda_+ = \frac{340 - 20}{800} = 0.400 \text{ m}, \quad f_o = \frac{340 - (-10)}{0.400} = 875 \text{ Hz}.$$

The frequency increase has two causes: the source compresses the fronts and the observer moves into them. With $v_s = -20 \text{ m s}^{-1}$ and $v_o = +10 \text{ m s}^{-1}$ instead,

$$\lambda_+ = 0.450 \text{ m}, \quad f_o = 800 \frac{330}{360} = 733.3 \text{ Hz}.$$

The source now increases the forward spacing and the observer reduces the encounter rate. As checks, the units are speed divided by length, and setting both velocities to zero gives $f_o = f_s$.

17.6.2 Signed propagation in either direction

Keep the same fixed axis and let $n = +1$ or $n = -1$ identify the direction of the relevant sound ray. Distance measured along that ray is positive. In one source period the along-ray spacing is $(c_w - n v_s)T_s$, while the along-ray encounter speed is $c_w - n v_o$. The general one-dimensional

expression is therefore

$$f_o = f_s \frac{c_w - nv_o}{c_w - nv_s}, \quad n \in \{-1, +1\}. \quad (17.39)$$

For a receiver behind a source moving in $+x$, take $n = -1$: the denominator is $c_w + v_s$ and the backward wavelength is larger. A frequency describes repeated arrivals along the selected ray; it is not a signed quantity. A negative result from a formula used outside its subsonic assumptions signals a failed model.

The exact acoustic shift depends on source and observer velocities separately. For small $|v_s|/c_w$ and $|v_o|/c_w$, use $(1 - z)^{-1} = 1 + z + O(z^2)$ to obtain

$$\frac{f_o - f_s}{f_s} \simeq \frac{n(v_s - v_o)}{c_w}. \quad (17.40)$$

Only at this first order does the shift depend solely on relative approach rate. For Example 17.15, the approximation gives 870.6 Hz, 4.4 Hz below the exact result; the discrepancy is a second-order effect, not rounding.

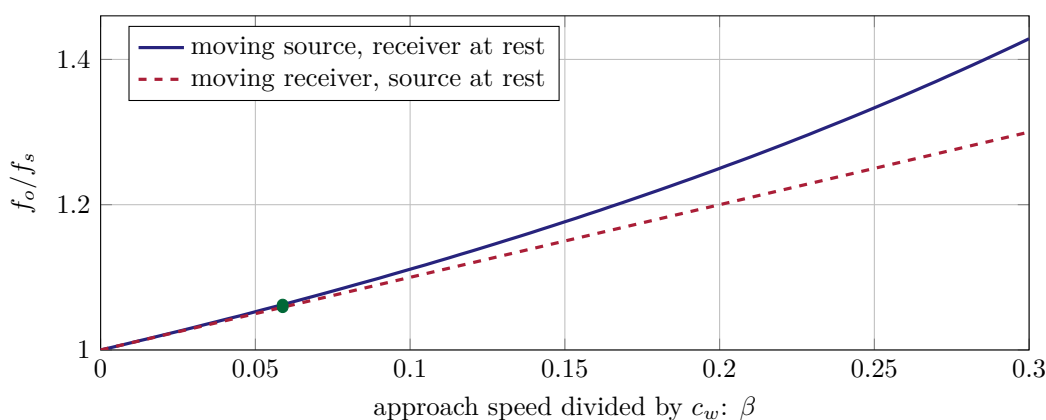


Figure 17.24: Two equal approach speeds produce different exact sound Doppler shifts. For the blue curve the source approaches a stationary receiver at speed βc_w ; for the red curve the receiver approaches a stationary source at the same speed. Their common low-speed tangent is $1 + \beta$. The dots mark $20/340$. The difference grows with speed because only source motion changes the medium-frame wavelength.

Example 17.16: Equal closing rates and a uniform wind

Task. With $f_s = 800 \text{ Hz}$ and $c_w = 340 \text{ m s}^{-1}$, compare (a) a source approaching at 20 m s^{-1} and a receiver at rest with (b) a receiver approaching at 20 m s^{-1} and a source at rest. Then find the shift for ground velocities $V_s = +20$, $V_o = -10$, and uniform wind velocity $U = +15$, all in m s^{-1} , along $+x$.

Solution. The first two closing rates are equal, but

$$f_a = 800 \frac{340}{320} = 850.0 \text{ Hz}, \quad f_b = 800 \frac{360}{340} = 847.1 \text{ Hz}.$$

Source motion changes spacing; receiver motion changes counting rate. The 2.9 Hz difference is the effect shown in Fig. 17.24. In the wind case transform both velocities into the medium frame:

$$v_s = V_s - U = +5, \quad v_o = V_o - U = -25.$$

Thus $f_o = 800(365/335) = 871.6 \text{ Hz}$. The value c_w remains the speed relative to the air, rather than being replaced by a ground speed in the same formula. Equivalently, a forward front has ground speed $c_w + U$, giving the same ratio $(c_w + U - V_o)/(c_w + U - V_s)$.

Check and limit. Two bodies fixed on the ground in the same steady uniform wind have $v_s = v_o = -U$ and no shift. Wind changes travel time, but does not change the interval between

arrivals in that case. Gusts, shear, and curved rays require a more detailed propagation model than this one-dimensional case.

17.6.3 A moving reflector produces two shifts

An echo is received first by the reflector, then by the original listener. For a stationary emitter and listener in the medium frame, place a reflector ahead with signed velocity v_r . The outgoing sound propagates in $+x$, so the frequency at the reflector is $f_r = f_s(c_w - v_r)/c_w$. An ideal reflector preserves this frequency in its own moving frame: successive incident cycles generate successive reflected cycles. It then acts as a moving source of a backward wave, for which $n = -1$. The listener at rest receives

$$f_{\text{echo}} = f_r \frac{c_w}{c_w + v_r} = f_s \frac{c_w - v_r}{c_w + v_r}. \quad (17.41)$$

This model assumes specular, one-dimensional reflection, steady motion, and no intrinsic modulation by the target. A single Doppler factor misses half of the first-order echo shift.

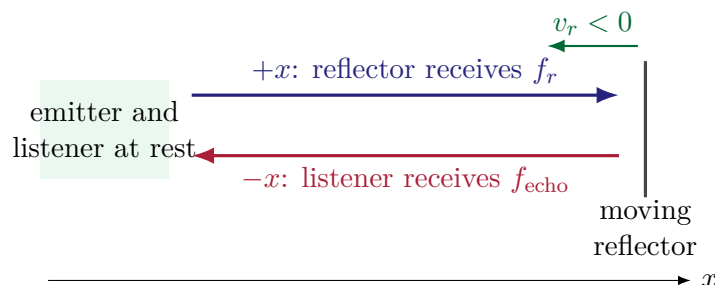


Figure 17.25: The same moving target is a receiver on the outward path and an effective source on the return path. The direction parameter changes from $n = +1$ to $n = -1$ between paths. The drawing is schematic; arrow lengths do not represent wavelengths or speeds.

Example 17.17: Infer a target speed from a double-shift echo

Task. An ideal stationary acoustic emitter sends a 2000 Hz signal through a stipulated medium with $c_w = 340 \text{ m s}^{-1}$. A planar target approaches at 8.00 m s^{-1} . Calculate the echo frequency and show how it determines the signed target speed.

Solution. Here $v_r = -8.00 \text{ m s}^{-1}$. The target first receives $f_r = 2000(348/340) = 2047.1 \text{ Hz}$. Its backward emission then gives

$$f_{\text{echo}} = 2000 \frac{348}{332} = 2096.4 \text{ Hz}.$$

To invert the observation, multiply Eq. (17.41) by its denominator: $f_{\text{echo}}c_w + f_{\text{echo}}v_r = f_sc_w - f_sv_r$. Hence

$$v_r = c_w \frac{f_s - f_{\text{echo}}}{f_s + f_{\text{echo}}} = -8.00 \text{ m s}^{-1}.$$

The negative sign identifies approach under the fixed-axis convention. The first-order estimate $\Delta f/f_s \simeq -2v_r/c_w$ gives $\Delta f \simeq 94.1 \text{ Hz}$ instead of 96.4 Hz. For $v_r = 0$ both stages reduce to unity; for recession both stages lower the frequency. These checks also identify the mistaken use of only one shift.

Caution

The acoustic Doppler equations describe transmission through a material medium. Do not substitute the speed of light to obtain a relativistic Doppler formula. Also, the apparent divergence of Eq. (17.38) as $v_s \rightarrow c_w$ does not predict an infinitely high pitched linear sound: the orderly subsonic front construction is then reaching its boundary of applicability.

17.7 Supersonic sources and wavefronts

17.7.1 The common tangent is an envelope of past emissions

Let a source move in $+x$ at constant speed $v_s > c_w$ through a homogeneous medium. A front emitted at time t_e is centred at the past position $x_e = v_s t_e$, not at the current source position. At observation time t its radius is $R_e = c_w(t - t_e)$. The front equation in a plane through the path is

$$(x - v_s t_e)^2 + y^2 = c_w^2(t - t_e)^2, \quad t_e \leq t. \quad (17.42)$$

These expanding circles share two tangent lines behind the source. Their three-dimensional counterparts share a conical envelope. The envelope is a geometrical construction; computing a real shock's pressure jump requires additional nonlinear physics.

Define the positive Mach number

$$\text{Ma} = \frac{v_s}{c_w} > 1. \quad (17.43)$$

In the triangle joining the current source, an old emission centre, and the tangent point, the radius to the tangent point is perpendicular to the tangent line. The source-centre distance is $v_s(t - t_e)$ and the radius is $c_w(t - t_e)$. Therefore

$$\boxed{\sin \mu = \frac{c_w}{v_s} = \frac{1}{\text{Ma}}}, \quad \text{Ma} > 1. \quad (17.44)$$

The half-angle μ is measured from the backward extension of the motion axis to either side of the cone. As $\text{Ma} \rightarrow 1^+$ it approaches 90° ; at high Mach number it approaches zero. Neither limit gives the shock strength.

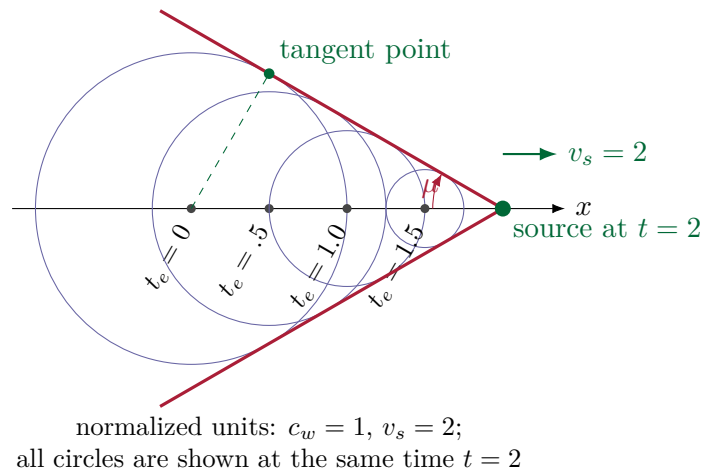


Figure 17.26: A checked Mach-envelope construction with $\text{Ma} = 2$. The four labelled emission centres are $x_e = 2t_e$, and their radii are $2 - t_e$. The red lines through the current source at $(4, 0)$ have $\mu = 30^\circ$ and touch every circle. The dashed radius reaches the first circle's tangent point $(1, \sqrt{3})$. Later fronts are smaller because they have had less time to propagate; the picture is not a sequence of snapshots.

Example 17.18: Construct and test the Mach envelope

Task. Compare the half-angles for $Ma = 1.50$ and 2.00 . Verify the tangency in Fig. 17.26 without relying on its appearance. For an ideal $Ma = 2$ source travelling horizontally at height $h = 1.00$ km with $c_w = 340$ m s⁻¹, find the delay between its passage directly overhead and the cone reaching an observer below.

Angles and tangency. The respective angles are $\arcsin(1/1.50) = 41.8^\circ$ and $\arcsin(1/2) = 30.0^\circ$. For the upper red line, $x + \sqrt{3}y - 4 = 0$. The perpendicular distance from an emission centre $(2t_e, 0)$ to this line is

$$d_e = \frac{|2t_e - 4|}{\sqrt{1+3}} = 2 - t_e = R_e \quad (0 \leq t_e \leq 2).$$

Distance equals radius, proving tangency for the entire continuous family, including the four displayed circles.

Arrival geometry. At cone arrival, the horizontal source-observer separation satisfies $\tan \mu = h/d$, so $d = h \cot 30^\circ = 1.732$ km. With source speed $v_s = 2c_w = 680$ m s⁻¹, the delay is $d/v_s = 2.55$ s. This assumes steady flight established sufficiently early for the relevant wavefronts to exist. The source is already beyond the observer when its backward cone reaches that observer. The calculation gives an arrival geometry, not a pressure amplitude or structural effect.

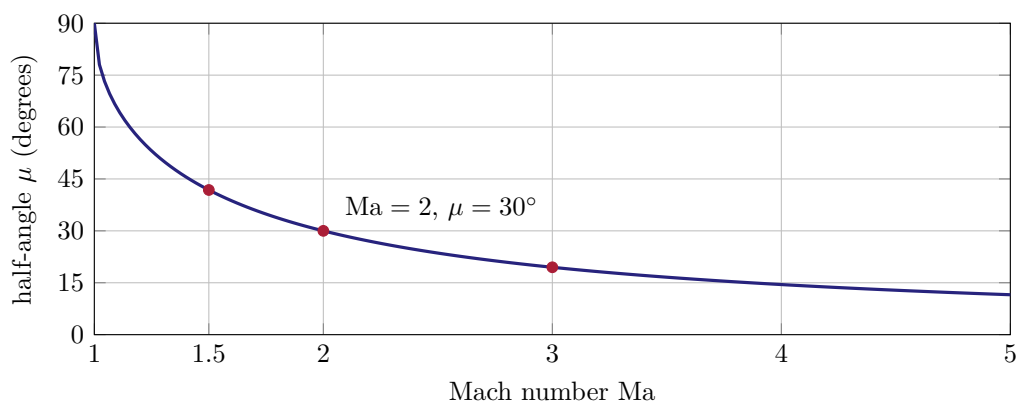


Figure 17.27: The cone narrows as source speed relative to the medium rises. The plotted function is $\mu = \arcsin(1/Ma)$ in degrees. The point at $Ma = 1$, $\mu = 90^\circ$ represents the limiting value from the supersonic side; it is not a finite-angle cone calculation for a subsonic source.

Pedagogical boundary

The common tangent determines where emissions can accumulate geometrically. A real shock is nonlinear and dissipative. Its thickness, heating, pressure jumps, and interaction with a vehicle or structure are outside this chapter's linear-wave model. A persistent supersonic source has a travelling envelope; it does not produce only one disturbance at the instant it first passes Mach 1.

17.8 Engineering, computation, and measurement

17.8.1 What is observed, and what is inferred?

A sensor measures a response related to a wave variable through a model. A pressure microphone produces voltage; a digitizer produces counts; a frequency estimator finds a peak. A displayed number is not automatically a material property. A useful inverse calculation states the observed

quantity, the unknown sought, and the physical and calibration relations linking them. For example, a tube resonance estimates c_w only after its end conditions and effective length have been identified.

A proposed low-amplitude resonance investigation is a useful teaching model. Vary a driving frequency near several resonances, use an adequately responsive sensor to locate peaks, and test the mode sequence before inferring speed. For an ideal closed–open tube,

$$f_j = a q_j, \quad q_j = 2j - 1, \quad a = \frac{c_w}{4L_{\text{eff}}}. \quad (17.45)$$

Each ratio f_j/q_j estimates the same quantity a . For equal independent frequency uncertainties, a least-squares fit through the origin minimizes $\sum_j (f_j - a q_j)^2$. Differentiating with respect to a gives

$$\hat{a} = \frac{\sum_j q_j f_j}{\sum_j q_j^2}, \quad \sigma_a = \frac{\sigma_f}{\sqrt{\sum_j q_j^2}}. \quad (17.46)$$

The uncertainty expression follows by propagating independent equal errors through this linear combination. A clock-scale error affecting every frequency is correlated and is not reduced by this formula. Residuals $r_j = f_j - \hat{a} q_j$ test whether the assumed mode sequence and linear model are compatible with the supplied data precision. Small residuals do not prove that the physical length equals the acoustic length.

A fitted line can also require a free intercept: $y_i = m x_i + b$. For equal independent uncertainties, minimizing $\sum_i (y_i - m x_i - b)^2$ with respect to both parameters first gives $b = \bar{y} - m \bar{x}$. Substitution into the remaining derivative gives

$$\hat{m} = \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sum_i (x_i - \bar{x})^2}, \quad \hat{b} = \bar{y} - \hat{m} \bar{x}. \quad (17.47)$$

Here bars denote arithmetic means. A nonzero intercept may represent an instrument offset or a physical model parameter; its meaning must be decided from the experiment. In the optional tube-length investigation, plotting $1/f$ against L permits both the slope and end correction to be inferred.

Example 17.19: A synthetic resonance fit: scatter, model bias, and clock scale

Task and data. A synthetic teaching data set represents four closed–open resonances. It was constructed for analysis practice; no tube was measured. Use $L = 0.4000 \pm 0.0005$ m (standard uncertainty), an explicitly stipulated end correction $\delta = 0.0060$ m, and independent standard frequency uncertainty $\sigma_f = 0.35$ Hz for each peak. Treat δ as exact within this exercise. Infer c_w , assess the residuals, and quantify the effect of ignoring δ .

q_j	1	3	5	7
f_j (Hz)	210.2	630.0	1049.8	1470.4

Fit and physical inference. Since $\sum q_j^2 = 84$ and $\sum q_j f_j = 17642.0$ Hz,

$$\hat{a} = 210.0238 \text{ Hz}, \quad \sigma_a = 0.0382 \text{ Hz}, \quad \hat{c}_w = 4(0.4060)\hat{a} = 341.08 \text{ m s}^{-1}.$$

The residuals are $+0.176$, -0.071 , -0.319 , and $+0.233$ Hz, all smaller than the stipulated single-peak uncertainty. The sum $\sum (r_j/\sigma_f)^2 = 1.57$ is compatible with this model for three residual degrees of freedom. It is not evidence of a universally valid end correction.

Propagating the independent length and fitted-frequency uncertainties gives

$$\sigma_c = \sqrt{(4L_{\text{eff}}\sigma_a)^2 + (4\hat{a}\sigma_L)^2} = 0.42 \text{ m s}^{-1}.$$

Thus this model yields $c_w = (341.08 \pm 0.42) \text{ m s}^{-1}$ as a standard measurement uncertainty, with the stipulated boundary correction held fixed. Using physical length alone gives $4L\hat{a} =$

336.04 m s^{-1} , a 1.48% underestimate. This unchanged bias survives averaging. An uncorrected sampling clock that makes all reported frequencies 0.20% high would likewise make the inferred speed 0.20% high, about 0.68 m s^{-1} . It must be corrected or reported separately.

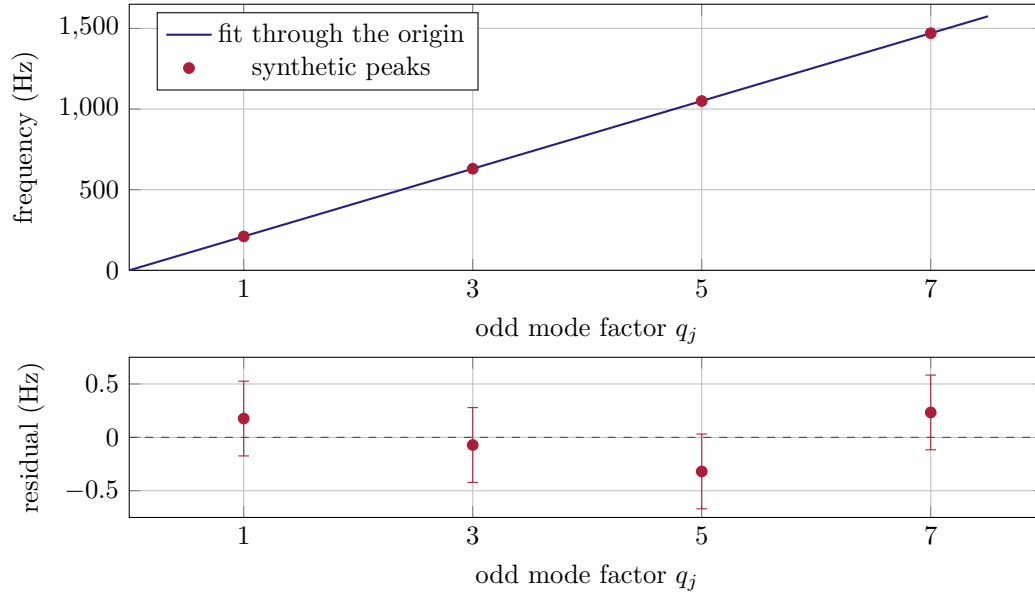


Figure 17.28: Synthetic resonance frequencies and their residuals for Example 17.19. The upper graph tests the expected odd-mode sequence; the lower graph shows residuals with the stipulated independent 0.35 Hz standard error bars. The large frequency scale would hide these residuals in the upper panel. Neither panel tests an omitted boundary correction or a common clock-scale error.

17.8.2 Calibration and the reported wave quantity

If a pressure microphone has sensitivity S_m in volts per pascal, an RMS voltage V_{rms} yields

$$p_{\text{rms}} = \frac{V_{\text{rms}}}{S_m}, \quad (17.48)$$

provided its sensitivity, bandwidth, gain, and lack of overload apply to the record. Frequency-dependent sensitivity must be treated frequency by frequency; a single conversion factor is then insufficient. A pressure-only measurement in a standing or reverberant field does not uniquely determine energy flux. The intensity inference $I = p_{\text{rms}}^2 / (\rho c_w)$ additionally requires the progressive plane-wave model taught earlier.

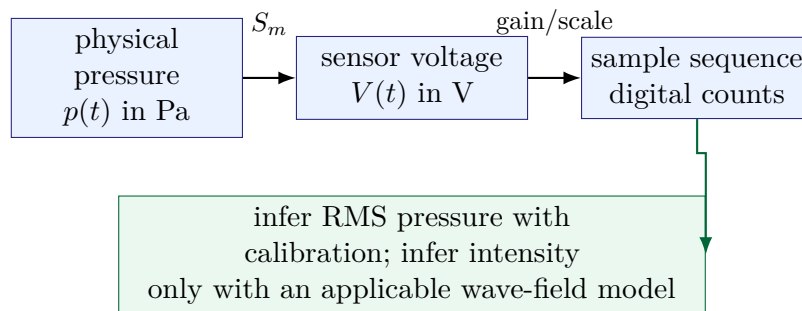


Figure 17.29: A calibrated measurement chain connects digital output to a physical quantity. The sampling rate calibrates the time axis; it does not calibrate pressure amplitude. Additional wave-field assumptions are needed to convert pressure to intensity. This is a model diagram, not a device specification.

Example 17.20: Uncertainty and calibration priorities

Task. Let $p_{\text{rms}} = KD$, where D is an RMS digital amplitude and K converts counts to pascals. Independent relative standard uncertainties are $\sigma_K/K = 5\%$ and $\sigma_D/D = 2\%$. Calculate the pressure uncertainty and its contribution to an inferred progressive-wave intensity. Evaluate a stipulated case $K = 0.0200 \text{ Pa/count}$, $D = 1.50 \text{ count}$, $\rho = 1.20 \text{ kg m}^{-3}$, and $c_w = 340 \text{ m s}^{-1}$.

Solution. Linear propagation of independent product factors gives

$$\frac{\sigma_p}{p} = \sqrt{.05^2 + .02^2} = 0.0539, \quad p_{\text{rms}} = 0.0300 \text{ Pa}, \quad \sigma_p = 0.00162 \text{ Pa}.$$

Holding ρc_w fixed for this exercise,

$$I = \frac{(0.0300)^2}{1.20(340)} = 2.21 \times 10^{-6} \text{ W m}^{-2}, \quad \frac{\sigma_I}{I} \simeq 2 \frac{\sigma_p}{p} = 10.8\%.$$

For a stated pressure reference $p_{\text{ref}} = 20 \mu\text{Pa}$, $L_p = 20 \log_{10}(p_{\text{rms}}/p_{\text{ref}}) = 63.52 \text{ dB}$ and $\sigma_{L_p} \simeq (20/\ln 10)(\sigma_p/p) = 0.47 \text{ dB}$. Calibration dominates the independent pressure uncertainty. Reducing the digital amplitude's scatter cannot remove an unchanged calibration bias or make a standing-wave pressure measurement determine intensity. The propagation is a first-order approximation justified here by small fractional uncertainties.

17.8.3 A reproducible computational benchmark

A synthetic signal tests an algorithm because the input frequencies are known:

$$s(t) = \sin(2\pi 440t) + 0.8 \sin(2\pi 446t). \quad (17.49)$$

Here t is in seconds and s is dimensionless. The computation represents neither recorded air pressure nor a calibrated voltage. A useful algorithm samples at known times, computes a spectrum with a declared transform and window, and checks that long records retain the 6 Hz tone separation. The advanced section supplies its complete construction and expected results.

The distinction among random scatter, common calibration error, and physical model bias applies to the whole chapter. Repeated measurements can reduce independent scatter. A better clock can improve frequency scale. A corrected boundary model can improve an inferred wave speed. These interventions solve different problems and should be selected from the actual inference chain.

17.9 Advanced Topics for Technical and Engineering Students

The following optional extensions connect a continuous wave model to a finite digital record and connect an ideal tube boundary to a stipulated effective length. Prerequisites are trigonometric functions, finite sums, elementary complex numbers, and first-order sensitivity reasoning. The required bridges are provided here. Ordinary Doppler and air-column calculations do not require a Fourier transform.

17.9.1 Sampling and the finite Fourier basis

Uniform sampling at rate f_{samp} in hertz means $t_n = n/f_{\text{samp}}$, with $n = 0, \dots, N-1$. The last sample is at $(N-1)/f_{\text{samp}}$. The record interval convention used for Fourier analysis is

$$T_{\text{rec}} = \frac{N}{f_{\text{samp}}}, \quad \Delta t = \frac{1}{f_{\text{samp}}}. \quad (17.50)$$

Thus N sample intervals define the periodically extended record, although there are only $N-1$ intervals between the first and last sample centres. Report the actual integer sample count when a requested duration is rounded.

Euler's identity $\exp(i\theta) = \cos \theta + i \sin \theta$ provides a compact basis for measuring the phase and magnitude of oscillations. In our discrete Fourier transform (DFT) convention,

$$X_k = \sum_{n=0}^{N-1} s_n \exp\left(-i \frac{2\pi kn}{N}\right), \quad s_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k \exp\left(i \frac{2\pi kn}{N}\right). \quad (17.51)$$

The forward transform has no normalization factor and the inverse has $1/N$. To see why the inverse works, sum z^n for $z = \exp[2\pi i(k - \ell)/N]$. The sum is $(1 - z^N)/(1 - z) = 0$ when $k \neq \ell$ modulo N , and N when $k = \ell$. This orthogonality selects the matching discrete sinusoid. The same orthogonality gives the energy identity

$$\sum_{n=0}^{N-1} |s_n|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X_k|^2, \quad \langle |s|^2 \rangle_N = \frac{1}{N^2} \sum_k |X_k|^2. \quad (17.52)$$

To derive it, multiply the inverse transform by its complex conjugate and sum over n . All cross terms vanish by orthogonality; each equal-index term contributes N , leaving the first equality. Dividing by N gives the sample mean square. For real data, count both mirrored bins, with DC and even- N Nyquist counted once. This is Parseval's identity for the stated normalization; a pressure record yields pressure squared, and conversion to progressive-wave intensity still requires the appropriate impedance.

The frequency grid and its spacing are

$$f_k = \frac{k f_{\text{samp}}}{N}, \quad \Delta f_{\text{bin}} = \frac{f_{\text{samp}}}{N} = \frac{1}{T_{\text{rec}}}. \quad (17.53)$$

For real data the negative-frequency information repeats the positive-frequency information through $X_{N-k} = X_k^*$. A one-sided peak-amplitude spectrum of an unwindowed record uses $2|X_k|/N$ for $0 < k < N/2$; the DC and even- N Nyquist bins use $|X_k|/N$ instead. For an isolated sinusoid exactly on a grid bin, this convention returns its peak amplitude, not its RMS amplitude.

17.9.2 Aliasing: two continuous curves can give identical samples

At the sample times, adding an integer multiple of f_{samp} to a frequency does not change the phase modulo 2π :

$$\cos[2\pi(f + m f_{\text{samp}})n/f_{\text{samp}}] = \cos(2\pi f n/f_{\text{samp}}), \quad m \in \mathbb{Z}.$$

Also $\cos[2\pi(f_{\text{samp}} - f)n/f_{\text{samp}}] = \cos(2\pi fn/f_{\text{samp}})$. The Nyquist frequency $f_{\text{Nyq}} = f_{\text{samp}}/2$ marks the upper end of the non-redundant band for real uniformly sampled data. Sampling alone cannot tell which member of an alias family existed before sampling. A physical anti-alias filter must suppress unwanted high-frequency content beforehand; digital filtering after the samples have been taken cannot undo that ambiguity.

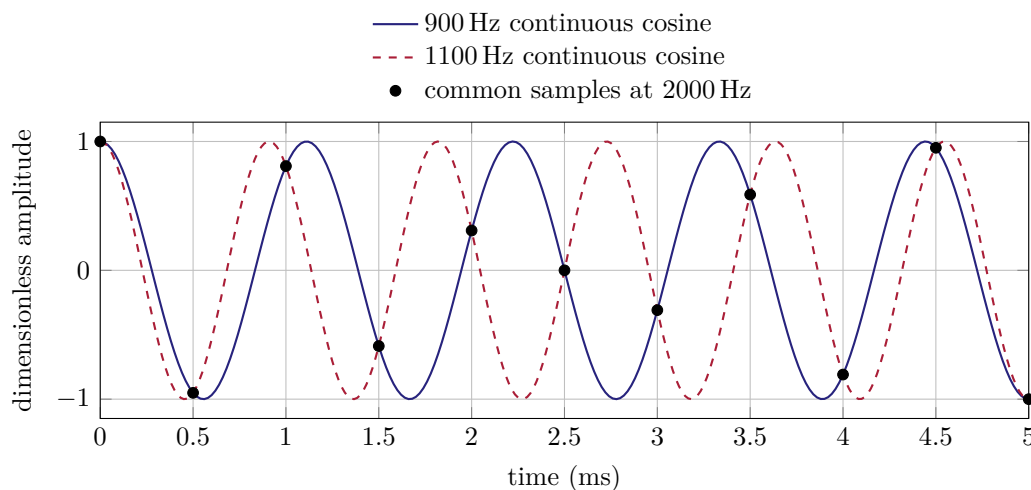


Figure 17.30: The 900 Hz and 1100 Hz continuous cosines differ between samples, but agree exactly at $t_n = n/2000$. The markers show their common sample values. This is a constructive aliasing example, with unit peak amplitude and zero starting phase for both cosines.

Example 17.21: A sampling diagnostic before taking a spectrum

Task. A recorder takes $N = 1000$ samples at $f_{\text{samp}} = 2000$ Hz. Find the record duration, bin spacing, Nyquist frequency, and aliases of 990 Hz and 1100 Hz tones. Explain what Fig. 17.30 establishes.

Solution. The duration is $1000/2000 = 0.500$ s; $\Delta f_{\text{bin}} = 2.00$ Hz; and $f_{\text{Nyq}} = 1000$ Hz. The 990 Hz tone is within the non-redundant band and lies at bin 495. The 1100 Hz tone folds to $|1100 - 2000| = 900$ Hz, bin 450. For the zero-phase cosines in the figure,

$$\cos(2\pi 1100n/2000) = \cos(2\pi n - 2\pi 900n/2000) = \cos(2\pi 900n/2000).$$

Their identical samples prove that no estimator based only on these samples can choose uniquely between the two continuous cosines. A sine at the folded frequency may differ by a sign, which is a phase shift, rather than making the alias identifiable. Prior information or appropriate filtering before sampling is needed. A large sample count improves the frequency grid; it does not remove aliasing caused by an inadequate sampling rate.

17.9.3 Windows and an explicitly normalized two-tone spectrum

A finite record truncates a signal. When its ends fail to match under periodic extension, the finite Fourier representation needs energy in many bins. This spectral leakage is the response of a finite window, not new physical tones emitted by the source. A taper reduces distant sidelobes while broadening the response near each tone.

We use the *periodic Hann window*

$$w_n = \frac{1}{2}[1 - \cos(2\pi n/N)], \quad n = 0, \dots, N - 1. \quad (17.54)$$

Its coherent gain is $G = (\sum_n w_n)/N = 1/2$ for $N > 2$. The symmetric Hann window instead uses $N - 1$ in the cosine denominator and has zero values at both displayed endpoints. It is a

different finite sequence; do not mix its normalization or plotted data with the periodic choice. Define $X_k^{(w)} = \sum_n w_n s_n \exp(-i2\pi kn/N)$ and the one-sided coherent-gain-corrected peak-amplitude convention

$$A_k = \frac{2|X_k^{(w)}|}{\sum_n w_n}, \quad 0 < k < N/2. \quad (17.55)$$

DC and Nyquist bins again omit the factor two. This removes the amplitude loss for an isolated on-bin tone. It does not promise an exact amplitude estimate for off-bin tones or overlapping lobes. It is also not a power spectral density: the latter needs a different normalization and units per hertz.

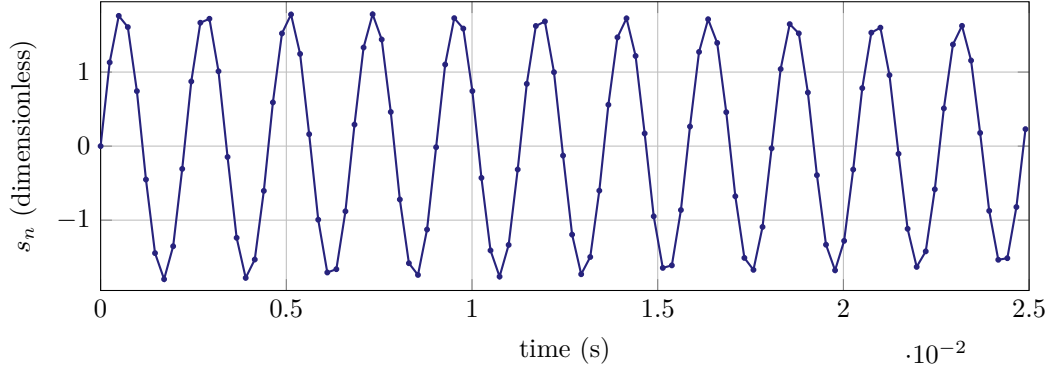


Figure 17.31: The first 103 actual samples of the synthetic signal in Eq. (17.49), at $t_n = n/4096$ for $0 \leq n \leq 102$. The last displayed sample is at 0.0249023 s. Markers are the sampled values; connecting segments guide the eye and do not define the continuous signal between samples. The full benchmark record has $N = 4096$ samples, not 103.

Example 17.22: Inspect the synthetic two-tone benchmark

Task. Reconcile the two spectra in Fig. 17.32 with their sample counts, normalization, and true frequencies. Determine whether their amplitudes permit an absolute acoustic level calculation.

Construction and results. Evaluate Eq. (17.49) at $t_n = n/4096$, multiply by the periodic Hann sequence, and apply Eq. (17.51). For $N = 4096$, $T_{\text{rec}} = 1.000$ s and $\Delta f_{\text{bin}} = 1.000$ Hz. The 440-Hz tone is at bin 440, the 446-Hz tone at bin 446. Their coherent-gain-corrected peak amplitudes are respectively 1.00 and 0.80. The periodic Hann spreads each isolated on-bin tone into its central bin and adjacent bins, with adjacent amplitudes one-half of the central amplitude. The two triples do not overlap here.

For $N = 410$, the actual record interval is $410/4096 = 0.10009765625$ s, with bin spacing 9.9902439 Hz. It is inaccurate to call these exactly 0.10 s and 10 Hz without acknowledging rounding. The strongest displayed bin is near 439.57 Hz, with $A_k \simeq 0.995$; overlapping responses prevent reading two separate tone amplitudes from their two largest bins.

Interpretation and checks. The long-record spectrum verifies the 6 Hz separation and stipulated amplitude ratio, while the short record shows the limits of a naive peak-bin method. Doubling the sampling rate while holding duration fixed leaves $1/T_{\text{rec}}$ unchanged. The signal is dimensionless, so neither spectrum yields pascals or decibels without a physical calibration model. A known sampling clock supplies seconds, not pressure units.

17.9.4 Main-lobe width, leakage, and zero-padding

For a rectangular record, the finite sum at frequency offset ν from a tone is a geometric series. Up to a phase factor, its normalized magnitude is

$$\frac{|W_{\text{rect}}(\nu)|}{N} = \left| \frac{\sin(\pi N \nu / f_{\text{samp}})}{N \sin(\pi \nu / f_{\text{samp}})} \right|, \quad (17.56)$$

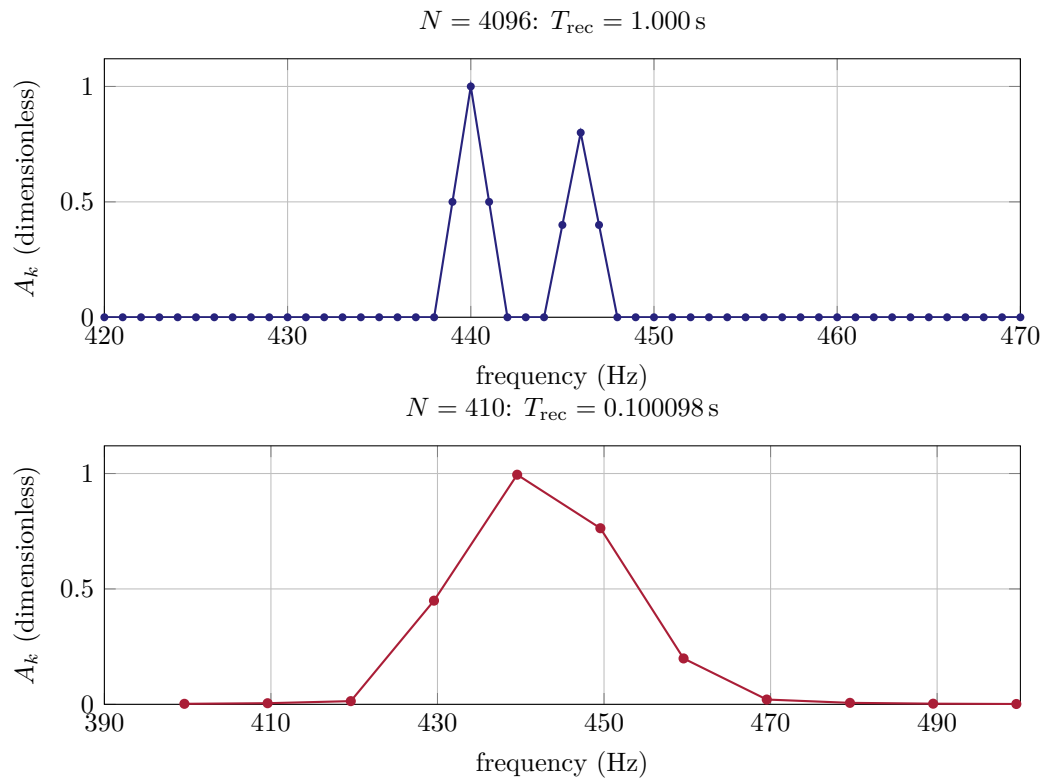


Figure 17.32: Periodic-Hann DFT amplitude spectra of the same two-tone benchmark at 4096 Hz sampling rate. Both use $A_k = 2|X_k^{(w)}|/\sum w_n$; these are dimensionless peak-amplitude conventions, not normalized sound-pressure levels. The long record has 1 Hz bin spacing and distinct on-bin peaks at 440 and 446 Hz. The short record has actual spacing $4096/410 = 9.99024$ Hz and strongly overlapping lobes. Lines join DFT-bin samples; they are not additional data.

with limiting value one at $\nu = 0$. The first zeros near the central lobe are at $\nu = \pm f_{\text{samp}}/N$, giving a null-to-null main-lobe width $2/T_{\text{rec}}$. For the periodic Hann window the first zeros are at $\nu = \pm 2f_{\text{samp}}/N$, giving width $4/T_{\text{rec}}$. The Hann response has lower distant sidelobes, with the cost visible in its broader central lobe. These are window-response widths, rather than universal tone-resolution thresholds.

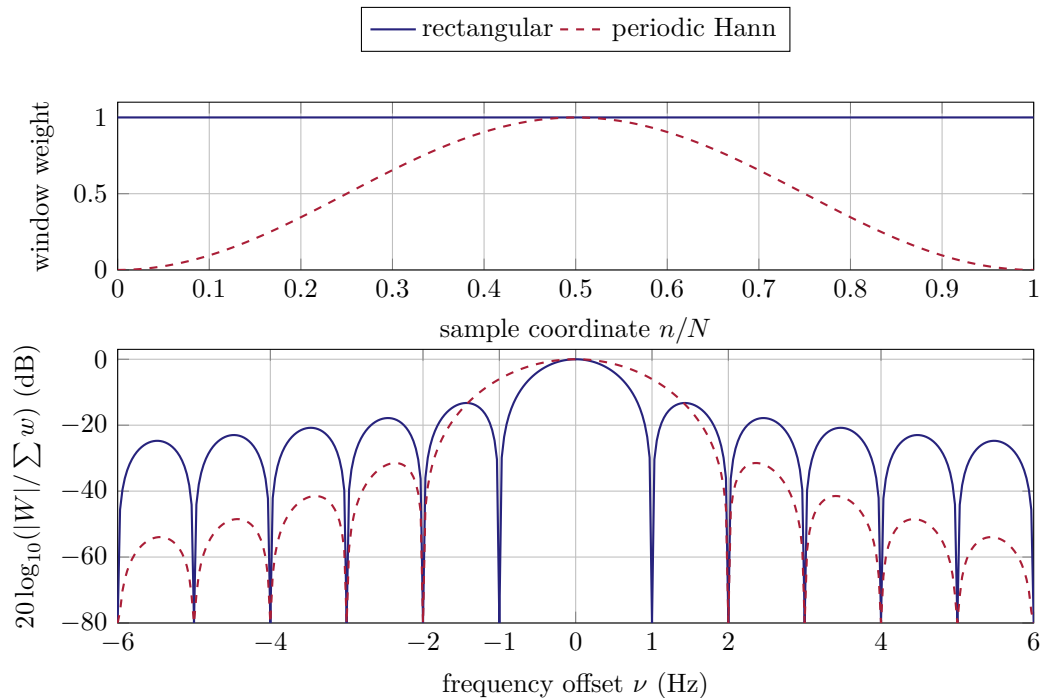


Figure 17.33: Window shape and finite-sum frequency response for $N = 4096$ at 4096 Hz, hence $T_{\text{rec}} = 1$ s. The lower curves evaluate $W(\nu) = \sum_n w_n \exp(-i2\pi\nu n/f_{\text{samp}})$ and divide by $\sum w_n$; values below -80 dB are clipped for display. The rectangular first zeros are at ± 1 Hz and the Hann first zeros at ± 2 Hz. The top Hann curve shows its continuous shape; actual periodic samples stop at $n/N = 4095/4096$, just short of the endpoint.

Example 17.23: Choose record duration before choosing a frequency grid

Task. Compare rectangular and periodic-Hann main-lobe widths for the $N = 4096$ and $N = 410$ records. A user zero-pads the short record to 4096 entries and claims that its two tones are now resolved because the plotted frequency grid is 1 Hz. Diagnose the claim.

Solution. For the long record, the rectangular null-to-null width is $2/T = 2$ Hz and the Hann width is $4/T = 4$ Hz. The tones are separated by 6 Hz, exceeding either isolated-window main-lobe width. Their visible separation in this noise-free example is consistent with those scales. For the short record the corresponding widths are 19.98 Hz and 39.96 Hz, much larger than 6 Hz. Its spectral responses overlap strongly.

Zero-padding evaluates the same 410-sample windowed sum at more frequency locations. Appending zeros to reach 4096 entries gives a $4096/4096 = 1$ Hz display grid, but the weights, nonzero sample times, and main-lobe widths still belong to the original 0.100098 s record. No additional oscillations were observed. The smooth curve can support interpolation or a model-based estimator, but grid densification alone does not establish the presence of two resolved peaks.

Check. Directly evaluating the finite sum at these denser frequency points gives exactly the same values as the zero-padded DFT. A longer physical observation changes the sum itself; zero-padding changes only its evaluation grid. Window choice, relative amplitudes, noise, and the stated estimator must all accompany a claim about resolving two tones.

A complete optional reproduction algorithm is short enough to state directly. Compute s_n from its declared frequencies, compute the chosen w_n , sum the DFT, set $f_k = kf_{\text{samp}}/N$, and divide its magnitude by the stated normalization. The following pseudocode makes the sample-count convention explicit:

```
fs = 4096.0
for N in (4096, 410):
    t = [n/fs for n in range(N)]
    s = [sin(2*pi*440*x) + 0.8*sin(2*pi*446*x) for x in t]
    w = [0.5*(1-cos(2*pi*n/N)) for n in range(N)]
    X = [sum(w[n]*s[n]*exp(-2j*pi*k*n/N)
            for n in range(N)) for k in range(N//2+1)]
    A = [2*abs(z)/sum(w) for z in X]
    A[0] /= 2
    A[-1] /= 2 # even N: Nyquist bin
    frequency = [k*fs/N for k in range(N//2+1)]
```

This direct sum is instructional; an FFT evaluates the same DFT more efficiently. Verification can compare selected direct sums with FFT bins, check the exact on-bin amplitudes, and verify alias identities at every sample. All data needed to read the chapter's graphs are embedded in the source; running this algorithm is optional.

17.9.5 A qualified effective-length model for an open tube end

The ideal pressure node lies at the geometrical opening. Radiation into the exterior shifts the effective acoustic boundary, so real mode frequencies need not use the physical tube length alone. For a stated teaching model of one approximately unflanged circular open end of radius r , stipulate

$$\delta = 0.60r, \quad L_{\text{eff}} = L + \delta. \quad (17.57)$$

The closed–open fundamental then becomes

$$f_1 \simeq \frac{c_w}{4(L + 0.60r)}. \quad (17.58)$$

The approximate radius dependence is discussed for thin-walled open ends in OpenStax [3]. Here the coefficient is an explicit model input, not a universal constant for every flange, aperture, or frequency. A different geometry or significant losses can require a different boundary model.

Since $f \propto L_{\text{eff}}^{-1}$, differentiation gives

$$\frac{\Delta f}{f} \simeq -\frac{\Delta L_{\text{eff}}}{L_{\text{eff}}} \quad (17.59)$$

for small changes. For the actual finite correction from L to $L + \delta$, the exact ratio within this stipulated model is $f_{\text{corr}}/f_{\text{ideal}} = L/(L + \delta)$. This distinction avoids treating a first-order sensitivity expression as an exact finite difference.

Example 17.24: Quantify a stipulated end correction

Task. A closed–open tube has physical length $L = 0.500$ m and circular radius $r = 0.010$ m. Take $c_w = 343$ m s⁻¹ and the stipulated correction $\delta = 0.60r$. Compare the ideal and corrected fundamentals. Then bound the prediction if the chosen coefficient is allowed to range from 0.55 to 0.65.

Solution. The geometrical-length idealization gives $f_{\text{ideal}} = 343/[4(0.500)] = 171.5$ Hz. The correction is 0.0060 m and

$$f_{\text{corr}} = \frac{343}{4(0.506)} = 169.47 \text{ Hz}, \quad \frac{f_{\text{corr}} - f_{\text{ideal}}}{f_{\text{ideal}}} = \frac{0.500}{0.506} - 1 = -1.186\%.$$

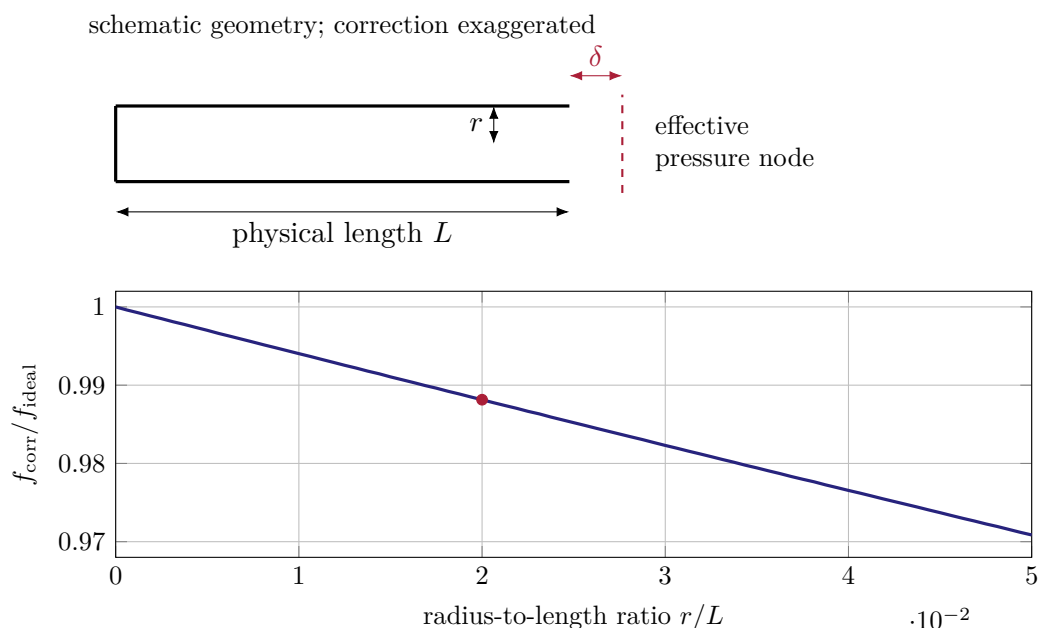


Figure 17.34: An effective pressure node beyond one circular open end, and the frequency consequence of the stipulated $\delta = 0.60r$ model. The geometry is schematic, with exaggerated correction for visibility; the lower curve is quantitative and equals $1/(1 + 0.60r/L)$. The marked ratio $r/L = 0.020$ belongs to Example 17.24.

The first-order estimate $-\delta/L = -1.20\%$ is close and slightly larger in magnitude. The correction is small geometrically yet relevant to a one-percent frequency target.

With coefficient 0.55, $L_{\text{eff}} = 0.5055$ m and $f_1 = 169.63$ Hz; with coefficient 0.65, $L_{\text{eff}} = 0.5065$ m and $f_1 = 169.30$ Hz. This 169.30–169.63 Hz range is a stipulated model-variation bound, not a statistically measured confidence interval. More repeated peak measurements do not make the chosen coefficient exact. The frequency returns to the ideal prediction as $r/L \rightarrow 0$, agreeing with the plotted limit.

17.9.6 Observation time, pulses, and an honest inference

A short pulse is a superposition of frequencies rather than an infinite single sinusoid. The finite-window calculation already demonstrates a reciprocal relation between temporal duration and spectral width. Timing a pulse arrival estimates propagation delay; a long coherent record supports narrow spectral discrimination. These are different observations. A sensor adequate for a narrow carrier band may distort a transient whose spectrum extends well beyond that band.

An acoustic inference can fail at several distinct layers: the medium and boundary model, the sensor calibration, the sampling clock and anti-alias filter, or the chosen spectral estimator. A finite-bin spacing is not an amplitude calibration. A standard uncertainty from random scatter is not a correction for a systematically wrong length. Keeping those mechanisms explicit turns a numerical answer into a useful physical result and indicates which improved observation or model would actually improve it.

17.10 Chapter summary and a working formula sheet

Summary

Sound in a fluid is a longitudinal mechanical wave. Individual material elements oscillate locally while a pattern of compression and rarefaction propagates. In the small-amplitude continuum model, compressibility and inertia combine to give $c_w = \sqrt{B/\rho}$.

A progressive sinusoid can be described by longitudinal displacement, particle velocity, and acoustic pressure. The displacement gradient produces pressure: $p' = -B\partial_x\xi$. For a right-going sinusoid, pressure and particle velocity are in phase while displacement is shifted by one quarter cycle. Peak and RMS pressure must be distinguished before intensity is calculated.

Sound level is logarithmic and always relative to a stated reference. Independent co-propagating mean intensities add; coherent equal-frequency pressure amplitudes add with phase. For two equal co-propagating plane components the resultant intensity ranges from ideal cancellation to four times one component. For counter-propagating or crossing waves, calculate the flux from pressure and particle velocity instead of pressure squared alone.

Standing modes in air columns follow from acoustic boundary conditions. Ideal open ends are pressure nodes; rigid closed ends are displacement nodes. The Doppler effect must be formulated in the medium frame, with a declared propagation direction. Supersonic wavefront geometry produces a Mach cone with $\sin\mu = 1/\text{Ma}$.

Formula summary

Wave speed and kinematics

$$c_w = \sqrt{\frac{B}{\rho}}, \quad c_w = f\lambda = \frac{\omega}{k_w}, \quad k_w = \frac{2\pi}{\lambda}, \quad \omega = 2\pi f.$$

Plane progressive sinusoid

$$\xi = \xi_{\text{amp}} \cos(k_w x - \omega t), \quad p' = -B \frac{\partial \xi}{\partial x}, \quad p_{\text{amp}} = \rho c_w \omega \xi_{\text{amp}},$$

$$p_{\text{rms}} = \frac{p_{\text{amp}}}{\sqrt{2}}, \quad I_{\text{ac}} = \frac{p_{\text{rms}}^2}{\rho c_w} = \frac{p_{\text{amp}}^2}{2\rho c_w}.$$

Intensity level with explicit reference

$$L_I = 10 \log_{10} \left(\frac{I_{\text{ac}}}{I_0} \right), \quad I_0 = 10^{-12} \text{ W m}^{-2} \quad \text{in this chapter.}$$

Pressure level and model distinction

$$L_p = 20 \log_{10}(p_{\text{rms}}/p_{\text{ref}}), \quad p_{\text{ref}} = 20 \times 10^{-6} \text{ Pa.}$$

A pressure level does not alone establish a net acoustic energy flux. The comparison with L_I requires the specified impedance and progressive-wave geometry.

Interference and beats For equal initial source phases, $\Delta\phi$ below denotes the path phase lag.

$$\Delta\phi = 2\pi \frac{\Delta L}{\lambda}, \quad I_R = 2I(1 + \cos \Delta\phi) \quad \text{for equal co-propagating coherent waves,}$$

$$f_b = |f_2 - f_1|, \quad f_c = \frac{f_1 + f_2}{2}.$$

Ideal air-column modes

$$f_n^{\text{open-open}} = \frac{nc_w}{2L}, \quad f_n^{\text{closed-open}} = \frac{(2n-1)c_w}{4L}.$$

One-dimensional Doppler, signal propagating along $+x$

$$f_o = f_s \frac{c_w - v_o}{c_w - v_s}.$$

Supersonic geometry

$$\text{Ma} = \frac{v_s}{c_w} > 1, \quad \sin \mu = \frac{1}{\text{Ma}}.$$

Before accepting a numerical answer, ask: Is the wave variable clear? Are peak and RMS values distinguished? Is the logarithmic reference stated? Does a source combination require coherent pressure addition or independent intensity addition? Are the acoustic end conditions attached to the correct variable? Are Doppler velocities defined in the medium frame? Does the chosen model apply to the measurement geometry?

17.11 Exercises and problems

Unless stated otherwise, use the ideal small-amplitude acoustic models developed in the chapter. When adding sound-source intensities in Problems 6, 17 and 18, assume the arriving waves share a common local propagation direction and impedance; independence removes the averaged interference term. When a sound level is requested, use $I_0 = 10^{-12} \text{ W m}^{-2}$. Do not introduce hearing or safety interpretations. For Doppler problems, state the propagation direction and define all velocities in the medium frame before substituting numbers. Problems 33–34, 36–40, 47–48 use the advanced enrichment. Problems 41–48 require graphical or data-based reasoning; their synthetic data are teaching models, not reported measurements.

A. Concepts, variables, and model choice

1. A compression pulse travels to the right in a gas. Explain why the gas molecules do not have to travel with the pulse. Distinguish propagation speed, particle velocity, and source velocity.
2. At one instant a fluid element has positive longitudinal displacement $\xi > 0$. Can you conclude that the local pressure perturbation is positive? Use $p' = -B\partial_x \xi$ to justify the answer.
3. A student says, “Sound travels faster in a less dense medium because $c \propto 1/\sqrt{\rho}$.” Diagnose the statement. What additional property must be compared?
4. A pressure microphone reports an RMS pressure. Another instrument specification quotes peak pressure. What conversion is valid for a pure sinusoid? Why can it fail for an arbitrary transient?
5. Explain why a local pressure amplitude in a standing wave cannot automatically be inserted into $I = p_{\text{rms}}^2/(\rho c)$ to obtain the net time-averaged energy flux.
6. Two acoustic sources each produce 60 dB at a point. List two physically different assumptions under which their combined level could differ, even though the individual readings are identical.
7. At an ideal open end of an air column, which acoustic variable has a node: pressure perturbation or longitudinal displacement? What is the corresponding condition at a rigid closed end?

8. A Doppler problem gives only the relative speed between source and observer. Explain why that information is generally insufficient for sound in a material medium.

B. Wave speed, pressure, and intensity

9. A stipulated medium has $\rho = 1.50 \text{ kg m}^{-3}$ and $B = 1.80 \times 10^5 \text{ Pa}$. Find the ideal sound speed.
10. Medium A has $(\rho, B) = (900 \text{ kg m}^{-3}, 1.44 \times 10^9 \text{ Pa})$ and medium B has $(1200 \text{ kg m}^{-3}, 1.92 \times 10^9 \text{ Pa})$. Compare their ideal sound speeds. What does the result illustrate?
11. A progressive sinusoid has $\rho = 1.18 \text{ kg m}^{-3}$, $c = 345 \text{ m s}^{-1}$, $f = 800 \text{ Hz}$, and $\xi_{\text{amp}} = 1.5 \times 10^{-7} \text{ m}$. Find p_{amp} and the mean intensity.
12. A progressive sinusoid has $p_{\text{amp}} = 0.30 \text{ Pa}$, $\rho = 1.20 \text{ kg m}^{-3}$, $c = 343 \text{ m s}^{-1}$, and $f = 1.00 \text{ kHz}$. Find ξ_{amp} , p_{rms} , and I_{ac} .
13. A plane progressive wave has $p_{\text{rms}} = 0.100 \text{ Pa}$ in a medium with $\rho = 1.20 \text{ kg m}^{-3}$ and $c = 343 \text{ m s}^{-1}$. Find its intensity. If the same source can be modelled as isotropic in a lossless free field and the observation distance is doubled, what happens to intensity and RMS pressure?
14. An ideal isotropic source emits acoustic power $P = 0.050 \text{ W}$. Find the intensity and intensity level at $r = 5.0 \text{ m}$ in the lossless spherical-spreading model.
15. Show dimensionally that $p_{\text{amp}} = \rho c \omega \xi_{\text{amp}}$ has units of pressure. Then show that $p_{\text{rms}}^2/(\rho c)$ has units of intensity.

C. Levels, interference, and beats

16. What intensity corresponds to 50.0 dB relative to $I_0 = 10^{-12} \text{ W m}^{-2}$?
17. Two independent sources produce levels of 58.0 dB and 62.0 dB at one point. Convert to intensities, add them, and find the combined level.
18. A source produces 65.0 dB at a point. Find the intensity. A second statistically independent source produces the same intensity there. Find the new level.
19. Two coherent equal-amplitude waves are locally co-propagating plane waves in the same lossless medium; each would carry intensity I separately. Their phase difference at a point is 60° . Find the resultant intensity in units of I and the level increment relative to one component.
20. Two in-phase coherent sources have a path difference $\Delta L = 0.75 \text{ m}$ at a point. The wavelength is 0.60 m and the arriving pressure amplitudes are equal. Approximate the waves as locally co-propagating plane waves with the same impedance. Find the phase difference modulo 2π , the resultant pressure-amplitude ratio p_R/p_a , and the resultant intensity in units of one component intensity.
21. Two tones have frequencies 256 Hz and 260 Hz. Find the beat frequency, interval between successive beat maxima, and carrier frequency.
22. Beat maxima are observed every 0.200 s. If two equal-amplitude tones are centred on 800 Hz, find the two frequencies assuming they straddle the centre symmetrically.
23. Explain the difference between (a) the signed cosine factor in Eq. (17.26), (b) its absolute magnitude, and (c) the time-averaged intensity modulation. Which recurrence defines the usual beat frequency?

D. Air-column modes, Doppler shift, and Mach geometry

24. An ideal open–open tube has length 0.80 m and $c = 344 \text{ m s}^{-1}$. Find the fundamental and third-harmonic frequencies.
25. An ideal tube of length 0.80 m has resonances at approximately 107.5, 322.5, and 537.5 Hz when $c = 344 \text{ m s}^{-1}$. Which endpoint model is consistent with this sequence? State the relevant pressure boundary conditions.
26. Sketch the fundamental pressure and displacement patterns for a closed–open tube. Mark nodes and antinodes. Explain why the two variables do not share the same node locations.
27. Sound travels along $+x$ with $c = 343 \text{ m s}^{-1}$. A source emits 700 Hz and moves toward a stationary observer ahead at $v_s = +25 \text{ m s}^{-1}$. Find the observed frequency.
28. A source emits 900 Hz into a medium with $c = 343 \text{ m s}^{-1}$. The signal propagates along $+x$. The source moves at $v_s = +15 \text{ m s}^{-1}$ and the observer ahead moves at $v_o = +10 \text{ m s}^{-1}$. Find the observed frequency and explain the competing effects of the two motions.
29. A stationary source emits 1000 Hz in a medium with $c = 343 \text{ m s}^{-1}$. An observer moves toward the source at 20 m s^{-1} . Choose $+x$ along propagation from source to observer and find the observed frequency.
30. A supersonic source has Mach number 1.80. Find the Mach-cone half-angle.
31. A source travels at 450 m s^{-1} through a medium with $c = 340 \text{ m s}^{-1}$. Find its Mach number and the kinematic Mach angle. State one physical feature that this simple calculation does not determine.

E. Measurement and computation

32. A closed–open resonance tube of physical length 0.400 m has an observed fundamental at 210 Hz. If the physical length is treated as the ideal effective length, infer the sound speed. Name two reasons the inference might be biased.
33. A recorder samples at 2000 Hz for 0.50 s. Find N , the DFT bin spacing, and the Nyquist frequency. Where would a 1100 Hz sinusoid alias if no anti-alias filter removed it?
34. The two-tone signal $440 \text{ Hz} + 446 \text{ Hz}$ is recorded with $N = 200$ samples and then $N = 2000$ samples, both at $f_{\text{samp}} = 2000 \text{ Hz}$. Thus the actual DFT durations are exactly 0.100 s and 1.000 s. Compare the DFT bin spacings. Which record gives a more credible direct spectral separation, and why does increasing sampling rate alone not replace a longer record?
35. A calibrated inference has $p_{\text{rms}} = KS$, with independent relative standard uncertainties 5% in K and 2% in S . Find the first-order relative standard uncertainty in p_{rms} . If $I \propto p_{\text{rms}}^2$ and all other quantities are treated as exact, estimate the pressure-induced relative uncertainty in I .

F. Advanced challenges

36. Starting from a finite rectangular record of a pure sinusoid, explain qualitatively why a tone not exactly on a DFT bin leaks into neighbouring bins. Show that increasing T_{rec} narrows the natural frequency scale $1/T_{\text{rec}}$.
37. A closed–open tube has $L = 0.500 \text{ m}$, radius $r = 0.010 \text{ m}$, and $c = 343 \text{ m s}^{-1}$. Compare the fundamental predicted with no end correction to the result from the explicitly stipulated model $\delta = 0.60r$ at the single open end. Find the percentage shift.

38. A stationary observer receives 900 Hz from a source emitting 850 Hz in a medium with $c = 340 \text{ m s}^{-1}$. The source is approaching along the propagation direction. Infer the source speed using the exact one-dimensional model.
39. Reproduce the tangency check for Fig. 17.26. In normalized units let $c = 1$, $v_s = 2$, and the observation time be $t = 2$. Show that the line through the current source point $(4, 0)$ at half-angle 30° is tangent to every earlier wavefront circle centred at $(2t_e, 0)$ with radius $2 - t_e$.
40. The synthetic two-tone benchmark is sampled at 4096 Hz. Compare a short record with exactly $N = 410$ samples and a long record with exactly $N = 4096$ samples. Use the DFT duration $T = N/f_{\text{samp}}$ in both cases. Calculate T , Δf_{bin} , and Nyquist frequency in each case. Explain what zero-padding would and would not change, and why a normalized spectrum cannot establish an absolute acoustic level.

G. Graphical and data-based synthesis

41. Figure 17.35 gives a snapshot of a small longitudinal displacement field in a fluid with $B = 1.40 \times 10^5 \text{ Pa}$. Treat the straight segments as idealizations and do not assign a derivative exactly at a corner. (a) Find p' on each of the four intervals. (b) Identify a region where $\xi > 0$ while $p' < 0$, and explain why this is consistent. (c) Evaluate $\int_0^{0.09 \text{ m}} p' dx$ and connect it to the endpoint displacements. Does the graph alone establish the direction of propagation?

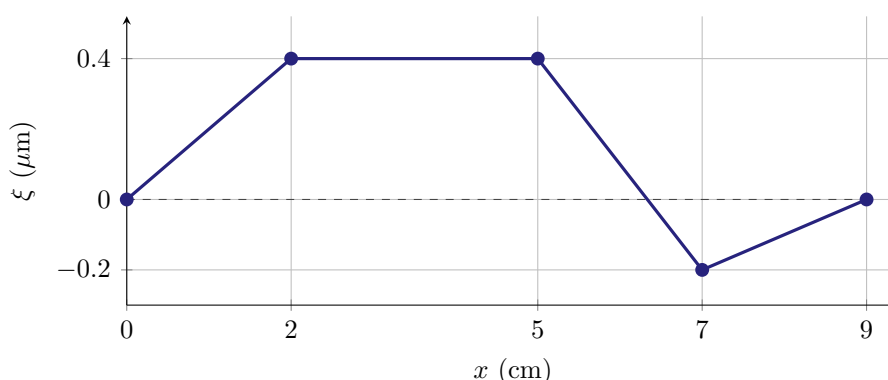


Figure 17.35: C17-P01: displacement snapshot for Problem 41. Coordinate values and slopes, rather than displacement signs alone, determine compression. The corners are idealized transitions.

42. A lossless standing plane wave has

$$p'(x, t) = 2p_a \cos(kx) \cos(\omega t), \quad u(x, t) = \frac{2p_a}{Z} \sin(kx) \sin(\omega t), \quad Z = \rho c,$$

- with $\omega = ck$, as represented in Fig. 17.36. (a) Locate the pressure and velocity nodes within $0 \leq x \leq \lambda$. (b) At $x = \lambda/8$ calculate the pressure and velocity amplitudes, then derive the instantaneous flux $j = p'u$ and its average. (c) Compare the pressure-equivalent quantity p_{rms}^2/Z there with the intensity $I_a = p_a^2/(2Z)$ of one travelling component. Explain the distinction between that quantity and net flux.
43. Two coherent locally co-propagating sinusoidal waves have the same frequency and impedance, but $p_{2,\text{amp}} = 0.60p_{1,\text{amp}}$. Figure 17.37 shows the resultant pressure amplitude. (a) Derive p_R/p_1 and I_R/I_1 as functions of the phase difference ϕ . (b) Find the maximum and minimum intensity ratios and their level changes relative to component 1. (c) Find both phases within $0 \leq \phi < 360^\circ$ for which $I_R = I_1$. Explain why complete cancellation is impossible.
44. The signed spatial pressure profile of a tube mode is shown in Fig. 17.38; the overall temporal factor is suppressed. Assume ideal acoustic endpoints, tube length $L = 0.750 \text{ m}$, and $c =$

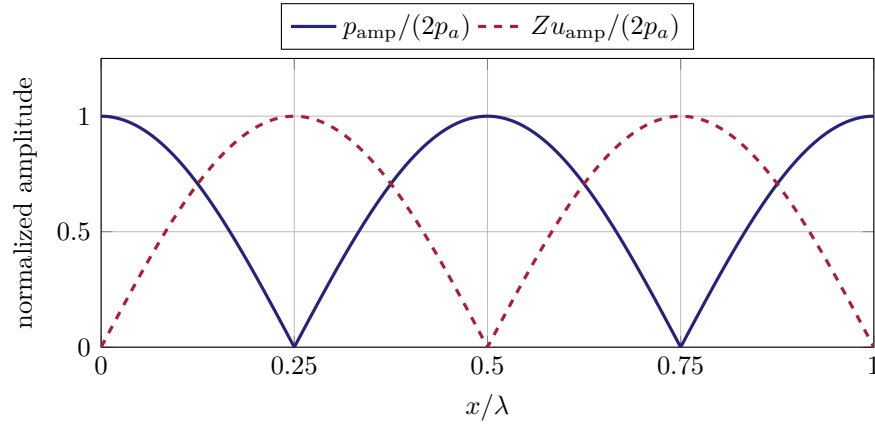


Figure 17.36: C17-P02: complementary standing-wave amplitudes for Problem 42. Positive amplitude envelopes do not display the temporal phase; the stated fields supply that information.

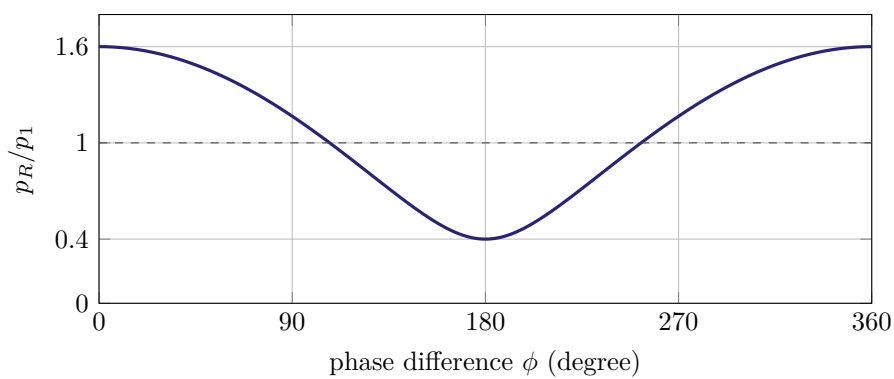


Figure 17.37: C17-P03: unequal-amplitude coherent addition for Problem 43. The ordinate is pressure amplitude; intensity ratios require its square.

340 m s^{-1} . (a) Identify the endpoint conditions and pressure nodes. (b) Infer kL , wavelength, frequency, and the mode's position in the allowed resonance sequence. (c) Sketch a compatible longitudinal-displacement profile and mark its nodes and antinodes. Explain the sign or phase choices that cannot be fixed by this pressure snapshot alone.

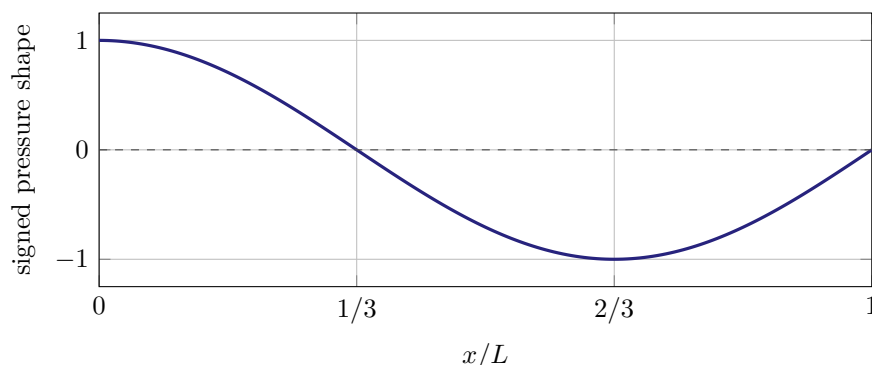


Figure 17.38: C17-P04: pressure-mode shape for Problem 44. Normalized position and the signed interior profile allow the endpoint conditions and resonance to be inferred. The temporal factor is suppressed.

45. Figure 17.39 is a medium-frame wavefront diagram for a source moving along $+x$ at 40 m s^{-1} , emitting at 100 Hz into a medium with $c = 340 \text{ m s}^{-1}$. The observer is stationary at $x = 5.00 \text{ m}$. (a) Find the positions of the first three emission events and write their wavefront trajectories. (b) Find the first three reception times and infer the observed frequency from their separation. (c) Now let the observer follow $x_o = 5.00 \text{ m} + (10 \text{ m s}^{-1})t$. Find the new reception spacing and frequency; distinguish wavefront slope, source slope, and observer slope.

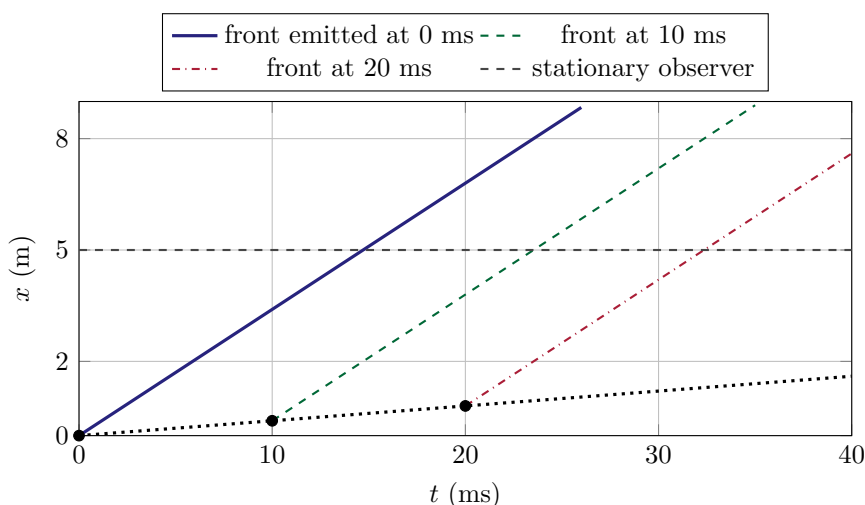


Figure 17.39: C17-P05: emission and reception geometry for Problem 45. Time increases horizontally and position vertically. The dotted low-slope line is the source trajectory. The rising fronts all have the same medium-frame speed.

46. A synthetic microphone calibration gives the following RMS voltages and known RMS pressures:

S (mV)	2.00	3.00	4.00	5.00
p_{rms} (Pa)	0.0380	0.0620	0.0810	0.0990

Assume the zero-offset model $p = KS$ with equal absolute uncertainty in the pressure observations. (a) Fit K by least squares through the origin and calculate each pressure residual. State what extra test is needed to support the zero-offset assumption. (b) A new progressive-wave reading is $S = 4.50 \text{ mV}$. Find p_{rms} , intensity and intensity level for $\rho = 1.20 \text{ kg m}^{-3}$ and $c = 343 \text{ m s}^{-1}$. (c) The independent relative standard uncertainties are 4% in K , 3% in the

new S , 2% in ρ and 1% in c . Propagate them to pressure, intensity, and the small-uncertainty approximation for level. Treat the given 4% calibration uncertainty as supplied, rather than inferring it from four residuals.

47. A synthetic closed–open resonance investigation gives

L (m)	0.300	0.400	0.500	0.600
$1/f_1$ (s)	0.0035652174	0.0047246377	0.0058840580	0.0070434783

The model is $f_1 = c/[4(L + \delta)]$, with constant c and a common end correction δ . (a) Show that fitting $1/f_1 = mL + b$ separates c from δ , and fit both parameters. (b) For tube radius $r = 0.0125$ m, find δ/r . Does that fitted ratio establish a universal open-end correction law? (c) At $L = 0.500$ m, compare the inferred speed if end correction is ignored with the fitted speed, and quantify the bias. Explain one diagnostic that would challenge the constant- δ model.

48. An exactly periodic synthetic pressure record contains $N = 8$ samples at $f_s = 8$ Hz:

$$p_n = 0.300 \cos(2\pi n/8) + 0.200 \sin(4\pi n/8) \quad \text{Pa}, \quad n = 0, \dots, 7.$$

Use the unnormalized DFT $X_k = \sum_{n=0}^{N-1} p_n \exp(-2\pi i kn/N)$; Fig. 17.40 gives its magnitudes. (a) Identify the positive-frequency tones, recover their one-sided peak amplitudes, and explain the mirrored bins. (b) Find RMS pressure both from the sinusoidal amplitudes and from discrete Parseval's identity. (c) Assuming these tones are locally co-propagating progressive waves in the same medium with $\rho = 1.20 \text{ kg m}^{-3}$ and $c = 343 \text{ m s}^{-1}$, calculate total intensity and level. Explain what would be lost if the graph were normalized to its highest bin without recording that bin's pressure scale. This deliberately low-frequency short record is a mathematical benchmark, not a microphone-performance test.

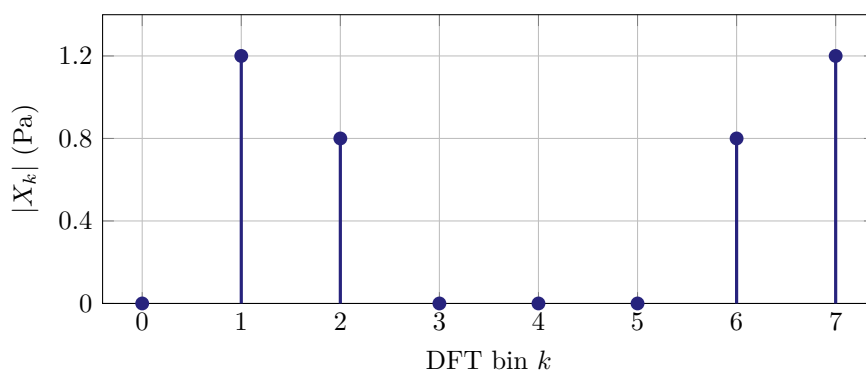


Figure 17.40: C17-P06: unnormalized two-sided DFT magnitudes for Problem 48. The discrete-transform convention fixes the amplitudes; X_k has pressure units because the finite sum is dimensionless apart from its pressure samples.

17.12 Complete reasoned solutions

These solutions retain the assumptions of the stated problems. Pressure amplitudes are peak amplitudes unless marked RMS. A pressure-derived intensity is identified as a progressive-wave result or explicitly distinguished from standing-wave net flux. DFT durations are N/f_s , so every sample count below is an integer.

Problem 1: transport of the pattern and motion of the medium

A compression increases the pressure on neighbouring fluid elements. Their response compresses the next region while the earlier elements return toward equilibrium. This sequence transports a pressure/displacement pattern without requiring each molecule to accompany that pattern over its entire path. The propagation speed c is the speed of a phase or pulse through the medium; $u = \partial_t \xi$ is the local particle velocity and reverses during an oscillation. The source velocity describes the emitter's motion in the medium frame. All three have units of speed, but they describe different physical objects. For a small-amplitude sinusoid, $u_{\text{amp}} = \omega \xi_{\text{amp}} \ll c$ is a consistency check on linear acoustics.

Problem 2: displacement does not set pressure by its sign

The stated relation is $p' = -B\partial_x \xi$, with $B > 0$. If displacement decreases with x , neighbouring elements approach one another and $p' > 0$; if it increases with x , their separation increases and $p' < 0$. For example, both $\xi = a - bx$ and $\xi = a + bx$ can be positive near $x = 0$ when $a, b > 0$, yet their pressure perturbations are $+Bb$ and $-Bb$, respectively. A spatially uniform positive displacement has $p' = 0$: it shifts elements without locally compressing the fluid. The missing information is the gradient, not the absolute displacement.

Problem 3: compare stiffness as well as density

The linear bulk-wave speed is $c = \sqrt{B/\rho}$. The student's conclusion is valid only while B is held fixed. Across different media the bulk modulus generally changes along with density, so the relevant comparison is B/ρ . A dense but very stiff fluid can have a greater speed than a dilute compressible gas. The derivative $\partial c / \partial \rho|_B = -c/(2\rho)$ describes a controlled variation; it is not a rule for arbitrary materials. For a gas the appropriate modulus also depends on the thermodynamic response during compression, a topic developed after this chapter.

Problem 4: peak and RMS pressure

For $p' = p_a \cos \omega t$, an average over whole cycles gives

$$p_{\text{rms}}^2 = \langle p'^2 \rangle = p_a^2 \langle \cos^2 \omega t \rangle = p_a^2/2.$$

Thus $p_{\text{rms}} = p_a/\sqrt{2}$ and $p_a = \sqrt{2}p_{\text{rms}}$. For an arbitrary transient use the reported averaging interval:

$$p_{\text{rms}} = \left[\frac{1}{T} \int_0^T p'^2 dt \right]^{1/2}.$$

For a rectangular pulse of peak p_a present for a fraction d of that interval, $p_{\text{rms}} = p_a\sqrt{d}$. The conversion factor therefore depends on waveform and averaging interval; a peak specification alone does not fix a transient's RMS pressure.

Problem 5: the standing-wave intensity trap

The energy flux is $j = p'u$, so its average needs both pressure and particle velocity. For a right-moving plane wave, $u = p'/Z$, where $Z = \rho c$, and therefore $\langle j \rangle = p_{\text{rms}}^2/Z$. For equal counterpropagating

waves one may write $p' = 2p_a \cos kx \cos \omega t$ and $u = 2p_a \sin kx \sin \omega t / Z$. Their product has a factor $\cos \omega t \sin \omega t$, whose period average vanishes. There can be a large pressure antinode while the mean net flux is zero. Inserting that pressure into the travelling-wave formula yields a pressure-equivalent quantity, not the transported net power per area.

Problem 6: two different meanings of addition

A 60 dB component corresponds to $I_1 = I_0 10^6 = 10^{-6} \text{ W m}^{-2}$. If the sources are statistically independent and the averaging suppresses their cross term, intensities add: $I_R = 2I_1$, giving 63.01 dB. For coherent locally co-propagating waves of equal amplitude, phase must be included. In phase, the pressure doubles and the intensity is $4I_1$, giving 66.02 dB. At a phase difference π , the ideal pressures and velocities cancel, giving zero flux at that location in this model; the logarithmic level of exactly zero intensity is not a finite number. Partial coherence or another phase produces other results. The separate level readings do not determine phase or correlation.

Problem 7: complementary endpoint conditions

At an ideal open pressure-release end, the acoustic pressure perturbation is constrained to zero, so pressure has a node. For a standing tube mode the longitudinal displacement and particle velocity then have antinodes there. At an ideal rigid end, particles cannot move normally through the wall: displacement and velocity have nodes, while pressure has an antinode. These conditions refer to the oscillating acoustic variables, not to the equilibrium atmospheric pressure. A real open aperture has radiation and an effective boundary outside the physical opening, so the ideal endpoint is an approximation.

Problem 8: why relative velocity is insufficient

Choose $+x$ along sound propagation. The exact one-dimensional formula is

$$f_o = f_s \frac{c - v_o}{c - v_s},$$

where both velocities are relative to the medium. Two pairs can share $v_s - v_o = 20 \text{ m s}^{-1}$: $(v_s, v_o) = (20, 0)$ and $(0, -20)$. For $c = 343 \text{ m s}^{-1}$, their ratios are $343/323 = 1.06192$ and $363/343 = 1.05831$, respectively. They differ because source motion changes wavefront spacing while observer motion changes encounter rate. At first order in speeds small compared with c , both give $f_o/f_s \simeq 1 + (v_s - v_o)/c$; that approximate agreement does not make the exact medium-frame information redundant.

Problem 9: bulk-wave speed

The ideal continuum model gives

$$c = \sqrt{B/\rho} = \sqrt{\frac{1.80 \times 10^5}{1.50}} = 346.41 \text{ m s}^{-1} \simeq 346 \text{ m s}^{-1}.$$

The ratio B/ρ has units $\text{m}^2 \text{s}^{-2}$; its positive square root has the required speed units. This result assumes the stipulated modulus is the one appropriate to the acoustic perturbation and that losses and finite-amplitude effects are negligible.

Problem 10: equal stiffness-to-density ratios

Calculate the ratio before taking either root:

$$\frac{B_A}{\rho_A} = \frac{1.44 \times 10^9}{900} = 1.60 \times 10^6 \text{ m}^2 \text{s}^{-2}, \quad \frac{B_B}{\rho_B} = \frac{1.92 \times 10^9}{1200} = 1.60 \times 10^6 \text{ m}^2 \text{s}^{-2}.$$

Hence $c_A = c_B = 1264.91 \text{ m s}^{-1} \simeq 1.26 \times 10^3 \text{ m s}^{-1}$. The greater density of B is exactly offset by its greater modulus. The equality tests the ratio law directly and illustrates why a density-only comparison fails.

Problem 11: convert displacement to pressure and energy flux

For the stipulated progressive sinusoid, pressure amplitude follows from the spatial displacement gradient and $k = \omega/c$:

$$p_a = Bk\xi_a = \rho c \omega \xi_a.$$

Here $\omega = 2\pi(800) = 5026.55 \text{ s}^{-1}$ and $Z = (1.18)(345) = 407.1 \text{ Pa s m}^{-1}$, so

$$p_a = 407.1(5026.55)(1.5 \times 10^{-7}) = 0.306946 \text{ Pa}, \quad I = \frac{p_a^2}{2Z} = 1.15716 \times 10^{-4} \text{ W m}^{-2}.$$

The factor $1/2$ is the sinusoidal peak-to-mean-square conversion. As a check, $u_a = \omega \xi_a = 7.54 \times 10^{-4} \text{ m s}^{-1} \ll c$; the particle speed is appropriately small.

Problem 12: recover displacement from pressure

Use $Z = \rho c = 411.6 \text{ Pa s m}^{-1}$ and $\omega = 2\pi(1000) \text{ s}^{-1}$. Then

$$\xi_a = \frac{p_a}{Z\omega} = \frac{0.30}{411.6(2\pi \times 1000)} = 1.16002 \times 10^{-7} \text{ m},$$

$$p_{\text{rms}} = \frac{0.30}{\sqrt{2}} = 0.212132 \text{ Pa}, \quad I = \frac{0.30^2}{2(411.6)} = 1.09329 \times 10^{-4} \text{ W m}^{-2}.$$

The displacement is about $0.116 \mu\text{m}$, much smaller than the wavelength $c/f = 0.343 \text{ m}$. Both the amplitude scale and $u_a = p_a/Z = 7.29 \times 10^{-4} \text{ m s}^{-1}$ support the linear approximation.

Problem 13: progressive impedance and spherical spreading

For the local progressive plane approximation,

$$I = \frac{0.100^2}{(1.20)(343)} = 2.42954 \times 10^{-5} \text{ W m}^{-2}.$$

The separate lossless isotropic-source model says $P = 4\pi r^2 I$ is constant. Therefore $I(2r) = I(r)/4 = 6.07386 \times 10^{-6} \text{ W m}^{-2}$. In the same medium, the local progressive impedance relation gives $p_{\text{rms}} \propto \sqrt{I}$, so $p_{\text{rms}}(2r) = 0.0500 \text{ Pa}$. This joins spatial spreading to a local pressure/velocity model; an arbitrary reverberant field would not support the same inference.

Problem 14: isotropic power and level

The emitted power crosses a sphere of area $4\pi r^2$, giving

$$I = \frac{0.050}{4\pi(5.0)^2} = 1.59155 \times 10^{-4} \text{ W m}^{-2}.$$

The intensity level is

$$L_I = 10 \log_{10} \left(\frac{1.59155 \times 10^{-4}}{10^{-12}} \right) = 82.0182 \text{ dB} \simeq 82.0 \text{ dB}.$$

Multiplying the intensity by the sphere area recovers 0.050 W , an energy-conservation check. Directionality, absorption or nearby reflecting surfaces would require a different spreading model.

Problem 15: two dimensional checks

First,

$$[\rho c \omega \xi] = \text{kg m}^{-3} \text{m s}^{-1} \text{s}^{-1} \text{m} = \text{kg m}^{-1} \text{s}^{-2} = \text{Pa}.$$

Second, since $[\rho c] = \text{kg m}^{-2} \text{s}^{-1}$,

$$\left[\frac{p_{\text{rms}}^2}{\rho c} \right] = \frac{\text{kg}^2 \text{m}^{-2} \text{s}^{-4}}{\text{kg m}^{-2} \text{s}^{-1}} = \text{kg s}^{-3} = \text{W m}^{-2}.$$

Units are therefore consistent with pressure and intensity. The checks do not prove applicability: the second relation still needs a progressive field with the stated impedance, as Problem 5 demonstrates.

Problem 16: invert the logarithmic level

Solve $L_I = 10 \log_{10}(I/I_0)$ for the underlying positive intensity:

$$I = I_0 10^{L_I/10} = 10^{-12} 10^{50.0/10} = 1.00 \times 10^{-7} \text{ W m}^{-2}.$$

Substitution gives an intensity ratio 10^5 and hence $10 \log_{10} 10^5 = 50.0 \text{ dB}$. The level is a logarithmic comparison to the specified reference, not an intensity measured in decibels.

Problem 17: independent sources must add before taking the logarithm

Independence means the mean cross term is absent. Convert each level to intensity:

$$I_1 = 10^{-12} 10^{5.8} = 6.30957 \times 10^{-7}, \quad I_2 = 10^{-12} 10^{6.2} = 1.58489 \times 10^{-6} \text{ W m}^{-2}.$$

Then

$$I_R = I_1 + I_2 = 2.21585 \times 10^{-6} \text{ W m}^{-2}, \quad L_R = 10 \log_{10}(I_R/I_0) = 63.4554 \text{ dB}.$$

The combined value lies above either individual level but only 1.46 dB above the stronger source. That scale is sensible because the weaker intensity is about 0.398 times the stronger one. Adding 58 + 62 would not perform the required energy sum.

Problem 18: doubling an independent intensity

The first source gives

$$I = 10^{-12} 10^{6.5} = 3.16228 \times 10^{-6} \text{ W m}^{-2}.$$

The independent equal contribution doubles this to $6.32456 \times 10^{-6} \text{ W m}^{-2}$. Thus

$$L_R = 65.0 + 10 \log_{10} 2 = 68.0103 \text{ dB}.$$

The increment is 3.0103 dB because intensity doubles. Coherent in-phase addition would instead double pressure and quadruple intensity; independence is the decisive assumption here.

Problem 19: coherent equal amplitudes

In the stated locally co-propagating approximation, pressure phasors and velocity phasors obey the same impedance relation. Two equal phasors with relative phase ϕ give

$$p_R^2 = p_a^2 + p_a^2 + 2p_a^2 \cos \phi, \quad \frac{I_R}{I} = 2(1 + \cos \phi).$$

At $\phi = 60^\circ$, $I_R/I = 2(1 + 1/2) = 3$. The increment is $10 \log_{10} 3 = 4.77121 \text{ dB}$. The result falls between independent addition ($2I$) and the in-phase maximum ($4I$). If the arriving propagation directions differed substantially, pressure summation alone would not determine the vector net energy flux; the stated co-propagating approximation avoids that ambiguity.

Problem 20: path difference determines relative phase

Equal initial source phase makes the path contribution $\Delta\phi = k\Delta L$ up to the sign chosen for which path is subtracted. Its magnitude is

$$2\pi \frac{0.75}{0.60} = 2.5\pi \equiv \frac{\pi}{2} \pmod{2\pi}.$$

Changing the path-order convention replaces $+\pi/2$ by $-\pi/2$; both have the same cosine and resultant amplitude. The phasor sum gives

$$\frac{p_R}{p_a} = \sqrt{2 + 2\cos(\pi/2)} = \sqrt{2}, \quad \frac{I_R}{I} = 2.$$

These are pressure amplitude and intensity ratios, respectively. The intensity conversion uses the explicitly stipulated common impedance and locally co-propagating arrival directions.

Problem 21: beat recurrence and carrier

Adding equal sinusoids produces a rapidly varying factor at the average frequency and a slowly varying signed factor at half the difference. The observable amplitude magnitude has maxima at the full difference frequency:

$$f_b = |260 - 256| = 4.00 \text{ Hz}, \quad T_b = 1/f_b = 0.250 \text{ s}, \quad f_c = (260 + 256)/2 = 258 \text{ Hz}.$$

The signed slow factor reverses after one beat interval; its absolute magnitude, and the squared intensity modulation, return to a maximum. This explains why counting audible or pressure-amplitude maxima gives 4 Hz rather than 2 Hz.

Problem 22: reconstruct a symmetric tone pair

The interval between magnitude maxima gives $f_b = 1/(0.200) = 5.00 \text{ Hz}$. Symmetry about 800 Hz means $f_1 + f_2 = 1600 \text{ Hz}$, while $f_2 - f_1 = 5.00 \text{ Hz}$. Solving these two equations yields $f_1 = 797.5 \text{ Hz}$ and $f_2 = 802.5 \text{ Hz}$. Their average is 800 Hz and their difference is 5 Hz, checking both constraints. Without the stated symmetry/centre information, the beat interval would determine only the separation, not the two frequencies individually.

Problem 23: signed envelope, magnitude, and mean-square envelope

For equal pressures,

$$p' = 2p_a \cos(\pi\Delta f t) \cos(2\pi f_c t), \quad \Delta f = f_2 - f_1.$$

(a) The signed factor $\cos(\pi\Delta f t)$ has period $2/|\Delta f|$. A negative factor represents a reversal of the carrier phase by π . (b) Its absolute magnitude has period $1/|\Delta f|$, because reversing the envelope sign leaves its magnitude unchanged. (c) Averaging the fast carrier while retaining the slowly varying beat envelope gives $I(t) \simeq 4I_a \cos^2(\pi\Delta f t)$ for locally co-propagating tones. Its period is also $1/|\Delta f|$. The ordinary beat frequency is the recurrence of magnitude or intensity maxima, $f_b = |\Delta f|$. An average over many complete beats would instead be constant, $2I_a$.

Problem 24: an open–open resonance sequence

Pressure nodes at both ideal open ends require an integer number of half-wavelengths: $L = n\lambda_n/2$. Hence $f_n = nc/(2L)$, and

$$f_1 = \frac{344}{2(0.80)} = 215 \text{ Hz}, \quad f_3 = 3f_1 = 645 \text{ Hz}.$$

The third harmonic is also the third resonance in this endpoint model. An end correction would replace the geometric length by an appropriate effective length, but none is stipulated here.

Problem 25: infer endpoint conditions from odd harmonics

The listed ratios are 1 : 3 : 5, characteristic of one pressure antinode and one pressure node, that is, a closed–open ideal tube. Its sequence is

$$f_n = \frac{(2n-1)c}{4L}, \quad \frac{344}{4(0.80)} = 107.5 \text{ Hz.}$$

The next values are $3(107.5) = 322.5 \text{ Hz}$ and $5(107.5) = 537.5 \text{ Hz}$, matching the observations. At the rigid end $u = \xi = 0$ and pressure has an antinode; at the ideal open end $p' = 0$ and displacement has an antinode. The second resonance here is the third harmonic, not the second harmonic. An open–open tube of the same length would begin at 215 Hz.

Problem 26: compatible fundamental tube profiles

Place the rigid end at $x = 0$ and the ideal open end at $x = L$. A compatible pair is

$$\xi(x, t) = A \sin\left(\frac{\pi x}{2L}\right) \cos \omega t, \quad p'(x, t) = -BA \frac{\pi}{2L} \cos\left(\frac{\pi x}{2L}\right) \cos \omega t.$$

The sketch in Fig. 17.41 has displacement rising from a node at 0 to an antinode at L ; pressure magnitude falls from an antinode at 0 to a node at L . For $A \cos \omega t > 0$, the signed pressure profile above is negative; multiplying both spatial profiles by a common phase factor gives equivalent snapshots. The derivative relation changes sine into cosine, so the spatial node sets are complementary. These pressure/displacement profiles share a temporal phase in this representation; velocity, obtained by differentiating displacement in time, is in quadrature.

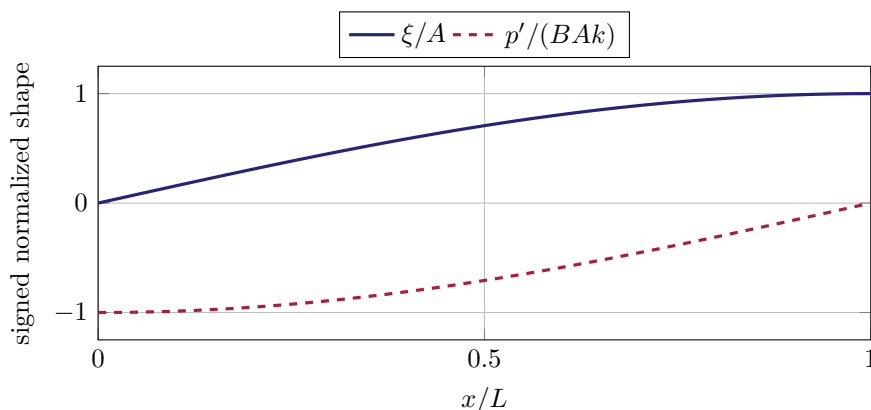


Figure 17.41: C17-P07: compatible signed fundamental profiles for the sketch requested in Problem 26. This is the instant $\cos \omega t = 1$ for the stated solution convention, with the rigid end at $x = 0$ and the open end at $x = L$.

Problem 27: an approaching source compresses wavefront spacing

Take $+x$ along propagation from the source toward the observer ahead. Both speeds are in the medium frame: $v_s = +25 \text{ m s}^{-1}$ and $v_o = 0$. The source advances during one emission period, giving $\lambda = (c - v_s)/f_s$. Therefore

$$f_o = \frac{c}{\lambda} = 700 \frac{343}{343 - 25} = 755.031 \text{ Hz.}$$

The upward shift is consistent with a shortened wavelength. Because $v_s < c$, the denominator is positive and the ordinary subsonic wavefront-spacing model applies.

Problem 28: competing source and observer motions

With $+x$ along propagation, $v_s = +15 \text{ m s}^{-1}$ and $v_o = +10 \text{ m s}^{-1}$. The wavelength is $(343-15)/900 \text{ m}$; the observer meets fronts at relative speed $343 - 10 \text{ m s}^{-1}$. Consequently

$$f_o = 900 \frac{343 - 10}{343 - 15} = 913.720 \text{ Hz}.$$

The source motion shortens the wavelength, raising the encounter rate; observer motion away in the propagation direction reduces it. Here the source effect is larger, so the net change is upward. If both velocities were equal, numerator and denominator would cancel and there would be no frequency shift in this one-dimensional model.

Problem 29: an observer moves against propagation

Choose $+x$ from the stationary source to the observer. Motion toward the source then means $v_o = -20 \text{ m s}^{-1}$, not $+20$. The source has $v_s = 0$, so its wavelength is $\lambda = c/f_s$. The encounter rate is

$$f_o = 1000 \frac{343 - (-20)}{343} = 1058.31 \text{ Hz}.$$

The observer crosses fronts faster than a stationary observer and therefore measures a higher frequency. Explicitly assigning the signed velocity before substitution avoids confusing “toward” with the positive coordinate direction.

Problem 30: Mach-cone half-angle

For constant supersonic speed, the wavefront-envelope geometry yields $\sin \mu = c/v_s = 1/\text{Ma}$. Thus

$$\mu = \sin^{-1}(1/1.80) = 33.74899^\circ \simeq 33.75^\circ.$$

The cone narrows as Mach number increases and tends toward a right angle as the speed approaches the sound speed from above. These limits check both the inverse and the choice of half-angle rather than a complementary angle.

Problem 31: speed ratio and shock limitations

The speed ratio is $\text{Ma} = 450/340 = 1.323529 > 1$, so a supersonic envelope exists. Its half-angle is

$$\mu = \sin^{-1}(340/450) = 49.07394^\circ \simeq 49.1^\circ.$$

The construction uses constant source speed and a constant linear propagation speed in the surrounding medium. It determines the wavefront envelope’s geometry. It does not determine a shock pressure jump, shock thickness, heating, attenuation, or the force on a structure: any of those requires additional dynamics and material information.

Problem 32: infer speed and diagnose modelling bias

For the ideal closed–open fundamental, $f_1 = c/(4L)$, giving

$$c = 4Lf_1 = 4(0.400)(210) = 336 \text{ m s}^{-1}.$$

Two possible biases are a positive open-end correction and a misidentified resonance. With $L_{\text{eff}} = L + \delta$, the true model gives $c = 4f_1(L + \delta)$, so ignoring a positive δ underestimates the speed. If the observed peak were a higher odd harmonic rather than the fundamental, the relation used would be inappropriate. Temperature/state variations, uncertain boundary geometry and appreciable loss can also matter. These are mechanisms, not numerical error bars determined by the single observation.

Problem 33: integer samples and aliasing

The sample count and DFT duration are $N = f_s T = 2000(0.50) = 1000$ and $T = N/f_s = 0.50$ s. Hence

$$\Delta f_{\text{bin}} = f_s/N = 2.00 \text{ Hz}, \quad f_{\text{Nyq}} = f_s/2 = 1000 \text{ Hz}.$$

For cosine samples, $\cos[2\pi(1100)n/2000] = \cos[2\pi(900)n/2000]$, because the first argument differs from the negative of the second by $2\pi n$. Thus the observed alias lies at 900 Hz. Sampling alone cannot reveal which analogue tone generated those samples. An analogue anti-alias filter must suppress such out-of-band input before sampling.

Problem 34: duration and spectral separation

At the stated $f_s = 2000$ Hz, $N = 200$ gives $T = 0.100$ s and $\Delta f = 10.0$ Hz; $N = 2000$ gives $T = 1.000$ s and $\Delta f = 1.00$ Hz. The tones differ by 6 Hz. A rectangular short record has a first-zero scale of 10 Hz on either side of each ideal spectral line, so the two responses overlap strongly. The one-second record narrows that scale to 1 Hz, and both 440 and 446 Hz lie exactly on its bins. It therefore gives more credible direct separation in this noiseless benchmark. A higher f_s at fixed T raises the Nyquist limit and increases N , but $f_s/N = 1/T$ remains unchanged. Window shape, noise and amplitude ratio still affect practical resolving power.

Problem 35: first-order uncertainty propagation

For $p = KS$, linearizing gives $dp/p = dK/K + dS/S$. Independent standard uncertainties therefore add in variance, not directly in magnitude:

$$\frac{\sigma_p}{p} = \sqrt{0.05^2 + 0.02^2} = 0.0538516 = 5.39\%.$$

Since I is proportional to p^2 while other quantities are exact, $dI/I = 2dp/p$, so the pressure-induced uncertainty is $\sigma_I/I \simeq 0.107703 = 10.8\%$. These are small-uncertainty standard deviations; a common calibration bias or correlated inputs would require different treatment.

Problem 36: leakage and finite observation time

A finite rectangular record multiplies an infinite sinusoid by a window $w(t)$ that is nonzero for a duration T . For a complex tone, its continuous transform contains

$$\int_0^T e^{2\pi i(f_0 - f)t} dt = T e^{i\pi(f_0 - f)T} \text{sinc}[(f_0 - f)T],$$

where $\text{sinc } y = \sin(\pi y)/(\pi y)$. An off-bin tone is sampled at DFT frequencies that do not coincide with the sinc zeros, so neighbouring coefficients are generally nonzero. The discrete counterpart is a finite geometric sum with the same reciprocal-duration scale. Increasing T moves the first zeros to $f_0 \pm 1/T$ and halves their separation when T doubles. A taper reduces sidelobes while changing the main lobe; zero-padding adds display points but does not replace physical observation time.

Problem 37: an explicitly stipulated open-end correction

The uncorrected prediction is

$$f_{\text{ideal}} = \frac{343}{4(0.500)} = 171.5 \text{ Hz}.$$

The prescribed model gives $\delta = 0.60(0.010) = 0.0060$ m and $L_{\text{eff}} = 0.5060$ m, so

$$f_{\text{corr}} = \frac{343}{4(0.5060)} = 169.4664 \text{ Hz}.$$

Relative to the uncorrected result, the percentage shift is $100(f_{\text{corr}}/f_{\text{ideal}} - 1) = -1.18577\% \simeq -1.19\%$. The negative sign is required: a longer effective length lowers frequency. The coefficient 0.60 is stipulated for this problem and is not a universal rule for arbitrary openings.

Problem 38: invert an exact source Doppler shift

Choose $+x$ from the source toward the stationary observer ahead. Then $v_o = 0$ and the approaching source has $v_s > 0$. From

$$900 = 850 \frac{340}{340 - v_s}$$

we obtain

$$v_s = 340 \left(1 - \frac{850}{900} \right) = 18.8889 \text{ m s}^{-1}.$$

Substitution recovers the measured 900 Hz. The result is subsonic and has the correct sign for an upward shift. Using only the first-order approximation would produce a slightly different estimate; the exact one-dimensional model was requested.

Problem 39: the Mach envelope is tangent

At $t = 2$, the current source is $S = (4, 0)$. For any prior emission time $t_e < 2$, a front has centre $C = (2t_e, 0)$ and radius $R = 2 - t_e$. The upper and lower cone lines extend backward from S , with equations

$$y = \pm \frac{4 - x}{\sqrt{3}}.$$

Their angle to the backward axis is 30° . The perpendicular distance from C to either line is

$$d = |SC| \sin 30^\circ = (4 - 2t_e)/2 = 2 - t_e = R.$$

A line at exactly one radius from a circle's centre is tangent, proving the claim for every prior front. For the upper line, the perpendicular foot is $Q = (1 + 1.5t_e, \sqrt{3}(1 - 0.5t_e))$; it lies on the backward ray for $t_e < 2$. At $t_e = 2$ the circle degenerates to the source point, so ordinary nonzero-radius tangency is understood as a limit.

Problem 40: actual sample counts, duration, and calibration

The sampling rate is 4096 Hz. The exact short sample count gives

$$T_{\text{short}} = 410/4096 = 0.10009765625 \text{ s}, \quad \Delta f_{\text{short}} = 4096/410 = 9.99024390 \text{ Hz}.$$

The long record has $T_{\text{long}} = 4096/4096 = 1.000 \text{ s}$ and $\Delta f_{\text{long}} = 1.000 \text{ Hz}$. Both Nyquist frequencies are 2048 Hz. The short DFT duration is not exactly 0.10 s; its first-to-last sample span is $(N - 1)/f_s$, a different quantity from the DFT's N/f_s convention.

Zero-padding to a larger transform size samples the finite record's spectral response on a denser displayed grid. It can improve peak interpolation, but it preserves the short record's window broadening and does not supply the extra acquired cycles of a one-second record. Dividing by a spectrum's maximum removes its absolute scale. Without the microphone calibration and the original normalization, a unit-height peak cannot determine p_{rms} , intensity, or an absolute acoustic level.

Problem 41: infer pressure from a displacement graph

(a) Convert the graph's micrometres and centimetres before finding slopes. On successive intervals $(0, 2)$, $(2, 5)$, $(5, 7)$ and $(7, 9)$ cm,

$$\partial_x \xi = \frac{0.40 \times 10^{-6}}{0.020}, \quad 0, \quad \frac{-0.60 \times 10^{-6}}{0.020}, \quad \frac{0.20 \times 10^{-6}}{0.020} = 2.0 \times 10^{-5}, \quad 0, \quad -3.0 \times 10^{-5}, \quad 1.0 \times 10^{-5}.$$

Multiplying each by $-B$ gives $p' = -2.80, 0, +4.20, -1.40 \text{ Pa}$, respectively. A pressure sketch therefore has four horizontal segments, with undefined ideal derivatives at the sharp corners. In a smooth physical field those transitions would be rounded.

(b) Throughout $0 < x < 2$ cm the displacement is positive but increases with position, so neighbouring fluid elements are farther apart and the pressure perturbation is negative. The sign of displacement does not encode compression; its gradient does.

(c) The signed area of the pressure graph is

$$\int_0^{0.09 \text{ m}} p' dx = (-2.80)(0.020) + (0)(0.030) + (4.20)(0.020) + (-1.40)(0.020) = 0 \text{ Pa m}.$$

It also equals $-B[\xi(0.09 \text{ m}) - \xi(0)] = 0$, since both endpoint displacements vanish. This integral agreement independently checks the slopes and signs. A single spatial snapshot lacks time evolution or particle velocity and cannot distinguish rightward propagation, leftward propagation, or a superposition.

Problem 42: local standing-wave pressure versus transported flux

(a) Pressure nodes occur where $\cos kx = 0$: $x/\lambda = 1/4, 3/4$. Velocity nodes occur where $\sin kx = 0$: $x/\lambda = 0, 1/2, 1$. The locations alternate, as Fig. 17.36 shows. A node of one acoustic variable is an antinode of the other.

(b) At $x = \lambda/8$, $kx = \pi/4$, so $p_{\text{amp}} = \sqrt{2}p_a$ and $u_{\text{amp}} = \sqrt{2}p_a/Z$. Multiplying the stated fields gives

$$j(x, t) = \frac{4p_a^2}{Z} \cos kx \sin kx \cos \omega t \sin \omega t = \frac{p_a^2}{Z} \sin(2kx) \sin(2\omega t).$$

At the requested position this becomes $j = (p_a^2/Z) \sin 2\omega t$, alternating between positive and negative flux. Its average over any integer number of periods is zero. Energy moves locally back and forth while equal incident and reflected mean fluxes cancel.

(c) Here $p_{\text{rms}} = p_{\text{amp}}/\sqrt{2} = p_a$, so $p_{\text{rms}}^2/Z = p_a^2/Z = 2I_a$. That positive pressure-equivalent value is not the zero mean net flux. The two arriving travelling components each carry mean intensity I_a in opposite directions. This explicit multiplication verifies why a pressure reading alone is insufficient in a standing field.

Problem 43: coherent interference with unequal amplitudes

(a) Add pressure phasors with component 1 as the phase origin:

$$\tilde{p}_R = p_1(1 + 0.60e^{i\phi}), \quad \frac{p_R}{p_1} = |1 + 0.60e^{i\phi}| = \sqrt{1.36 + 1.20 \cos \phi}.$$

Since the waves share impedance and propagation direction, $I_R/I_1 = (p_R/p_1)^2 = 1.36 + 1.20 \cos \phi$. The square root, rather than the intensity ratio itself, is the plotted curve.

(b) Constructive addition at $\phi = 0$ gives $p_R/p_1 = 1.60$ and $I_R/I_1 = 2.56$, a level increment $10 \log_{10} 2.56 = +4.0824$ dB. Destructive addition at $\phi = 180^\circ$ gives $p_R/p_1 = 0.40$ and $I_R/I_1 = 0.16$, a change $10 \log_{10} 0.16 = -7.9588$ dB. The plot's extrema agree with the triangle inequality: the resultant amplitude lies between the amplitude difference and sum.

(c) Requiring $1.36 + 1.20 \cos \phi = 1$ gives $\cos \phi = -0.30$. Within the specified interval, $\phi = 107.4576^\circ$ or 252.5424° . Complete cancellation would require equal amplitude as well as opposite phase. Here the stronger component retains 40% of its pressure amplitude even at the destructive minimum.

Problem 44: read a mode's nodes and infer its frequency

(a) The left pressure antinode is consistent with a rigid closed end; the right pressure node is consistent with an ideal open end. Pressure nodes occur at $x/L = 1/3$ and 1, and pressure antinodes at 0 and $2/3$.

(b) The graph matches $\cos(3\pi x/2L)$, so $kL = 3\pi/2$. Because $k = 2\pi/\lambda$, $L = 3\lambda/4$ and $\lambda = 4L/3 = 1.000$ m. The frequency is $f = c/\lambda = 340$ Hz. It is the second allowed resonance of a closed–open tube, namely its third harmonic. The fundamental would be $c/(4L) = 113.333$ Hz.

(c) A compatible signed displacement shape is $-\sin(3\pi x/2L)$: its nodes lie at 0 and $2L/3$, and its antinodes at $L/3$ and L . Differentiation gives the required cosine pressure shape. More precisely, if the plotted signed pressure at the chosen instant is $P \cos kx$ with $P > 0$, the corresponding displacement is $-P \sin kx/(Bk)$, up to the physical endpoint condition. A snapshot fixes these relative spatial signs but cannot fix particle velocity or the future temporal phase: a positive pressure at that instant does not reveal whether pressure is rising or falling. The middle panel of Fig. 17.21 supplies the requested compatible signed pressure/displacement sketch.

Problem 45: reception times from wavefront trajectories

(a) The source emits every $T_s = 1/100 = 0.0100$ s. For $t_e = 0, 0.0100, 0.0200$ s, its positions are 0, 0.400, 0.800 m. Each front travels at $c = 340$ m s⁻¹ after its own emission:

$$x(t; t_e) = v_s t_e + c(t - t_e) = 340t - 300t_e, \quad t \geq t_e.$$

Thus the three trajectories are $340t$, $340t - 3.00$, and $340t - 6.00$, with metres and seconds understood. Their common slope is the medium's sound speed, not the source speed.

(b) Setting $x = 5.00$ m gives $t_a = (5 + 300t_e)/340$. The first three reception times are 0.0147059, 0.0235294, and 0.0323529 s. Their equal separation is $3/340 = 0.00882353$ s, so $f_o = 113.333$ Hz. These intersections are visible on the stationary observer line in Fig. 17.39. Equivalently, emission shortens the forward wavelength to $(340 - 40)/100 = 3.00$ m.

(c) For the moving observer, set $340t - 300t_e = 5 + 10t$, giving $t_a = (5 + 300t_e)/330$. Consecutive arrivals are separated by $3/330 = 0.00909091$ s and $f_o = 110.0$ Hz. The observer's slope is $+10$ m s⁻¹, the source's is $+40$ m s⁻¹, and every emitted front has slope $+340$ m s⁻¹. Motion away from the source reduces encounter rate, partly offsetting the approaching-source increase.

Problem 46: calibration, residuals, and propagated uncertainty

(a) For equal absolute pressure uncertainties, minimize $Q(K) = \sum_i (p_i - K S_i)^2$. Differentiation gives $\partial Q/\partial K = -2 \sum S_i (p_i - K S_i) = 0$ and hence

$$K = \frac{\sum_i S_i p_i}{\sum_i S_i^2} = \frac{0.001081 \text{ Pa V}}{0.000054 \text{ V}^2} = 20.0185185 \text{ Pa V}^{-1}.$$

The voltage units must be volts rather than millivolts in this evaluation. The four residuals $p_i - K S_i$ are -0.00203704 , $+0.00194444$, $+0.00092593$, and -0.00109259 Pa. They need not sum to zero for a through-origin fit; the least-squares condition is $\sum S_i r_i = 0$. A zero-input measurement, and preferably low-signal measurements checked with a free-intercept model, are needed to justify negligible offset. Four scattered residuals alone do not establish it.

(b) The new pressure is $p = 20.0185185(0.00450) = 0.09008333$ Pa. For the stipulated progressive field,

$$I = \frac{p^2}{\rho c} = \frac{0.09008333^2}{411.6} = 1.971576 \times 10^{-5} \text{ W m}^{-2}, \quad L_I = 72.9481 \text{ dB}.$$

The intensity is an inference from calibration and the local wave model, rather than an independent measurement of power flow.

(c) The independent pressure uncertainty is $\sigma_p/p = \sqrt{0.04^2 + 0.03^2} = 0.0500 = 5.00\%$. For $I = K^2 S^2/(\rho c)$, logarithmic differentiation gives

$$\frac{\sigma_I}{I} = \sqrt{(2 \times 0.04)^2 + (2 \times 0.03)^2 + 0.02^2 + 0.01^2} = 0.102470 = 10.2470\%.$$

Finally, $\sigma_L \simeq (10/\ln 10)(\sigma_I/I) = 0.4450$ dB. The absolute pressure and intensity standard uncertainties are about 0.00450 Pa and $2.02 \times 10^{-6} \text{ W m}^{-2}$. This small-uncertainty propagation uses the supplied calibration uncertainty and independence assumptions; it does not estimate calibration quality from an undersized synthetic data set.

Problem 47: separate speed from an effective-length correction

(a) Invert the resonance model:

$$\frac{1}{f_1} = \frac{4}{c}L + \frac{4\delta}{c} = mL + b, \quad c = \frac{4}{m}, \quad \delta = \frac{b}{m}.$$

Equal-weight linear fitting to the displayed rounded values gives $m = 0.0115942030 \text{ s m}^{-1}$ and $b = 0.0000869565 \text{ s}$. Therefore $c = 344.999997 \text{ m s}^{-1} \simeq 345.0 \text{ m s}^{-1}$ and $\delta = 0.007499998 \text{ m} \simeq 7.50 \text{ mm}$. The tabulated $1/f_1$ values have a constant increment of 0.0011594203 s for every 0.100 m , independently checking the slope. Their final digit is rounded; extra fitted digits are arithmetic diagnostics, not experimental precision.

(b) With the given radius, $\delta/r = 0.00750/0.0125 = 0.600$. This ratio describes the particular synthetic model. A universal law cannot be established by several lengths sharing one opening and radius; end corrections may depend on aperture geometry, flanging and frequency. Varying those controls is needed to assess a broader model.

(c) At $L = 0.500 \text{ m}$, the fitted model gives $f_1 = 345/[4(0.5075)] = 169.95074 \text{ Hz}$. Ignoring the correction would infer

$$c_{\text{naive}} = 4Lf_1 = 339.90148 \text{ m s}^{-1}, \quad 100 \left(\frac{c_{\text{naive}}}{c} - 1 \right) = 100 \left(\frac{L}{L + \delta} - 1 \right) = -1.47783\%.$$

The bias is downward because the physical length underestimates the acoustic length. In a real investigation, curvature or systematic residual trends in a plot of $1/f_1$ against L would challenge constant c and constant δ . Repeated resonance families or controlled temperature readings could help distinguish boundary-model variation from medium-state variation.

Problem 48: transform normalization and Parseval's identity

(a) Here $T = N/f_s = 1.00 \text{ s}$, so the bin spacing is 1.00 Hz . The positive-frequency tones occupy $k = 1$ and 2 , giving $f = 1$ and 2 Hz . Under the stated DFT convention an interior sinusoid of peak amplitude A gives $|X_k| = NA/2$, so

$$A_1 = \frac{2(1.20)}{8} = 0.300 \text{ Pa}, \quad A_2 = \frac{2(0.80)}{8} = 0.200 \text{ Pa}.$$

The nonzero complex coefficients are $X_1 = X_7 = 1.20 \text{ Pa}$, $X_2 = -0.80i \text{ Pa}$ and $X_6 = +0.80i \text{ Pa}$. Reality of p_n gives $X_{N-k} = X_k^*$; bins 6, 7 represent the negative-frequency partners, not additional physical tones. The factor of two is used for interior one-sided bins, not for DC or the Nyquist bin.

(b) Orthogonality over this exact periodic record makes the cross term vanish, so

$$p_{\text{rms}}^2 = \frac{0.300^2 + 0.200^2}{2} = 0.0650 \text{ Pa}^2, \quad p_{\text{rms}} = 0.254951 \text{ Pa}.$$

With an unnormalized forward DFT, Parseval reads $\sum_n |p_n|^2 = N^{-1} \sum_k |X_k|^2$. Dividing again by N gives

$$p_{\text{rms}}^2 = \frac{1}{N^2} \sum_k |X_k|^2 = \frac{2(1.20)^2 + 2(0.80)^2}{8^2} = \frac{4.16}{64} = 0.0650 \text{ Pa}^2.$$

The agreement independently checks the DFT normalization and mirrored-bin counting.

(c) With the stated local progressive-wave assumption, $u = p'/Z$ for the combined field and $Z = 411.6 \text{ Pa s m}^{-1}$. Thus

$$I = \frac{0.0650}{411.6} = 1.5792031 \times 10^{-4} \text{ W m}^{-2}, \quad L_I = 10 \log_{10}(I/I_0) = 81.9844 \text{ dB}.$$

Normalizing the graph by 1.20 Pa while forgetting that scale would retain the tone frequencies and relative amplitudes, but lose absolute pressure, RMS pressure, intensity and absolute level. The benchmark verifies mathematical bookkeeping; its low sampling rate does not imply any claim about actual microphone bandwidth or audible sound.

17.13 Bridge to Chapter 18

Sound has connected wave motion to compression, pressure, and energy transfer. The medium has been represented through mechanical properties such as density and the appropriate bulk modulus. Chapter 18, *Temperature, Heat, and the First Law of Thermodynamics*, develops temperature, thermal expansion, heat transfer, internal energy, and the energy balance of a thermodynamic system. Chapter 19 then explains how a gas equation of state and molecular motion determine its acoustic response. These connections supply the thermodynamic reason behind choosing a modulus appropriate to the acoustic timescale.

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